FAVOURABLE MODULES: FILTRATIONS, POLYTOPES, NEWTON-OKOUNKOV BODIES AND FLAT DEGENERATIONS

EVGENY FEIGIN, GHISLAIN FOURIER AND PETER LITTELMANN

ABSTRACT. We introduce the notion of a favourable module for a complex unipotent algebraic group, whose properties are governed by the combinatorics of an associated polytope. We describe two filtrations of the module, one given by the total degree on the PBW basis of the corresponding Lie algebra, the other by fixing a homogeneous monomial order on the PBW basis.

In the favourable case a basis of the module is parametrized by the lattice points of a normal polytope. The filtrations induce flat degenerations of the corresponding flag variety to its abelianized version and to a toric variety, the special fibres of the degenerations being projectively normal and arithmetically Cohen-Macaulay. The polytope itself can be recovered as a Newton-Okounkov body. We conclude the paper by giving classes of examples for favourable modules.

INTRODUCTION

Let $\mathbb{U}$ be a complex algebraic unipotent group acting on a cyclic finite dimensional complex vector space M, so for the nilpotent Lie algebra $\mathfrak{N} = \operatorname{Lie}\mathbb{U}$ and a cyclic vector v_M we have $M = \mathrm{U}(\mathfrak{N})v_M$. A well known example we have in mind is a maximal unipotent subgroup $\mathbb{U}$ of a reductive algebraic group G acting on an irreducible finite dimensional representation of G.

We call such a module M *favourable* if important properties of the module are governed by polytope combinatorics. More precisely, starting with an ordered basis $\{f_1, \dots, f_N\}$ of $\mathfrak{N}$ and an induced homogeneous monomial ordering "$\leq$" on the monomials in $U(\mathfrak{N})$, consider the induced filtration of M defined by

$$M_{\mathbf{q}} = \mathbb{C}\text{-span of } \{\mathbf{f}^{\mathbf{p}} v_M = f_1^{p_1} \dots f_N^{p_N} v_M \text{ for all } \mathbf{f}^{\mathbf{p}} \leq \mathbf{f}^{\mathbf{q}}\}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{N}^N$ are multi-indices. In the associated graded module $\operatorname{gr}^t M$ every homogeneous component $\operatorname{gr}^t M(\mathbf{p})$ is at most one-dimensional. Following Vinberg, we call a monomial $\mathbf{f}^{\mathbf{p}} \in U(\mathfrak{N})$ *essential for* M if $\operatorname{gr}^t M(\mathbf{p})$ is non-zero, the exponent $\mathbf{p}$ is called an essential multi-index for M. The set $\operatorname{es}(M)$ of all essential multi-indices is a finite subset of $\mathbb{Z}^N$.

The first condition for M to be favourable is that there exists a normal lattice polytope $P(M) \subset \mathbb{R}^N$ such that the lattice points $S(M)$ in $P(M)$ are exactly $\operatorname{es}(M)$.

Recall the definition of the Cartan component in the n-fold tensor product of M:

$$M^{\odot n} = \mathrm{U}(\mathfrak{N})(v_M \otimes \dots \otimes v_M) \subseteq M^{\otimes n}.$$

The second condition is: $\dim M^{\odot n} = \sharp nS(M)$ for all $n \in \mathbb{N}$, it concerns the comparison of the number of points in the Minkowski sum $nS(M)$ with the dimension of $M^{\odot n}$.

Recall the PBW-filtration of $\mathrm{U}(\mathfrak{N})$ given by the span of all monomials up to a fixed total degree. Since M is cyclic, we get a natural induced filtration on M, which is coarser than the filtration above. The associated graded space is denoted by $\mathrm{gr}^a M$. Note that $\mathrm{gr}^a M$ and $\mathrm{gr}^t M$ are not anymore $U(\mathfrak{N})$-modules, but they are cyclic modules with generator v_{M^a} respectively v_{M^t} for the commutative algebra $U(\mathfrak{N}^a)$. Here $\mathfrak{N}^a$ is the abelian Lie algebra with the same underlying vector space as $\mathfrak{N}$. Similarly on the group level, we have a commutative unipotent group $\mathbb{U}^a$ with Lie algebra $\mathfrak{N}^a$ acting on $\mathrm{gr}^a M$ and $\mathrm{gr}^t M$.

Main Theorem. *Let M be a favourable $\mathbb{U}$-module and let $P(M)$ be the associated normal lattice polytope.*

(i) *Let $S(M)$ be the lattice points in $P(M)$. The set $\{\mathbf{f}^{\mathbf{p}} v_M^{\otimes n} \mid \mathbf{p} \in nS(M)\}$ is a basis for $M^{\odot n}$ as well as for its graded versions $gr^a M^{\odot n}$ and $gr^t M^{\odot n}$.*

To an action of a unipotent group, we associate a projective variety, which we call a flag variety by analogy with the classical highest weight orbits of reductive groups:

$$\mathcal{F}_{\mathbb{U}}(M) = \overline{\mathbb{U}.[v_M]} \subseteq \mathbb{P}(M), \quad \mathcal{F}_{\mathbb{U}^a}(\mathrm{gr}^a M) = \overline{\mathbb{U}^a.[v_{M^a}]} \subseteq \mathbb{P}(\mathrm{gr}^a M)$$

respectively $\mathcal{F}_{\mathbb{U}^a}(\mathrm{gr}^t M) = \overline{\mathbb{U}^a.[v_{M^t}]} \subset \mathbb{P}(\mathrm{gr}^t M)$.

Main Theorem. *Let M be a favourable $\mathbb{U}$-module and let $P(M)$ be the associated normal lattice polytope.*

(ii) *$\mathcal{F}_{\mathbb{U}^a}(gr^t M) \subseteq \mathbb{P}(gr^t M)$ is the toric variety defined by the polytope $P(M)$.*

(iii) *There exists a flat degeneration of $\mathcal{F}_{\mathbb{U}}(M)$ into $\mathcal{F}_{\mathbb{U}^a}(gr^a M)$, and for both there exists a flat degeneration into $\mathcal{F}_{\mathbb{U}^a}(gr^t M)$.*

(iv) *The projective flag varieties $\mathcal{F}_{\mathbb{U}}(M) \subseteq \mathbb{P}(M)$, $\mathcal{F}_{\mathbb{U}^a}(gr^a M) \subseteq \mathbb{P}(gr^a M)$ and its toric version $\mathcal{F}_{\mathbb{U}^a}(gr^t M) \subseteq \mathbb{P}(gr^t M)$ are projectively normal and arithmetically Cohen-Macaulay varieties.*

By construction, $\mathbb{U}$ as well as its abelianized version $\mathbb{U}^a$ has a dense and open orbit in the corresponding flag varieties. For a projective variety X (and a fixed valuation ν of its function field), let $\Delta(X)$ denote its corresponding Newton-Okounkov body (for details see Section 9).

Main Theorem. *Let M be a favourable $\mathbb{U}$-module and let $P(M)$ be the associated normal lattice polytope. Let v_M be a cyclic generator and let $\mathbb{V} \subset \mathbb{U}$ be the stabilizer of $[v_M] \in \mathbb{P}(M)$. If $\mathbb{U}$ satisfies some mild conditions (see Section 9) and M is favourable, then we have for the Newton-Okounkov bodies:*

(v) *The polytope $P(M)$ is the Newton-Okounkov body for the flag variety, its abelianized version and its toric version, i.e.*

$$\Delta(\mathcal{F}_{\mathbb{U}}(M)) = P(M) = \Delta(\mathcal{F}_{\mathbb{U}^a}(gr^a M)).$$

Our main example and motivation for the study of these polytopes is our ongoing research on PBW-filtrations and the associated degenerate flag varieties for the classical algebraic groups. Let G be a simply connected simple algebraic group with Lie algebra $\mathfrak{g}$. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$, and let $\mathbb{U}$ be the maximal unipotent subgroup of G with Lie algebra $\mathfrak{N} = \mathfrak{n}^-$. As an immediate consequence of the results in [FFoL1, FFoL2, G] we see that

Corollary. *For $G = SL_n(\mathbb{C})$, $G = Sp_{2n}(\mathbb{C})$ and $G = G_2$, there exists an ordering of the positive roots and a homogeneous ordering on the monomials in $U(\mathfrak{N})$ such that all irreducible finite dimensional G-modules are favourable for $\mathbb{U}$ with a highest weight vector as cyclic generator.*

The projective normality and the Cohen-Macaulay property of the flag variety $\mathcal{F}_{\mathbb{U}^a}(\operatorname{gr}^a M)$ were proved in [FF] and [FFiL] for $G = SL_n(\mathbb{C}), Sp_{2n}(\mathbb{C})$ using an explicit desingularization; the same result can be derived via the realization of the degenerate flag varieties in types A and C as Schubert varieties [CL], [CLL]. For an explicit description of the corresponding polytopes and more examples see Section 11, where we also discuss the relation to Gelfand-Tsetlin and other string polytopes.

In Section 1 we describe the coordinate ring of the flag varieties. In Section 2 we recall some generalities about filtrations and introduce the fundamental notions, in Section 3 we discuss the connection between filtrations and degenerations of flag varieties to toric varieties. In Section 4 we make a link to the toric geometry and in Sections 5 and 6 we study the homogeneous coordinate rings. In Section 7 we introduce the notion of a favourable module and prove some first properties, in Section 8 we show part (iv) of the Main Theorem. In Section 9 we show part (v) of the Main Theorem. In Section 10 we discuss a multi-cone version of the Main Theorem. Examples are discussed in Section 11.

1. The coordinate ring of an orbit closure

Let M be a cyclic finite dimensional module for a connected complex algebraic group H. Fix a cyclic generator $m_0 \in M$. In the following we identify the ℓ-th symmetric power $S^\ell(M)$ of M with the symmetric tensors in the ℓ-fold tensor product of M. Another way to identify $S^\ell(M)$ with a subspace of $M^{\otimes \ell}$ is to view $S^\ell(M)$ as the linear span $\langle GL(M).m_0^{\otimes \ell}\rangle \subset M^{\otimes \ell}$ of the orbit $GL(M).m_0^{\otimes \ell}$.

By the Cartan component $M^{\odot \ell}$ of the ℓ-th tensor product $M^{\otimes \ell}$ we mean the H-submodule

$$M^{\odot \ell} = \langle H.m_0^{\otimes \ell}\rangle \subseteq \langle GL(M).m_0^{\otimes \ell}\rangle = S^\ell(M) \subseteq M^{\otimes \ell}.$$

Note that $M^{\odot \ell}$ is a subspace of $S^\ell(M)$.

1.1. **The ℓ-tuple embedding.** We have a natural $GL(M)$-equivariant map of degree ℓ:

$$\phi : M \to S^\ell(M) \subset M^{\otimes \ell}, \quad m \mapsto m^{\otimes \ell},$$

which induces the ℓ-tuple embedding:

$$\bar{\phi} : \mathbb{P}(M) \to \mathbb{P}(S^\ell(M)), \quad [m] \mapsto [m^{\otimes \ell}].$$

Next consider the varieties $X_1 := \overline{H.[m_0]} \subseteq \mathbb{P}(M)$ and $X_\ell := \overline{H.[m_0^{\otimes \ell}]} \subseteq \mathbb{P}(S^\ell(M))$. The map $\bar{\phi}$ is $GL(M)$-equivariant and an isomorphism onto the image, so

$$\bar{\phi}(X_1) = \bar{\phi}(\overline{H.[m_0]}) = \overline{H.[m_0]^{\otimes \ell}} = X_\ell.$$

The orbit closure X_ℓ is nothing but the image of X_1 with respect to the ℓ-tuple embedding. Let $\hat{X}_\ell \subseteq S^\ell(M)$ be the affine cone over X_ℓ, then

$$M^{\odot \ell} = \langle \hat{X}_\ell \rangle \subseteq S^\ell(M).$$

1.2. **Coordinate rings.** Let $\mathbb{C}[M]$ be the ring of polynomial functions. Endowed with the standard grading $\mathbb{C}[M] = \bigoplus_{p\geq 0}\mathbb{C}[M]_p$ we view the ring also as the homogeneous coordinate ring of $\mathbb{P}(M)$.

Similarly, let $\mathbb{C}[X_1]$ be the coordinate ring of the affine cone $\hat{X}_1$ over X_1. Endowed with the standard grading $\mathbb{C}[X_1] = \bigoplus_{p\geq 0}\mathbb{C}[X_1]_p$ we view the ring as the homogeneous coordinate ring of the embedded variety $X_1 \hookrightarrow \mathbb{P}(M)$.

We let $R_\ell(M) = (M^{\odot \ell})^*$ be the dual space. The vector space

$$R(M) = \bigoplus_{\ell\geq 0} R_\ell(M) \tag{1.1}$$

is then naturally endowed with a ring structure, where the multiplication maps $R_n \otimes R_k \to R_{n+k}$ are induced by the embeddings

$$M^{\odot(n+k)} \hookrightarrow M^{\odot n}\otimes M^{\odot k}.$$

Proposition 1.1. $\mathbb{C}[X_1] \simeq R(M)$.

Proof. Recall that the ℓ-tuple embedding induces an isomorphism:

$$\phi^*_{1,\ell} : \mathbb{C}[S^\ell(M)]_1 \to \mathbb{C}[M]_\ell.$$

Combining $\phi^*_{1,\ell}$ with the restriction homomorphisms induced by the embeddings $X_1 \subset \mathbb{P}(M)$ and $X_\ell \subset \mathbb{P}(S^\ell(M))$, we get the following commutative diagram, for which the rows are isomorphisms and the down arrows are epimorphisms:

$$\begin{array}{ccc} \phi^* : \mathbb{C}[S^\ell(M)]_1 & \to & \mathbb{C}[M]_\ell \\ \downarrow \mathrm{res}_{X_\ell} & & \downarrow \mathrm{res}_{X_1} \\ \phi^* : \mathbb{C}[X_\ell]_1 & \to & \mathbb{C}[X_1]_\ell \end{array}, \qquad \begin{array}{ccc} f & \mapsto & f\circ\phi \\ \downarrow & & \downarrow \\ f|_{X_\ell} & \mapsto & (f\circ\phi)|_{X_1} = f|_{X_\ell}\circ\phi|_{X_1}. \end{array}$$

The restriction of a linear function on $S^\ell(M)$ to the affine cone $\hat{X}_\ell$ vanishes if and only if it vanishes on the linear span $\langle \hat{X}_\ell\rangle = M^{\odot\ell}$, and hence $\mathbb{C}[X_\ell]_1 = (M^{\odot\ell})^*$. It follows that

$$\mathbb{C}[X_1] = \bigoplus_{\ell\geq 0}\mathbb{C}[X_1]_\ell = \bigoplus_{\ell\geq 0}(M^{\odot\ell})^* = \bigoplus_{n\geq 0} R_n(M) = R(M).$$

□

2. Representations and filtrations

2.1. **PBW filtration.** Let $\mathbb{U}$ be a complex algebraic unipotent group acting on a cyclic finite dimensional complex vector space M, so for the nilpotent Lie algebra $\mathfrak{N} = \operatorname{Lie}\mathbb{U}$ and a cyclic vector v_M we have $M = \mathrm{U}(\mathfrak{N})v_M$. The PBW filtration of $\mathrm{U}(\mathfrak{N})$ is defined by $\mathrm{U}(\mathfrak{N})_s = \mathrm{span}\{x_1\dots x_l,\ l\leq s, x_i\in\mathfrak{N}\}$, the associated graded algebra is the symmetric algebra $S(\mathfrak{N})$ over $\mathfrak{N}$. Let $\mathfrak{N}^a$ be the same vector space as $\mathfrak{N}$ but endowed with the trivial Lie bracket. Then $S(\mathfrak{N}) = U(\mathfrak{N}^a)$ is the enveloping algebra of the abelianized version $\mathfrak{N}^a$ of $\mathfrak{N}$. The increasing filtration

$$\mathrm{U}(\mathfrak{N})_0 = \mathbb{C}1 \subseteq \mathrm{U}(\mathfrak{N})_1 \subseteq \mathrm{U}(\mathfrak{N})_2 \subseteq \dots$$

of $\mathrm{U}(\mathfrak{N})$ defines an induced increasing filtration on M

$$PF_0(M) = \mathbb{C}v_M \subseteq PF_1(M) \subseteq PF_2(M) \subseteq \dots,$$

where

$$PF_s(M) = \mathrm{U}(\mathfrak{N})_s v_M = \mathrm{span}\{x_1 \dots x_l v_M,\ l \le s, x_i \in \mathfrak{N}\}.$$

The *associated graded module*

$$M^a = \bigoplus_{s\ge 0} PF_s(M)/PF_{s-1}(M)$$

is naturally endowed with the structure of a graded $U(\mathfrak{N}^a)$-module, each element $x \in \mathfrak{N} \setminus \{0\}$ induces an operator of degree 1 on M^a. We denote by v_{M^a} the image of the cyclic generator v_M in M^a. It is clear that

Proposition 2.1. *M^a is a cyclic $U(\mathfrak{N}^a)$-module with v_{M^a} as a generator.*

Remark 2.2. The construction of the PBW-degeneration is non-trivial even if the initial algebra $\mathfrak{N}$ is abelian. In fact, the $U(\mathfrak{N}^a)$-module M^a is graded by non-negative integers and each non-trivial operator from $\mathfrak{N}$ has degree one. So if the initial $\mathfrak{N}$-module M is not graded, the modules M^a and M are not isomorphic.

2.2. **Essential monomials.** In this section we follow the approach due to Vinberg (see [V], [G]). Let $\mathbb{U}, \mathfrak{N}, M$ and v_M be as above. We fix an ordered basis of $\mathfrak{N} \supset \{f_1 > \dots > f_N\}$ and let "$>$" be an induced homogeneous monomial order (for example the homogeneous reverse lexicographic order, the homogeneous lexicographic order, ...) on the monomials in $\{f_1, \dots, f_N\}$. In other words, when we compare two monomials, we first compare their total degree.

To a collection of non-negative integers p_i, $i = 1, \dots, N$, we attach a vector

$$v_M(\mathbf{p}) = \mathbf{f}^{\mathbf{p}} v_M = f_1^{p_1} \dots f_N^{p_N} v_M \in M.$$

Here and below we denote a multi-exponent $(p_1, \dots, p_N)$ simply by $\mathbf{p}$, for example, $v_M = v_M(\mathbf{0})$. The degree of $\mathbf{f}^{\mathbf{p}}$ is denoted by $|\mathbf{p}| = p_1 + \dots + p_N$. The sum of multi-exponents $\mathbf{p} + \mathbf{q} = (p_1 + q_1, \dots, p_N + q_N)$ is defined componentwise. Since we have a monomial order, $\mathbf{p} \ge \mathbf{q}$ and $\mathbf{p}' \ge \mathbf{q}'$ implies $\mathbf{p} + \mathbf{p}' \ge \mathbf{q} + \mathbf{q}'$. Here and below we write $\mathbf{p} \ge \mathbf{q}$ iff $\mathbf{f}^{\mathbf{p}} \ge \mathbf{f}^{\mathbf{q}}$.

Definition 2.3. A pair $(M, \mathbf{p})$ is said to be *essential* if

$$v_M(\mathbf{p}) \notin \mathrm{span}\{v_M(\mathbf{q}) :\ \mathbf{q} < \mathbf{p}\}.$$

If $(M, \mathbf{p})$ is essential, then we say that $\mathbf{p}$ is an essential multi-exponent, $\mathbf{f}^{\mathbf{p}}$ is an essential monomial in M and we call the vector $v_M(\mathbf{p})$ an essential vector.

Definition 2.4. We denote by $\mathrm{es}(M) \subset \mathbb{Z}^N_{\ge 0}$ the set of essential multi-exponents for the module M.

Remark 2.5. Since the chosen order is homogeneous and the PBW filtration is given by the total degree, if $(M, \mathbf{p})$ is essential, then $f^{\mathbf{p}} v_M$ is non-zero in the PBW degenerate representation M^a.

We use sometimes the notation $S[f_1, \dots, f_N]$ for $U(\mathfrak{N}^a)$ if we want to emphasize that we have fixed a vector space basis for $\mathfrak{N}$. We define subspaces $F_{<\mathbf{p}}(M) \subseteq F_{\mathbf{p}}(M) \subseteq M$:

$$F_{<\mathbf{p}}(M) = \mathrm{span}\{v_M(\mathbf{q}) :\ \mathbf{q} < \mathbf{p}\}, \quad F_{\mathbf{p}}(M) = \mathrm{span}\{v_M(\mathbf{q}) :\ \mathbf{q} \le \mathbf{p}\}.$$

These subspaces define an increasing filtration on M, which is finer than the PBW filtration:

$$PF_{|\mathbf{p}|-1}(M) \subseteq F_{\mathbf{p}}(M) \subseteq PF_{|\mathbf{p}|}(M). \tag{2.1}$$

For $i = 1, \dots, N$ let $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the only nonzero entry is at the i-th place. Then one has

$$f_i(F_{\mathbf{p}}(M)/F_{<\mathbf{p}}(M)) \subseteq F_{\mathbf{p}+\mathbf{e}_i}(M)/F_{<\mathbf{p}+\mathbf{e}_i}(M).$$

The inclusion in (2.1) implies

$$f_i(F_{\mathbf{p}}(M)/PF_{|\mathbf{p}|-1}(M)) \subseteq F_{\mathbf{p}+\mathbf{e}_i}(M)/PF_{|\mathbf{p}|}(M). \tag{2.2}$$

By construction we have $F_{\mathbf{p}}(M) \subseteq F_{\mathbf{q}}(M)$ if $\mathbf{p} < \mathbf{q}$. We denote the associated graded space by M^t (t is for toric, this notation will be justified later). The image of v_M in M^t is denoted by v_{M^t}. The space M^t is $\mathbb{Z}^N_{\geq 0}$-graded:

$$M^t = \bigoplus_{\mathbf{p} \in \mathbb{Z}^N_{\geq 0}} M^t(\mathbf{p}), \text{ where } M^t(\mathbf{p}) = F_{\mathbf{p}}(M)/F_{<\mathbf{p}}(M).$$

Proposition 2.6. (i) *The cyclic $U(\mathfrak{N})$-module structure on M induces the structure of a cyclic $U(\mathfrak{N}^a)$-module on M^t.*

(ii) *The annihilating ideal of $v_{M^t} \in M^t$ is a monomial ideal.*

(iii) *Given $\mathbf{p} \in \mathbb{Z}^N_{\geq 0}$, the homogeneous component $M^t(\mathbf{p})$ is at most one-dimensional, and $\dim M^t(\mathbf{p}) = 1$ if and only if $(M, \mathbf{p})$ is essential.*

(iv) *The vectors $v_{M^t}(\mathbf{p})$, $\mathbf{p} \in \mathrm{es}(M)$, form a basis of M^t.*

Proof. Part (i) follows by (2.2). The essential vectors $\{v_M(\mathbf{q}) = \mathbf{f}^{\mathbf{p}} v_M \mid \mathbf{p} \in \mathrm{es}(M)\}$ form a basis of M by construction. Since $\mathbf{f}^{\mathbf{q}} v_{M^t} = v_{M^t}(\mathbf{q}) = 0$ in M^t for $\mathbf{q}$ **not** essential, it follows by dimension reason that $\{v_{M^t}(\mathbf{q}) = \mathbf{f}^{\mathbf{p}} v_M \mid \mathbf{p} \in \mathrm{es}(M)\}$ is a basis of M^t and the $\{\mathbf{f}^{\mathbf{p}} \mid \mathbf{p} \notin \mathrm{es}(M)\}$ form a basis of the annihilating ideal.

Finally, since any two multi-exponents are comparable, the dimension of $M^t(\mathbf{p})$ is at most one, and it is one if and only if $(M, \mathbf{p})$ is essential. □

Remark 2.7. Each operator f_i on M^t is homogeneous with respect to the $\mathbb{Z}^N$-grading and has degree $\mathbf{e}_i$.

The next corollary just summarizes the nice behaviour of the essential vectors with respect to the filtrations:

Corollary 2.8. *The vectors $\{v_M(\mathbf{p}) \mid \mathbf{p} \in \mathrm{es}(M)\} \subset M$ as well as $\{v_{M^a}(\mathbf{p}) \mid \mathbf{p} \in \mathrm{es}(M)\} \subset M^a$, $\{v_{M^t}(\mathbf{p}) \mid \mathbf{p} \in \mathrm{es}(M)\} \subset M^t$ form a basis of the corresponding space.*

Remark 2.9. If $\mathfrak{N}$ is abelian, after fixing a basis $\{f_1, \dots, f_N\}$ we can identify $U(\mathfrak{N})$ with the symmetric algebra $S[f_1, \dots, f_N]$, and a cyclic module M is of the form $S[f_1, \dots, f_N]/I$, where I is the annihilating ideal. The general procedure described above gives a degeneration of the ideal I to a monomial ideal.

Remark 2.10. Since the filtration induced by a homogeneous order is a refinement of the PBW filtration, it is easy to see that even if $\mathfrak{N}$ is not abelian, then $(M^a)^t$ is isomorphic to M^t as $\mathfrak{N}^a$-modules.

For two cyclic $\mathrm{U}(\mathfrak{N})$-modules M_1 and M_2 with cyclic generators $v_{M_i} \in M_i$, $i = 1, 2$, we denote by $M_1 \odot M_2$ the Cartan component in $M_1 \otimes M_2$, i.e.

$$M_1 \odot M_2 = \mathrm{U}(\mathfrak{N})(v_{M_1} \otimes v_{M_2}) \subset M_1 \otimes M_2.$$

Proposition 2.11. *If $(M_1, \mathbf{p})$ and $(M_2, \mathbf{q})$ are essential, then $(M_1 \odot M_2, \mathbf{p} + \mathbf{q})$ is essential as well.*

Proof. We denote v_{M_1} by v_1, v_{M_2} by v_2 and $v_{M_1\odot M_2}$ by v_{12} (we have $v_{12}=v_1\otimes v_2$). Similarly, we set $v_1(\mathbf{p})=v_{M_1}(\mathbf{p})$, $v_2(\mathbf{p})=v_{M_2}(\mathbf{p})$, $v_{12}(\mathbf{p})=v_{M_1\odot M_2}(\mathbf{p})$. We have to show that

$$f_1^{p_1+q_1}\dots f_N^{p_N+q_N}v_{12}\notin \mathrm{span}\{v_{12}(\mathbf{r}),\ \mathbf{r}<\mathbf{p}+\mathbf{q}\}.$$

In fact, note that if $\mathbf{r}<\mathbf{p}+\mathbf{q}$, then

$$\mathbf{f}^{\mathbf{r}}(v_1\otimes v_2)\in M_1\otimes \mathrm{span}\{v_2(\mathbf{q}'):\ \mathbf{q}'<\mathbf{q}\}+\mathrm{span}\{v_1(\mathbf{p}'):\ \mathbf{p}'<\mathbf{p}\}\otimes M_2 \tag{2.3}$$

(acting by a monomial in the f_i's on the tensor product $v_1\otimes v_2$ means we simply distribute the factors among v_1 and v_2). However, $v_{12}(\mathbf{p}+\mathbf{q})$ does not belong to the right hand side of (2.3). To prove this, it suffices to show that

$$f_1^{p_1+q_1}\dots f_N^{p_N+q_N}(v_1\otimes v_2)=C\cdot f_1^{p_1}\dots f_N^{p_N}v_1\otimes f_1^{q_1}\dots f_N^{q_N}v_2+\text{ rest },\tag{2.4}$$

where C is a non-zero constant and the remaining terms *rest* belong to

$$M_1\otimes \mathrm{span}\{v_2(\mathbf{q}'):\ \mathbf{q}'<\mathbf{q}\}+\mathrm{span}\{v_1(\mathbf{p}'):\ \mathbf{p}'<\mathbf{p}\}\otimes M_2.\tag{2.5}$$

Recall that f_i acts as $f_i\otimes 1+1\otimes f_i$. The left hand side of (2.4) is a sum of many terms, among which there are (possibly) several of the following form:

$$f_1^{p_1}\dots f_N^{p_N}v_1\otimes f_1^{q_1}\dots f_N^{q_N}v_2.$$

Note that while distributing f_i as $f_i\otimes 1$ and $1\otimes f_i$ we do not care about the order because we assume the order on the monomials to be homogeneous – and hence we can assume that all f_i's commute because the additional terms coming up during the reordering process are automatically elements of (2.5). Now consider the terms

$$f_1^{p_1'}\dots f_N^{p_N'}v_1\otimes f_1^{q_1'}\dots f_N^{q_N'}v_2,$$

where some p_i' differ from p_i. Then we have necessarily

$$\text{either } (p_1,\dots,p_N)>(p_1',\dots,p_N') \text{ or } (q_1,\dots,q_N)>(q_1',\dots,q_N'),$$

since $\mathbf{p}+\mathbf{q}=\mathbf{p}'+\mathbf{q}'$ (i.e. $p_i+q_i=p_i'+q_i'$ for $i=1,\dots,N$). $\square$

The proposition above implies that the set $\Gamma_{\mathbb{U}}(M):=\bigcup_{n\geq 1}(n,\mathrm{es}(M^{\odot n}))\subset\mathbb{Z}\times\mathbb{Z}^N$ is naturally endowed with the structure of a semigroup.

Definition 2.12. The semigroup $\Gamma_{\mathbb{U}}(M)$ is called the *essential semigroup* of M.

3. Coordinate rings and flag varieties

3.1. **The unipotent case.** Let $\mathbb{U}$ be a complex algebraic unipotent group acting on a cyclic finite dimensional complex vector space M, so for the nilpotent Lie algebra $\mathfrak{N}=\mathrm{Lie}\,\mathbb{U}$ and a cyclic vector v_M we have $M=\mathrm{U}(\mathfrak{N})v_M$. We define the $\mathbb{U}$-flag variety $\mathcal{F}_{\mathbb{U}}(M)$ in $\mathbb{P}(M)$ as the closure of the $\mathbb{U}$-orbit through the line $\mathbb{C}v_M$ inside the projective space $\mathbb{P}(M)$:

$$\mathcal{F}_{\mathbb{U}}(M)=\overline{\mathbb{U}\cdot\mathbb{C}v_M}\subseteq\mathbb{P}(M).\tag{3.1}$$

Example 3.1. Let $M=V(\lambda)$ be a finite dimensional highest weight representation of a simple Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{n}^-\oplus\mathfrak{h}\oplus\mathfrak{n}^+$ be the Cartan decomposition. Let $\mathbb{U}=U^-$ be the maximal unipotent subgroup with Lie algebra $\mathfrak{n}^-$ of the corresponding Lie group G, then the orbit closure $\mathcal{F}_{U^-}(M)$ is the (possibly partial) flag variety G/P_λ, where P_λ is the parabolic subgroup stabilizing the highest weight line.

By Proposition 1.1 we have the following representation theoretic description of the homogeneous coordinate ring of the embedded variety $\mathcal{F}_{\mathbb{U}}(M) \subset \mathbb{P}(M)$:

$$\mathbb{C}[\mathcal{F}_{\mathbb{U}}(M)] = \bigoplus_{\ell \geq 0} \mathbb{C}[\mathcal{F}_{\mathbb{U}}(M)]_\ell = \bigoplus_{\ell \geq 0} (M^{\odot \ell})^* = \bigoplus_{n \geq 0} R_n(M) = R(M).$$

3.2. **Abelianized version.** Instead of starting with $\mathbb{U}, \mathfrak{N}$ and M in Section 3.1 we can start with the abelian Lie algebra $\mathfrak{N}^a$, the module M^a and $\mathbb{U}^a = \exp(\mathfrak{N}^a) \subset \mathrm{GL}(M^a)$, the Lie group associated to $\mathfrak{N}^a$. We call the orbit closure

$$\mathcal{F}_{\mathbb{U}^a}(M^a) = \overline{\mathbb{U}^a \cdot \mathbb{C}v_M} \subseteq \mathbb{P}(M^a)$$

the PBW-degeneration of $\mathcal{F}_{\mathbb{U}}(M)$. We have the following representation theoretic description of the homogeneous coordinate ring of the embedded variety:

$$\mathbb{C}[\mathcal{F}_{\mathbb{U}^a}(M^a)] = R(M^a).$$

3.3. **Toric version.** Instead of starting with $\mathbb{U}, \mathfrak{N}$ and M in Section 3.1 we can start with $\mathbb{U}^a, \mathfrak{N}^a$ and the module M^t. We call the orbit closure

$$\mathcal{F}_{\mathbb{U}^a}(M^t) = \overline{\mathbb{U}^a \cdot \mathbb{C}v_M} \subseteq \mathbb{P}(M^t)$$

the toric degeneration of $\mathcal{F}_{\mathbb{U}}(M)$. Again we have the following representation theoretic description of the homogeneous coordinate ring of the embedded variety:

$$\mathbb{C}[\mathcal{F}_{\mathbb{U}^a}(M^t)] = R(M^t).$$

We are interested to find conditions which ensure that the abelian respectively toric degeneration of $\mathcal{F}_{\mathbb{U}}(M)$ are obtained by a flat degeneration. A first step is to describe the structure of $\mathcal{F}_{\mathbb{U}^a}(M^t)$ as a toric variety.

4. The structure of $\mathcal{F}_{\mathbb{U}^a}(M^t)$ as a toric variety

4.1. **Polytopes.** A *convex lattice polytope* is a polytope P in a Euclidean space $\mathbb{R}^m$ which is the convex hull of finitely many points in the integer lattice $\mathbb{Z}^m \subset \mathbb{R}^m$. A convex lattice polytope P is called *normal* if it has the following property: given any positive integer n, every lattice point of the dilation nP, obtained from P by scaling its vertices by the factor n and taking the convex hull of the resulting points, can be written as the sum of exactly n lattice points in P. Another way of formulating this property is: the set of lattice points in nP is the n-fold Minkowski sum of the lattice points in P (recall that the Minkowski sum of two subsets $A + B$ is the set of all possible sums $a + b$, $a \in A$, $b \in B$).

4.2. **Toric varieties.** Let us fix some notation (see [CLS], Section 2). Let $S \subset \mathbb{Z}^N$ be a finite set, $S = (\mathbf{s}^1, \ldots, \mathbf{s}^k)$. For $\mathbf{z} = (z_1, \ldots, z_N) \in \mathbb{C}^N$ we set $\mathbf{z}^{\mathbf{s}} = \prod_{i=1}^N z_i^{s_i}$. The variety $X(S) \subset \mathbb{P}^{k-1}$ is defined as the closure of the set

$$\{(\mathbf{z}^{\mathbf{s}^1} : \mathbf{z}^{\mathbf{s}^2} : \cdots : \mathbf{z}^{\mathbf{s}^k}) \mid \mathbf{z} \in (\mathbb{C}^*)^N\} \subset \mathbb{P}^{k-1}.$$

The variety $X(S)$ is a toric variety, it admits a dense orbit by the torus $T = (\mathbb{C}^*)^N$, which acts by scaling the variables z_i.

In general the homogeneous coordinate ring of $X(S)$ is the semigroup algebra of the graded semigroup in $\mathbb{N} \times \mathbb{Z}^N$ generated by $\{1\} \times S$. Now assume that S is the set of lattice points inside a normal polytope P. Let us consider the polyhedral cone $C(S)$ consisting of elements of the form $(n, \mathbf{s})$, $\mathbf{s} \in nP$. The set of lattice points in $C(S)$ is equal to the set $(n, \mathbf{s})$, $\mathbf{s} \in nS$. Clearly, this set forms a semigroup. We denote by $R(S)$ the complex group algebra of this semigroup. We have $R(S) = \bigoplus_{n\geq 0} R_n(S)$ and the dimension of $R_n(S)$ is given by the cardinality of nS. The ring $R(S)$ is the homogeneous coordinate ring of the projective variety $X(S)$.

4.3. **The toric degeneration $\mathcal{F}_{\mathbb{U}^a}(M^t)$ of $\mathcal{F}_{\mathbb{U}}(M)$.** We make the same assumptions and we use the same notation as in Section 3. We use the notation $\mathbb{G}_a$ for the one dimensional algebraic group $(\mathbb{C}, +)$. Recall that a commutative unipotent group is isomorphic to a product of several copies of $\mathbb{G}_a$.

Proposition 4.1. *The variety $\mathcal{F}_{\mathbb{U}^a}(M^t)$ is both a $\mathbb{G}_a^N$-variety and a toric variety. The toric variety $\mathcal{F}_{\mathbb{U}^a}(M^t)$ is isomorphic to $X(\mathrm{es}(M))$.*

Proof. We need to prove that there exists a torus acting on $\mathcal{F}_{\mathbb{U}^a}(M^t)$ with an open orbit. Consider the N-dimensional torus $T = (\mathbb{C}^*)^N$ acting on $\mathbb{C}[f_1, \ldots, f_N]$ by

$$(t_1, \ldots, t_N) \cdot f_1^{p_1} \ldots f_N^{p_N} = \big(\prod_{i=1}^N t_i^{p_i}\big)\mathbf{f}^{\mathbf{p}}.$$

Since $M^t = \mathbb{C}[f_1, \ldots, f_N]/I$ for some monomial ideal I, we obtain a T-action on M^t and hence on $\mathbb{P}(M^t)$. We are left to show that T acts on $\mathbb{U} \cdot v_M = \mathbb{G}_a^N \cdot \mathbb{C}v_M$ with an open orbit. Let $\mathrm{es}(M) \subset \mathbb{Z}_{\geq 0}^N$ be the set of essential multi-exponents for M. Then in M^t one has

$$\exp(\sum_{i=1}^N a_i f_i) v_{M^t} = \sum_{\mathbf{p}\in \mathrm{es}(M)} \frac{1}{\prod_{i=1}^N p_i!} a_1^{p_1} \ldots a_N^{p_N} v_{M^t}(\mathbf{p}),\ a_i \in \mathbb{C}.$$

Since $(t_1, \ldots, t_N) \cdot v_{M^t}(\mathbf{p}) = \prod_{i=1}^N t_i^{p_i} v_{M^t}(\mathbf{p})$, we obtain

$$(t_1, \ldots, t_N) \cdot \exp(\sum_{i=1}^N a_i f_i) v_{M^t} = \exp(\sum_{i=1}^N a_i t_i f_i) v_{M^t}.$$

Fix the basis $\{m_{\mathbf{p}} = \frac{1}{\prod_{i=1}^N p_i!} v_{M^t}(\mathbf{p}) \mid \mathbf{p} \in \mathrm{es}(M)\}$ of M^t and let $x_0 \in \mathbb{P}(M)$ be the point $x_0 = [\exp(\sum_{i=1}^N f_i) v_{M^t}]$. Now $\{\exp(\sum_{i=1}^N z_i f_i) \mid z_i \in \mathbb{C}^*\}$ is an open and dense subset of $\mathbb{U}^a \simeq \mathbb{G}_a^N$, and $x_0 \in \mathbb{U}^a[v_{M^t}(\mathbf{p})]$. It follows that

$$\begin{aligned}\mathcal{F}_{\mathbb{U}^a}(M^t) &= \overline{\mathbb{U}^a.[v_{M^t}]}\\ &= \overline{\{\exp(\textstyle\sum_{i=1}^N z_i f_i).[v_{M^t}] \mid z_1, \ldots, z_N \in \mathbb{C}^*\}}\\ &= \overline{\{(z^{\mathbf{p}^1} : z^{\mathbf{p}^2} : \ldots : z^{\mathbf{p}^N}) \mid z = (z_1, \ldots, z_N) \in (\mathbb{C}^*)^N\}},\end{aligned}$$

where $\mathrm{es}(M) = \{\mathbf{p}^1, \ldots \mathbf{p}^N\}$, which proves the proposition. ☐

5. A basis of the homogeneous coordinate ring of $\mathcal{F}_{\mathbb{U}}(M)$

Let $\mathbb{U}$ be a complex algebraic unipotent group with Lie algebra $\mathfrak{N} = \mathrm{Lie}\,\mathbb{U}$. Let $\mathbb{U}$ act on a cyclic finite dimensional complex vector space M with cyclic vector v_M. We consider the basis $\{v_M(\mathbf{p}) \mid \mathbf{p} \in es(M)\}$ of M and denote the elements of the dual basis in M^* by $\{\xi_{\mathbf{p}} \mid \mathbf{p} \in es(M)\}$.

Lemma 5.1. *Let $\mathbf{q} = (q_i)_{i=1}^N$ be a multi-exponent (not necessarily essential). Then for any essential $\mathbf{p}$ such that $\mathbf{q} < \mathbf{p}$ we have $\xi_{\mathbf{p}}(v_M(\mathbf{q})) = 0$.*

Proof. The vector $v_M(\mathbf{q})$ can be expressed as a linear combination of vectors $v_M(\mathbf{q}')$ with $\mathbf{q}'$ essential and $\mathbf{q}' \leq \mathbf{q} < \mathbf{p}$, which proves the lemma. ☐

Recall the description of the homogeneous coordinate ring

$$\mathbb{C}[\mathcal{F}_{\mathbb{U}}(M)] = R(M) = \bigoplus_{n\geq 0} (M^{\odot n})^*.$$

Consider the structure constants $c^{\mathbf{r}}_{\mathbf{p},\mathbf{q}}$, defined by

$$\xi_{\mathbf{p}}\xi_{\mathbf{q}} = \sum_{\mathbf{r}\in \mathrm{es}(M\odot M)} c^{\mathbf{r}}_{\mathbf{p},\mathbf{q}}\xi_{\mathbf{r}}, \quad \mathbf{p},\mathbf{q}\in \mathrm{es}(M).$$

Corollary 5.2. *The structure constant $c^{\mathbf{r}}_{\mathbf{p},\mathbf{q}}$ vanishes if $\mathbf{r} < \mathbf{p}+\mathbf{q}$, but $c^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}$ does not vanish.*

Proof. We have

$$c^{\mathbf{r}}_{\mathbf{p},\mathbf{q}} = (\xi_{\mathbf{p}}\otimes\xi_{\mathbf{q}})(\mathbf{f}^{\mathbf{r}}(v_M\otimes v_M)) = \sum_{\mathbf{r}'+\mathbf{r}''=\mathbf{r}} d_{\mathbf{r}',\mathbf{r}''}\xi_{\mathbf{p}}(v_M(\mathbf{r}'))\xi_{\mathbf{q}}(v_M(\mathbf{r}'')),$$

where $d_{\mathbf{r}',\mathbf{r}''}$ are some nonvanishing constants (multiplicities of the corresponding terms). Now if $\mathbf{r}<\mathbf{p}+\mathbf{q}$, then either $\mathbf{r}'<\mathbf{p}$ or $\mathbf{r}''<\mathbf{q}$ and by Lemma 5.1 we are done. If $\mathbf{r}=\mathbf{p}+\mathbf{q}$ and $\mathbf{r}'\neq\mathbf{p}$, then again either $\mathbf{r}'<\mathbf{p}$ and $\xi_{\mathbf{p}}(v_M(\mathbf{r}'))=0$, or $\mathbf{r}'>\mathbf{p}$ and then $\mathbf{r}''<\mathbf{q}=\mathbf{r}-\mathbf{p}$, so $\xi_{\mathbf{p}}(v_M(\mathbf{r}''))=0$. Hence only the terms with $\mathbf{r}'=\mathbf{p}$, $\mathbf{r}''=\mathbf{q}$ contribute to $c^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}$. □

Lemma 5.3. *We can renormalize $\xi_{\mathbf{p}}$ in such a way that $c^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}=1$ for all essential $\mathbf{p}$ and $\mathbf{q}$.*

Proof. We note that $c^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}} = \prod_{i=1}^N \frac{(p_i+q_i)!}{p_i!q_i!}$. In fact, according to the proof of Corollary 5.2, $c^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}$ is equal to the product over all i of the coefficients of $f_i^{p_i}\otimes f_i^{q_i}$ in $(f_i\otimes 1+1\otimes f_i)^{p_i+q_i}$. Now the desired renormalization is simply $\xi_{\mathbf{p}}\to\xi_{\mathbf{p}}\prod_{i=1}^N p_i!$. □

6. Abelianization and tensor product

Let $\mathbb{U}$ be a complex algebraic unipotent group with Lie algebra $\mathfrak{N}=\mathrm{Lie}\,\mathbb{U}$. Let $\mathbb{U}$ act on two cyclic finite dimensional complex vector spaces V and W.

Lemma 6.1. *There exists a surjective homomorphism of $\mathfrak{N}^a$ modules*

$$(V\odot W)^a\to V^a\odot W^a.$$

Proof. Let $v\in V$ and $w\in W$ be the cyclic vectors and let $V_0\subset V_1\subset\cdots\subset V$ and $W_0\subset W_1\subset\cdots\subset W$ be the induced PBW filtrations on V and W respectively. We also denote by $\mathrm{U}(\mathfrak{N})_s(v\otimes w)=(V\odot W)_s\subset V\odot W$ the s-th space of the induced PBW filtration on $V\odot W$. Then by definition

$$(V\odot W)^a = \bigoplus_{s\geq 0}\frac{(V\odot W)_s}{(V\odot W)_{s-1}}.$$

We have an obvious embedding $(V\odot W)_s=\mathrm{U}(\mathfrak{N})_s(v\otimes w)\hookrightarrow\sum_{i+j=s}V_i\otimes W_j$ inducing a natural homomorphism of $\mathfrak{N}^a$ modules

$$\Psi:\bigoplus_{s\geq 0}\frac{(V\odot W)_s}{(V\odot W)_{s-1}}\to\bigoplus_{s\geq 0}\frac{\sum_{i+j=s}V_i\otimes W_j}{\sum_{p+q=s-1}V_p\otimes W_q}. \tag{6.1}$$

Since $(V \odot W)^a$ is cyclic, $\operatorname{Im}\Psi$ is generated by $v \otimes w \in V_0 \otimes W_0$. Given k, l such that $k + l = s$, the natural linear map

$$\tilde{\Phi}_{k,l} : V_k \otimes W_l \to \frac{\sum_{i+j=s} V_i \otimes W_j}{\sum_{p+q=s-1} V_p \otimes W_q}, \ u \otimes u' \mapsto \overline{u \otimes u'},$$

has kernel $V_k \otimes W_{l-1} + V_{k-1} \otimes W_l$ and hence gives rise to an injective map:

$$\Phi_{k,l} : \frac{V_k}{V_{k-1}} \otimes \frac{W_l}{W_{l-1}} \to \frac{\sum_{i+j=s} V_i \otimes W_j}{\sum_{p+q=s-1} V_p \otimes W_q}, \ \bar{u} \otimes \bar{u}' \mapsto \overline{u \otimes u'}.$$

Combining these maps for all $k + l = s$, we get a natural isomorphism of vector spaces:

$$\Phi_s : \bigoplus_{i+j=s} \frac{V_i}{V_{i-1}} \otimes \frac{W_j}{W_{j-1}} \to \frac{\sum_{i+j=s} V_i \otimes W_j}{\sum_{p+q=s-1} V_p \otimes W_q},$$

which gives rise to an isomorphism of $\mathfrak{N}^a$ modules:

$$\Phi : \bigoplus_{s \geq 0} \bigoplus_{i+j=s} \frac{V_i}{V_{i-1}} \otimes \frac{W_j}{W_{j-1}} \simeq \bigoplus_{s \geq 0} \frac{\sum_{i+j=s} V_i \otimes W_j}{\sum_{i+j=s-1} V_i \otimes W_j}, \tag{6.2}$$

sending the tensor product of classes of two vectors to the class of their tensor product. Now the composition $\Phi^{-1}\Psi$ gives the desired surjective homomorphism, since the $\mathfrak{N}^a$ submodule of the left hand side of (6.2), generated by the product of the cyclic vectors, is equal to $V^a \odot W^a$. $\square$

In the setting of sections 2 and 3 one gets:

Corollary 6.2. *i) For any two positive integers n and m there is a natural map $(M^{\odot n+m})^a \to (M^{\odot n})^a \otimes (M^{\odot m})^a$.*
ii) For any n there exists a natural surjective map $(M^{\odot n})^a \to (M^a)^{\odot n}$.

So we can attach two rings to $\mathcal{F}_{\mathbb{U}^a}(M^a)$: its homogeneous coordinate ring

$$\mathbb{C}[\mathcal{F}_{\mathbb{U}^a}(M^a)] = R(M^a) = \bigoplus_{n \geq 0} ((M^a)^{\odot n})^* \text{ and } R^a(M) = \bigoplus_{n \geq 0} ((M^{\odot n})^a)^*.$$

It is natural to compare these rings and ask under which conditions these are isomorphic.

Example 6.3. In the settings of Example 3.1 the orbit closure $\mathcal{F}_{\mathbb{U}^a}(M^a)$ is the PBW-degeneration of flag varieties. The rings $R(M^a)$ and $R^a(M)$ turn out to be isomorphic in types A, C and G_2 (see [F1],[FFiL] and section 11).

7. Favourable modules

Let $\mathbb{U}$ be a complex algebraic unipotent group with Lie algebra $\mathfrak{N} = \operatorname{Lie}\mathbb{U}$, and let M be a finite dimensional cyclic $\mathbb{U}$-module with cyclic generator v_M. By Proposition 2.11 we know that

$$\operatorname{es}(M^{\odot n}) \supseteq \underbrace{\operatorname{es}(M) + \cdots + \operatorname{es}(M)}_{n}, \tag{7.1}$$

we are interested in the case where we have equality for all n, or, to put it differently, in the case when the essential semigroup $\Gamma_{\mathbb{U}}(M)$ of M is generated by $(1, \operatorname{es}(M))$.

Definition 7.1. We say that a finite dimensional cyclic $\mathbb{U}$-module M is favourable if there exists an ordered basis $f_1, \dots, f_N$ of $\mathfrak{N}$ and an induced homogeneous monomial order on the PBW basis such that

- There exists a normal polytope $P(M) \subset \mathbb{R}^N$ such that $\mathrm{es}(M)$ is exactly the set $S(M)$ of lattice points in $P(M)$.
- $\forall\, n \in \mathbb{N} : \dim M^{\odot n} = \sharp n S(M)$.

Remark 7.2. Since $P(M)$ is normal and $S(M) = \mathrm{es}(M)$, we know that $nS(M)$ is the n-fold Minkowski sum of $\mathrm{es}(M)$. Since $\sharp\mathrm{es}(M^{\odot n}) = \dim M^{\odot n}$, the two conditions in the definition above ensure that we have equality in (7.1) for all $n \geq 1$.

Remark 7.3. The normality of the polytope $P(M)$ depends on the choice of the induced homogeneous monomial order! So the property of the module M to be favourable strongly depends on the choice of the basis and the orderings.

Proposition 7.4. *If M is a finite dimensional favourable module, then $(M^{\odot n})^a \simeq (M^a)^{\odot n}$ as $S(\mathfrak{N})$-modules for all $n \geq 0$. In particular, the rings $R(M^a)$ and $R^a(M)$ coincide.*

Proof. We have a natural surjective map $(M^{\odot n})^a \to (M^a)^{\odot n}$ of $S(\mathfrak{N})$-modules by Corollary 6.2. In the favourable situation the dimensions of the modules coincide and hence they are isomorphic. $\square$

Corollary 7.5. *If M is favourable, then one can naturally identify the two essential semigroups $\Gamma_{\mathbb{U}}(M)$ and $\Gamma_{\mathbb{U}^a}(M^a)$.*

Corollary 7.6. *If M is favourable, then $M^{\odot n}$ is favourable for all $n \geq 1$.*

Proof. Since M is favorable, the Minkowski sum of m copies of $\mathrm{es}(M^{\odot n})$ coincides with $\mathrm{es}(M^{\odot mn}) = \mathrm{es}((M^{\odot n})^{\odot m})$, the polytope $P(M^{\odot n}) := nP(M) \subset \mathbb{R}^N$ is obviously normal, and the set $\mathrm{es}(M^{\odot n})$ is the set of lattice points in $P(M^{\odot n})$. $\square$

8. Flat degenerations

Let $\mathbb{U}$ be a complex algebraic unipotent group acting on a cyclic finite dimensional complex vector space M, so for the nilpotent Lie algebra $\mathfrak{N} = \mathrm{Lie}\,\mathbb{U}$ and a cyclic vector v_M we have $M = \mathrm{U}(\mathfrak{N})v_M$.

Theorem 8.1. *Let M be a favourable module.*

i) There exists a flat degeneration of the affine cone $\hat{\mathcal{F}}_{\mathbb{U}}(M)$ into the affine cone $\hat{\mathcal{F}}_{\mathbb{U}^a}(M^a)$, and for both there exists a flat degeneration into $\hat{\mathcal{F}}_{\mathbb{U}^a}(M^t)$.
ii) There exists a flat degeneration of $\mathcal{F}_{\mathbb{U}}(M)$ into $\mathcal{F}_{\mathbb{U}^a}(M^a)$, and for both there exists a flat degeneration into $\mathcal{F}_{\mathbb{U}^a}(M^t)$.

The corresponding flat families are equipped with a $\mathbb{C}^*$-action such that the projection onto $\mathbb{A}^1$ is equivariant with respect to the t^{-1}-multiplication action on $\mathbb{A}^1$.

Remark 8.2. Using the connection with Newton-Okounkov polytopes proved in Section 9, the degenerations into the toric variety can also be deduced from [A]. Nevertheless, we state below a full proof because a slight variation immediately implies also the flat degeneration of $\mathcal{F}_{\mathbb{U}}(M)$ in the PBW-degenerate variety $\mathcal{F}_{\mathbb{U}^a}(M^a)$.

Proof. We adapt the arguments in [AB], Proposition 2.2, respectively [C], 3.2, and define a decreasing filtration on the coordinate ring $R(M)$: given $\mathbf{p} \in \mathbb{N}^N$, set

$$R_n(M)^{>\mathbf{p}} = \bigoplus_{\substack{\mathbf{q}>\mathbf{p} \\ \mathbf{q}\in \mathrm{es}(M^{\odot n})}} \mathbb{C}\xi_{\mathbf{q}} \text{ and } R_n(M)^{\geq\mathbf{p}} = \bigoplus_{\substack{\mathbf{q}\geq\mathbf{p} \\ \mathbf{q}\in \mathrm{es}(M^{\odot n})}} \mathbb{C}\xi_{\mathbf{q}},$$

and

$$R(M)^{>\mathbf{p}} = \bigoplus_{n\geq 0} R_n(M)^{>\mathbf{p}}, \text{ and } R(M)^{\geq\mathbf{p}} = \bigoplus_{n\geq 0} R_n(M)^{\geq\mathbf{p}}.$$

Corollary 5.2 implies that $R(M)^{\geq\mathbf{p}}$ and $R(M)^{>\mathbf{p}}$ are ideals in $R(M)$, let $\mathrm{gr}\,\mathrm{R(M)}$ be the associated graded ring:

$$\mathrm{gr}\,\mathrm{R(M)} = \bigoplus_{\mathrm{n}\geq 0}\Bigg(\bigoplus_{\mathbf{p}\in \mathrm{es}(\mathrm{M}^{\odot \mathrm{n}})} \frac{\mathrm{R_n(M)}^{\geq\mathbf{p}}}{\mathrm{R_n(M)}^{>\mathbf{p}}}\Bigg).$$

Since the structure constants $c_{\mathbf{p},\mathbf{q}}^{\mathbf{p}+\mathbf{q}}$ can be fixed as those after renormalization, we conclude that $\mathrm{gr}\,\mathrm{R(M)}$ is the coordinate ring of the toric variety defined by $\mathrm{es}(M)$, i.e. $\mathrm{gr}\,\mathrm{R(M)}$ is the $\mathbb{C}$-algebra of the essential semigroup $\Gamma_{\mathbb{U}}(M)$. For $\mathbf{r} = (\mathbf{r}', n) \in \Gamma_{\mathbb{U}}(M)$ we just write $\xi_{\mathbf{r}}$ for the corresponding element $\xi_{\mathbf{r}'} \in (M^{\odot n})^*$.

By assumption, $\mathrm{es}(M)\times 1$ is a minimal set of generators for $\Gamma_{\mathbb{U}}(M)$, the corresponding elements $\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell} \in M^*$ generate $R(M)$ by Proposition 1.1. Let $\mathcal{S}$ be the polynomial ring $\mathbb{C}[x_1,\dots,x_\ell]$. We call the usual grading of $\mathcal{S}$ the *standard grading* and define a $\Gamma_{\mathbb{U}}(M)$-grading on the ring by setting $\deg_\Gamma x_i = \mathbf{r}^i$. Let $I = \mathrm{Ker}\Psi$ be the kernel of the surjective map of $\Gamma_{\mathbb{U}}(M)$-graded rings

$$\Psi : \mathcal{S} \to \mathrm{gr}\,\mathrm{R(M)}, \quad \mathrm{x_i} \mapsto \bar{\xi}_{\mathbf{r}^{\mathrm{i}}}.$$

The image of a monomial $x_{i_1}\cdots x_{i_q}$ is $\bar{\xi}_{\mathbf{p}}$, where $\mathbf{p} = \mathbf{r}_{i_1}+\ldots+\mathbf{r}_{i_q} \in \mathrm{es}(M^{\odot q})\times q$. The elements $\bar{\xi}_{\mathbf{p}}$, $\mathbf{p}\in\Gamma_{\mathbb{U}}(M)$, are linearly independent, so I is linearly spanned by binomials $x_{i_1}\cdots x_{i_q} - x_{j_1}\cdots x_{j_q}$, where $\mathbf{r}_{i_1}+\ldots+\mathbf{r}_{i_q} = \mathbf{r}_{j_1}+\ldots+\mathbf{r}_{j_q}$. We choose generators $\bar{g}_1,\dots,\bar{g}_m \in \mathcal{S}$ of the ideal I of this form, i.e. $\bar{g}_k = x_{i_1}\cdots x_{i_{q_k}} - x_{j_1}\cdots x_{j_{q_k}}$. Let $\mathbf{q}^k = (\mathbf{q}'^k, q_k)$ be the $\Gamma_{\mathbb{U}}(M)$-degree of $\bar{g}_k$. Since $\bar{g}_k(\bar{\xi}_{\mathbf{r}^1},\dots,\bar{\xi}_{\mathbf{r}^\ell}) = 0$, it follows that $\bar{g}_k(\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell}) \in R(M)^{>\mathbf{q}'^k}$. More precisely, by Corollary 5.2,

$$\bar{g}_k(\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell}) = \sum_{\substack{\mathbf{t}\in \mathrm{es}(M^{\odot q_k}) \\ \mathbf{t}>\mathbf{q}'^k}} a_{\mathbf{t}}\xi_{(\mathbf{t},q_k)}$$

with possibly non-zero coefficients $a_{\mathbf{t}}$. Since the $\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell}$ generate $R(M)$, we can find monomials $g_{k,\mathbf{t}}$ of the same standard degree as $\bar{g}_k$ such that $g_{k,\mathbf{t}}(\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell}) = \xi_{(\mathbf{t},q_k)}$ plus a sum of elements $\xi_{(\mathbf{s},q_k)}$, $\mathbf{s}\in \mathrm{es}(M^{\odot q_k})$, such that $\mathbf{s} > \mathbf{t}$. Since $\mathrm{es}(M^{\odot q_k})$ is a finite set, this implies that we can find a polynomial

$$g_k = \bar{g}_k + \sum_{j=1}^{r_k} g_{k,j} \tag{8.1}$$

such that g_k is homogeneous of standard degree q_k, each $g_{k,j}$ is homogeneous of $\Gamma_{\mathbb{U}}(M)$-degree $\mathbf{q}^{k,j} = (\mathbf{q}'^{k,j}, q_k)$ such that $\mathbf{q}'^{k,j} > \mathbf{q}'^k$, and

$$g_k(\xi_{\mathbf{r}^1},\dots,\xi_{\mathbf{r}^\ell}) = 0.$$

To deal with the degeneration into the abelianized flag variety, one uses as above the $\xi_{\mathbf{p}}$ to define a filtration of $R(M)$ with respect to the total PBW-degree, the associated graded ring $\mathrm{gr}^a R(M)$ is the homogeneous coordinate ring of the PBW-degenerate flag variety $\mathcal{F}_{\mathbb{U}^a}(M^a)$ (Proposition 7.4). Rewrite the sum $\sum_{j=1}^{r_i} g_{i,j}$ above as $\sum_{j=0}^{t_i} h_{i,j}$, where $h_{i,j}$ is a sum of $\Gamma_{\mathbb{U}}(M)$-homogeneous elements such that the total PBW-degree of $h_{i,j}$ is equal to (total PBW-degree of $\bar{g}_i$) $+ j$. Set $g_i^a = \bar{g}_i + h_{i,0}$.

Lemma 8.3. *The natural surjective maps below are isomorphisms:*

$$\begin{array}{ll} a) & \Phi : \mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S}) \to R(M), \quad x_i \mapsto \xi_{\mathbf{r}^i}, \\ b) & \Phi^a : \mathcal{S}/(g_1^a\mathcal{S} + \ldots + g_m^a\mathcal{S}) \to gr^a R(M), \quad x_i \mapsto \xi_{\mathbf{r}^i}. \end{array}$$

Proof. (of the lemma) To prove that the map Φ is an isomorphism we define a filtration and show that the associated graded map is injective. For $\mathbf{q} \in \mathbb{N}^N$ let $\mathcal{S}_n^{\geq \mathbf{q}}$ be the span of all monomials $x_1^{a^1} \ldots x_\ell^{a^\ell}$ in $\mathcal{S}$ of standard degree n such that $a^1\mathbf{r}'^1 + \ldots + a^\ell \mathbf{r}'^\ell \geq \mathbf{q}$. Then $\mathcal{S}^{\geq \mathbf{q}} = \oplus_{n\geq 0}\mathcal{S}_n^{\geq \mathbf{q}}$ is obviously an ideal, we define $\mathcal{S}_n^{>\mathbf{q}}$ and $\mathcal{S}^{>\mathbf{q}}$ similarly. Let $\mathrm{gr}\,\mathcal{S}$ be the associated graded algebra:

$$\mathrm{gr}\,\mathcal{S} = \bigoplus_{n\geq 0}\Bigg(\bigoplus_{\mathbf{p}\in \mathrm{es}(M^{\odot n})} \frac{\mathcal{S}_n^{\geq \mathbf{p}}}{\mathcal{S}_n^{>\mathbf{p}}}\Bigg).$$

Note that g_i and $\bar{g}_i$ are representatives in $\mathcal{S}$ of the same class in $\mathrm{gr}\,\mathcal{S}$. Let $p : \mathcal{S} \to \mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S})$ be the projection. The algebra $\mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S})$ is filtered by the images $p(\mathcal{S}^{\geq \mathbf{q}})$ of the ideals, let $\mathrm{gr}\,\big(\mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S})\big)$ be the associated graded algebra. The filtration of $R(M)$ induced by the images $\Phi \circ p(\mathcal{S}^{\geq \mathbf{q}})$ is exactly the filtration of $R(M)$ we started with, so we get induced morphisms:

$$\begin{array}{ccc} \mathrm{gr}\,\mathcal{S} & & \\ \Big\downarrow \mathrm{gr}\,p & \searrow \mathrm{gr}\Psi & \\ \mathrm{gr}\,\big(\mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S})\big) & \longrightarrow_{\mathrm{gr}\Phi} & \mathrm{gr}\,R(M) \end{array}.$$

The classes of g_i and $\bar{g}_i$ coincide in the associated graded algebra, so we have a surjective map $\mathcal{S}/(\bar{g}_1\mathcal{S} + \ldots + \bar{g}_m\mathcal{S}) \to \mathrm{gr}\,\big(\mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S})\big)$. The isomorphism $\mathcal{S}/(\bar{g}_1\mathcal{S} + \ldots + \bar{g}_m\mathcal{S}) \simeq \mathrm{gr\,R(M)}$, implies that $\mathrm{gr}\Phi$, and hence also Φ, is injective, and thus Φ is an isomorphism. The proof for Φ^a is similar. $\square$

(*Continuation of the proof of Theorem 8.1*) We consider now first the degeneration of the affine cone $\hat{\mathcal{F}}_{\mathbb{U}}(M)$ into the affine cone $\hat{\mathcal{F}}_{\mathbb{U}^a}(M^t)$. By the lemma above we know:

$$R(M) = \mathcal{S}/(g_1\mathcal{S} + \ldots + g_m\mathcal{S}) \tag{8.2}$$

where the generators g_i are homogeneous with respect to the standard grading and have a decomposition into homogeneous parts for the $\Gamma(M)$-grading such that

$$g_i = \bar{g}_i + \sum_{j=1}^{r_i} g_{i,j}, \quad \text{where } \deg g_{i,j} = \mathbf{q}^{i,j} > \mathbf{q}^i = \deg \bar{g}_i. \tag{8.3}$$

The set $\{\mathbf{q}^i, \mathbf{q}^{i,j}) \mid i = 1, \ldots, m,\ j = 1, \ldots, r_i\} \subset \mathbb{N}^N$ is finite, so by [C], Lemma 3.2, there exists a linear map $e : \mathbb{R}^N \to \mathbb{R}$ such that $e(\mathbb{N}^N) \subseteq \mathbb{N}$ and $e(\mathbf{q}^i) < e(\mathbf{q}^{i,j})$. Let

$$\mathcal{R} = \mathcal{S}[x_0]/\mathcal{I}, \tag{8.4}$$

where $\mathcal{I}$ is the ideal generated by the elements

$$\bar{g}_i + \sum_{j=1}^{r_i} x_0^{e(\mathbf{q}^{i,j})-e(\mathbf{q}^i)} g_{i,j} \tag{8.5}$$

for $i = 1, \ldots, m$. Let X be the variety

$$X = \left\{ v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_\ell \end{pmatrix} \in \mathbb{C}^{\ell+1} \mid f(v) = 0 \ \forall f \in \mathcal{I} \right\}. \tag{8.6}$$

The projection $\pi : \mathbb{C}^{\ell+1} \to \mathbb{A}^1$, $v \mapsto v_0$ onto the first coordinate induces a projection (denoted by the same letter)

$$\pi : X \to \mathbb{A}^1.$$

The construction implies for the fibre $\pi^{-1}(1)$ that $x_0 = 1$ and hence $X_1 = \pi^{-1}(1)$ is isomorphic to $\hat{\mathfrak{F}}_{\mathbb{U}}(M)$. Similarly, for $X_0 = \pi^{-1}(0)$ we have $x_0 = 0$ and hence X_0 is isomorphic to $\hat{\mathfrak{F}}_{\mathbb{U}^a}(M^t)$, the affine cone over the toric variety. We define a $\mathbb{C}^*$-action on $\mathbb{C}^{\ell+1}$ by

$$t \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_\ell \end{pmatrix} = \begin{pmatrix} t^{-1} v_0 \\ t^{e(\mathbf{r}^1)} v_1 \\ \vdots \\ t^{e(\mathbf{r}^\ell)} v_\ell \end{pmatrix}. \tag{8.7}$$

Note that

$$\begin{aligned} (\bar{g}_i + \textstyle\sum_{j=1}^{r_i} \quad & x_0^{e(\mathbf{q}^{i,j})-e(\mathbf{q}^i)} g_{i,j})(t \cdot v) \\ = \quad & t^{e(\mathbf{q}^i)} \bar{g}_i(v) + \textstyle\sum_{j=1}^{r_i} t^{(-e(\mathbf{q}^{i,j})+e(\mathbf{q}^i))} x_0^{e(\mathbf{q}^{i,j})-e(\mathbf{q}^i)}(v) t^{e(\mathbf{q}^{i,j})} g_{i,j}(v) \\ = \quad & t^{e(\mathbf{q}^i)} (\bar{g}_i + \textstyle\sum_{j=1}^{r_i} x_0^{e(\mathbf{q}^{i,j})-e(\mathbf{q}^i)} g_{i,j})(v). \end{aligned}$$

As an immediate consequence we see that X is stable under the $\mathbb{C}^*$-action, and the map π is $\mathbb{C}^*$-equivariant with respect to the t^{-1}-multiplication action of $\mathbb{C}^*$ on $\mathbb{C}$.

By the $\mathbb{C}^*$-action we know that all fibres over a point different from 0 are isomorphic to $\hat{\mathfrak{F}}_{\mathbb{U}}(M)$, and the special fibre over 0 is isomorphic to $\hat{\mathfrak{F}}_{\mathbb{U}^a}(M^t)$. It follows that $X = \overline{\mathbb{C}^* \cdot (\pi^{-1}(1))}$ is irreducible (the $\overline{\mathbb{C}^* \cdot (\pi^{-1}(1))}$ contains the special fiber, since the dimension of the special fiber coincides with the dimension of the general fibers). Since π is surjective, it follows that π is flat ([H], Chapter III, Proposition 9.7).

Using part *b)* of Lemma 8.3, one can proceed as in the toroidal case to prove the existence of a flat degeneration of the affine cone $\hat{\mathfrak{F}}_{\mathbb{U}}(M)$ into the affine cone $\hat{\mathfrak{F}}_{\mathbb{U}^a}(M^a)$. To prove the existence of a flat degeneration of $\hat{\mathfrak{F}}_{\mathbb{U}^a}(M^a)$ into $\hat{\mathfrak{F}}_{\mathbb{U}^a}(M^t)$, one proceeds as above, the only difference being that instead of starting with g_k as in (8.1) one starts with $g_k^a = \bar{g}_k + h_{k,0}$ and decomposes $h_{k,0}$ into its $\Gamma_{\mathbb{U}}(M)$-homogeneous parts.

To prove the second part of the theorem we extend the natural standard grading on $\mathcal{S}$ to $\mathcal{S}[x_0]$ by setting $\deg x_0 = 0$, the ideal $\mathcal{I}$ (see (8.5)) is homogeneous with respect to this grading, so the algebra $\mathcal{R} = \mathcal{S}[x_0]/\mathcal{I}$ inherits a natural grading. Due to the canonical isomorphism $\mathcal{R}_0 \simeq \mathbb{C}[x_0]$, the variety $Y = \operatorname{Proj} \mathcal{R}$ comes naturally equipped with a projective morphism $\theta : Y \to \mathbb{A}^1$. By replacing the arguments above by the appropriate ones for proper maps one proves part *ii)* of the theorem. □

Corollary 8.4. *If M is a finite dimensional favourable module, then $\mathcal{F}_{\mathbb{U}}(M)$), its PBW-degeneration $\mathcal{F}_{\mathbb{U}^a}(M^a)$ and its toric degeneration $\mathcal{F}_{\mathbb{U}^a}(M^t)$ are projective varieties which for the given embeddings are projectively normal and arithmetically Cohen-Macaulay.*

Proof. The condition on M to be favourable implies that the polytope $P(M)$ is normal and hence the variety $\mathcal{F}_{\mathbb{U}^a}(M^t) \simeq X(\mathrm{es}(M)) \subseteq \mathbb{P}(M^t)$ is projectively normal and arithmetically Cohen-Macaulay (see Theorem 9.2.9 and Exercise 9.2.8 [CLS]), i.e. the affine cone $\hat{\mathcal{F}}_{\mathbb{U}^a}(M^t) \subseteq M^t$ over the projective variety is normal and Cohen-Macaulay.

Since one knows now that the special fibre has the claimed good properties, it is a standard fact for flat families which are trivial away from "0" that all fibres have these nice properties, see for example [Knu]. □

9. Newton-Okounkov bodies and filtrations

Our general reference for more details on Newton-Okounkov bodies is [KK]. Let $\mathbb{U}$ be a complex algebraic unipotent group acting on a cyclic finite dimensional complex vector space M. Without loss of generality assume the action to be faithful. Denote by $\mathfrak{N} = \mathrm{Lie}\,\mathbb{U}$ its Lie algebra, let $v_M \in M$ be a fixed cyclic vector and denote by $\mathbb{V}$ the stabilizer in $\mathbb{U}$ of $[v_M] \in \mathbb{P}(M)$.

The field $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ of rational functions on $\mathcal{F}_{\mathbb{U}}(M)$ coincides with the field $\mathbb{C}(\mathbb{U}.[v_M]))$ of rational functions on the orbit. The orbit of a unipotent group is an affine space. To determine the Newton-Okounkov body of $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ (associated to a valuation), we want to fix an appropriate parameterization of this affine space.

Example 9.1. Let the notation be as in Example 3.1. For a dominant weight λ let P_λ be the stabilizer of the highest weight line in $V(\lambda)$ and let P_λ^- be the opposite parabolic subgroup. Denote by $\mathbb{U}'$ the unipotent radical of P_λ^-, the stabilizer $\mathbb{V}$ of $[v_\lambda]$ in $\mathbb{U} = U^-$ is the intersection $\mathbb{U} \cap L_\lambda$ with the Levi subgroup $L_\lambda \subset P_\lambda$. Then $\mathbb{U} = \mathbb{U}'\mathbb{V}$, $\mathbb{U}' \cap \mathbb{V} = \{id\}$, $\mathbb{U}.[v_M] = \mathbb{U}'.[v_M]$, and the stabilizer in $\mathbb{U}'$ is trivial. It follows that $\mathbb{U}.[v_M] \simeq \mathbb{U}'$ as an affine variety.

9.1. **The decomposition case.** Suppose first that $\mathbb{U}$ admits a subgroup $\mathbb{U}'$ such that $\mathbb{U} = \mathbb{U}'\mathbb{V}$ and $\mathbb{U}' \cap \mathbb{V} = \{id\}$ (as in Example 9.1 above). We fix a basis $\mathbb{B}_{\mathfrak{N}'}$ of $\mathfrak{N}' = \mathrm{Lie}\,\mathbb{U}'$, a basis $\mathbb{B}_{\mathbb{V}} = \{f_{N+1}, \dots, f_r\}$ of $\mathrm{Lie}\,\mathbb{V}$ and we set $\mathbb{B}_{\mathfrak{N}} = \mathbb{B}_{\mathbb{U}'} \cup \mathbb{B}_{\mathbb{V}}$.

Fix a homogeneous monomial ordering on the monomials in the elements of $\mathbb{B}_{\mathfrak{N}}$. Since $\mathbb{V}$ is the stabilizer of v_M, it is obvious that $(M^{\odot n}, \mathbf{p})$ is essential only if $p_{N+1} = \ldots = p_r = 0$. The essential semigroup $\Gamma_{\mathbb{U}}(M)$ is by definition a subset of $\mathbb{N} \times \mathbb{Z}^r$, but since the last entries are necessarily identically zero for an essential multi-exponent, we omit in the following these components and view $\Gamma_{\mathbb{U}}(M)$ as a subset of $\mathbb{N} \times \mathbb{Z}^N$. By abuse of notation we write $\mathbf{p} \geq \mathbf{q}$ for $\mathbf{p}, \mathbf{q} \in \mathbb{N}^N$ if $\mathbf{p}' \geq \mathbf{q}'$, where $\mathbf{p}', \mathbf{q}' \in \mathbb{Z}^r$ are the tuples obtained from $\mathbf{p}, \mathbf{q}$ by adding zero entries.

Let $x_1, \dots, x_N$ be the basis of $(\mathfrak{N}')^*$ dual to the basis $\{f_1, \dots, f_N\}$ of $\mathfrak{N}'$. The exponential map

$$\exp : \mathfrak{N}' \to \mathbb{U}', \quad X \mapsto \exp(X),$$

is an isomorphism of affine varieties. So the field of rational functions $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ can be identified with $\mathbb{C}(x_1, \ldots, x_N)$ and the x_j, $j = 1, \ldots, N$, form a system of parameters. We write $\mathbf{x}^{\mathbf{p}}$ for a monomial $x_1^{p_1} \cdots x_N^{p_N}$.

We define a $\mathbb{Z}^N$-valued valuation on $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ as follows: given a polynomial $g(\mathbf{x}) = \sum a_{\mathbf{p}} \mathbf{x}^{\mathbf{p}}$, we define

$$\nu(g(\mathbf{x})) = \min\{\mathbf{p} \mid a_{\mathbf{p}} \neq 0\}. \tag{9.1}$$

For a rational function $h = \frac{g}{g'}$ we define $\nu(h(\mathbf{x})) = \nu(g(\mathbf{x})) - \nu(g'(\mathbf{x}))$. The valuation ν is called the *lowest term valuation with respect to the parameters* $x_1, \ldots, x_N$ *and the total order "$\geq$"*.

Let $\xi_{\mathbf{0}}$ (section 5) be the dual vector of the fixed cyclic generator $v_M \in M$. Consider the homogeneous coordinate ring $A = \mathbb{C}[\mathcal{F}_{\mathbb{U}}(M)] = \bigoplus_{n \geq 0} A_n$ of the embedded variety $\mathcal{F}_{\mathbb{U}}(M) \hookrightarrow \mathbb{P}(M)$. We associate to A the valuation semigroup S_A as follows:

$$S_A = S(A, \nu, \xi_{\mathbf{0}}) = \bigcup_{n \geq 1} \{(n, \nu(\frac{g}{\xi_{\mathbf{0}}^n})) \mid g \in A_n - \{0\}\} \subseteq \mathbb{N} \times \mathbb{Z}^N. \tag{9.2}$$

The fact that we have a valuation implies that this is a semigroup. Recall that we view the essential semigroup $\Gamma_{\mathbb{U}}(M)$ as a subset of $\mathbb{N} \times \mathbb{Z}^N$.

Proposition 9.2. *The essential semigroup and the valuation semigroup coincide:*

$$S_A = \Gamma_{\mathbb{U}}(M).$$

Proof. Let $(n, \mathbf{p}) \in \{n\} \times \mathrm{es}(M^{\odot n})$, we want to evaluate

$$\xi_{\mathbf{p}}\bigg(\exp(x_1 f_1 + \ldots + x_N f_N) v_{M^{\odot n}}\bigg).$$

Since the argument of the exponential is a nilpotent operator, the sum is finite. It is necessary to reorder the factors within certain monomials. So new terms may occur, but they are of lower total degree and hence strictly smaller with respect to the homogeneous monomial order. So the exponential can be written as a linear combination of ordered monomials as follows:

$$\sum_{k \geq 0} \frac{1}{k!}(x_{\ell_1} f_{\ell_1} + \ldots + x_{\ell_N} f_{\ell_N})^k v_{M^{\odot n}} = \sum_{k \geq 0} \Bigg(\sum_{\substack{\mathbf{q} \in \mathbb{N}^N \\ |\mathbf{q}| = k}} (c_{\mathbf{q}} \mathbf{x}^{\mathbf{q}} \mathbf{f}^{\mathbf{q}} + \sum_{\substack{\mathbf{q}' \in \mathbb{N}^N \\ |\mathbf{q}'| < k}} a_{\mathbf{q}', \mathbf{q}} \mathbf{x}^{\mathbf{q}} \mathbf{f}^{\mathbf{q}'}) v_{M^{\odot n}} \Bigg),$$

where $c_{\mathbf{q}} \neq 0$. If $\mathbf{f}^{\mathbf{r}} v_{M^{\odot n}}$ is not an essential vector, then the vector can be rewritten as a linear combination of smaller essential vectors:

$$\mathbf{x}^{\mathbf{q}} \mathbf{f}^{\mathbf{r}} v_{M^{\odot n}} = \sum_{\substack{\mathbf{s} < \mathbf{r} \leq \mathbf{q} \\ (n, \mathbf{s}) \in (n, \mathrm{es}(M^{\odot n}))}} \mathbf{x}^{\mathbf{q}} b_{\mathbf{s}} \mathbf{f}^{\mathbf{s}} v_{M^{\odot n}}.$$

So we can rewrite the sum above as a linear combination of essential vectors:

$$\exp(x_{\ell_1} f_{\ell_1} + \ldots + x_{\ell_N} f_{\ell_N}) v_{M^{\odot n}} = \sum_{(n, \mathbf{q}) \in (n, \mathrm{es}(M^{\odot n}))} \Bigg(\frac{c_{\mathbf{q}}}{|\mathbf{q}|} \mathbf{x}^{\mathbf{q}} + \sum_{\mathbf{r} > \mathbf{q}} a'_{\mathbf{q}, \mathbf{r}} \mathbf{x}^{\mathbf{r}} \Bigg) \mathbf{f}^{\mathbf{q}} v_{M^{\odot n}}.$$

It follows that

$$\xi_{\mathbf{p}}\bigg(\exp(x_{\ell_1} f_{\ell_1} + \ldots + x_{\ell_N} f_{\ell_N}) v_{M^{\odot n}}\bigg) = \frac{c_{\mathbf{p}}}{|\mathbf{p}|} \mathbf{x}^{\mathbf{p}} + \sum_{\mathbf{r} > \mathbf{p}} a'_{\mathbf{p}, \mathbf{r}} \mathbf{x}^{\mathbf{r}}.$$

Since the coefficient $c_{\mathbf{p}} \neq 0$, we get $\nu\left(\frac{\xi_{\mathbf{p}}}{\xi_{\mathbf{0}}^n}\right) = \mathbf{p}$. It follows that $\Gamma_{\mathbb{U}}(M) \subseteq S_A$. Since the $\xi_{\mathbf{p}}$ form a basis of A with pairwise different evaluations, we have equality. □

9.2. **The general case.** If we do not have the decomposition as in 9.1, then the possible choices for a basis of $\mathfrak{N}$ are more restrictive. We fix a sequence of subgroups $\mathbb{U} = \mathbb{U}_1 \supset \ldots \supset \mathbb{U}_r \supset \mathbb{U}_{r+1} = \{id\}$ such that $\mathbb{U}_{i+1}$ is normal in $\mathbb{U}_i$ for $i \geq 1$ and of codimension 1. Since $\mathbb{U}$ is unipotent, such a sequence always exists. We get an induced filtration for $\mathbb{V} = \mathbb{V}_{i_1} \supset \ldots \supset \mathbb{V}_{i_s} \supset \mathbb{V}_{i_{s+1}} = \{id\}$ by subgroups, i.e. for $j = 1, \ldots, s$ we have

$$\mathbb{V}_{i_j} = \mathbb{U}_{i_j} \cap \mathbb{V} = \mathbb{U}_{i_j+1} \cap \mathbb{V} = \ldots = \mathbb{U}_{i_{j+1}-1} \cap \mathbb{V} \supsetneq \mathbb{V}_{i_{j+1}} = \mathbb{U}_{i_{j+1}} \cap \mathbb{V}$$

and $\operatorname{codim}_{\mathbb{V}_{i_j}} \mathbb{V}_{i_{j+1}} = 1$. Fix a basis $\mathbb{B}_{\mathfrak{N}} = \{f_1, \ldots, f_r\}$ of $\mathfrak{N}$ compatible with the filtrations above, so $\{f_i, \ldots, f_r\}$ is a basis of $\operatorname{Lie} \mathbb{U}_i$ and for all $j = 1, \ldots, s$, the subset $\{f_{i_j}, f_{i_{j+1}}, \ldots, f_{i_s}\}$ is a basis of $\operatorname{Lie} \mathbb{V}_j$. In particular, $\mathbb{B}_{\mathbb{V}} = \{f_{i_1}, , \ldots, f_{i_s}\}$ is a basis of $\operatorname{Lie} \mathbb{V}$. We fix an induced homogeneous monomial order "$\geq$" on the monomials in $\mathbb{B}_{\mathfrak{N}}$.

Lemma 9.3. *i)* $(M^{\odot n}, \mathbf{p})$ *is essential only if* $p_{i_1} = \ldots = p_{i_s} = 0$.

ii) Let $\mathbb{G}(f_i)$ *be the subgroup* $\{\exp(tf_i) \mid t \in \mathbb{C}\} \subset \mathbb{U}$. *The product map*

$$m_{\mathbb{U}} : \mathbb{G}(f_1) \times \mathbb{G}(f_2) \times \cdots \times \mathbb{G}(f_r) \to \mathbb{U}$$

is an isomorphism of affine varieties.

iii) Let $\{f_{\ell_1}, \ldots, f_{\ell_N}\}$ *be the complement in* $\mathbb{B}_{\mathfrak{N}}$ *of* $\mathbb{B}_{\mathbb{V}}$. *The product map*

$$m_{\mathbb{U}/\mathbb{V}} : \mathbb{G}(f_{\ell_1}) \times \mathbb{G}(f_{\ell_2}) \times \cdots \times \mathbb{G}(f_{\ell_N}) \to \mathbb{U}/\mathbb{V}$$

induces an isomorphism of affine varieties.

Proof. Suppose $\mathbf{f}^{\mathbf{p}}$ is such that $p_{i_j} > 0$ for some $j \in \{1, \ldots, s\}$. By commuting f_{i_j} to the right one can rewrite $\mathbf{f}^{\mathbf{p}}$ as a linear combination of strictly smaller elements and a monomial that annihilates $v^{\odot n}$, so $(M^{\odot n}, \mathbf{p})$ is not essential.

Let $1 \leq i \leq r$. Since $\mathbb{G}(f_i)$ is a subgroup of $\mathbb{U}_i$, we have a natural map

$$m^i_{\mathbb{U}} : \mathbb{G}(f_i) \times \mathbb{U}_{i+1} \to \mathbb{U}_i.$$

Since $\mathbb{G}(f_i) \cap \mathbb{U}_{i+1}$ is a finite group and unipotent, the intersection is equal to $\{id\}$ and the map is hence injective. Now $\mathbb{U}_i, \mathbb{U}_{i+1}$ are unipotent and $\mathbb{U}_{i+1}$ is a normal subgroup in $\mathbb{U}_i$. Using the Zassenhaus formula one shows that the map is also surjective. Both varieties are smooth, so by Zariski's main theorem ([Kum], Theorem A.11) it follows that the map is in fact an isomorphism. Now *ii)* follows by induction.

To prove *iii)*, note that $\mathbb{U}_i = \mathbb{G}(f_i)\mathbb{U}_{i+1} = \mathbb{U}_{i+1}\mathbb{G}(f_i)$ because $\mathbb{U}_{i+1}$ is normal in $\mathbb{U}_i$. By applying this argument to the subgroups corresponding to $\mathbb{V}$, it follows by part *ii)* of the lemma:

$$\mathbb{U} = \mathbb{G}(f_1)\mathbb{G}(f_2)\cdots\mathbb{G}(f_r) = \mathbb{G}(f_{\ell_1})\cdots\mathbb{G}(f_{\ell_N})\mathbb{G}(f_{i_s})\cdots\mathbb{G}(f_{i_1})$$

By applying the same arguments to $\mathbb{V}$ as to $\mathbb{U}$ in *ii)* and above, we see

$$\mathbb{V} = \mathbb{G}(f_{i_1})\cdots\mathbb{G}(f_{i_s}) = \mathbb{G}(f_{i_s})\cdots\mathbb{G}(f_{i_1})$$

and hence $\mathbb{U} = \mathbb{G}(f_{\ell_1})\cdots\mathbb{G}(f_{\ell_N})\mathbb{V}$, which finishes the proof of the lemma. □

Let $\{x_{\ell_1},\dots,x_{\ell_N}\}$ be the basis of $(\mathfrak{N}/\mathrm{Lie}\,\mathbb{V})^*$ dual to the basis $\{\bar f_{\ell_1},\dots,\bar f_{\ell_N}\}$ of $\mathfrak{N}/\mathrm{Lie}\,\mathbb{V}$. Since $\mathbb{U}\cdot[v_M]\subset\mathcal{F}_{\mathbb{U}}(M)$ is a smooth open affine subset isomorphic to $\mathbb{G}(f_{\ell_1})\times\cdots\times\mathbb{G}(f_{\ell_N})$, the field of rational functions $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ can be identified with $\mathbb{C}(x_{\ell_1},\dots,x_{\ell_N})$ and the x_{ℓ_j}, $j=1,\dots,N$, form a system of parameters. We write $\mathbf{x}^{\mathbf{p}}$ for a monomial $x_{\ell_1}^{p_{\ell_1}}\cdots x_{\ell_N}^{p_{\ell_N}}$, where $\mathbf{p}\in\mathbb{N}^N$. We use the same convention as above: $\mathbf{p}\geq\mathbf{q}$ if $\mathbf{p}'\geq\mathbf{q}'$, where $\mathbf{p}',\mathbf{q}'\in\mathbb{Z}^{r=N+s}$ are the tuples obtained from $\mathbf{p},\mathbf{q}$ by adding zero entries in the places $i_1,\dots,i_s$. Let $\geq$ also denote the induced monomial order on $\mathbb{C}[x_{\ell_1},\dots,x_{\ell_N}]$, i.e. $\mathbf{x}^{\mathbf{p}}\geq\mathbf{x}^{\mathbf{q}}$ if and only if $\mathbf{p}\geq\mathbf{q}$.

Now we define the $\mathbb{Z}^N$-valued valuation on $\mathbb{C}(\mathcal{F}_{\mathbb{U}}(M))$ and the valuation semigroup S_A as in (9.1) and (9.2).

Proposition 9.4. *The essential semigroup and the valuation semigroup coincide:*

$$S_A=\Gamma_{\mathbb{U}}(M).$$

Proof. We have to evaluate $\xi_{\mathbf{p}}\Big(\exp(x_{\ell_1}f_{\ell_1})\cdots\exp(x_{\ell_N}f_{\ell_N})v_{M^{\odot n}}\Big)$ for an element $(n,\mathbf{p})\in\{n\}\times\mathrm{es}(M^{\odot n})$. Now

$$\exp(x_{\ell_1}f_{\ell_1})\cdots\exp(x_{\ell_N}f_{\ell_N})v_{M^{\odot n}}=\Big(\sum_{\mathbf{q}\in\mathbb{N}^N}a_{\mathbf{q}}\mathbf{x}^{\mathbf{q}}\mathbf{f}^{\mathbf{q}}\Big)v_{M^{\odot n}}\tag{9.3}$$

for some constants $a_{\mathbf{q}}\neq0$. The same arguments as in the proof of Proposition 9.2 apply: after rewriting non essential vectors as a linear combination of smaller essential vectors one gets:

$$\xi_{\mathbf{p}}\Big(\exp(x_{\ell_1}f_{\ell_1})\cdots\exp(x_{\ell_N}f_{\ell_N}))v_{M^{\odot n}}\Big)=a_{\mathbf{p}}\mathbf{x}^{\mathbf{p}}+\sum_{\mathbf{p}'>\mathbf{p}}a'_{\mathbf{p}'}\mathbf{x}^{\mathbf{p}'}.$$

Since the coefficient $a_{\mathbf{p}}\neq0$, we get $\nu\big(\frac{\xi_{\mathbf{p}}}{\xi_{\mathbf{0}}^n}\big)=\mathbf{p}$. It follows that $\Gamma_{\mathbb{U}}(M)\subseteq S_A$. Since the $\xi_{\mathbf{p}}$ form a basis of A with pairwise different evaluations, we have equality. □

9.3. **Newton-Okounkov body.** One associates to A also the cone C generated by $S_A=\Gamma_{\mathbb{U}}(M)$ in $\mathbb{R}\times\mathbb{R}^N$:

$$C=\text{smallest closed convex cone centered at 0 containing } S_A.$$

Definition 9.5. The *Newton-Okounkov body* $\Delta(\mathcal{F}_{\mathbb{U}}(M),\nu,\geq)$ of $\mathcal{F}_{\mathbb{U}}(M)$ in $\mathbb{R}^N$ is the projection of the intersection $C\cap 1\times\mathbb{R}^N$ on $\mathbb{R}^N$. In other words, the Newton-Okounkov body is the closure of the convex hull of the rescaled exponents:

$$\Delta(\mathcal{F}_{\mathbb{U}}(M))=\Delta(\mathcal{F}_{\mathbb{U}}(M),\nu,\geq)=\overline{\mathrm{convex}(\bigcup_{n\geq1}\{\frac{\mathbf{p}}{n}\mid(n,\mathbf{p})\in S_A\})}.$$

Theorem 9.6. *Let $\mathbb{U}$ be a be a complex algebraic unipotent group with Lie algebra $\mathfrak{N}$ and let M be a finite dimensional cyclic $\mathbb{U}$-module with cyclic vector v_M and stabilizer $\mathbb{V}\subset\mathbb{U}$ of $[v_M]\in\mathbb{P}(M)$. Assume that either $\mathbb{U}$ admits a decomposition $\mathbb{U}=\mathbb{U}'\mathbb{V}$ as in section 9.1 or the basis of $\mathfrak{N}$ has been chosen as in section 9.2. If M is favourable and $P(M)$ is the associated lattice polytope (Definition 7.1), then we have for the Newton-Okounkov bodies:*

$$\Delta(\mathcal{F}_{\mathbb{U}}(M))=P(M)=\Delta(\mathcal{F}_{\mathbb{U}^a}(M^a)).$$

10. The multicone version

Let G be a simple simply connected complex algebraic group G. We fix a Cartan decomposition of its Lie algebra $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, an ordered basis $\underline{\mathfrak{n}}^- := \{f_1, \ldots, f_N\}$ of $\mathfrak{n}^-$, and we fix a homogeneous monomial order on the PBW basis. Let $\mathbb{U} \subset G$ be the maximal unipotent subgroup with Lie algebra $\mathfrak{n}^-$.

Let $\omega_1, \ldots, \omega_n$ be the fundamental weights for G. For a dominant integral weight $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ we denote by the support $\sup \lambda$ the set of fundamental weights

$$\sup \lambda = \{\omega_i \mid a_i \neq 0\}. \tag{10.1}$$

By Proposition 2.11 we know that

(10.2)

$$\mathrm{es}(V(\lambda)) \supseteq \underbrace{\mathrm{es}(V(\omega_1)) + \cdots + \mathrm{es}(V(\omega_1))}_{a_1} + \cdots + \underbrace{\mathrm{es}(V(\omega_n)) + \cdots + \mathrm{es}(V(\omega_n))}_{a_n},$$

we are interested in the case when we have equality for all dominant weights:

Definition 10.1. The pair $(G, \underline{\mathfrak{n}}^-)$ is called *favourable* for the fixed order if

- for each fundamental weight ω_i there exists a normal polytope P_i such that the lattice points $S_i \subset P_i$ index the essential monomials for $V(\omega_i)$,
- for a dominant weight $\lambda = a_1\omega_1 + \ldots + a_n\omega_n$ let $P_\lambda := a_1P_1 + \ldots + a_nP_n$ be the corresponding Minkowski sum of the polytopes P_i. Let $S_\lambda \subset P_\lambda$ be the set of lattice points. Then:

$$\forall \text{ dominant weights } \lambda : \ \dim V(\lambda) = \sharp S_\lambda = \sharp(\sum_{i=1}^n a_iS_i).$$

Remark 10.2. As in Definition 7.1, the conditions are split into two parts of rather different kind: the first deals with the structure of the fundamental representations as $\mathfrak{n}^-$-modules, the second is of more combinatorial nature comparing dimension formulas for representations with formulas counting lattice points in polytopes.

The conditions imply equality in (10.2) (compare Remark 7.2):

$$\sharp \mathrm{es}(V(\lambda)) = \dim V(\lambda) = \sharp\left(\sum_{i=1}^n a_iS(\omega_i)\right) = \sharp(\sum_{i=1}^n a_i \mathrm{es}(V(\omega_i)))$$

which, by the inclusion in (10.2) is only possible if

$$\mathrm{es}(V(\lambda)) = \underbrace{\mathrm{es}(V(\omega_1)) + \cdots + \mathrm{es}(V(\omega_1))}_{a_1} + \cdots + \underbrace{\mathrm{es}(V(\omega_n)) + \cdots + \mathrm{es}(V(\omega_n))}_{a_n}.$$

Remark 10.3. The condition for $(G, \underline{\mathfrak{n}}^-)$ to be favourable implies that the global Okounkov body (see for example [Gon]) for G/B is a polyhedral cone, and one gets an explicit description of the cone. More precisely, let

$$GOB(G/B) = \{(p, \lambda) \in \mathbb{R}^N \times X_\mathbb{R} \mid \lambda = \sum a_i\omega_i, a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}, p \in \sum a_iP_i\},$$

where $X_\mathbb{R}$ is the real span of the weight lattice. It is easy to see that this is a convex cone such that if λ is a regular dominant integral weight, then the fibre $\pi^{-1}(\lambda)$ of the projection map

$$\Psi : GOB(G/B) \hookrightarrow \mathbb{R}^N \times X_\mathbb{R} \longrightarrow X_\mathbb{R}$$

is the polytope P_λ, so $GOB(G/B)$ is the global Okounkov body for G/B. Consider the vertices $\{p^i_j \mid j = 1, \dots, r_i\}$ of the polytope P_i for $i = 1, \dots, n$. A simple calculation shows that $GOB(G/B)$ is the cone spanned by

$$\{(p^i_j, \omega_i) \mid i = 1, \dots, n; j = 1, \dots, r_i\}.$$

This strong condition has some beautiful consequences:

Theorem 10.4. *If $(G, \underline{\mathfrak{n}}^-)$ is favourable, then*

i) $V(\lambda+\mu)^a \simeq V(\lambda)^a \odot V(\mu)^a$ as $S(n^-)$-modules,

ii) the representations $V(\lambda)$, λ a dominant weight, are favourable for $\mathfrak{n}^-$,

iii) for all dominant integral weights λ, the projective varieties $\mathcal{F}_{\mathbb{U}^a}(V^a_\lambda) \subseteq \mathbb{P}(V^a_\lambda)$ and $\mathcal{F}_{\mathbb{U}^a}(V^t_\lambda) \subseteq \mathbb{P}(V^t_\lambda)$ are projectively normal and arithmetically Cohen-Macaulay,

iv) there are embeddings $\mathcal{F}_{\mathbb{U}^a}(V(\lambda+\mu)^a) \hookrightarrow \mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a) \times \mathcal{F}_{\mathbb{U}^a}(V(\mu)^a)$,

v) the variety $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a)$ depends on the support $\sup\lambda$ of λ only (see (10.1)),

vi) the vectors $\{v_\lambda(\mathbf{p}) \mid \mathbf{p} \in S(\lambda)\}$ form a basis for $V(\lambda)$, $V(\lambda)^a$ respectively $V(\lambda)^t$ (depending on whether one chooses $v_\lambda(0)$ to be the cyclic generator of $V(\lambda)$, $V(\lambda)^a$ or $V(\lambda)^t$).

Proof. We have a natural surjective map $V(\lambda+\mu)^a \to V(\lambda)^a \odot V(\mu)^a$. Since $(G, \underline{\mathfrak{n}}^-)$ is favourable, one has

$$\mathrm{es}(V(\lambda+\mu)) = S(\lambda+\mu) = S(\lambda) + S(\mu) = \mathrm{es}(V(\lambda)) + \mathrm{es}(V(\mu)),$$

so by Proposition 2.11 this map is injective and hence an isomorphism. The Minkowski sum of lattice polytopes is a lattice polytope, so the condition being favourable implies that all the polytopes $P(\lambda)$ are normal. As an immediate consequence we see: all representations $V(\lambda)$ are favourable and the varieties $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a)$ and $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^t)$ are projectively normal and arithmetically Cohen-Macaulay (Corollary 8.4). Similarly, part *vi)* is a consequence of Corollary 2.8.

The Segre embedding $\mathbb{P}(V(\lambda)^a) \times \mathbb{P}(V(\mu)^a) \hookrightarrow \mathbb{P}(V(\lambda)^a \otimes V(\mu)^a)$ and the isomorphism $V(\lambda+\mu)^a \simeq V(\lambda)^a \odot V(\mu)^a$ implies that the image of $\mathcal{F}_{\mathbb{U}^a}(V(\lambda+\mu)^a)$ in $\mathbb{P}(V(\lambda)^a \otimes V(\mu)^a)$ lies in the embedded product $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a) \times \mathcal{F}_{\mathbb{U}^a}(V(\mu)^a)$. By embedding $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a)$ in the corresponding product of degenerate flag varieties for fundamental weights, it is easy to see that $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a) \simeq \mathcal{F}_{\mathbb{U}^a}(V(\nu)^a)$ for $\nu = \sum_{\omega \in \sup\lambda} \omega$, and hence $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a)$ depends only on the support of λ. □

Remark 10.5. In the next section we will see that $(G, \underline{\mathfrak{n}}^-)$ is favourable for type $\mathtt{A}_n$, $\mathtt{C}_n$ and $\mathtt{G}_2$.

11. The classical examples

In this section we illustrate the construction of the previous section on the example of flag varieties of classical groups. Let $\mathfrak{g}$ be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ and let G be the corresponding semisimple, simply connected complex algebraic group. As before, $\mathbb{U}$ denotes the maximal unipotent subgroup with Lie algebra $\mathfrak{n}^-$. Let $\triangle_+$ be the set of positive roots of $\mathfrak{g}$ and let $\alpha_1, \dots, \alpha_n$ be the simple roots. Let $f_\beta \in \mathfrak{n}^-$, $\beta \in \triangle_+$, be a root basis of $\mathfrak{n}^-$.

Let λ be a dominant integral weight for the Lie algebra $\mathfrak{g}$ and let $V(\lambda)$ be the corresponding irreducible $\mathfrak{g}$-module of highest weight λ. Fix a highest weight vector $v_\lambda \in V(\lambda)$; in particular, $\mathfrak{n}v_\lambda = 0$ and $V(\lambda) = \mathrm{U}(\mathfrak{n}^-)v_\lambda$. We will be interested in the degenerate modules $V(\lambda)^a$ and $V(\lambda)^t$ introduced above. To apply Theorem 10.4

we need to introduce an ordering $\beta_1, \ldots, \beta_N$ of the positive roots and fix a homogeneous monomial order. Then the set of essential monomials is fixed and we give a combinatorial description in terms of a normal polytope.
In the following we will consider only orderings having the following special property (we give examples of such orderings below):
Let "$\succ$" be the standard partial order on the set of positive roots. We assume that

$$\beta_i \succ \beta_j \text{ implies } i < j.$$

An ordering with this property (the larger roots come first) will be called a *good ordering*. Once we fix such a good ordering, this induces an ordering on the basis vectors f_β. As monomial order on the PBW basis we fix the induced homogeneous reverse lexicographic order.

Example 11.1. In type A_n ($\mathfrak{g} = \mathfrak{sl}_{n+1}$) the positive roots are of the form $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$ for $1 \le i \le j \le n$. Here is an example of a good ordering in type A_n:

$$\begin{gathered} \beta_1 = \alpha_{1,n}, \\ \beta_2 = \alpha_{1,n-1},\ \beta_3 = \alpha_{2,n}, \\ \ldots, \\ \beta_{(n-1)n/2+1} = \alpha_1, \ldots, \beta_{n(n+1)/2} = \alpha_n. \end{gathered}$$

Example 11.2. In type C_n ($\mathfrak{g} = \mathfrak{sp}_{2n}$) the positive roots are of the form

$$\begin{gathered} \alpha_{i,j} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j,\ 1 \le i \le j \le n, \\ \alpha_{i,\overline{j}} = \alpha_i + \alpha_{i+1} + \ldots + \alpha_n + \alpha_{n-1} + \ldots + \alpha_j,\ 1 \le i \le j \le n \end{gathered}$$

(note that $\alpha_{i,n} = \alpha_{i,\overline{n}}$). Here is an example of a good ordering in type C_n:

$$\begin{gathered} \beta_1 = \alpha_{1,\overline{1}}, \\ \beta_2 = \alpha_{1,\overline{2}},\ \beta_3 = \alpha_{2,\overline{2}}, \\ \ldots, \\ \beta_{(n-1)n/2+1} = \alpha_{1,\overline{n}}, \ldots, \beta_{n(n+1)/2} = \alpha_{n,\overline{n}}, \\ \beta_{n(n+1)/2+1} = \alpha_{1,n-1}, \ldots, \beta_{n(n+1)/2+n-1} = \alpha_{n-1,n-1}, \\ \ldots, \\ \beta_{n^2} = \alpha_{1,1}. \end{gathered}$$

We now recall the polytopes describing the basis of the PBW graded modules in types A and C ([FFoL1], [FFoL2], [FFoL3]) and the basis in type G_2 from [G]. We just write $P(\lambda)$ instead of $P(V(\lambda))$.

11.1. **Type $\mathtt{A}_n$.** Let $\omega_1, \ldots, \omega_n$ be the fundamental weights, we denote $f_{\alpha_{i,j}}$ by $f_{i,j}$. Note that $N = \dim \mathfrak{n}^- = n(n+1)/2$.

For a dominant integral weight $\lambda = \sum_{k=1}^n m_k \omega_k$, $m_k \in \mathbb{Z}_{\ge 0}$, we define a polytope $P(\lambda) \subset \mathbb{R}^N_{\ge 0}$ and the set $S(\lambda) \subset \mathbb{Z}^N_{\ge 0}$ as follows:

A sequence $\mathbf{b} = (\beta_1, \ldots, \beta_r)$ of positive roots is called a *Dyck path* if the first and the last roots are simple roots ($\beta_1 = \alpha_{i,i}$, $\beta_r = \alpha_{j,j}$, $i \le j$), and if $\beta_m = \alpha_{p,q}$, then $\beta_{m+1} = \alpha_{p+1,q}$ or $\beta_{m+1} = \alpha_{p,q+1}$.

Definition 11.3. The polytope $P(\lambda) \subset \mathbb{R}^N_{\ge 0}$ is defined as the set of points $\mathbf{p} = (p_\beta)_{\beta \in \triangle_+}$ in $\mathbb{R}^N_{\ge 0}$ satisfying the following inequalities (with integer coefficients): for all Dyck paths $\mathbf{b}$ with $\beta_1 = \alpha_{i,i}$, $\beta_r = \alpha_{j,j}$ one has

$$p_{\beta_1} + p_{\beta_2} + \cdots + p_{\beta_r} \le m_i + \cdots + m_j.$$

The set $S(\lambda) = P(\lambda) \cap \mathbb{Z}^N_{\ge 0}$ is the set of lattice points in $P(\lambda)$. We proved in [FFoL1] that $\{\mathbf{f}^{\mathbf{p}} v_\lambda \mid \mathbf{p} \in S(\lambda)\}$ forms a basis of $V(\lambda)^a$ and hence of $V(\lambda)$ itself.

11.2. **Type $\mathtt{C}_n$.** Let $\omega_1, \ldots, \omega_n$ be the fundamental weights, we will use the following abbreviations for the roots and the operators:

$$\alpha_i = \alpha_{i,i},\ \alpha_{\overline{i}} = \alpha_{i,\overline{i}},\ f_{i,j} = f_{\alpha_{i,j}},\ f_{i,\overline{j}} = f_{\alpha_{i,\overline{j}}}.$$

Note that $N = \dim \mathfrak{n}^- = \#\triangle_+ = n^2$. We recall the usual order on the alphabet $A = \{1, \ldots, n, \overline{n-1}, \ldots, \overline{1}\}$

$$1 < 2 < \ldots < n-1 < n < \overline{n-1} < \ldots < \overline{1}.$$

A *symplectic Dyck path* is a sequence $\mathbf{b} = (\beta_1, \ldots, \beta_r)$ of positive roots such that: the first root is a simple root, $\beta_1 = \alpha_{i,i}$; the last root is either a simple root $\beta_r = \alpha_j$ or $p(k) = \alpha_{\overline{j}}$ $(i \le j \le n)$; if $\beta_m = \alpha_{r,q}$ with $r, q \in A$ then β_{m+1} is either $\alpha_{r,q+1}$ or $\alpha_{r+1,q}$, where $x+1$ denotes the smallest element in A which is bigger than x.

Definition 11.4. The polytope $P(\lambda) \subset \mathbb{R}^N_{\ge 0}$ is defined as the set of points $\mathbf{p} = (p_\beta)_{\beta \in \triangle_+}$ in $\mathbb{R}^N_{\ge 0}$ satisfying the following inequalities (with integer coefficients): for all Dyck paths $\mathbf{b}$ with $\beta_1 = \alpha_{i,i}$, $\beta_r = \alpha_{j,j}$ one has

$$p_{\beta_1} + p_{\beta_2} + \cdots + p_{\beta_r} \le m_i + \cdots + m_j;$$

for all Dyck paths $\mathbf{b}$ with $\beta_1 = \alpha_{i,i}$, $\beta_r = \alpha_{j,\overline{j}}$ one has

$$p_{\beta_1} + p_{\beta_2} + \cdots + p_{\beta_r} \le m_i + \cdots + m_n.$$

The set $S(\lambda) = P(\lambda) \cap \mathbb{Z}^N_{\ge 0}$ is the set of lattice points in $P(\lambda)$. We proved in [FFoL2] that the set $\{\mathbf{f}^{\mathbf{p}} v_\lambda \mid \mathbf{p} \in S(\lambda)\}$ forms a basis of $V(\lambda)^a$ and hence of $V(\lambda)$ itself.

11.3. **Type $\mathtt{G}_2$.** Let α_1, α_2 be simple roots. The six positive roots are as follows:

$$\beta_1 = 3\alpha_1 + 2\alpha_2,\ \beta_2 = 3\alpha_1 + \alpha_2,\ \beta_3 = 2\alpha_1 + \alpha_2,\ \beta_4 = \alpha_1 + \alpha_2,\ \beta_5 = \alpha_2,\ \beta_6 = \alpha_1.$$

We note that this ordering is good. Let $\lambda = k\omega_1 + l\omega_2$, $k, l \ge 0$.

Definition 11.5. The polytope $P(\lambda) \subset \mathbb{R}^6_{\ge 0}$ is defined as the set of points $\mathbf{p} = (p_\beta)_{\beta \in \triangle_+}$ in $\mathbb{R}^6_{\ge 0}$ satisfying the following inequalities (with integer coefficients):

$$p_5 \le l,\ p_6 \le k,$$
$$p_2 + p_3 + p_6 \le k + l, \quad p_3 + p_4 + p_6 \le k + l, \quad p_4 + p_5 + p_6 \le k + l,$$
$$p_1 + p_2 + p_3 + p_4 + p_5 \le k + 2l, \qquad p_2 + p_3 + p_4 + p_5 + p_6 \le k + 2l.$$

The set $S(\lambda) = P(\lambda) \cap \mathbb{Z}^6_{\ge 0}$ is the set of lattice points in $P(\lambda)$. It is proved in [G] that the set $\{\mathbf{f}^{\mathbf{p}} v_\lambda \mid \mathbf{p} \in S(\lambda)\}$ is a basis of $V(\lambda)^t$ and hence of $V(\lambda)^a$ and $V(\lambda)$.

11.4. **Essential sets and $S(\lambda)$.** To apply Theorem 10.4 we need to prove that the ordering gives a favourable pair. So we need to show that $P(\lambda)$ is the Minkowski sum of the polytopes for the fundamental weights and furthermore $P(\lambda)$ is normal. We recall the following proposition ([FFoL3], [G]):

Proposition 11.6. *Let $\mathfrak{g}$ be of type A_n, C_n or G_2. Then for any two dominant weights λ and μ one has $S(\lambda+\mu) = S(\lambda) + S(\mu)$.*

Lemma 11.7. *The polytopes $P(\lambda)$ defined above for $\mathfrak{g}$ of type $\mathtt{A_n}$, $\mathtt{C_n}$ or $\mathtt{G_2}$ are normal.*

Proof. The polytopes are defined by inequalities with integer coefficients, hence the vertices have rational coordinates. Let now $v \in P(\lambda)$ be a point with rational coordinates. Fix $q \in \mathbb{N}$ such that qv has integral coordinates, so $qv \in S(q\lambda)$. By Proposition 11.6, qv is an element of the q-fold Minkowski sum of $S(\lambda)$, so one can write $qv = s_1 + s_2 + \ldots + s_q$, where $s_1, s_2, \ldots, s_q \in S(\lambda)$, and hence $v = \frac{1}{q}s_1 + \ldots \frac{1}{q}s_q$ is in the convex hull of the lattice points of $P(\lambda)$. It follows that $P(\lambda)$ is a lattice polytope, which is normal by Proposition 11.6. $\square$

Theorem 11.8. *Let $\mathfrak{g}$ be of type A_n, C_n or G_2. Assume that the positive roots are ordered and the ordering is good. Then the pair $(G, \underline{\mathfrak{n}}^-)$ is favourable.*

Proof. By Proposition 11.6 and Lemma 11.7, it remains to show that the set $\mathrm{es}(V(\lambda))$ coincides with $S(\lambda)$. The case of G_2 is worked out in [G], where it is proved that $S(\lambda)$ indexes a basis of $V(\lambda)^t$.

Let $\mathfrak{g} = \mathfrak{sl}_n$. First, we prove the theorem for fundamental weights $\lambda = \omega_k$. Then $V(\lambda) = \Lambda^k(V(\omega_1))$ and $V(\omega_1)$ is the n-dimensional vector representation. Fix the standard basis $w_1, \ldots, w_n$ of $V(\omega_1)$. We denote by $w_{i_1,\ldots,i_k}$ the wedge product $w_{i_1} \wedge \cdots \wedge w_{i_k}$. For example, $v_\lambda = w_{1,2,\ldots,k}$. Let

$$I = \{i_1, \ldots, i_k\} \quad \text{with} \quad 1 \le i_1 < \cdots < i_s \le k < i_{s+1} < \cdots < i_k \le n.$$

Then we set $w_I := w_{i_1} \wedge \cdots \wedge w_{i_k}$ and note that $w_I \neq 0$ in $V(\lambda)$. Further, let $J = \{1, \ldots, k\} \setminus \{i_1, \ldots, i_s\}$. We write $J = (j_1, \ldots, j_{k-s})$, where $k < j_1 < \cdots < j_{k-s} \le n$. There might be several multi-exponents $\mathbf{p}$ such that $v_M(\mathbf{p}) = w_I$. We claim that the minimal monomial (and hence the essential one) is

$$f_{j_1,i_k-1} f_{j_2,i_{k-1}-1} \cdots f_{j_{k-s},i_{s+1}-1} v_\lambda \tag{11.1}$$

(recall that $f_{i,j} w_i = w_{j+1}$). In fact, first of all the minimal length of a monomial is exactly $k-s$. Now a monomial $\mathbf{f}^{\mathbf{p}}$ such that $\mathbf{f}^{\mathbf{p}} v_\lambda$ is proportional to w_I is of the form

$$f_{j_{\sigma(1)},i_k-1} f_{j_{\sigma(2)},i_{k-1}-1} \cdots f_{j_{\sigma(k-s)},i_{s+1}-1} v_\lambda \tag{11.2}$$

for some permutation $\sigma \in S_{k-s}$. We claim that the minimal monomial (11.2) corresponds to $\sigma = \mathrm{id}$. In fact, the minimal root vector among all $f_{j_\sigma(\ell),i_\ell-1}$ is f_{j_1,i_k-1} (see the ordering in Example 11.1).
This implies that there is a factor f_{j_1,i_k-1} in the minimal monomial with $f^{\mathbf{p}} v_\lambda = w_I$. By proceeding in the same way we obtain the claim by downwards induction. It suffices now to note that (11.1) belongs to $S(\omega_k)$. We thus obtain the inclusion $es(V(\omega_k)) \subseteq S(\omega_k)$. Since the cardinalities of these sets coincide, we have the equality.

Similarly we check that for fundamental weights of the symplectic algebras $\mathrm{es}(V(\omega_k)) = S(\omega_k)$.

Now let us consider the general λ. Thanks to Proposition 2.11 we know that for dominant weights λ and μ we have $\mathrm{es}(V(\lambda)) + \mathrm{es}(V(\mu)) \subset \mathrm{es}(V(\lambda+\mu))$. But

$$\sharp(\mathrm{es}(V(\lambda))+\mathrm{es}(V(\mu))) = \sharp(S(\lambda)+S(\mu)) = \sharp S(\lambda+\mu) = \dim V(\lambda+\mu) = \sharp\mathrm{es}(V(\lambda+\mu))$$

(here, the third equality is proved in [FFoL1]). We conclude that the equalities $\mathrm{es}(V(\omega_k)) = S(\omega_k)$ for $k = 1,\dots,n$ imply $\mathrm{es}(V(\lambda)) = S(\lambda)$ for any dominant weight λ.

The proof in the case of $\mathfrak{g} = \mathfrak{sp}_{2n}$ is similar, it suffices to consider the fundamental weights. $\square$

As in Section 9 we deduce:

Corollary 11.9. *For all dominant weights λ, there exists an appropriate evaluation on the field $\mathbb{C}(\mathbb{U}.[v_\lambda])$ respectively $\mathbb{C}(\mathbb{U}^a.[v_\lambda])$, such that the polytope $P(\lambda)$ is the Newton-Okounkov body of $\mathcal{F}_{\mathbb{U}}(V(\lambda))$ and of $\mathcal{F}_{\mathbb{U}^a}(V(\lambda)^a)$.*

11.5. **Automorphism group.** Let us turn again to the $G = SL_{n+1}$-case and regular λ, and use that we have an explicit description of the polytope $P(\lambda)$ in terms of inequalities defined by Dyck paths. Then $P(\lambda) \subset \mathbb{R}^N_{\geq 0}$ where $N = n(n+1)/2$, we denote $F \subset \mathbb{R}^N$ the associated normal fan. The rays $F(1)$ of F are given by

(i) $\mathbb{R}.e_{i,j}$, $1 \leq i \leq j \leq n$,
(ii) $\mathbb{R}.(-\sum_{k=1}^r e_{i_k,j_k})$ for all possible Dyck paths $(i_1,j_1),\dots,(i_r,j_r)$.

Now following [Nil, Cox], we define the Demazure roots of the toric variety

$$\mathcal{R} = \{m \in \mathbb{Z}^N \mid \exists\tau \in F(1) \text{ such that } \langle v_\tau, m\rangle = -1, \langle v_\eta, m\rangle \geq 0\ \forall \eta \in F(1), \eta \neq \tau\}.$$

It is easy to see that if τ is a ray of type (ii), then there is a unique Demazure root for this τ, namely m with $m_{1,n} = 1$ and $m_{i,j} = 0$ else and this is the unique semisimple root in $\mathcal{R}$.
For $\tau = \mathbb{R}e_{k,l}$, one has the following Demazure roots:

- For $l = 1$: $m = (m_{i,j})$ where $m_{k,1} = -1$ and $\exists k' > k : m_{k',1} = 1$ and $m_{i,j} = 0\ \forall (i,j) \neq (k,1),(k',1)$,
- for $k = n$: $m = (m_{i,j})$ where $m_{n,l} = -1$ and $\exists l' > l : m_{n,l'} = 1$ and $m_{i,j} = 0\ \forall (i,j) \neq (n,k),(n,k')$.,
- and else $m = (m_{i,j})$ where $m_{k,l} = -1$ and $m_{i,j} = 0\ \forall (i,j) \neq (k,l)$.

Then the number of Demazure roots is exactly

$$1 + \frac{1}{2}n(n+1) + \frac{1}{2}n(n-1) + \frac{1}{2}n(n-1) = \frac{3}{2}n^2 - \frac{1}{2}n + 1.$$

We conclude that the connected component of the automorphism group of the toric variety is the semidirect product of a reductive group with a N dimensional torus, a semisimple part isomorphic to SL_2 or PSL_2, and a $\frac{3}{2}n^2 - \frac{1}{2}n - 1$-dimensional unipotent radical.

11.6. **More examples.** For $G = SL_3$, the toric variety obtained in this way is isomorphic to the one constructed in [AB] via the Gelfand-Tsetlin polytope (see also [KM, GL]), but even for $G = SL_4$ the two are not isomorphic in general. To be more precise, it has been shown in [BF, Fou2] that for $G = SL_n$, the toric varieties associated to the polytopes $P(\lambda)$ described in this paper and the Gelfand-Tsetlin-polytopes are isomorphic if and only if λ is supported only on the first two nodes or on the last two nodes or on the first and last node. Recall that in [ABS] a bijection between the integral points in the two polytopes has been provided. It

would be quite interesting to have a geometric interpretation of this bijection.
Next take for $G = SL_4$ the total order on the positive roots obtained by the following reduced decomposition $w_0 = s_2s_1s_3s_2s_1s_3$ of the longest Weyl group element (see [AB, Lit]). In this case our toric variety is isomorphic to the one in [AB] associated to this reduced decomposition (for most of these computations we used the program *polymake* ([GJ]). One can show, again using *polymake*, that for $G = SL_6$, that there is no reduced decomposition of the longest Weyl group element, such that for regular λ our toric variety is isomorphic to the one obtained from the corresponding string polytope.
Our construction of the polytope and the toric variety is very explicit, for example we are able to compute the rays of the associated normal fan, the Demazure roots and the automorphism group of the toric variety (for $G = SL_n$ and λ regular) (see Section 11.5).
Some other special cases are investigated in [BD]. Let ω be a fundamental weight for G such that $\langle\omega, \theta^\vee\rangle = 1$, where θ is the highest root (this includes all minuscule and cominuscule fundamental weights).

Corollary. *There exists an ordering of the positive roots and a homogeneous ordering on the monomials in $U(\mathfrak{n}^-)$ such that for all $m \geq 1 : V(m\omega)$ is favourable for $\mathbb{U}$ with a highest weight vector as cyclic generator.*

Instead of working with the full G-representation it is natural to look also at special submodules stable under a maximal unipotent subgroup $\mathbb{U}$ of G, the standard example being Demazure submodules. One can show for $G = SL_n$ and $\lambda = m\omega$ a multiple of a fundamental weight that all Demazure submodules are favourable $\mathbb{U}$-modules ([BF]). For arbitrary λ a partial answer was provided in [Fou], while the general question remains open even in the SL_n-case.

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Evgeny Feigin:
Department of Mathematics,
National Research University Higher School of Economics,
Vavilova str. 7, 117312, Moscow, Russia,
and
Tamm Theory Division, Lebedev Physics Institute
E-mail address: evgfeig@gmail.com

Ghislain Fourier:
Mathematisches Institut, Universität zu Köln,
Weyertal 86-90, D-50931 Köln,Germany
and
School of Mathematics and Statistics
University of Glasgow, UK
E-mail address: gfourier@math.uni-koeln.de, Ghislain.Fourier@glasgow.ac.uk

Peter Littelmann:
Mathematisches Institut, Universität zu Köln,
Weyertal 86-90, D-50931 Köln, Germany
E-mail address: peter.littelmann@math.uni-koeln.de