THE DEGREE OF THE HILBERT-POINCARÉ POLYNOMIAL OF PBW-GRADED MODULES

TEODOR BACKHAUS AND LARA BOSSINGER AND CHRISTIAN DESCZYK AND GHISLAIN FOURIER

Abstract. In this note, we study the Hilbert-Poincaré polynomials for the associated PBW-graded modules of simple modules for a simple complex Lie algebra. The computation of their degree can be reduced to modules of fundamental highest weight. We provide these degrees explicitly.

Nous étudions les polynômes de Hilbert-Poincaré pour les modules PBW-gradués associés aux modules simples d'une algèbre de Lie simple complexe. Le calcul de leur degré peut être restreint aux modules de plus haut poids fondamental. Nous donnons une formule explicite pour ces degrés.

1. Introduction

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Then the PBW filtration on $U(\mathfrak{n}^-)$ is given as $U(\mathfrak{n}^-)_s := \operatorname{span}\{x_{i_1} \cdots x_{i_l} \mid x_{i_j} \in \mathfrak{n}^-, l \leq s\}$. The associated graded algebra is isomorphic to $S(\mathfrak{n}^-)$. Let $V(\lambda)$ be a simple finite-dimensional module of highest weight λ and v_λ a highest weight vector. Then we have an induced filtration on $V(\lambda) = U(\mathfrak{n}^-)v_\lambda$, denoted $V(\lambda)_s := U(\mathfrak{n}^-)_s v_\lambda$. The associated graded module $V(\lambda)^a$ is a $S(\mathfrak{n}^-)$-module generated by v_λ.

These modules have been studied in a series of papers. Monomial bases of the graded modules and the annihilating ideals have been provided for the $\mathfrak{sl}_n, \mathfrak{sp}_n$ [FFL11a, FFL11b, FFL13b], for cominuscule weights and their multiples in other types [BD14], for certain Demazure modules in the $\mathfrak{sl}_n$-case in [Fou14b, BF14]. In type G_2 there is a monomial basis provided by [Gor11].

The degenerations of the corresponding flag varieties have been studied in [Fei12, FFL13a, CIL14, CILL14]. Further, it turned out ([Fou14a]), that these PBW degenerations have an interesting connection to fusion product for current algebras. The study of the characters of PBW-graded modules has been initiated in [CF13, FM14].

In the present paper we will compute the maximal degree of PBW-graded modules in full generality (for all simple complex Lie algebras), where there have been partial answers in the above series of paper for certain cases.

We denote the Hilbert-Poincaré series of the PBW-graded module, often referred to as the *q-dimension of the module*, by

$$p_\lambda(q) = \sum_{s=0}^{\infty} \left(\dim V(\lambda)_s / V(\lambda)_{s-1}\right) q^s.$$

Since $V(\lambda)$ is finite-dimensional, this is obviously a polynomial in q. In this note we want to study further properties of this polynomial. We see immediately that the constant term of $p_\lambda(q)$ is always 1 and the linear term is equal to

$$\dim(\mathfrak{n}^-) - \dim \operatorname{Ker}\left(\mathfrak{n}^- \longrightarrow \operatorname{End}(V(\lambda))\right).$$

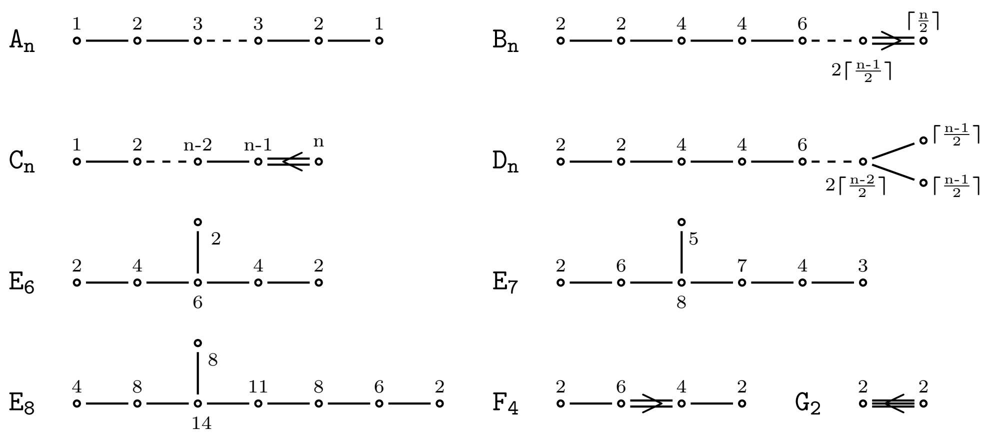
Our main goal is to compute the degree of $p_\lambda(q)$ and the first step is the following reduction [CF13, Theorem 5.3 ii)]:

Theorem. Let $\lambda_1, \ldots, \lambda_s \in P^+$ and set $\lambda = \lambda_1 + \ldots + \lambda_s$. Then

$$\deg p_\lambda(q) = \deg p_{\lambda_1}(q) + \ldots + \deg p_{\lambda_s}(q).$$

It remains to compute the degree of $p_\lambda(q)$ where λ is a fundamental weight. We have done this for all fundamental weights of simple complex finite-dimensional Lie algebras:

Theorem 1. The degree of $p_{\omega_i}(q)$ is equal to the label of the i-th node in the following diagrams:



The paper is organized as follows: In Section 2 we introduce definitions and basic notations, in Section 3 we prove Theorem 1.

Acknowledgements T.B. was funded by the DFG-priority program 1388 "Representation Theory", L.B., C.D. and G.F. were partially funded by this program. The main work of this article has been conducted during a workshop organized at the University of Glasgow, and all authors would like to thank the Glasgow Mathematical Journal Trust Fund, the Edinburgh Mathematical Journal and especially the University of Glasgow for this opportunity.

Mathematisches Institut, Universität zu Köln, Germany (T.B., L.B., C.D.)
Mathematisches Institut, Universität Bonn, Germany, School of Mathematics and Statistics, University of Glasgow, UK (G.F.)

2. Preliminairies

Let $\mathfrak{g}$ be a simple Lie algebra of rank n. We fix a Cartan subalgebra $\mathfrak{h}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The set of roots (resp. positive roots) of $\mathfrak{g}$ is denoted R (resp. R^+), θ denotes the highest root. Let α_i, ω_i $i = 1, ..., n$ be the simple roots and the fundamental weights. Let W be the Weyl group associated to the simple roots and $w_0 \in W$ the longest element. For $\alpha \in R^+$ we fix a $\mathfrak{sl}_2$ triple $\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\}$. The integral weights and the dominant integral weights are denoted P and P^+.
Let $\{x_1, x_2, ...\}$ be an ordered basis of $\mathfrak{g}$, then $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ with PBW basis $\{x_{i_1} \cdots x_{i_m} \mid m \in \mathbb{Z}_{\geq 0}, i_1 \leq i_2 \leq ... \leq i_m\}$.

2.1. **Modules.** For $\lambda \in P^+$ we consider the irreducible $\mathfrak{g}$-Module $V(\lambda)$ with highest weight λ. Then $V(\lambda)$ admits a decomposition into $\mathfrak{h}$-weight spaces, $V(\lambda) = \bigoplus_{\tau \in P} V(\lambda)_\tau$ with $V(\lambda)_\lambda$ and $V(\lambda)_{w_0(\lambda)}$, the highest and lowest weight spaces, being one dimensional. Let v_λ denote the highest weight vector, $v_{w_0(\lambda)}$ denote the lowest weight vector satisfying

$$e_\alpha v_\lambda = 0,\ \forall\, \alpha \in R^+ \ ;\ f_\alpha v_{w_0(\lambda)} = 0,\ \forall\, \alpha \in R^+.$$

We have $U(\mathfrak{n}^-).v_\lambda \cong V(\lambda) \cong U(\mathfrak{n}^+).v_{w_0(\lambda)}$.
The comultiplication ($x \mapsto x \otimes 1 + 1 \otimes x$) provides a $\mathfrak{g}$-module structure on $V(\lambda) \otimes V(\mu)$. This module decomposes into irreducible components, where the Cartan component generated by the highest weight vector $v_\lambda \otimes v_\mu$ is isomorphic to $V(\lambda + \mu)$.

2.2. **PBW-filtration.** The Hilbert-Poincaré series of the PBW-graded module $V(\lambda)^a := \bigoplus_{s \geq 0} V(\lambda)_s / V(\lambda)_{s-1}$ is the polynomial

$$\begin{aligned} p_\lambda(q) &= \sum\nolimits_{s \geq 0} \dim(V(\lambda)_s / V(\lambda)_{s-1}) q^s \\ &= 1 + \dim(V(\lambda)_1 / V(\lambda)_0) q + \dim(V(\lambda)_2 / V(\lambda)_1) q^2 + \ldots \end{aligned}$$

and we define the PBW-degree of $V(\lambda)$ to be $\deg(p_\lambda(q))$.

It is easy to see that $\mathfrak{n}^+.(U(\mathfrak{n}^-)_s.v_\lambda) \subseteq U(\mathfrak{n}^-)_s.v_\lambda \ \forall\, s \geq 0$ (see also [FFL11a]) and hence $U(\mathfrak{n}^+).V(\lambda)_s \subseteq V(\lambda)_s$. Let s_λ be minimal such that $v_{w_0(\lambda)} \in V(\lambda)_{s_\lambda}$. Then $V(\lambda) = U(\mathfrak{n}^+).v_{w_0(\lambda)} \subseteq V(\lambda)_{s_\lambda}$ and

Corollary. $s_\lambda = \deg(p_\lambda(q))$ and

$$V(\lambda) = V(\lambda)_{s_\lambda}.$$

2.3. **Graded weight spaces.** The PBW filtration is compatible with the decomposition into $\mathfrak{h}$-weight spaces:

$$\dim V(\lambda)_\tau = \sum_{s \geq 0} \dim \left(V(\lambda)_s / V(\lambda)_{s-1} \right) \cap V(\lambda)_\tau .$$

So we can define for every weight τ the Hilbert-Poincaré polynomial:

$$p_{\lambda,\tau}(q) = \sum_{s \geq 0} \dim \left(V(\lambda)_s / V(\lambda)_{s-1} \right)_\tau q^s \text{ and then } p_\lambda(q) = \sum_{\tau \in P} p_{\lambda,\tau}(q).$$

A natural question is, if we can extend our results to these polynomials? If the weight space $V(\lambda)_\tau$ is one-dimensional, then $p_{\lambda,\tau}(q)$ is a power of q. For $\tau = \lambda$ this is constant 1, for $\tau = w_0(\lambda)$, the lowest weight, this is $q^{\deg p_\lambda(q)}$ as we have seen in Corollary 2.2. A first approach to study these polynomials can be found in [CF13].

2.4. **Graded Kostant partition function.** For the readers convienience we recall here the *graded Kostant partition function* (see [Kos59]), which counts the number of decompositions of a fixed weight into a sum of positive roots, and how it is related to our study. We consider the power series and its expansion:

$$\prod_{\alpha > 0} \frac{1}{(1 - q e^\alpha)}, \quad \sum_{\nu \in P} P_\nu(q) e^\nu .$$

We have immediately $\operatorname{char} S(\mathfrak{n}^-) = \sum_{\nu \in P} P_\nu(q) e^{-\nu}$.

Remark. For a polynomial $p(q) = \sum_{i=0}^{n} a_i q^i$, we denote $\operatorname{mindeg} p(q)$ the minimal j such that $a_j \neq 0$. Then we have obviously

$$\operatorname{mindeg} p_{\lambda,\nu}(q) \geq \operatorname{mindeg} P_{\lambda-\nu}(q). \tag{2.1}$$

We will use this inequality for the very special case $\nu = w_0(\lambda)$ in the proof of Theorem 1.
We see from Theorem 1 that this inequality is a proper inequality for certain cases in exceptional type as well as $B_{,}D_n$ (this has been noticed also in [CF13]).

3. Proof of Theorem 1

In this section we will provide a proof of Theorem 1. For a fixed $1 \leq i \leq \operatorname{rank} \mathfrak{g}$, we will give a monomial $u \in U(\mathfrak{n}^-)$ of the predicted degree mapping the highest weight vector v_{ω_i} to the lowest weight vector $v_{w_0(\omega_i)}$. We then show that there is no monomial of smaller degree satisfying this.
To write down these monomials explicitly, let us denote θ_{X_n} the highest root of a Lie algebra of type X_n. We set further (using the indexing from [Hum72]):

- In the A_n-case, Y_{n-2} the type of the Lie algebra generated by the simple roots $\{\alpha_2, \ldots, \alpha_{n-1}\}$.
- In the B_n, D_n-case, Y_{n-k} the type of the Lie algebra generated by the simple roots $\{\alpha_{k+1}, \ldots, \alpha_n\}$.
- In the exceptional and symplectic cases, $\theta_{X_n} = c_k \omega_k$ for some k, Y_{n-1} the type of the Lie algebra generated by the simple roots $\{\alpha_1, \ldots, \alpha_n\} \setminus \{\alpha_k\}$.

Let $u \in U(\mathfrak{n}^-)$ be one of the monomials in Figure 1. It can be seen easily from Figure 1 that $u = f_{\theta_{X_n}}^{a_i^\vee} u_1$, where $a_i^\vee = w_i(h_{\theta_{X_n}})$ and u_1 is the monomial in Figure 1 corresponding to the restriction of ω_i to the Lie subalgebra of type $Y_{n-\ell}$. If we denote $\mathfrak{n}_1^-$ the lower part in the triangular decomposition of the Lie subalgebra of type $Y_{n-\ell}$, then $u_1 \in U(\mathfrak{n}_1^-)$.
Let $u = f_{\theta_1}^{b_1} f_{\theta_2}^{b_2} \ldots f_{\theta_r}^{b_r}$. Note that all f_{θ_j} commute and it is easy to see that $\theta_j(h_{\theta_{j+p}}) = 0, \ \forall p \geq 0$ (since θ_j is a sum of fundamental weights, which are all orthogonal to the simple roots of the Lie algebra with highest root θ_{j+p}) and $b_j = \omega_i(h_{\theta_j})$.
The Weyl group W acts on $V(\omega_i)$ and if v is an extremal weight vector of weight μ, then $w.v$ is a nonzero extremal weight vector of weight $w(\mu)$. Further if $w = s_\alpha$ (reflection at a root α) and $\mu(h_\alpha) \geq 0$, then $w.v = c^* f_\alpha^{\mu(h_\alpha)}.v$ for some $c^* \in \mathbb{C}^*$.
Now consider $w = s_{\theta_r} \ldots s_{\theta_1}$, where s_{θ_j} is the reflection at the root θ_j. Then we have $w.v_{\omega_i} = v_{w_0(\omega_i)} = u.v_{\omega_i} \neq 0$ in $V(\omega_i)$. So we obtain an upper estimate for the degree.
In general the degree of u is bigger than the minimal degree coming from Kostant's graded partition function (2.1). For A_n, C_n the degrees coincide and hence we are done in these cases.
We will prove Theorem 1 for the remaining cases X_n by induction on the rank of the Lie algebra. So we want to prove that if $p \in U(\mathfrak{n}^-)$ with $p.v_{\omega_i} = v_{w_0(\omega_i)}$ then $\deg(p) \geq \deg(u)$, where u is from Figure 1.
Consider the induction start, e.g. $\omega_i = \theta_{X_n}$, then the minimal degree is obviously 2. The maximal non-vanishing power of $f_{\theta_{X_n}}$ is certainly $a_i^\vee$ and $f_{\theta_{X_n}}^{a_i^\vee}.v_{\omega_i}$ is the highest weight vector of a simple module of fundamental weight for the Lie algebra Y_{n-l} defined as above. By induction we know that if $q \in U(\mathfrak{n}_1^-)$ with $q.(f_{\theta_{X_n}}^{a_i^\vee}.v_{\omega_i}) = v_{w_0(\omega_i)}$ then $\deg(q) \geq \deg(u_1)$.

$$
\begin{array}{lll}
X_n & \omega_i = \theta_{X_n} & f_{\theta_{X_n}}^2 \\
A_n & \omega_i & f_{\theta_{A_n}} f_{\theta_{A_{n-2}}} \cdots f_{\theta_{A_{n+2-2\min\{i,n-i\}}}} \\
C_n & \omega_i & f_{\theta_{C_n}} f_{\theta_{C_{n-1}}} \cdots f_{\theta_{C_{n+1-i}}} \\
B_n & \omega_{2i} & f_{\theta_{B_n}}^2 f_{\theta_{B_{n-2}}}^2 \cdots f_{\theta_{B_{n+2-2i}}}^2 \\
B_n & \omega_{2i+1} & f_{\theta_{B_n}}^2 f_{\theta_{B_{n-2}}}^2 \cdots f_{\theta_{B_{n-2i}}}^2 f_{\alpha_{2i+1}} \\
B_n & n \text{ even}, \omega_n & f_{\theta_{B_n}} f_{\theta_{B_{n-2}}} \cdots f_{\theta_{B_2}} \\
B_n & n \text{ odd}, \omega_n & f_{\theta_{B_n}} f_{\theta_{B_{n-2}}} \cdots f_{\theta_{B_2}} f_{\alpha_n} \\
D_n & \omega_{2i} & f_{\theta_{D_n}}^2 f_{\theta_{D_{n-2}}}^2 \cdots f_{\theta_{D_{n+2-2i}}}^2 \\
D_n & \omega_{2i+1} & f_{\theta_{D_n}}^2 f_{\theta_{D_{n-2}}}^2 \cdots f_{\theta_{D_{n-2i}}} f_{\alpha_{2i+1}} \\
D_n & n \text{ even}, \omega_i, i = n-1, n & f_{\theta_{D_n}} f_{\theta_{D_{n-2}}} \cdots f_{\theta_{D_4}} f_{\alpha_i} \\
D_n & n \text{ odd}, \omega_i, i = n-1, n & f_{\theta_{D_n}} f_{\theta_{D_{n-2}}} \cdots f_{\theta_{D_5}} f_{\theta_{A_4}} \\
E_6 & \omega_1, \omega_6 & f_{\theta_{E_6}} f_{\theta_{A_5}} \\
E_6 & \omega_3, \omega_5 & f_{\theta_{E_6}}^2 f_{\theta_{A_5}} f_{\theta_{A_3}} \\
E_6 & \omega_4 & f_{\theta_{E_6}}^3 f_{\theta_{A_5}} f_{\theta_{A_3}} f_{\alpha_4} \\
E_7 & \omega_2 & f_{\theta_{E_7}}^2 f_{\theta_{D_6}} f_{\theta_{D_4}} f_{\alpha_2} \\
E_7 & \omega_3 & f_{\theta_{E_7}}^3 f_{\theta_{D_6}} f_{\theta_{D_4}} f_{\alpha_3} \\
E_7 & \omega_4 & f_{\theta_{E_7}}^4 f_{\theta_{D_6}}^2 f_{\theta_{D_4}}^2 \\
E_7 & \omega_5 & f_{\theta_{E_7}}^3 f_{\theta_{D_6}}^2 f_{\theta_{D_4}} f_{\alpha_5} \\
E_7 & \omega_6 & f_{\theta_{E_7}}^2 f_{\theta_{D_6}}^2 \\
E_7 & \omega_7 & f_{\theta_{E_7}} f_{\theta_{D_6}} f_{\alpha_7} \\
E_8 & \omega_1 & f_{\theta_{E_8}}^2 f_{\theta_{E_7}}^2 \\
E_8 & \omega_2 & f_{\theta_{E_8}}^3 f_{\theta_{E_7}}^2 f_{\theta_{D_6}} f_{\theta_{D_4}} f_{\alpha_2} \\
E_8 & \omega_3 & f_{\theta_{E_8}}^4 f_{\theta_{E_7}}^3 f_{\theta_{D_6}} f_{\theta_{D_4}} f_{\alpha_3} \\
E_8 & \omega_4 & f_{\theta_{E_8}}^6 f_{\theta_{E_7}}^4 f_{\theta_{D_6}}^2 f_{\theta_{D_4}}^2 \\
E_8 & \omega_5 & f_{\theta_{E_8}}^5 f_{\theta_{E_7}}^3 f_{\theta_{D_6}}^2 f_{\theta_{D_4}} f_{\alpha_5} \\
E_8 & \omega_6 & f_{\theta_{E_8}}^4 f_{\theta_{E_7}}^2 f_{\theta_{D_6}}^2 \\
E_8 & \omega_7 & f_{\theta_{E_8}}^3 f_{\theta_{E_7}} f_{\theta_{D_6}} f_{\alpha_7} \\
F_4 & \omega_2 & f_{\theta_{F_4}}^3 f_{\theta_{C_3}} f_{\theta_{A_2}} f_{\alpha_2} \\
F_4 & \omega_3 & f_{\theta_{F_4}}^2 f_{\theta_{C_3}} f_{\theta_{C_2}} \\
F_4 & \omega_4 & f_{\theta_{F_4}} f_{\theta_{C_3}} \\
G_2 & \omega_1 & f_{\theta_{G_2}} f_{\alpha_1}
\end{array}
$$

FIGURE 1.

First we suppose $f_{\theta_{X_n}}^{a_i^\vee}.v_{\omega_i}$ is a factor of p, so $p = f_{\theta_{X_n}}^{a_i^\vee} p'$ and then by weight considerations $p' \in U(\mathfrak{n}_1^-)$. Then $p'.(f_{\theta_{X_n}}^{a_i^\vee}.v_{\omega_i}) = v_{w_0(\omega_i)}$ (the lowest weight vector in $V(\omega_i)$ as well as in the simple submodule). Therefore $\deg(p') \geq \deg(u_1)$ which implies $\deg(p) \geq \deg(u)$.

Suppose now the maximal power of $f_{\theta_{X_n}}$ in p is $f_{\theta_{X_n}}^{a_i^\vee - k}$, $k \geq 0$ and $\deg(p) < \deg(u)$. Let X_n be of type B_n, D_n or exceptional, then $\theta_{X_n} = \omega_j$ and we denote

$$R_s^+ = \{\alpha \in R^+ \mid w_j(h_\alpha) = s\},$$

Then $R_2^+ = \{\theta_{X_n}\}$ and if $\beta \in R_1^+$ then $\theta_{X_n} - \beta \in R_1^+$. By weight reasons $p = f_{\theta_{X_n}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1$ for some $\beta_1, \ldots, \beta_{2k} \in R_1^+$ and some polynomial p_1 in root vectors of roots in R_0^+. We have to show that $p.v_{\omega_i} = 0 \in V(\omega_i)^a$ and we will use induction on k for that: The induction start is $k = 0$. The induction step is for $k \geq 1$:

$$
\begin{aligned}
0 = p_1 f_{\theta_{X_n}}^{a_i^\vee + k}.v_{\omega_i} &= (e_{\theta_{X_n} - \beta_1}) \cdots (e_{\theta_{X_n} - \beta_{2k}}) p_1 f_{\theta_{X_n}}^{a_i^\vee + k}.v_{\omega_i} \\
&= c f_{\theta_{X_n}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1.v_{\omega_i} + \textstyle\sum_{\ell > 0}^{k} f_{\theta_{X_n}}^{a_i^\vee - k + \ell} q_\ell.v_{\omega_i}
\end{aligned}
$$

for some $c \in \mathbb{C}^*, q_\ell \in U(\mathfrak{n}^-)$. For $0 \leq \ell < k$ all the summands are equals to zero by induction (on k). For $\ell = k$, we recall our assumption $\deg(p) < \deg(u)$ and so $\deg(q_k) < \deg(u_1)$ which implies $f^{a_i^\vee} q_k.v_{\omega_i} = 0$. So we can conclude $f_{\theta_{X_n}}^{a_i^\vee - k} f_{\beta_1} \cdots f_{\beta_{2k}} p_1.v_{\omega_i} = 0$.

References

[BD14] T. Backhaus and C. Desczyk. PBW filtration: Feigin-Fourier-Littelmann modules via Hasse diagrams. arXiv:1407.73664, 2014.

[BF14] R. Biswal and G. Fourier. Minuscule Schubert varieties: Poset polytopes, PBW-degenerated Demazure modules, and Kogan faces. Preprint, 2014.

[CF13] I. Cherednik and E. Feigin. Extremal part of the PBW-filtration and E-polynomials. Preprint: arXiv:1306.3146, 2013.

[CIL14] G. Cerulli Irelli and M. Lanini. Degenerate flag varieties of type A and C are Schubert varieties. Preprint arXiv:1403.2889, 2014.

[CILL14] G. Cerulli Irelli, M. Lanini, and P. Littelmann. Degenerate flag varieties and Schubert varieties. Preprint, 2014.

[Fei12] Evgeny Feigin. $\mathbb{G}_a^M$ degeneration of flag varieties. *Selecta Math. (N.S.)*, 18(3):513–537, 2012.

[FFL11a] E. Feigin, G. Fourier, and P. Littelmann. PBW filtration and bases for irreducible modules in type A_n. *Transform. Groups*, 16(1):71–89, 2011.

[FFL11b] E. Feigin, G. Fourier, and P. Littelmann. PBW filtration and bases for symplectic Lie algebras. *Int. Math. Res. Not. IMRN*, 1(24):5760–5784, 2011.

[FFL13a] E. Feigin, G. Fourier, and P. Littelmann. Favourable modules: Filtrations, polytopes, Newton-Okounkov bodies and flat degenerations. arXiv:1306.1292v3, 2013.

[FFL13b] E. Feigin, G. Fourier, and P. Littelmann. PBW-filtration over $\mathbb{Z}$ and compatible bases for $v(\lambda)$ in type A_n and C_n. *Springer Proceedings in Mathematics and Statistics*, 40:35–63, 2013.

[FM14] E. Feigin and I. Makedonskyi. Nonsymmetric Macdonald polynomials, Demazure modules and PBW filtration. Preprint arXiv:1407.6316, 2014.

[Fou14a] G. Fourier. New homogeneous ideals for current algebras: Filtrations, fusion products and Pieri rules. Preprint: arXiv:1403.4758, 2014.

[Fou14b] G. Fourier. PBW-degenerated Demazure modules and Schubert varieties for triangular elements. arXiv:1408.6939, 2014.

[Gor11] A. Gornitsky. Essential signatures and canonical bases in irreducible representations of the group G_2. Diploma thesis, 2011.

[Hum72] James E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9.

[Kos59] B. Kostant. A formula for the multiplicity of a weight. *Trans. Amer. Math. Soc.*, 93:53–73, 1959.