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THE DEGREE OF THE HILBERT-POINCARÉ POLYNOMIAL
OF PBW-GRADED MODULES

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Abstract. In this note, we study the Hilbert-Poincaré polynomials for the
associated PBW-graded modules of simple modules for a simple complex Lie
algebra. The computation of their degree can be reduced to modules of funda-
mental highest weight. We provide these degrees explicitly.

Nous étudions les polynômes de Hilbert-Poincaré pour les modules PBW-
gradués associés aux modules simples d’une algèbre de Lie simple complexe.
Le calcul de leur degré peut être restreint aux modules de plus haut poids
fondamental. Nous donnons une formule explicite pour ces degrés.

1. Introduction

Let \( g \) be a simple complex finite-dimensional Lie algebra with triangular de-
composition \( g = n^+ \oplus h \oplus n^- \). Then the PBW filtration on \( U(n^-) \) is given as
\[
U(n^-)_s := \text{span}\{x_{i_1} \cdots x_{i_t} \mid x_{i_j} \in n^-, t \leq s\}.
\]
The associated graded algebra is isomorphic to \( S(n^-) \). Let \( V(\lambda) \) be a simple finite-dimensional module of highest
weight \( \lambda \) and \( v_\lambda \) a highest weight vector. Then we have an induced filtration on
\( V(\lambda) = U(n^-)v_\lambda \), denoted \( V(\lambda)_s := U(n^-)_sv_\lambda \). The associated graded module
\( V(\lambda)^a \) is a \( S(n^-) \)-module generated by \( v_\lambda \).

These modules have been studied in a series of papers. Monomial bases of the
graded modules and the annihilating ideals have been provided for the \( \mathfrak{sl}_n, \mathfrak{sp}_n \)
[FFL11a, FFL11b, FFL13b], for cominuscule weights and their multiples in other
types [BD14], for certain Demazure modules in the \( \mathfrak{sl}_n \)-case in [Fou14b, BF14].
In type \( G_2 \) there is a monomial basis provided by [Gor11].
The degenerations of the corresponding flag varieties have been studied in [Fei12,
FFL13a, CIL14, CILL14]. Further, it turned out ([Fou14a]), that these PBW
degenerations have an interesting connection to fusion product for current alge-
bras. The study of the characters of PBW-graded modules has been initiated in
[CF13, FM14].

In the present paper we will compute the maximal degree of PBW-graded mod-
ules in full generality (for all simple complex Lie algebras), where there have been
partial answers in the above series of paper for certain cases.
We denote the Hilbert-Poincaré series of the PBW-graded module, often referred
to as the \( q \)-dimension of the module, by
\[
p_\lambda(q) = \sum_{s=0}^{\infty} \left( \frac{\dim V(\lambda)_s}{V(\lambda)_{s-1}} \right) q^s.
\]
Since \( V(\lambda) \) is finite-dimensional, this is obviously a polynomial in \( q \). In this note
we want to study further properties of this polynomial. We see immediately that
the constant term of \( p_\lambda(q) \) is always 1 and the linear term is equal to
\[
\dim(n^-) - \dim \text{Ker } (n^- \rightarrow \text{End}(V(\lambda))).
\]
Our main goal is to compute the degree of $p_\lambda(q)$ and the first step is the following reduction [CF13, Theorem 5.3 ii]):

**Theorem.** Let $\lambda_1, \ldots, \lambda_s \in P^+$ and set $\lambda = \lambda_1 + \ldots + \lambda_s$. Then

$$\deg p_\lambda(q) = \deg p_{\lambda_1}(q) + \ldots + \deg p_{\lambda_s}(q).$$

It remains to compute the degree of $p_\lambda(q)$ where $\lambda$ is a fundamental weight. We have done this for all fundamental weights of simple complex finite-dimensional Lie algebras:

**Theorem 1.** The degree of $p_{\omega_i}(q)$ is equal to the label of the $i$-th node in the following diagrams:

\[
\begin{align*}
A_n & \quad B_n \\
1 & \quad 2 & 3 & \ldots & 3 & 2 & 1 \\
& \quad 2 & 2 & 4 & 4 & 6 & \ldots & \Rightarrow \frac{[8]}{2[\frac{m}{2}]} \\
C_n & \quad D_n \\
1 & \quad 2 & \cdots & n-2 & n-1 & n \\
& \quad 2 & 2 & 4 & 4 & 6 & \ldots & \Rightarrow \frac{[m+1]}{2[\frac{m}{2}]} \\
E_6 & \quad E_7 \\
2 & \quad 4 & \cdots & 6 \\
& \quad 2 & 4 & 2 \\
& \quad 2 & 6 & 8 & 7 & 4 & 3 \\
E_8 & \quad \Rightarrow \frac{8}{14} \\
& \quad 6 & 11 & 8 & 6 & 2 \\
& \quad 2 & 6 & 4 & 2 \\
F_4 & \quad G_2 \\
& \quad 2 & 6 & 4 & 2 \\
& \quad 2 & 2 \\
\end{align*}
\]

The paper is organized as follows: In Section 2 we introduce definitions and basic notations, in Section 3 we prove Theorem 1.

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2. Preliminaries

Let $\mathfrak{g}$ be a simple Lie algebra of rank $n$. We fix a Cartan subalgebra $\mathfrak{h}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The set of roots (resp. positive roots) of $\mathfrak{g}$ is denoted $R$ (resp. $R^+$), $\theta$ denotes the highest root. Let $\alpha_i, \omega_i$ $i = 1, \ldots, n$ be the simple roots and the fundamental weights. Let $W$ be the Weyl group associated to the simple roots and $w_0 \in W$ the longest element. For $\alpha \in R^+$ we fix a $\mathfrak{sl}_2$ triple $\{e_\alpha, f_\alpha, h_\alpha = [e_\alpha, f_\alpha]\}$. The integral weights and the dominant integral weights are denoted $P$ and $P^+$.

Let $\{x_1, x_2, \ldots\}$ be an ordered basis of $\mathfrak{g}$, then $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ with PBW basis $\{x_{i_1} \cdots x_{i_m} | m \in \mathbb{Z}_{\geq 0}, i_1 \leq i_2 \leq \ldots \leq i_m\}$. 
2.1. Modules. For $\lambda \in P^+$ we consider the irreducible $\mathfrak{g}$-module $V(\lambda)$ with highest weight $\lambda$. Then $V(\lambda)$ admits a decomposition into $\mathfrak{h}$-weight spaces, 

$$V(\lambda) = \bigoplus_{\tau \in P} V(\lambda)_\tau$$

with $V(\lambda)_\lambda$ and $V(\lambda)_{w_0(\lambda)}$, the highest and lowest weight spaces, being one dimensional. Let $v_\lambda$ denote the highest weight vector, $v_{w_0(\lambda)}$ denote the lowest weight vector satisfying

$$e_\alpha v_\lambda = 0, \forall \alpha \in R^+; f_\alpha v_{w_0(\lambda)} = 0, \forall \alpha \in R^+.$$ 

We have $U(n^-).v_\lambda \cong V(\lambda) \cong U(n^+).v_{w_0(\lambda)}$.

The comultiplication $(x \mapsto x \otimes 1 + 1 \otimes x)$ provides a $\mathfrak{g}$-module structure on $V(\lambda) \otimes V(\mu)$. This module decomposes into irreducible components, where the Cartan component generated by the highest weight vector $v_\lambda \otimes v_\mu$ is isomorphic to $V(\lambda + \mu)$.

2.2. PBW-filtration. The Hilbert-Poincaré series of the PBW-graded module $V(\lambda)^0 := \bigoplus_{s \geq 0} V(\lambda)_s/V(\lambda)_{s-1}$ is the polynomial

$$p_\lambda(q) = \sum_{s \geq 0} \dim(V(\lambda)_s/V(\lambda)_{s-1})q^s$$

$$= 1 + \dim(V(\lambda)_1/V(\lambda)_0)q + \dim(V(\lambda)_2/V(\lambda)_1)q^2 + ...$$

and we define the PBW-degree of $V(\lambda)$ to be $\deg(p_\lambda(q))$.

It is easy to see that $n^+(U(n^-)_s.v_\lambda) \subseteq U(n^-)_s.v_\lambda \forall s \geq 0$ (see also [FFL11a]) and hence $U(n^+).V(\lambda)_s \subseteq V(\lambda)_s$. Let $s_\lambda$ be minimal such that $v_{w_0(\lambda)} \in V(\lambda)_{s_\lambda}$.

Then $V(\lambda) = U(n^+).v_{w_0(\lambda)} \subseteq V(\lambda)_{s_\lambda}$ and

Corollary. $s_\lambda = \deg(p_\lambda(q))$ and

$$V(\lambda) = V(\lambda)_{s_\lambda}.$$ 

2.3. Graded weight spaces. The PBW filtration is compatible with the decomposition into $\mathfrak{h}$-weight spaces:

$$\dim V(\lambda)_\tau = \sum_{s \geq 0} \dim(V(\lambda)_s/V(\lambda)_{s-1}) \cap V(\lambda)_\tau.$$ 

So we can define for every weight $\tau$ the Hilbert-Poincaré polynomial:

$$p_{\lambda,\tau}(q) = \sum_{s \geq 0} \dim (V(\lambda)_s/V(\lambda)_{s-1})_\tau q^s$$

and then $p_\lambda(q) = \sum_{\tau \in P} p_{\lambda,\tau}(q)$.

A natural question is, if we can extend our results to these polynomials? If the weight space $V(\lambda)_\tau$ is one-dimensional, then $p_{\lambda,\tau}(q)$ is a power of $q$. For $\tau = \lambda$ this is constant 1, for $\tau = w_0(\lambda)$, the lowest weight, this is $q^{\deg(p_\lambda(q))}$ as we have seen in Corollary 2.2. A first approach to study these polynomials can be found in [CF13].

2.4. Graded Kostant partition function. For the readers convenience we recall here the graded Kostant partition function (see [Kos59]), which counts the number of decompositions of a fixed weight into a sum of positive roots, and how it is related to our study. We consider the power series and its expansion:

$$\prod_{\alpha > 0} \frac{1}{1 - qe^\alpha}, \sum_{\nu \in P} P_\nu(q)e^{\nu}.$$ 

We have immediately $\text{char} S(n^-) = \sum_{\nu \in P} P_\nu(q)e^{-\nu}$. 
Remark. For a polynomial \( p(q) = \sum_{i=0}^{n} a_i q^i \), we denote \( \mindeg p(q) \) the minimal \( j \) such that \( a_j \neq 0 \). Then we have obviously
\[
(2.1) \quad \mindeg p_{\lambda, \nu}(q) \geq \mindeg P_{\lambda, \nu}(q).
\]
We will use this inequality for the very special case \( \nu = w_0(\lambda) \) in the proof of Theorem 1.

We see from Theorem 1 that this inequality is a proper inequality for certain cases in exceptional type as well as \( B_7 D_7 \) (this has been noticed also in [CF13]).

3. Proof of Theorem 1

In this section we will provide a proof of Theorem 1. For a fixed \( 1 \leq i \leq \text{rank } \mathfrak{g} \), we will give a monomial \( u \in U(n^-) \) of the predicted degree mapping the highest weight vector \( v_{\omega_i} \) to the lowest weight vector \( v_{w_0(\omega_i)} \). We then show that there is no monomial of smaller degree satisfying this.

To write down these monomials explicitly, let us denote \( \theta_{X_n} \) the highest root of a Lie algebra of type \( X_n \). We set further (using the indexing from [Hum72]):

- In the \( A_n \)-case, \( Y_{n-2} \) the type of the Lie algebra generated by the simple roots \( \{\alpha_2, \ldots, \alpha_{n-1}\} \).
- In the \( B_n, D_n \)-case, \( Y_{n-k} \) the type of the Lie algebra generated by the simple roots \( \{\alpha_{k+1}, \ldots, \alpha_n\} \).
- In the exceptional and symplectic cases, \( \theta_{X_n} = \epsilon_k \omega_k \) for some \( k \), \( Y_{n-1} \) the type of the Lie algebra generated by the simple roots \( \{\alpha_1, \ldots, \alpha_n\} \setminus \{\alpha_k\} \).

Let \( u \in U(n^-) \) be one of the monomials in Figure 1. It can be seen easily from Figure 1 that \( u = f_{\theta_X}^{\alpha} \cdot u_1 \), where \( a_i^{\check{\gamma}} = w_i(h_{\theta_{X_n}}) \) and \( u_1 \) is the monomial in Figure 1 corresponding to the restriction of \( \omega_i \) to the Lie subalgebra of type \( Y_{n-\ell} \). If we denote \( \pi^- \) the lower part in the triangular decomposition of the Lie subalgebra of type \( Y_{n-\ell} \), then \( u_1 \in U(\pi^-) \).

Let \( u = f_{\theta_1}^{b_1} \cdot f_{\theta_2}^{b_2} \cdots f_{\theta_r}^{b_r} \cdot u_1 \). Note that all \( f_{\theta_i} \) commute and it is easy to see that \( \theta_j(h_{\theta_{j+p}}) = 0, \forall p \geq 0 \) (since \( \theta_j \) is a sum of fundamental weights, which are all orthogonal to the simple roots of the Lie algebra with highest root \( \theta_{j+p} \)) and \( b_j = \omega_i(h_{\theta_j}) \).

The Weyl group \( W \) acts on \( V(\omega_i) \) and if \( v \) is an extremal weight vector of weight \( \mu \), then \( w.v \) is a nonzero extremal weight vector of weight \( w(\mu) \). Further if \( w = s_\alpha \) (reflection at a root \( \alpha \)) and \( \mu(h_\alpha) \geq 0 \), then \( w(\mu) = c^* f^\mu(h\alpha) \cdot v \) for some \( c^* \in \mathbb{C}^* \).

Now consider \( w = s_{\theta_j} \cdots s_{\theta_1} \), where \( s_{\theta_j} \) is the reflection at the root \( \theta_j \). Then we have \( w.v_{\omega_i} = v_{w_0(\omega_i)} = u.v_{\omega_i} \neq 0 \) in \( V(\omega_i) \). So we obtain an upper estimate for the degree.

In general the degree of \( u \) is bigger than the minimal degree coming from Kostant’s graded partition function \( (2.1) \). For \( A_n, C_n \) the degrees coincide and hence we are done in these cases.

We will prove Theorem 1 for the remaining cases \( X_n \) by induction on the rank of the Lie algebra. So we want to prove that if \( p \in U(n^-) \) with \( p.v_{\omega_i} = v_{w_0(\omega_i)} \) then \( \deg(p) \geq \deg(u) \), where \( u \) is from Figure 1.

Consider the induction start, e.g. \( \omega_i = \theta_{X_n} \), then the minimal degree is obviously 2. The maximal non-vanishing power of \( f_{\theta_{X_n}}^{\alpha} \) is certainly \( a_i^{\check{\gamma}} \) and \( f_{\theta_{X_n}}^{\alpha} \cdot v_{\omega_i} \) is the highest weight vector of a simple module of fundamental weight for the Lie algebra \( Y_{n-\ell} \) defined as above. By induction we know that if \( q \in U(\pi^-) \) with \( q.f_{\theta_{X_n}}^{\alpha} \cdot v_{\omega_i} = v_{w_0(\omega_i)} \) then \( \deg(q) \geq \deg(u_1) \).
First we suppose \( f_{\theta X_n}^i \cdot v_{\omega_i} \) is a factor of \( p \), so \( p = f_{\theta X_n}^i \cdot p' \) and then by weight considerations \( p' \in U(n^-) \). Then \( p', (f_{\theta X_n}^i \cdot v_{\omega_i}) = v_{w_0(\omega_i)} \) (the lowest weight vector in \( V(\omega_i) \) as well as in the simple submodule). Therefore \( \deg(p') \geq \deg(u) \) which implies \( \deg(p) \geq \deg(u) \).

Suppose now the maximal power of \( f_{\theta X_n} \) in \( p \) is \( f_{\theta X_n}^{\alpha^\vee - k} \), \( k \geq 0 \) and \( \deg(p) < \deg(u) \).

Let \( X_n \) be of type \( B_n \), \( D_n \) or exceptional, then \( \theta X_n = \omega_j \) and we denote

\[ R^+_s = \{ \alpha \in R^+ | w_j(h_\alpha) = s \} , \]
Then $R^+ = \{ \theta_{X_n} \}$ and if $\beta \in R^+_{1}$ then $\theta_{X_n} - \beta \in R^+_{1}$. By weight reasons $p = f^\theta_{X_n}\beta_1 \cdots f^\theta_{X_n}\beta_{2k} \frac{p_1}{\beta_1 \cdots \beta_{2k}} \in R^+_0$ and some polynomial $p_1$ in root vectors of roots in $R^+_0$. We have to show that $p.v_{\omega_i} = 0 \in V(\omega_i)^a$ and we will use induction on $k$ for that: The induction start is $k = 0$. The induction step is for $k \geq 1$:

$$0 = p_1 f^\theta_{X_n}\beta_1 \cdots f^\theta_{X_n}\beta_{2k} \frac{p_1}{\beta_1 \cdots \beta_{2k}}.v_{\omega_i} = (e_{\theta_{X_n}} - \beta_1) \cdots (e_{\theta_{X_n}} - \beta_{2k}) p_1 f^\theta_{X_n}.v_{\omega_i} = c f^\theta_{X_n}\beta_1 \cdots f^\theta_{X_n}\beta_{2k} p_1. v_{\omega_i} + \sum_{\ell > 0} f^{\theta_{X_n}}_{\beta_1 \cdots \beta_{2k}} q_0^{\ell}.v_{\omega_i}$$

for some $c \in \mathbb{C}^*$, $q_0^{\ell} \in U(n^-)$. For $0 \leq \ell < k$ all the summands are equals to zero by induction (on $k$). For $\ell = k$ we recall our assumption $\deg(p) < \deg(u)$ and so $\deg(q_k) < \deg(u_1)$ which implies $f^{\theta_{X_n}}_{\beta_1 \cdots \beta_{2k}}.v_{\omega_i} = 0$. So we can conclude $f^{\theta_{X_n}}_{\beta_1 \cdots \beta_{2k}} p_1.v_{\omega_i} = 0$.

**References**


