



Donald, G.C., Davies, C.T.H., Koponen, J., and Lepage, G.P. (2014) Prediction of the  $D_s$  width from a calculation of its radiative decay in full lattice QCD. *Physical Review Letters*, 112 (212002). ISSN 0031-9007

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# Prediction of the $D_s^*$ width from a calculation of its radiative decay in full lattice QCD

G. C. Donald,<sup>1,\*</sup> C. T. H. Davies,<sup>2,†</sup> J. Koponen,<sup>2</sup> and G. P. Lepage<sup>3</sup>  
(HPQCD collaboration)<sup>‡</sup>

<sup>1</sup>*School of Mathematics, Trinity College, Dublin 2, Ireland*

<sup>2</sup>*SUPA, School of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK*

<sup>3</sup>*Laboratory of Elementary-Particle Physics, Cornell University, Ithaca, New York 14853, USA*

(Dated: May 6, 2014)

We determine the rate for  $D_s^* \rightarrow D_s \gamma$  for the first time from lattice QCD and include the full effect of  $u$ ,  $d$  and  $s$  sea quarks. The valence quarks are implemented using the Highly Improved Staggered Quark (HISQ) formalism and we normalise the vector current nonperturbatively. We obtain  $M(D_s^*) - M(D_s)$  of 148(4) MeV, in good agreement with experiment. The value of the decay constant of the  $D_s^*$  we find to be 274(6) MeV, so that  $f_{D_s^*}/f_{D_s} = 1.10(2)$ . For the radiative decay we find  $\Gamma(D_s^* \rightarrow D_s \gamma) = 0.066(26)$  keV. Given the experimental branching fraction for this decay we predict a total width for the  $D_s^*$  of 0.070(28) keV, making this the longest lived charged vector meson. We also determine the branching fraction for  $D_s^*$  leptonic decay to be  $3.4(1.4) \times 10^{-5}$ , putting it within the range of possible detection at next generation flavor factories.

## INTRODUCTION.

Lattice QCD calculations offer the prospect of increasingly accurate results for both hadron masses and simple hadronic matrix elements, decay constants and form factors, that give information about the hadron's internal structure. The hadronic matrix elements can be directly connected to experiment since they control the rate for simple weak and electromagnetic processes. For weak processes the connection to experiment involves an element from the Cabibbo-Kobayashi-Maskawa (CKM) matrix and we can use this to determine the CKM elements. For electromagnetic processes there is no CKM factor and so the comparison is more direct. Where electromagnetic decay rates have been measured experimentally, they then provide strong tests of the calculation of QCD matrix elements that are very similar to the ones that appear in weak decays.

For the electromagnetic decays of charged charmed vector mesons we have a particularly interesting situation in which there is significant destructive interference between radiation of the photon from the  $c$  quark and from the light quark. This makes the results very sensitive to the different contributions to the decay matrix element and therefore a stringent test of the calculation.

Here we study  $D_s^* \rightarrow D_s \gamma$  decay using lattice QCD for the first time. We are able to calculate the rate for the decay accurately by using gluon field configurations that include the full effect of  $u$ ,  $d$  and  $s$  quarks in the sea at two values of the lattice spacing, by having a formalism for the quarks with very small discretisation errors and because we are able to normalise the current that couples to the photon fully nonperturbatively. As described below, we find that the total form factor for the  $D_s^*$  decay is only 20% of that for photon emission from the  $s$  quark alone, so that the total rate for the electromagnetic decay is very highly suppressed. It nevertheless represents 94.2(7)% of the branching fraction [1] and so we find that

the total width of the  $D_s^*$  is the narrowest of any of the vector mesons containing a  $c$  quark. Only the  $B_s^*$  is expected to be narrower.

We also provide further important tests of QCD through our accurate determination of the  $D_s^*$  mass and its decay constant. The mass determination adds to the growing set of gold-plated meson masses from lattice QCD that are tested at the few MeV level against experiment. The decay constant allows us to determine the leptonic annihilation rate for the  $D_s^*$ , which is much larger than that of the  $D_s$  because it is not helicity suppressed. The rate is (only) 5 orders of magnitude smaller than the electromagnetic decay rate. Although small, this is the largest branching fraction for annihilation to a  $W$  boson for any vector meson.

## LATTICE CALCULATION

For the lattice QCD calculation we use the Highly Improved Staggered Quark action [2] for all the valence quarks. This action has very small discretisation errors, making it an excellent action for both  $c$  and  $s$  [2–5]. We calculate HISQ propagators on gluon field configurations generated by the MILC collaboration that include  $u$ ,  $d$  and  $s$  sea quarks using the asqtad formalism [6]. Table I gives the parameters of the ensembles of configurations we use, with two different lattice spacing values and two different  $u/d$  sea quark masses. Analysis of results from [4] shows that this is a good set of ensembles to capture at the few percent level the lattice spacing and sea quark mass effects expected here.

To tune the  $s$  and  $c$  quark masses to their correct physical values we use the pseudoscalar  $\eta_s$  and  $\eta_c$  meson masses [4, 8]. The  $\eta_s$  is a fictitious  $s\bar{s}$  pseudoscalar that is not allowed to decay in lattice QCD, and so does not correspond to a particle in the real world. It is useful because its mass can be accurately determined in lattice

Set	$r_1/a$	$au_0m_l^{asq}$	$au_0m_s^{asq}$	$L_s/a \times L_t/a$	$n_{cfg}$	$T/a$
1	2.647(3)	0.005	0.05	$24 \times 64$	2088	12,15,18
2	2.618(3)	0.01	0.05	$20 \times 64$	2259	12,15,18
3	3.699(3)	0.0062	0.031	$28 \times 96$	1911	16,20,23

TABLE I: Ensembles (sets) of MILC configurations used here. Sea (asqtad) quark masses  $m_\ell^{asq}$  ( $\ell = u/d$ ) and  $m_s^{asq}$  use the MILC convention where  $u_0$  is the plaquette tadpole parameter. The lattice spacing,  $a$ , is given in units of  $r_1$  after ‘smoothing’ [6]. We use  $r_1 = 0.3133(23)$  fm [7]. Sets 1 and 2 are ‘coarse’ ( $a \approx 0.12$  fm) and set 3, ‘fine’ ( $a \approx 0.09$  fm). The lattice size is given by  $L_s^3 \times L_t$ . We use 4 time sources on each of the  $n_{cfg}$  configurations and 3 values of  $T/a$  (Fig. 2).

QCD as  $0.6858(40)$  GeV [7]. For the  $c$  quark mass here we use the  $\eta_c$  mass [4] in a world without electromagnetism or  $c$  sea quarks:  $M_{\eta_c} = 2.985(3)$  GeV [9].

The HISQ  $s$  and  $c$  quark propagators calculated on these gluon fields are combined to make meson correlators for  $D_s^*$  and  $D_s$  mesons. For the  $D_s$  we use the local pseudoscalar operator which, in combination with the quark mass, is absolutely normalised [4]. For the  $D_s^*$  we use the local vector operator, whose normalisation can be determined fully nonperturbatively as described in [8]. The  $Z$  factors from [8] are reproduced in Table II. The correlators are fit to a multi-exponential form using a Bayesian approach [10] so that we can include systematic errors from the presence of higher excitations in the correlator when extracting the ground-state mass and amplitude. The excited state parameters are loosely constrained by prior values and widths; splittings between excited states are given a prior of  $600(300)$  MeV and amplitudes are given priors of  $0.01(1.0)$ . The ground-state masses obtained from a 6-exponential fit are given in Table II along with the decay constants determined from the ground-state amplitude as discussed in [4, 5].

The mass difference between  $D_s^*$  and  $D_s$  and decay constant ratio are plotted in Fig. 1, showing mild dependence on the lattice spacing,  $a$ , and almost none on the sea quark mass. We fit the lattice results to:

$$f(a^2, \delta x_m) = f_0 \times \left( 1 + \sum_{i=1}^5 c_i \left( \frac{am_c}{2} \right)^{2i} + \sum_{j=1}^2 \chi_j \left( \frac{\delta x_m}{10} \right)^j \right) \quad (1)$$

allowing for  $a$ -dependence controlled by  $m_c$  [4] (consistent with that seen) and sea-mass dependence expected from chiral perturbation theory.  $\delta x_m$  is the discrepancy between the sum of sea quark masses,  $2m_l + m_s$ , and their physical value in units of the physical  $s$  quark mass (from [4]). Dividing by 10 converts the denominator to be close to the scale of corrections in chiral perturbation theory ( $4\pi f_\pi \approx 1$  GeV). We have checked that allowing for separate dependence on sea  $m_l$  and  $m_s$  makes no difference. We take priors on  $c_j$  and  $\chi_j$  to be  $0.0(1.0)$  apart from  $c_1$  which we take to be  $0.0(0.3)$  since  $a^2$  errors are suppressed by  $\alpha_s$  for the HISQ action [2]. Constraining

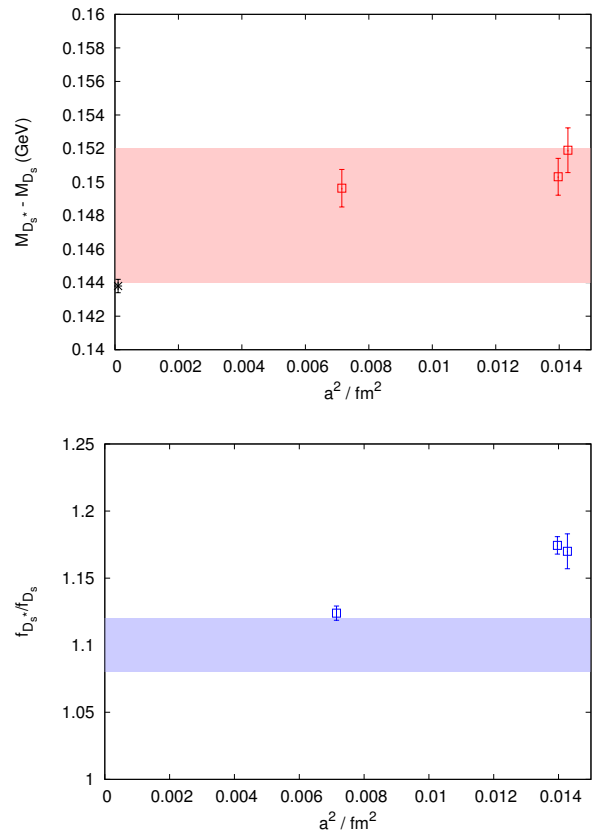


FIG. 1: Upper plot: The difference in mass between the  $c\bar{s}$  vector  $D_s^*$  and pseudoscalar  $D_s$  as a function of  $a^2$ . The experimental result [1] is shown as a black burst. The red shaded band gives the lattice result with its total error in the continuum and chiral limits from the fit described in the text. Lower plot: A similar plot for the ratio of  $D_s^*$  and  $D_s$  decay constants. The blue shaded band shows the physical result obtained from the corresponding fit.

ing the  $c_j$  allows us to include 5 terms in  $a^2$  and take fully into account systematic errors from higher order  $a$ -dependence; similar results (and errors) are obtained from only including two terms. The final error on the physical value,  $f_0$ , is dominated here by the  $a \rightarrow 0$  extrapolation error and is roughly proportional to the width of the  $c_j$  and independent of the other priors.

The physical result that we obtain for  $M_{D_s^*} - M_{D_s}$  is  $148(3)(2)$  MeV, where the first error is from the extrapolation and the second from the (correlated) uncertainty in the lattice spacing determination [7] which has a double impact on hyperfine splittings because of their sensitivity to tuning the quark masses [5]. This is in good agreement with the experimental average of  $143.8(4)$  MeV [1]. Note that there are no additional systematic uncertainties from missing electromagnetism or charm-in-the-sea because the leading effects of these, already small, cancel between  $D_s^*$  and  $D_s$  [4, 5].

Our physical result for  $f_{D_s^*} / f_{D_s}$  is  $1.10(2)$ , clearly

Set	$am_s$	$am_c$	$aM_{D_s}$	$aM_{D_s^*}$	$af_{D_s}$	$af_{D_s^*}/Z_{cs}$	$V_c(0)/Z_{cc}$	$V_s(0)/Z_{ss}$	$Z_{cs}$	$Z_{cc}$	$Z_{ss}$
1	0.0489	0.622	1.18976(17)	1.2800(7)	0.15435(18)	0.1765(9)	1.21(9)	3.02(15)	1.027(3)	0.9896(11)	1.007(12)
2	0.0496	0.630	1.20209(21)	1.2942(9)	0.15641(24)	0.1793(11)	1.33(6)	3.24(12)	1.020(10)	0.9894(8)	1.003(9)
3	0.0337	0.413	0.84701(12)	0.9112(5)	0.10790(11)	0.1202(5)	1.22(7)	2.95(18)	1.009(2)	1.0049(10)	1.009(11)

TABLE II: Results for the masses of the  $D_s$  and  $D_s^*$  mesons and the  $D_s$  and  $D_s^*$  decay constants in lattice units for the HISQ valence  $c$  and  $s$  masses given in columns 2 and 3. Columns 8 and 9 give the vector form factors at  $q^2 = 0$  for the cases where the photon couples to the  $c$  or  $s$  quarks. We also give the  $Z$  factors we need to include for the  $\bar{c}s$ ,  $\bar{c}\bar{c}$  and  $\bar{s}s$  vector currents.

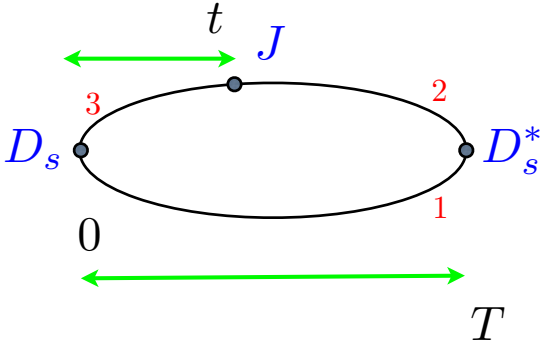


FIG. 2: A schematic diagram of the 3-point function for  $D_s^* \rightarrow D_s \gamma$  decay.  $J$  is a vector current which can couple either to the  $s$  quark or the  $c$  quark in the  $D_s^*$ .

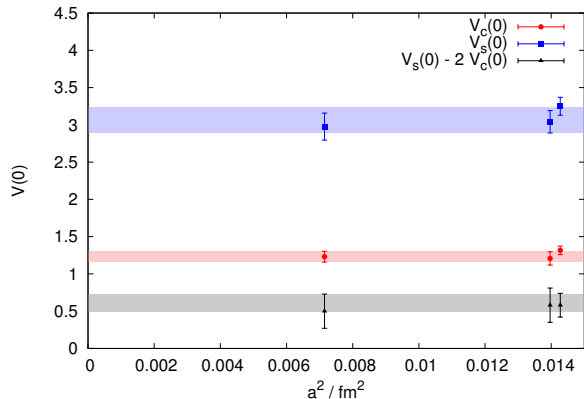


FIG. 3: The vector form factor for  $D_s^* \rightarrow D_s \gamma$  at  $q^2 = 0$  for a transition via a  $c\bar{c}$  current (denoted  $V_c(0)$ ) and via an  $s\bar{s}$  current (denoted  $V_s(0)$ ). The form factors are plotted against  $a^2$ . We also show  $V_s(0) - 2V_c(0)$  which is three times the total form factor which appears in the rate for  $D_s^* \rightarrow D_s \gamma$  (eq. 4).

greater than 1. Combined with our earlier result for  $f_{D_s}$  of 248.0(2.5) MeV [4] (with which we agree here but with larger errors) we predict  $f_{D_s^*} = 274(6)$  MeV.

A 3-point correlation function that allows us to calculate the  $D_s^*$  to  $D_s$  transition matrix element is sketched in Fig. 2, with  $J$  representing the vector current that couples to the photon. Writing the relevant piece of the

electromagnetic current as

$$J = \frac{2e}{3} \mathcal{V}_{c\bar{c}} - \frac{e}{3} \mathcal{V}_{s\bar{s}}, \quad (2)$$

shows that we have to consider two configurations. In one, propagators 2 and 3 are  $c$  quarks and 1 is an  $s$  quark. In the second, 2 and 3 are  $s$  quarks and 1 is a  $c$  quark. Since the photon produced in  $D_s^* \rightarrow D_s \gamma$  decay is real, we need to tune the momentum of the  $D_s$  in the rest frame of the  $D_s^*$  so that the square of the 4-momentum transferred,  $q^2$ , is zero. This is done by calculating propagator 3 with a ‘twisted boundary condition’ [11, 12], to give it a small, tuned spatial momentum.

When making correlation functions with staggered quarks we have a choice of operators because every meson comes in 16 ‘tastes’ that differ by effects proportional to  $a^2$  [2]. In a 3-point function the taste combinations at the 3 points must cancel. Here, for  $D_s^* \rightarrow D_s \gamma$ , we follow the procedure developed for  $J/\psi \rightarrow \eta_c \gamma$  [5]. We take the  $D_s$  to be the ‘Goldstone’ pseudoscalar (in taste-spin notation  $\gamma_5 \otimes \gamma_5$ ), the  $D_s^*$  uses a 1-link operator ( $\gamma_0 \gamma_i \otimes \gamma_0 \gamma_i \gamma_j$ ) and then the vector current is a local vector  $\gamma_k \otimes \gamma_k$ . We can normalise this vector current fully nonperturbatively using the techniques described in [5, 8].

The 3-point functions for  $D_s^* \rightarrow D_s$  are calculated for all  $t$  values from 0 to  $T$  and for 3 values of  $T$  (see Table I) so that the dependence of the function on  $t$  and  $T$  can be fully mapped out. Both 3-point functions are fit simultaneously with the 2-point functions for the  $D_s$  and  $D_s^*$  using the operators discussed above at source and sink. The fit functions have a multi-exponential form as given in [5], and we use the same Bayesian approach and priors described for the 2-point correlators above.

The quantity that we extract from the fit is the vector current matrix element  $\langle D_s^* | \mathcal{V} | D_s \rangle$  between the ground-state particles in the  $D_s$  and  $D_s^*$  channels for each current in eq. 2. The corresponding vector form factor  $V(q^2)$  is given by

$$Z \langle D_s^*(p', \varepsilon) | \mathcal{V}^\mu | D_s(p) \rangle = \frac{2e^{\mu\alpha\beta t}}{m_{D_s} + m_{D_s^*}} \varepsilon_\alpha^* p_\beta p'_t V(q^2), \quad (3)$$

where we have allowed for a renormalisation of the lattice vector current. Note that for a non-zero answer all the vectors have to point in different directions. The  $D_s^*$  is at rest so its momentum only has a component in

the  $t$  direction. Results for emission from the  $s$  quark,  $V_s(0)/Z_{s\bar{s}}$ , and from the  $c$  quark,  $V_c(0)/Z_{c\bar{c}}$ , are given separately in Table II along with the appropriate  $Z$  factors from [5, 8]. The  $D_s^*$  masses obtained from the fit are consistent with those from the local vector operator but with larger uncertainties.

To calculate the rate for  $D_s^* \rightarrow D_s \gamma$  decay we must combine  $V_s$  and  $V_c$  into the decay amplitude. There are two cancelling factors of (-1) in doing this. The  $D_s^*$  contains a quark and an antiquark with the same sign of electric charge (so changing the relative sign in eq. 2). The transition requires a spin-flip to convert a vector (with symmetric spin configuration) into a pseudoscalar (with antisymmetric spin configuration). This gives another relative minus sign between the two contributions. Thus the total form factor is  $V_{\text{tot}}(0) = [V_s(0) - 2V_c(0)]/3$  and the partial width for the decay is given by

$$\Gamma_{(D_s^* \rightarrow D_s \gamma)} = \alpha_{QED} \frac{4|\vec{q}|^3}{3(M_{D_s} + M_{D_s^*})^2} \frac{|V_s(0) - 2V_c(0)|^2}{9}. \quad (4)$$

Here  $|\vec{q}|$  is the magnitude of the momentum of the  $D_s$  in the  $D_s^*$  rest frame and takes value 138.9(6) MeV using the experimental masses for  $D_s$  and  $D_s^*$  [1].

Fig. 3 shows our results for  $V_s(0)$ ,  $V_c(0)$  and  $3V_{\text{tot}}(0)$  as a function of lattice spacing. Very little dependence on the lattice spacing or sea quark masses is seen. We fit the results as a function of  $a$  and  $\delta x_m$  to the form given in Eq. 1, obtaining physical results  $V_s(0)=3.07(17)$ ,  $V_c(0)=1.23(7)$  and  $3V_{\text{tot}}(0)=0.61(12)$ . Correlations between  $V_c(0)$  and  $V_s(0)$  reduce the error in  $V_{\text{tot}}(0)$ . Our result for  $V_{\text{tot}}(0)$  gives a partial width for the transition of 0.066(26) keV. The error is dominated by the lattice statistical error as the result of subtracting two form factors that almost cancel.

Fig. 3 shows several interesting features. One is the relative size of  $V_c(0)$  and  $V_s(0)$ . Since the transition is an M1 transition we expect the form factor to be proportional to the quark velocity in the meson. For a charm-strange meson, if each of the quarks has momentum of  $\mathcal{O}(\Lambda_{QCD})$ , the  $s$  quark will be relativistic but the  $c$  quark will have a velocity of  $\Lambda_{QCD}/m_c \approx 1/3$ . We can therefore readily explain the factor of around 3 between the two form factors. In fact the factor is less than 3 which means that the destructive interference between the two form factors is even more severe, given that the electric charge ratio is 2. We can also compare  $V_c(0)$  here to that for  $J/\psi \rightarrow \eta_c$  decay where we obtained a value of 1.90(7) [5]. Here we have a significantly lower result showing that the form factor is pushed down by the presence of a lighter (here  $s$ ) spectator quark. This is consistent with the velocity of a  $c$  quark in  $J/\psi$  being higher than in a  $D_s^*$ .

## DISCUSSION/CONCLUSIONS

We find the partial width for electromagnetic decay of the  $D_s^*$  to be very small, 0.066(26) keV. The branching fraction for this decay is measured to be 94.2(7)% [13], giving a total width for the  $D_s^*$  of 0.070(28) keV and a lifetime of  $9.4(3.8) \times 10^{-18}$ s. Estimates for the  $D_s^*$  radiative decay width using a variety of non-lattice techniques give a range of results [14] ranging from 0.06keV to 0.4keV. Our calculation shows that the width is at the lower end of this range. Only the  $B_s^*$  is likely to be longer-lived [14]. In that case the total form factor is a sum of  $V_s(0)$  and  $V_b(0)$  but the kinematic factors reduce the rate.

The only other measured decay rate for the  $D_s^*$  is that to  $D_s \pi^0$ , a Zweig-suppressed isospin-violating  $P$ -wave decay. This has the remaining 5.8% branching fraction which, given our total width, corresponds to a partial width of  $4.0 \times 10^{-3}$ keV. We can parameterise this hadronic decay in terms of a  $D_s^* D_s \pi$  coupling as

$$\Gamma_{(D_s^* \rightarrow D_s \pi^0)} = \frac{g_{D_s^* D_s \pi}^2}{24\pi M_{D_s^*}^2} p_\pi^3 \quad (5)$$

where  $p_\pi$  is the  $\pi$  momentum in the  $D_s^*$  rest frame (47.8(3.2) MeV [1]). This gives a result for  $g_{D_s^* D_s \pi}$  of 0.112(11)<sub>expt</sub>(24)<sub>latt</sub>, to be compared to that from the Zweig-allowed  $D^* \rightarrow D \pi^0$  transition of 12.7(1.3) [15].

For  $D^{*+}$  the electromagnetic decay has a much smaller rate than the hadronic decay. Its total width [16] and branching fraction [1] yield a partial width for the electromagnetic decay of 1.3(3) keV. This is 20(6) times larger than the result we find for the  $D_s^*$  and implies a much weaker cancellation of  $V_c(0)$  against  $V_d(0)$  than that between  $V_c(0)$  and  $V_s(0)$  in the  $D_s^*$ . Comparison of  $V_c(0)$  for  $D_s^*$  to that for  $J/\psi$  [5] indicate that  $V_c(0)$  for the  $D^*$  could be lower still. It also seems likely that  $V_d(0)$  for  $D^{*+}$  will be higher than  $V_s(0)$  for  $D_s^*$  based on similar arguments. 20% or larger shifts would be needed in both of the two directions to reach the experimental result. A direct calculation in lattice QCD for the  $D^{*+}$  can of course now be done following the method we have given here for the  $D_s^*$ .

Our result for the  $D_s^*$  decay constant shows it to be 10% higher than that for the  $D_s$ . The ratio of vector to pseudoscalar decay constants is predicted in Heavy Quark Effective Theory to be less than 1 [17] in the infinite heavy quark mass limit, but to be larger than 1 for  $c$  and  $b$  quarks when  $1/m_Q$  effects are included. A recent value for  $f_{D_s^*}/f_{D_s}$  from QCD sum rules is 1.32(10) [18], and from lattice QCD with  $u/d$  quarks (only) in the sea is 1.26(3) [19] (statistical errors only). See also [20].

Our ratio of 1.10(2) for  $D_s^*/D_s$  is closer to 1 than these earlier expectations, and therefore we can expect the ratio at  $b$  to be even smaller. Going down in mass to the  $s$  quark, results in lattice QCD for the  $\eta_s$  [7] and experiment for the  $\phi$  [1, 8] show a consistent picture with

a larger ratio:  $f_\phi/f_{\eta_s} = 1.26(2)$ . Our decay constant calculation is also complemented by the good agreement with experiment of our accurate result for the  $D_s^*$  mass, to be expected of lattice QCD calculations with  $u$ ,  $d$  and  $s$  sea quarks for a gold-plated particle.

Our result  $f_{D_s^*} = 274(6)$  MeV can be used to determine the weak leptonic decay rate from:

$$\Gamma_{(D_s^* \rightarrow \ell \nu)} = \frac{G_F^2}{12\pi} |V_{cs}|^2 f_{D_s^*}^2 M_{D_s^*}^3 \left(1 - \frac{m_\ell^2}{M_{D_s^*}^2}\right)^2 \left(1 + \frac{m_\ell^2}{2M_{D_s^*}^2}\right). \quad (6)$$

Our result for  $f_{D_s^*}$  gives a partial width of  $2.4(1) \times 10^{-6}$  keV for this decay (for  $e$  or  $\mu$ ) and hence branching fraction, using our determination of the total width, of  $3.4(1.4) \times 10^{-5}$ . This is only a factor of 200 smaller than the branching fraction for  $D_s \rightarrow \mu \nu$  [1], and so detection may be possible with the increased datasets expected at the next generation of flavor factories, for example Belle II [21]. This then offers potentially the best prospect of measuring a weak annihilation rate for a vector meson for the first time.

*Acknowledgements.* We are grateful to the MILC collaboration for the use of their gauge configurations and to R. Briere and E. Follana for useful discussions. We used the Darwin Supercomputer as part of STFC's DiRAC facility jointly funded by STFC, BIS and the Universities of Cambridge and Glasgow. This work was funded by STFC, the Royal Society and the Wolfson Foundation.

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\* donaldg@tcd.ie

† christine.davies@glasgow.ac.uk

‡ URL: <http://www.physics.gla.ac.uk/HPQCD>

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