

tt^* -GEOMETRY ON THE BIG PHASE SPACE

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Abstract: The big phase space, the geometric setting for the study of quantum cohomology with gravitational descendants, is a complex manifold and consists of an infinite number of copies of the small phase space. The aim of this paper is to define a Hermitian geometry on the big phase space.

Using the approach of Dijkgraaf and Witten [2], we lift various geometric structures of the small phase space to the big phase space. The main results of our paper state that various notions from tt^* -geometry are preserved under such liftings.

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1. INTRODUCTION

The big phase space M^∞ - the geometric arena for the study of quantum cohomology and topological quantum field theories with gravitational descendants - consists of an infinite number of copies of the small phase space

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M . Typically M is the cohomology ring of some smooth projective variety, so

$$M^\infty = \prod_{n \geq 0} H^*(V; \mathbb{C}).$$

The Poincaré pairing on $H^*(V; \mathbb{C})$ does not lift canonically to M^∞ and certain lifts of this pairing that can be defined on the big phase space are highly degenerate [17]. Thus from a differential geometric point of view the big phase space is hard to study. However, in a talk given at the 2006 ICM, Liu [16] defined a non-degenerate metric on M^∞ which is a natural lift of the Poincaré pairing, namely

$$\hat{\eta}(T^n(\gamma_\alpha), T^m(\gamma_\beta)) = \delta_{mn} \eta(\gamma_\alpha, \gamma_\beta) = \delta_{mn} \eta_{\alpha\beta}.$$

Note that this is defined in terms of the Poincaré pairing on M and a certain endomorphism T , which encapsulates the properties of the topological recursion operator.

The aim of this paper is to study Hermitian structures on M^∞ . This will be achieved by coupling the above idea of Liu with original ideas of Witten and Dijkgraaf [2], which relate two-point correlator functions on M to two-point correlator functions on M^∞ . As it turns out, this procedure can be used to lift structures such as the tt^* -equations of Cecotti and Vafa [1] from M to M^∞ . At the centre of the theory developed in this paper lies the following result:

Theorem 1. *Suppose that the tt^* -equations for a pseudo-Hermitian metric h and Higgs field C are satisfied on M , so*

$$\partial^D C = 0, \quad {}^D R + [C, C^\dagger] = 0.$$

Then there exists a natural lift of the pseudo-Hermitian metric and Higgs field, so that the tt^ -equations are satisfied on M^∞ .*

We also show that the Saito structure on $T^{1,0}M$ lifts to a Saito structure on $T^{1,0}M^\infty$ and that various other substructures on M , compatible with the Saito structure or governed by the tt^* -equations (such as real Saito structures, harmonic real Saito structures, harmonic potential real Saito structures, $DChk$ -structures and CV -structures), can be similarly lifted to M^∞ .

1.1. Background. The study of Gromov-Witten invariants and intersection theory on the moduli space of curves has provided the impetus for many recent developments in mathematics and mathematical physics. By studying integrals of products of ψ -classes over moduli spaces

$$\langle \tau_{a_1} \dots \tau_{a_n} \rangle_g := \int_{\overline{M}_{g,n}} \psi_1^{a_1} \dots \psi_n^{a_n}$$

Witten [22] derived three basic equations: the string equation, the dilaton equation and the topological recursion relation, and these, used recursively, enabled the invariants to be constructed. Such invariants may be combined into a generating function and it was conjectured by Witten, and later proved by Kontsevich [13], that this is a certain solution of the KdV hierarchy. These basic relations may then be generalized and raised to the status of axioms of a topological quantum field theory (TQFT).

Consider a smooth projective variety V with $H^{\text{odd}}(V; \mathbb{C}) = 0$, $\{\gamma_1, \dots, \gamma_N\}$ a basis for the cohomology ring $M := H^*(V; \mathbb{C})$ and let

$$\eta_{\alpha\beta} = \eta(\gamma_\alpha, \gamma_\beta) = \int_V \gamma_\alpha \cup \gamma_\beta$$

be the Poincaré pairing which defines a non-degenerate metric which may be used to raise and lower indices. Following the conventions of Liu and Tian [14, 15], a flat coordinate system $\{t_0^\alpha, \alpha = 1, \dots, N\}$ may be found on M so $\gamma_\alpha = \frac{\partial}{\partial t_0^\alpha}$, and in which the components of η are constant.

The big phase space consists of an infinite number of copies of the M , the small phase space, so

$$M^\infty = \prod_{n \geq 0} H^*(V; \mathbb{C}).$$

The coordinate system $\{t_0^\alpha\}$ induces, in a canonical way, a coordinate system $\{t_n^\alpha, n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, N\}$ on M^∞ . We denote by $\tau_n(\gamma_\alpha) = \frac{\partial}{\partial t_n^\alpha}$ (also abbreviated to $\tau_{n,\alpha}$) the associated fundamental vector fields, which represent various tautological line bundles over the moduli space of curves. A vector field $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha} \tau_m(\gamma_\alpha)$ is called a primary field if $f_{m,\alpha} = 0$ for $m > 0$ and a descendent field if $f_{0,\alpha} = 0$, for any α .

The descendent Gromov-Witten invariants

$$\langle \tau_{n_1}(\gamma_{\alpha_1}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g$$

may be combined into generating functions, called prepotentials, labeled by the genus g ,

$$\mathcal{F}_g = \sum_{k \geq 0} \frac{1}{k!} \sum_{n_1, \alpha_1 \dots n_k, \alpha_k} t_{n_1}^{\alpha_1} \dots t_{n_k}^{\alpha_k} \langle \tau_{n_1}(\gamma_{\alpha_1}) \dots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g,$$

and these in turn may be used to define k -tensor fields on the big phase space, via the formula

$$(1) \quad \langle \langle \mathcal{W}_1 \dots \mathcal{W}_k \rangle \rangle_g = \sum_{m_1, \alpha_1, \dots, m_k, \alpha_k} f_{m_1, \alpha_1}^1 \dots f_{m_k, \alpha_k}^k \frac{\partial^k \mathcal{F}_g}{\partial t_{m_1}^{\alpha_1} \dots \partial t_{m_k}^{\alpha_k}},$$

for any vector fields $\mathcal{W}_i = \sum_{m,\alpha} f_{m,\alpha}^i \frac{\partial}{\partial t_m^\alpha}$. The tensor field (1) has a physical interpretation as the k -point correlation function of the TQFT.

The basic relationships between these correlators may then be encapsulated in the following:

Definition 2. Let $\tilde{t}_n^\alpha = t_n^\alpha - \delta_{n,1} \delta_{\alpha,1}$ and let

$$\begin{aligned} \mathcal{S} &= - \sum_{n,\alpha} \tilde{t}_n^\alpha \tau_{n-1}(\gamma_\alpha), \\ \mathcal{D} &= - \sum_{n,\alpha} \tilde{t}_n^\alpha \tau_n(\gamma_\alpha) \end{aligned}$$

be the string and dilaton vector fields respectively. Then the prepotentials \mathcal{F}_g satisfy the following relations:

String Equation:

$$\langle\langle \mathcal{S} \rangle\rangle_g = \frac{1}{2} \delta_{g,0} \sum_{\alpha,\beta} \eta_{\alpha\beta} t_0^\alpha t_0^\beta;$$

Dilaton Equation:

$$\langle\langle \mathcal{D} \rangle\rangle_g = (2g - 2) \mathcal{F}_g - \frac{1}{24} \chi(V) \delta_{g,1};$$

Genus-zero Topological Recursion Relation:

$$\langle\langle \tau_{m+1}(\gamma_\alpha) \tau_n(\gamma_\beta) \tau_k(\gamma_\sigma) \rangle\rangle_0 = \langle\langle \tau_m(\gamma_\alpha) \gamma_\mu \rangle\rangle_0 \langle\langle \gamma^\mu \tau_n(\gamma_\beta) \tau_k(\gamma_\sigma) \rangle\rangle_0.$$

The Topological Recursion Relation in turn leads to the generalized WDVV equation:

$$\sum_{\mu,\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_m^\alpha \partial t_n^\beta \partial t_0^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_0^\nu \partial t_k^\gamma \partial t_l^\delta} = \sum_{\mu,\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_m^\alpha \partial t_k^\gamma \partial t_0^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_0^\nu \partial t_n^\beta \partial t_l^\delta}.$$

This may be written more succinctly by introducing the so-called quantum product between vector fields on the big phase space:

$$\mathcal{W}_1 \circ \mathcal{W}_2 = \langle\langle \mathcal{W}_1 \mathcal{W}_2 \gamma^\sigma \rangle\rangle_0 \gamma_\sigma$$

where $\gamma^\sigma = \eta^{\sigma\beta} \gamma_\beta$ and $(\eta^{\alpha\beta})$ is the inverse of $(\eta_{\alpha\beta})$. With this the generalized WDVV equation just becomes the associativity condition

$$(\mathcal{W}_1 \circ \mathcal{W}_2) \circ \mathcal{W}_3 = \mathcal{W}_1 \circ (\mathcal{W}_2 \circ \mathcal{W}_3).$$

By restricting such theories to primary vector fields with coefficients in the small phase space one recovers a Frobenius manifold structure [4, 5] on the small phase space, with

$$F_0(t_0^1, \dots, t_0^N) = \mathcal{F}_0(\mathbf{t})|_{t_n^\alpha=0, n>0}$$

becoming the prepotential for the Frobenius manifold and multiplication given by

$$\tau_{0,\alpha} \bullet \tau_{0,\beta} = \langle\langle \tau_{0,\alpha} \tau_{0,\beta} \gamma^\sigma \rangle\rangle_0 |_{M} \gamma_\sigma.$$

Frobenius manifolds have turned out to be extremely ubiquitous structure appearing, for example, via the work of K. Saito in singularity theory [20] and in the theory of integrable systems, as well as in quantum cohomology and mirror symmetry.

The underlying manifolds, when studying Frobenius manifolds, are actually complex manifolds, and the metric η is an holomorphic (non-degenerate) metric, rather than a real-valued metric [4]. To define real objects one requires an anti-holomorphic involution which may then in turn be used to define Hermitian objects. This direction of research was started by Cecotti and Vafa [1] in their study of tt^* -geometry (topological-anti-topological fusion). The idea has since been developed by Dubrovin [6] (who studied the integrability of tt^* -equations and developed the connection with pluriharmonic maps), Hertling [11] (who connected tt^* -geometry with the work of Simpson on Higgs bundles and generalizations of variations of Hodge structures) and by Sabbah [19] (this stressing the actual construction of these objects). For a collection of articles on these subject, see [3].

The historical development outlined above may be summarized in the following diagram:

$$\begin{array}{ccc}
 \left\{ \begin{array}{c} \text{TQFT} \\ \text{big phase space} \\ \text{[Witten, Dijkgraaf]} \end{array} \right\} & & \\
 \updownarrow & & \\
 \left\{ \begin{array}{c} \text{Frobenius manifold} \\ \text{small phase space} \\ \text{[Dubrovin]} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{\textit{tt}^* - geometry, TERP - structures} \\ \text{Hermitian - Higgs bundles} \\ \text{[Cecotti - Vafa, Dubrovin,} \\ \text{Hertling, Sabbah]} \end{array} \right\}
 \end{array}$$

As the title indicates, the purpose of this paper is to introduce a Hermitian structure on the big phase space and to study the properties of such a structure, in particular its relationship with the standard, holomorphic, structures. It turns out that one may define a full, infinite dimensional, tt^* -geometry on the big phase space.

The key object in our treatment is a certain endomorphism of $T^{1,0}M^\infty$ introduced and studied in [14, 15],

$$(2) \quad T(\mathcal{W}) := \tau_+(\mathcal{W}) - \mathcal{S} \circ \tau_+(\mathcal{W}), \quad \mathcal{W} \in T^{1,0}M^\infty,$$

where

$$\tau_\pm \left(\sum_{n,\alpha} f_{n,\alpha} \tau_{n,\alpha} \right) := \sum_{n,\alpha} f_{n,\alpha} \tau_{n\pm 1,\alpha}.$$

With this the Topological Recursion Relation takes the compact form

$$(3) \quad T(\mathcal{W}_1) \circ \mathcal{W}_2 = 0 \quad \mathcal{W}_1, \mathcal{W}_2 \in T^{1,0}M^\infty.$$

The Poincaré pairing is an holomorphic metric on the small phase space, not the large phase space. An extension of this metric to the big phase space was given in [17], namely $\langle \mathcal{U}, \mathcal{V} \rangle = \langle \langle \mathcal{S}\mathcal{U}\mathcal{V} \rangle \rangle_0$, but this metric is degenerate, a fact that follows easily from the use of the Topological Recursion Relation. However, in [16] a non-degenerate holomorphic metric on the big phase space was defined, namely

$$(4) \quad \hat{\eta}(\mathcal{W}, \mathcal{V}) = \sum_{k=0}^{\infty} \langle \langle \mathcal{S}\tau_-^k(\mathcal{W})\tau_-^k(\mathcal{V}) \rangle \rangle_0.$$

On using properties of the endomorphism T it is easy to show that

$$(5) \quad \hat{\eta}(T^n(\gamma_\alpha), T^m(\gamma_\beta)) = \delta_{mn}\eta(\gamma_\alpha, \gamma_\beta) = \delta_{mn}\eta_{\alpha\beta}.$$

This last formula may be seen as a lift, using the endomorphism T , of η , defined on the small phase space, to the big phase space. Combined with a basic result of Witten and Dijkgraaf [2] on the use of constitutive relations, this gives a way to define Hermitian structures on M^∞ starting with finite dimensional structures on M .

1.2. Structure of Paper. The rest of the paper is laid out as follows. Section 2 is intended to fix notation. Here we briefly recall the basic facts we need about the big phase space, tt^* -geometry and the relations between tt^* -geometry and Frobenius manifolds (see Definitions 7 and 8).

In Section 3 we develop our main tool from this paper. We define the natural lift of functions and vector fields from the small phase space M to the big phase space M^∞ (see Definition 11) and we study their basic properties. The natural lift of a vector field from M to M^∞ is a primary vector field, whose coefficients in canonical flat coordinates are natural lifts of the coordinates of the initial vector field. We study how natural lifts of functions behave under derivations and in particular we show that any function on M^∞ , which is a natural lift of a function on M , is annihilated by the vector fields from the image of the operator T , defined by (2) (see Lemma 12). Lemma 12 is a main tool for our computations from the next sections. Finally, using the same ideas, we remark that any tensor field \mathcal{F} on M may be lifted to a tensor field $\hat{\mathcal{F}}$ on M^∞ , referred as the natural lift of \mathcal{F} (see Section 3.2). In the following sections we consider natural lifts of specific tensor fields on M , and show they are part of tt^* -structures on M^∞ .

In Section 4 we assume that the small phase space comes with a real structure k , compatible with η (i.e. $h := \eta(\cdot, k\cdot)$ is a pseudo-Hermitian metric). The natural lift \hat{k} of k is compatible with the natural lift $\hat{\eta}$ of η and we compute the Chern connection of the associated pseudo-Hermitian metric $\hat{h} = \hat{\eta}(\cdot, \hat{k}\cdot)$ on $T^{1,0}M^\infty$ and its curvature (see Lemmas 19 and 20). These computations will be used later on, in our study of the tt^* -equations for the extended structures.

In Section 5 we lift the Higgs field C on M to a (non-commutative) Higgs field \hat{C} on M^∞ (see Definition 21). The field \hat{C} is not the natural lift of C (in the sense of Section 3.2), but it reflects that 2-point functions lift trivially to the big phase space (see Remark 22). The definition of \hat{C} also mirrors the properties of the Chern connection of \hat{h} and turns out to be very suitable for the tt^* -geometry and the theory of Saito bundles on M^∞ . Other lifts of the multiplication from the small phase space to the big phase space (e.g. the quantum multiplication mentioned above or the Liu's multiplication \diamond , see Remark 4), appear in the literature, but they are not so well suited for the tt^* -geometry on the big phase space.

In Section 6 we gather all lifted structures defined in the previous sections and we prove that the basic notions from the theory of Frobenius manifolds and tt^* -geometry are preserved under these lifts. More precisely, let $(T^{1,0}M, \nabla, \eta, C, R_0, R_\infty)$ be the Saito bundle associated to the Frobenius manifold (M, \bullet, η, E) , where ∇ is the Levi-Civita connection of η , $C_X(Y) = X \bullet Y$ is the Higgs field, $R_0 = C_E$ and $R_\infty = \nabla E$, where E is the Euler field. Assume that k is a real structure on M , compatible with η , and let $h := \eta(\cdot, k\cdot)$ be the associated pseudo-Hermitian metric. On $T^{1,0}M^\infty$ we consider $\hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty, \hat{k}, \hat{h}$, where $\hat{\nabla}$ is a flat connection on $T^{1,0}M^\infty$, defined by the condition that all vector fields $T^n(\tau_{0,\alpha})$ ($n \geq 0, 1 \leq \alpha \leq N$) are $\hat{\nabla}$ -parallel, $\hat{\eta}, \hat{R}_0, \hat{R}_\infty, \hat{k}$ and \hat{h} are natural lifts of $\eta, R_0, R_\infty, k, h$, and \hat{C} is the lifted (non-commutative) Higgs field, as defined in Section 5 (see Definition 21). In a first stage we show that $(T^{1,0}M, \hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty)$

is a Saito bundle. Then we prove our main results from this paper: namely, we assume that various compatibility conditions between the real structure k and the Saito structure $(\nabla, \eta, C, R_0, R_\infty)$ hold (giving rise to a harmonic Higgs bundle, a real Saito bundle, a harmonic potential real Saito bundle, a $DChk$ -bundle or a CV-bundle), and we prove that all such conditions are inherited by the lifted structures on the big phase space (see Sections 6.2, 6.3, 6.4).

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2. PRELIMINARY MATERIAL

This section is intended to recall basic facts we need from the geometry of the big phase space, theory of Frobenius manifolds and tt^* -geometry. The manifolds we consider are complex and the vector bundles holomorphic. For a complex manifold M , we denote by $T^{1,0}M$ the holomorphic tangent bundle of M , by $\mathcal{T}_M, \mathcal{T}_M^{1,0}, C^\infty(M), C^\infty(M, \mathbb{R})$ the sheaf of holomorphic vector fields, vector fields of type $(1, 0)$ and complex, respectively real valued smooth functions on M . Vector fields on the small phase space will be denoted by X, Y, Z , etc, while vector fields on the big phase space will be usually denoted by $\mathcal{U}, \mathcal{V}, \mathcal{W}$. (Unless otherwise specified, vector fields are of type $(1, 0)$, not necessarily holomorphic, and functions are smooth and complex valued). For an holomorphic bundle $V \rightarrow M$, $\Omega^1(M, V), \Omega^{1,0}(M, V)$ and $\Omega^{0,1}(M, V)$ will denote, respectively, the sheaves of holomorphic 1-forms, forms of type $(1, 0)$ and forms of type $(0, 1)$ on M , with values in V .

2.1. The geometry of the big phase space. The material in this section is taken directly from [14, 15, 16] and no proofs will be given. The first lemma shows that the string vector field behaves like a unit for the quantum product restricted to primary fields, and the second lemma derives properties of a naturally defined covariant derivative on the big phase space.

Lemma 3. *For all primary vector fields \mathcal{W} and for all vectors fields \mathcal{U}, \mathcal{V} on M^∞ ,*

$$\begin{aligned} \mathcal{S} \circ \mathcal{W} &= \mathcal{W}, \\ \mathcal{S} \circ \mathcal{U} \circ \mathcal{V} &= \mathcal{U} \circ \mathcal{V}. \end{aligned}$$

Remark 4. Besides the quantum product, there are also other interesting multiplications on the big phase space, see e.g. [16]. The multiplication

$$T^n(\mathcal{U}) \diamond T^m(\mathcal{V}) = \delta_{mn} T^n(\mathcal{U} \circ \mathcal{V}),$$

where \mathcal{U} and \mathcal{V} are primary vector fields, is commutative, associative, with unit field

$$\hat{\mathcal{S}} = \sum_{k \geq 0} T^k(\mathcal{S} \circ \mathcal{S}).$$

One may also show that the metric $\hat{\eta}$ and the multiplication \diamond are compatible in the sense that

$$\hat{\eta}(\mathcal{W}_1 \diamond \mathcal{W}_2, \mathcal{W}_3) = \hat{\eta}(\mathcal{W}_1, \mathcal{W}_2 \diamond \mathcal{W}_3).$$

However one may check that $(M, \diamond, \hat{\mathcal{S}})$ is not an F -manifold [10].

Lemma 5. *Let ∇ be the covariant derivative defined by*

$$\nabla_{\mathcal{V}}\mathcal{W} = \sum_{m,\alpha} \mathcal{V}(f_{m,\alpha}) \tau_m(\gamma_\alpha)$$

where $\mathcal{W} = \sum_{m,\alpha} f_{m,\alpha} \tau_m(\gamma_\alpha)$ and \mathcal{V} is an arbitrary vector field on M^∞ . Then

$$(6) \quad \begin{aligned} \mathcal{W} \ll \mathcal{W}_1 \cdots \mathcal{W}_k \gg_0 &= \sum_{i=1}^k \ll \mathcal{W}_1 \cdots \mathcal{W}_{i-1} (\nabla_{\mathcal{W}} \mathcal{W}_i) \mathcal{W}_{i+1} \cdots \mathcal{W}_k \gg_0 \\ &+ \ll \mathcal{W} \mathcal{W}_1 \cdots \mathcal{W}_k \gg_0, \end{aligned}$$

for any vector fields $\mathcal{W}, \mathcal{W}_i$ on M^∞ , and

$$\begin{aligned} \nabla_{\mathcal{V}}(T^k(\mathcal{W})) &= T^k(\nabla_{\mathcal{V}}\mathcal{W}) - T^{k-1}(\mathcal{V} \circ \mathcal{W}), \\ [T^n(\gamma_\alpha), T^m(\gamma_\beta)] &= 0, \quad n, m \geq 1, \\ [T^n(\gamma_\alpha), \gamma_\beta] &= T^{n-1}(\gamma_\alpha \circ \gamma_\beta), \quad n \geq 1. \end{aligned}$$

Thus the vector fields $\{T^n(\gamma_\alpha), n \in \mathbb{Z}_{\geq 0}, \alpha = 1, \dots, N\}$ form a frame for $T^{1,0}M^\infty$, but not a coordinate frame, due to the last of the above equations.

Remark 6. The connection ∇ as defined above induces an holomorphic connection on $T^{1,0}M$ in the obvious way (if X and Y are vector fields on $T^{1,0}M$ then $\nabla_X Y$ is also a vector field on $T^{1,0}M$), for which $\{\tau_{0,\alpha}\}$ are flat. In particular, the induced connection coincides with the Levi-Civita connection of η , and will be also denoted by ∇ . It will be clear from the context when ∇ acts on $T^{1,0}M$ or $T^{1,0}M^\infty$.

2.2. Frobenius manifolds and tt^* -geometry. In this section we recall the basic definitions from Frobenius manifolds and tt^* -geometry, see e.g. [11, 19]. We begin with the definition of Saito bundles and we explain their relation with Frobenius manifolds. Then we add a real structure on a Saito bundle and we define various compatibility conditions between the Saito structure and the real structure. We work in the holomorphic category: all manifolds, bundles, tensor fields, connections etc from this section are holomorphic, unless otherwise stated.

Definition 7. A *Saito bundle* (of weight $w \in \mathbb{C}$) is a vector bundle

$$(\pi : V \rightarrow M, \nabla, g, C, R_0, R_\infty)$$

endowed with a connection ∇ , a metric g , a vector valued 1-form $C \in \Omega^1(M, \text{End}V)$ and two endomorphisms R_0 and R_∞ , such that the following conditions are satisfied:

$$(7) \quad R^\nabla = 0, \quad d^\nabla C = 0, \quad \nabla g = 0, \quad C \wedge C = 0, \quad C^* = C,$$

and

$$\begin{aligned}\nabla R_0 + C &= [C, R_\infty], \quad [R_0, C] = 0, \quad R_0^* = R_0; \\ \nabla R_\infty &= 0, \quad R_\infty^* + R_\infty = -w\text{Id}_V.\end{aligned}$$

Above $d^\nabla C$ and $C \wedge C$ are $\text{End}(V)$ -valued 2-forms, defined by

$$\begin{aligned}(d^\nabla C)_{X,Y} &:= \nabla_X(C_Y) - \nabla_Y(C_X) - C_{[X,Y]}, \\ (C \wedge C)_{X,Y} &:= C_X C_Y - C_Y C_X\end{aligned}$$

for any $X, Y \in \mathcal{T}_M$, and the superscript “*” denotes the g -adjoint (in particular, $C_X^* \in \text{End}(V)$ is the g -adjoint of $C_X \in \text{End}(V)$). Moreover, $[R_0, C]$ is an $\text{End}(V)$ -valued 1-form, which, on $X \in T^{1,0}M$, is equal to $[R_0, C_X]$.

A Frobenius manifold (M, \bullet, e, η, E) defines a Saito structure $(\nabla, \eta, C, R_0, R_\infty)$ on $T^{1,0}M$, where ∇ is the Levi-Civita connection of η , $C_X Y := X \bullet Y$ is the Higgs field, $R_0 := C_E$ is the multiplication by the Euler field and $R_\infty := \nabla E$. The weight of this Saito structure is d , where $L_E(g) = -dg$. Conversely, any Saito bundle whose rank is equal to the dimension of the base, together with a suitably chosen parallel section (usually called primitive homogeneous), gives rise to a Frobenius structure on the base of the bundle [20] (see also [18]).

We now add a real structure to a Saito bundle and define various compatibility conditions, which give rise to the notions of real Saito bundles, harmonic real Saito bundles and harmonic potential real Saito bundles.

Definition 8. 1) A real Saito bundle is a Saito bundle

$$(\pi : V \rightarrow M, \nabla, g, C, R_0, R_\infty)$$

together with a real structure $k : V \rightarrow V$ (i.e. k is a fiber-preserving smooth anti-linear involution) such that g, k are compatible (i.e. $h := g(\cdot, k\cdot)$ is a pseudo-Hermitian metric) and g, h are also compatible (i.e. $D(g) = 0$, where D is the Chern connection of h).

2) A harmonic real Saito bundle is a real Saito bundle

$$(\pi : V \rightarrow M, \nabla, g, C, R_0, R_\infty, k)$$

such that $(V, C, h = g(\cdot, k\cdot))$ is a harmonic Higgs bundle, i.e. the tt^* -equations

$$(\partial^D C)_{Z_1, Z_2} = 0, \quad R_{Z_1, \bar{Z}_2}^D + [C_{Z_1}, C_{\bar{Z}_2}^\dagger] = 0, \quad Z_1, Z_2 \in T^{1,0}M$$

hold, where

$$(\partial^D C)_{Z_1, Z_2} := D_{Z_1}(C_{Z_2}) - D_{Z_2}(C_{Z_1}) - C_{[Z_1, Z_2]},$$

$C^\dagger \in \Omega^{0,1}(M, \text{End}V)$ denotes the h -adjoint of $C \in \Omega^{1,0}(M, \text{End}V)$ and D is the Chern connection of h .

3) A harmonic potential real Saito bundle is a harmonic real Saito bundle

$$(\pi : V \rightarrow M, \nabla, g, C, R_0, R_\infty, k)$$

together with a smooth g -self adjoint endomorphism A of V (called the potential) such that the following conditions hold:

$$3a) D^{(1,0)}A = C, \text{ where } D \text{ is the Chern connection of } h := g(\cdot, k\cdot);$$

$$3b) D^{(1,0)} = \nabla - [A^\dagger, C], \text{ where } A^\dagger \text{ is the } h\text{-adjoint of } A;$$

$$3c) \text{ the endomorphism } R_\infty + [A^\dagger, R_0] \text{ is } h\text{-self adjoint.}$$

A Frobenius manifold with a suitably chosen real structure and a (not necessarily holomorphic) endomorphism of the tangent bundle gives rise to the notion of harmonic Frobenius manifold, defined as follows.

Definition 9. *A harmonic Frobenius manifold is a complex Frobenius manifold (M, \circ, e, g, E) such that $L_E(g) = dg$, with $d \in \mathbb{R}$, together with a real structure k on $T^{1,0}M$ and a smooth endomorphism A of $T^{1,0}M$ (called the potential of the Frobenius manifold), such that the associated Saito bundle $(T^{1,0}M, \nabla, g, C, R_0 = C_E, R_\infty = \nabla E)$ is a harmonic potential real Saito bundle, with real structure k and potential A .*

One may show that any harmonic potential real Saito bundle whose rank is equal to the dimension of the base, together with a parallel primitive real homogeneous section, gives rise to a harmonic Frobenius structure on the base of the bundle. For our purposes, we do not need this construction. For a precise statement and proof, see Corollary 1.31 of [19].

Remark 10. As will be proved in Section 6.1, the tangent bundle of the big phase space comes naturally equipped with a Saito structure, obtained by lifting the Saito structure of the small phase space. Thus it is natural to look for tt^* -structures on the big phase space, compatible with this lifted Saito structure. For this reason, in Definition 8 above only notions from tt^* -geometry, which admit an underlying Saito structure, were recalled. It is worth to remark however that other important notions exist in tt^* -geometry, which are build as an enrichment of the notion of harmonic Higgs bundle, rather than Saito bundle, and do not necessarily admit an underlying Saito structure. For example, in the language of [11], one may consider the notion of $DChk$ -bundle, which is a harmonic Higgs bundle (V, C, h) , together with a real structure k , such that $g := h(\cdot, k\cdot)$ is a symmetric (holomorphic) metric, compatible with h . A richer notion is the notion of CV-bundle, which is, by definition, a $DChk$ -bundle (V, C, h, k) , together with two endomorphisms \mathcal{U} and \mathcal{Q} (the latter not necessarily holomorphic), satisfying the following conditions:

$$i) \text{ for any } X \in T^{1,0}M, [C_X, \mathcal{U}] = 0;$$

$$ii) \text{ for any } X \in T^{1,0}M,$$

$$D_X \mathcal{U} + [C_X, \mathcal{Q}] - C_X = 0$$

and

$$D_X \mathcal{Q} - [C_X, k\mathcal{U}k] = 0$$

(as usual, D denotes the Chern connection of h).

iii) \mathcal{Q} is h -self adjoint and g -skew adjoint.

Such structures arise naturally in singularity theory [11]. While our primary interest is in the structures presented in Definition 8, it turns out that the notions of $DChk$ and CV -structures are preserved under our liftings to the big phase space. These facts will be referred to throughout the course of this paper, but no proofs will be given.

3. NATURAL LIFTS FROM M TO M^∞

In this section we describe a canonical way to lift tensor fields from the small phase space M to the big phase space M^∞ . This will be our main tool in the construction of a Hermitian geometry on the big phase space. The idea is an adoption of the use of constitutive equations and was originally introduced by Dijkgraaf and Witten [2]. We preserve the notation from the Introduction and Section 2.1.

3.1. Natural lifts of functions and vector fields.

Definition 11. *i) Let f be a function on M . The function \hat{f} on M^∞ , defined by*

$$(8) \quad \hat{f}(t_n^\alpha) := f\left(\eta^{\alpha\beta} \ll \tau_{0,1}\tau_{0,\beta} \gg_0\right),$$

is called the natural lift of f .

ii) Let $X \in \mathcal{T}_M^{1,0}$ a vector field on M , given in flat coordinates $\{t_0^\alpha\}$ by

$$X = \sum_{\alpha=1}^N f^\alpha \tau_{0,\alpha}.$$

The primary field

$$\hat{X} := \sum_{\alpha=1}^N \hat{f}^\alpha \tau_{0,\alpha}$$

is called the natural lift of X to M^∞ .

We now develop some simple properties of natural lifts, which will be useful in the next sections.

Lemma 12. *For any $\mathcal{W} \in \mathcal{T}_{M^\infty}^{1,0}$ and $f \in C^\infty(M)$,*

$$(9) \quad \mathcal{W}(\hat{f}) = \left[\frac{\partial f}{\partial t_0^\beta} \right]^\wedge \eta^{\beta\sigma} \ll \tau_{0,1}\tau_{0,\sigma} \mathcal{W} \gg_0.$$

In particular,

$$(10) \quad T(\mathcal{W})(\hat{f}) = 0.$$

Proof. Consider the vector valued function

$$\vec{u} := (u^1, \dots, u^N), \quad u^\alpha := \eta^{\alpha\beta} \ll \tau_{0,1}\tau_{0,\beta} \gg_0.$$

From (8), $\hat{f} = f \circ \vec{u}$ and

$$(11) \quad \frac{\partial \hat{f}}{\partial t_n^\alpha} = \left[\frac{\partial f}{\partial t_0^\beta} \right]^\wedge \frac{\partial u^\beta}{\partial t_n^\alpha} = \left[\frac{\partial f}{\partial t_0^\beta} \right]^\wedge \eta^{\beta\sigma} \ll \tau_{0,1} \tau_{0,\sigma} \tau_{n,\alpha} \gg_0,$$

where we used (6). Relation (9) follows. Relation (10) follows from (9), the Topological Recursion Relation (3) and the definition of the quantum product. □

Remark 13. In computations we shall often use

$$\mathcal{W}(\hat{f}) = \left[\frac{\partial f}{\partial t_0^\beta} \right]^\wedge \overline{\eta^{\beta\sigma} \ll \tau_{0,1} \tau_{0,\sigma} \overline{\mathcal{W}} \gg_0},$$

for any $f \in C^\infty(M)$ and $\mathcal{W} \in \mathcal{T}_{M^\infty}^{0,1}$, and also

$$(12) \quad \overline{T(\mathcal{W})}(\hat{f}) = 0, \quad \forall f \in C^\infty(M), \quad \forall \mathcal{W} \in \mathcal{T}_{M^\infty}^{1,0}.$$

These follow by taking conjugations of (9) and (10), and using that conjugations commute with natural lifts of functions.

3.2. Natural lifts of arbitrary tensor fields. Using the above ideas one may lift any tensor field, from M to M^∞ , in the following way. We first extend componentwise the endomorphism T , defined by (2), to products $T^{1,0}M^\infty \times \dots \times T^{1,0}M^\infty$ ($p \geq 0$ factors; when $p = 0$ the product reduces to the trivial bundle $M^\infty \times \mathbb{C}$ and T is the identity operator). Similarly, we define a map

$$(13) \quad \wedge : \mathcal{T}^{1,0}M \times \dots \times \mathcal{T}^{1,0}M \rightarrow \mathcal{T}^{1,0}M^\infty \times \dots \times \mathcal{T}^{1,0}M^\infty,$$

(p -factors in both products) which, for $p \geq 1$, is given by

$$(X_1, \dots, X_p)^\wedge := (\widehat{X}_1, \dots, \widehat{X}_p).$$

For $p = 0$, the map (13) is defined on $C^\infty(M)$ and is just the natural lift of functions. With this preliminary notation, let \mathcal{F} be a (p, q) -tensor field on M , i.e. a map

$$\mathcal{F} : T^{1,0}M \times \dots \times T^{1,0}M \rightarrow T^{1,0}M \times \dots \times T^{1,0}M,$$

(p -factors in the left hand side, q in the right hand side), which is $C^\infty(M)$ -linear (in all arguments) or $C^\infty(M, \mathbb{R})$ -linear and complex anti-linear (in all arguments). The natural lift $\hat{\mathcal{F}}$ of \mathcal{F} is a tensor field of the same type on M^∞ , defined by

$$\hat{\mathcal{F}}(T^{n_1}(\tau_{0,\alpha_1}), \dots, T^{n_p}(\tau_{0,\alpha_p})) = \delta_{n_1, \dots, n_p} T^{n_i} \left([\mathcal{F}(\tau_{0,\alpha_1}, \dots, \tau_{0,\alpha_p})]^\wedge \right),$$

where $\delta_{n_1, \dots, n_p} = 0$ unless all n_i are equal and $\delta_{n \dots n} = 1$.

Remark 14. From the definition of $\hat{\mathcal{F}}$, for any vector fields $X_1, \dots, X_p \in \mathcal{T}_M^{1,0}$,

$$(14) \quad \hat{\mathcal{F}}(T^{n_1}(\widehat{X}_1), \dots, T^{n_p}(\widehat{X}_p)) = \delta_{n_1, \dots, n_p} T^{n_i}([\mathcal{F}(X_1, \dots, X_p)]^\wedge).$$

Example 15. i) Liu's metric (5) (see [17]) already mentioned in the Introduction is the natural lift of the Poincaré metric η :

$$(15) \quad \hat{\eta}(T^m(\tau_{0,\alpha}), T^n(\tau_{0,\beta})) = \delta_{mn} \eta(\tau_{0,\alpha}, \tau_{0,\beta}), \quad \forall m, n$$

the right hand side of the above expression being constant, hence coincides with its natural extension.

ii) The natural lift \hat{A} of an endomorphism A of $T^{1,0}M$ (viewed as a $(1, 1)$ -tensor field) is given by

$$\hat{A}(T^m(\tau_{0,\alpha})) = T^m([A(\tau_{0,\alpha})]^\wedge).$$

Note that $[A, B]^\wedge = [\hat{A}, \hat{B}]$, for any endomorphisms A and B of $T^{1,0}M$.

4. THE LIFTED PSEUDO-HERMITIAN METRIC

We now consider a real structure $k : T^{1,0}M \rightarrow T^{1,0}M$ compatible with η , i.e. $h := \eta(\cdot, k\cdot)$ is a pseudo-Hermitian metric. It is easy to check that its natural lift $\hat{k} : T^{1,0}M^\infty \rightarrow T^{1,0}M^\infty$ is a real structure, compatible with the natural lift $\hat{\eta}$ of η , and that \hat{k} and $\hat{\eta}$ give rise to a pseudo-Hermitian metric $\hat{h} = \hat{\eta}(\cdot, \hat{k}\cdot)$, which is the natural lift of h . In this section we compute the Chern connection and the curvature of \hat{h} . For completeness of our exposition, we recall the expression of \hat{h} .

Definition 16. *The natural lift \hat{h} of h to M^∞ is defined by*

$$(16) \quad \hat{h}(T^n(\tau_{0,\alpha}), T^m(\tau_{0,\beta})) = \delta_{mn} [h(\tau_{0,\alpha}, \tau_{0,\beta})]^\wedge, \quad \forall m, n.$$

More generally,

$$(17) \quad \hat{h}(T^n(\hat{X}), T^m(\hat{Y})) = \delta_{mn} [h(X, Y)]^\wedge, \quad \forall X, Y \in \mathcal{T}_M^{1,0}.$$

To simplify the expression of the Chern connection of \hat{h} and other expressions we define the functions (following Getzler [7])

$$(18) \quad M_\alpha^\gamma := \eta^{\gamma\sigma} \langle\langle \tau_{0,1} \tau_{0,\sigma} \tau_{0,\alpha} \rangle\rangle_0, \quad 1 \leq \alpha, \gamma \leq N$$

and study some of their basic properties (see Remark 17 and Lemma 18).

Remark 17. With the above notation, relation (9) implies

$$(19) \quad \tau_{0,\alpha}(\hat{f}) = \left[\frac{\partial f}{\partial t_0^\beta} \right]^\wedge M_\alpha^\beta, \quad \forall f \in C^\infty(M).$$

Notice that relation (19) implies $M_\alpha^\sigma = \tau_{0,\alpha}(\widehat{f^\sigma})$, where $f^\sigma(t_0^1, \dots, t_0^N) = t_0^\sigma$. It also follows from the String Equation that

$$M_\alpha^\sigma|_M = \delta_\alpha^\sigma.$$

In what follows we shall use the derivatives of the functions M_α^σ along vector fields from the image of the operator T . We will compute these derivatives now, and then compute the Chern connection of \hat{h} .

Lemma 18. *The following relations hold: for any $1 \leq \alpha, \beta, \sigma \leq N$,*

$$T(\tau_{0,\beta})(M_\alpha^\sigma) = (\tau_{0,\alpha} \circ \tau_{0,\beta})(\widehat{f^\sigma}) = \eta^{\sigma\mu} \ll \tau_{0,1}\tau_{0,\mu}(\tau_{0,\alpha} \circ \tau_{0,\beta}) \gg_0$$

and

$$T^q(\tau_{0,\beta})(M_\alpha^\sigma) = 0, \quad \forall q \geq 2.$$

Proof. We use $M_\alpha^\sigma = \tau_{0,\alpha}(\widehat{f^\sigma})$ (from the previous remark) and that $\widehat{f^\sigma}$ is annihilated by $T^q(\tau_{0,\beta})$ for any $q \geq 1$ (see Lemma 12). Recall, also, that

$$[T^q(\tau_{0,\beta}), \tau_{0,\alpha}] = T^{q-1}(\tau_{0,\alpha} \circ \tau_{0,\beta}),$$

(see Lemma 5). We obtain: for any $q \geq 1$,

$$\begin{aligned} T^q(\tau_{0,\beta})(M_\alpha^\sigma) &= T^q(\tau_{0,\beta})\tau_{0,\alpha}(\widehat{f^\sigma}) \\ &= [T^q(\tau_{0,\beta}), \tau_{0,\alpha}](\widehat{f^\sigma}) = T^{q-1}(\tau_{0,\alpha} \circ \tau_{0,\beta})(\widehat{f^\sigma}), \end{aligned}$$

which vanishes when $q \geq 2$. When $q = 1$, we obtain

$$T(\tau_{0,\beta})(M_\alpha^\sigma) = (\tau_{0,\alpha} \circ \tau_{0,\beta})(\widehat{f^\sigma}) = \eta^{\sigma\mu} \ll \tau_{0,1}\tau_{0,\mu}(\tau_{0,\alpha} \circ \tau_{0,\beta}) \gg_0$$

where in the last equality we used (9), with $f := f^\sigma$ and $\mathcal{W} = \tau_{0,\alpha} \circ \tau_{0,\beta}$. \square

From the above lemma, $T^q(\tau_{0,\beta})(M_\alpha^\sigma)$ is symmetric in α and β .

Lemma 19. *Let D be the Chern connection of h . The Chern connection \hat{D} of \hat{h} is given by: for any $n \geq 0$ and $1 \leq \alpha, \beta \leq N$,*

$$(20) \quad \hat{D}_{\tau_{0,\alpha}}(T^n(\tau_{0,\beta})) = M_\alpha^\sigma T^n \left([D_{\tau_{0,\sigma}}(\tau_{0,\beta})]^\wedge \right)$$

and

$$(21) \quad \hat{D}_{T(\mathcal{W})}(T^n(\tau_{0,\alpha})) = 0, \quad \forall \mathcal{W} \in \mathcal{T}_{M^\infty}^{1,0}.$$

Proof. By definition, the Chern connection \hat{D} satisfies

$$(22) \quad \mathcal{W}_1 \hat{h}(\mathcal{W}_2, \mathcal{W}_3) = \hat{h}(\hat{D}_{\mathcal{W}_1} \mathcal{W}_2, \mathcal{W}_3), \quad \forall \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in \mathcal{T}_{M^\infty}.$$

Let $\mathcal{W}_1 := \tau_{0,\alpha}$, $\mathcal{W}_2 := T^n(\tau_{0,\beta})$, $\mathcal{W}_3 := T^m(\tau_{0,\gamma})$. With these arguments, equation (22) becomes

$$(23) \quad \delta_{nm} \tau_{0,\alpha}(\widehat{h_{\beta\gamma}}) = \hat{h} \left(\hat{D}_{\tau_{0,\alpha}}(T^n(\tau_{0,\beta})), T^m(\tau_{0,\gamma}) \right),$$

where, to simplify notation, we defined $h_{\beta\gamma} := h(\tau_{0,\beta}, \tau_{0,\gamma})$. But from (19),

$$(24) \quad \tau_{0,\alpha}(\widehat{h_{\beta\gamma}}) = \left[\frac{\partial h_{\beta\gamma}}{\partial t_0^\sigma} \right]^\wedge M_\alpha^\sigma = M_\alpha^\sigma [h(D_{\tau_{0,\sigma}}(\tau_{0,\beta}), \tau_{0,\gamma})]^\wedge.$$

Combining (23) with (24) we obtain: $\forall n, m, \alpha, \beta$,

$$M_\alpha^r \hat{h} \left(T^n \left([D_{\tau_{0,r}}(\tau_{0,\beta})]^\wedge \right), T^m(\tau_{0,\gamma}) \right) = \hat{h} \left(\hat{D}_{\tau_{0,\alpha}}(T^n(\tau_{0,\beta})), T^m(\tau_{0,\gamma}) \right).$$

From the non-degeneracy of \hat{h} , we obtain (20).

We now prove (21). With \mathcal{W}_2 and \mathcal{W}_3 as above, $\hat{h}(\mathcal{W}_2, \mathcal{W}_3)$ is the natural extension of $\delta_{mn} h_{\beta\gamma}$. Thus, if $\mathcal{W}_1 = T(\mathcal{W})$ in the image of T , it annihilates $\hat{h}(\mathcal{W}_2, \mathcal{W}_3)$ (see Lemma 12) and from (22),

$$(25) \quad \hat{h} \left(\hat{D}_{T(\mathcal{W})}(T^n(\tau_{0,\beta})), T^m(\tau_{0,\gamma}) \right) = 0.$$

Using the non-degeneracy of \hat{h} again, we obtain (21).

□

From Lemmas 12 and 19, the covariant derivatives with respect to \hat{D} of vector fields $T^n(\hat{X})$, in directions from the image of T , vanish:

$$(26) \quad \hat{D}_{T(\mathcal{W})} \left(T^n(\hat{X}) \right) = 0, \quad \forall \mathcal{W} \in \mathcal{T}_{M^\infty}, \quad X \in \mathcal{T}_M.$$

This fact will be used often in the computations from the next sections.

Lemma 20. *The curvature $\hat{D}R$ of \hat{D} is related to the curvature ${}^D R$ of D by:*

$$(27) \quad \hat{D}R_{\tau_{0,\alpha}, \overline{\tau_{0,\beta}}} (T^r(\tau_{0,\gamma})) = M_\alpha^\sigma \overline{M_\beta^\nu} T^r \left([{}^D R_{\tau_{0,\sigma}, \overline{\tau_{0,\nu}}}(\tau_{0,\gamma})]^\wedge \right),$$

where $1 \leq \alpha, \beta, \gamma \leq N$ and $r \geq 0$. If m or n is bigger than zero,

$$\hat{D}R_{T^n(\tau_{0,\alpha}), \overline{T^m(\tau_{0,\beta})}} (T^r(\tau_{0,\gamma})) = 0.$$

Proof. For any $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in \mathcal{T}_{M^\infty}$,

$$\begin{aligned} \hat{D}R_{\mathcal{W}_1, \overline{\mathcal{W}_2}}(\mathcal{W}_3) &= \hat{D}_{\mathcal{W}_1} \hat{D}_{\overline{\mathcal{W}_2}}(\mathcal{W}_3) - \hat{D}_{\overline{\mathcal{W}_2}} \hat{D}_{\mathcal{W}_1}(\mathcal{W}_3) - \hat{D}_{[\mathcal{W}_1, \overline{\mathcal{W}_2}]}(\mathcal{W}_3) \\ &= -\bar{\partial}_{\overline{\mathcal{W}_2}} \hat{D}_{\mathcal{W}_1}(\mathcal{W}_3), \end{aligned}$$

where we used

$$[\mathcal{W}_1, \overline{\mathcal{W}_2}] = 0, \quad \hat{D}_{\overline{\mathcal{W}_2}}(\mathcal{W}_3) = \bar{\partial}_{\overline{\mathcal{W}_2}}(\mathcal{W}_3) = 0$$

because \mathcal{W}_i are holomorphic and \hat{D} is a Chern connection (hence its $(0, 1)$ -part is the $\bar{\partial}$ operator). Let

$$\mathcal{W}_1 := T^n(\tau_{0,\alpha}), \quad \mathcal{W}_2 := T^m(\tau_{0,\beta}), \quad \mathcal{W}_3 := T^r(\tau_{0,\gamma}).$$

With these arguments,

$$(28) \quad \hat{D}R_{T^n(\tau_{0,\alpha}), \overline{T^m(\tau_{0,\beta})}} (T^r(\tau_{0,\gamma})) = -\bar{\partial}_{\overline{T^m(\tau_{0,\beta})}} \hat{D}_{T^n(\tau_{0,\alpha})} (T^r(\tau_{0,\gamma})).$$

From Lemma 19, if $n > 0$,

$$\hat{D}R_{T^n(\tau_{0,\alpha}), \overline{T^m(\tau_{0,\beta})}} (T^r(\tau_{0,\gamma})) = 0.$$

If $n = 0$, then, again from Lemma 19,

$$(29) \quad \begin{aligned} \hat{D}R_{\tau_{0,\alpha}, \overline{T^m(\tau_{0,\beta})}} (T^r(\tau_{0,\gamma})) &= -\bar{\partial}_{\overline{T^m(\tau_{0,\beta})}} \left(M_\alpha^\sigma T^r \left([D_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^\wedge \right) \right) \\ &= -M_\alpha^\sigma T^r \left(\bar{\partial}_{\overline{T^m(\tau_{0,\beta})}} \left([D_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^\wedge \right) \right) \end{aligned}$$

where in the last relation we used that T and M_α^σ are holomorphic. We define functions $f_{\sigma\gamma}^\mu$ on M by the formula

$$(30) \quad D_{\tau_{0,\sigma}}(\tau_{0,\gamma}) = f_{\sigma\gamma}^\mu \tau_{0,\mu}.$$

With this notation, relation (29) becomes

$$(31) \quad \hat{D}R_{\tau_{0,\alpha}, \overline{T^m(\tau_{0,\beta})}} (T^r(\tau_{0,\gamma})) = - \left\{ M_\alpha^\sigma \overline{T^m(\tau_{0,\beta})} \left(\widehat{f_{\sigma\gamma}^\mu} \right) \right\} T^r(\tau_{0,\mu}).$$

We now distinguish two cases:

1) If $m = 0$, then, from Remark 13,

$$\overline{\tau_{0,\beta}} \left(\widehat{f_{\sigma\gamma}^\mu} \right) = \left[\frac{\partial f_{\sigma\gamma}^\mu}{\partial t_0^\nu} \right]^\wedge \overline{M_\beta^\nu}$$

and

$$\begin{aligned} \hat{D}R_{\tau_0,\alpha,\overline{\tau_0,\beta}}(T^r(\tau_0,\gamma)) &= -M_\alpha^\sigma \overline{M_\beta^\nu} \left[\frac{\partial f_{\sigma\gamma}^\mu}{\partial t_0^\nu} \right]^\wedge T^r(\tau_0,\mu) \\ &= M_\alpha^\sigma \overline{M_\beta^\nu} T^r \left([DR_{\tau_0,\sigma,\overline{\tau_0,\nu}}(\tau_0,\gamma)]^\wedge \right), \end{aligned}$$

which implies (27).

2) If $m > 0$, the right hand side of (31) is zero, because $\widehat{f_{\sigma\gamma}^s}$ is annihilated by $\overline{T^m(\tau_0,\beta)}$ (again from Remark 13). \square

One of the fundamental properties of the pseudo-Hermitian metric h is that it must be compatible with the holomorphic metric η , namely $D\eta = 0$ [1, 6]. It will turn out that the lifted metrics satisfy this same condition on the big phase space (see Section 6.3).

5. THE LIFTED HIGGS FIELD

In this section we lift the Higgs field from the small phase space to the big phase space. Unlike other types of tensor fields involved in our constructions, the Higgs field on the big phase space is not obtained via the general lifting procedure developed in Section 3.2. One motivation for its definition is explained in Remark 22.

Definition 21. Let $C_X Y = X \bullet Y$ be the Higgs field on M . The Higgs field \hat{C} on M^∞ is defined by

$$(32) \quad \hat{C}_{\tau_0,\alpha}(T^n(\tau_0,\beta)) = M_\alpha^\sigma T^n \left([C_{\tau_0,\sigma}(\tau_0,\beta)]^\wedge \right)$$

and

$$(33) \quad \hat{C}_{T^m(\tau_0,\alpha)}(T^n(\tau_0,\beta)) = 0,$$

for any $n \geq 0$, $m \geq 1$ and $1 \leq \alpha, \beta \leq N$.

We make some comments on the above definition.

Remark 22. The Higgs field \hat{C} , restricted to primary vector fields, gives the quantum product. This fact relies on a property proved in [2], namely that 2-point functions $\langle\langle \tau_{0,\alpha}\tau_{0,\beta} \rangle\rangle_0$ on M^∞ are natural lifts of their restrictions to M . More precisely, to prove that $\hat{C}_{\tau_0,\alpha}(\tau_{0,\beta}) = \tau_{0,\alpha} \circ \tau_{0,\beta}$, we notice that

$$\begin{aligned} \langle\langle \tau_{0,\alpha}\tau_{0,\beta}\tau_{0,\gamma} \rangle\rangle_0 &= \tau_{0,\gamma}(\langle\langle \tau_{0,\alpha}\tau_{0,\beta} \rangle\rangle_0) \\ &= \tau_{0,\gamma}(\langle\langle \tau_{0,\alpha}\tau_{0,\beta} \rangle\rangle_0 |M)^\wedge = \left(\frac{\partial}{\partial t_0^\sigma} \langle\langle \tau_{0,\alpha}\tau_{0,\beta} \rangle\rangle_0 |M \right)^\wedge M_\gamma^\sigma, \end{aligned}$$

for any $1 \leq \alpha, \beta, \gamma \leq N$. We obtain:

$$(34) \quad \langle\langle \tau_{0,\alpha}\tau_{0,\beta}\tau_{0,\gamma} \rangle\rangle_0 = (\langle\langle \tau_{0,\alpha}\tau_{0,\beta}\tau_{0,\sigma} \rangle\rangle_0 |M)^\wedge M_\gamma^\sigma.$$

In particular, the right hand side of (34) is symmetric in α and γ and thus

$$\langle\langle \tau_{0,\alpha}\tau_{0,\beta}\tau_{0,\gamma} \rangle\rangle_0 = (\langle\langle \tau_{0,\gamma}\tau_{0,\beta}\tau_{0,\sigma} \rangle\rangle_0 |M)^\wedge M_\alpha^\sigma.$$

From the above relation we then obtain

$$\begin{aligned} \tau_{0,\alpha} \circ \tau_{0,\beta} &= \langle \langle \tau_{0,\alpha} \tau_{0,\beta} \tau_{0,\gamma} \rangle \rangle_0 \eta^{\gamma\nu} \tau_{0,\nu} \\ &= M_\alpha^\sigma (\langle \langle \tau_{0,\gamma} \tau_{0,\beta} \tau_{0,\sigma} \rangle \rangle_0 | M \eta^{\gamma\nu} \tau_{0,\nu})^\wedge \\ &= M_\alpha^\sigma [C_{\tau_{0,\sigma}}(\tau_{0,\beta})]^\wedge = \hat{C}_{\tau_{0,\alpha}}(\tau_{0,\beta}), \end{aligned}$$

as claimed. This is represented in Figure 1.

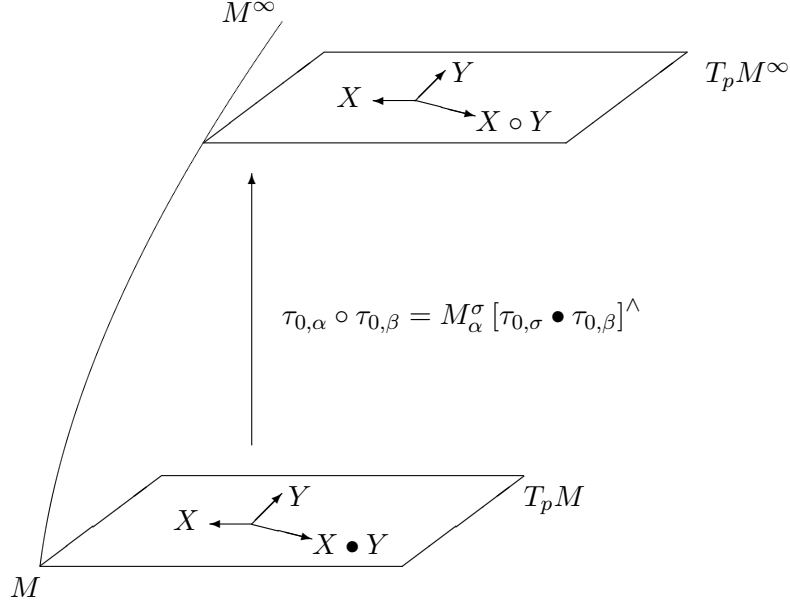


Figure 1: The lifting of the multiplication from M to M^∞ .

Remark 23. Note also that \hat{C} mirrors the properties of the Chern connection \hat{D} given in Lemma 19. Thus, each term in the second tt^* -equation

$$\hat{D}R_{Z_1, \bar{Z}_2} + [\hat{C}_{Z_1}, \hat{C}_{\bar{Z}_2}^\dagger] = 0,$$

behaves in the same manner. Owing to this, the second tt^* -equation is inherited from M to M^∞ (we shall give details of this fact in Section 6.2). This would not occur if Liu's multiplication \diamond or the quantum product \circ were used, instead of \hat{C} .

Lemma 24. *The metric $\hat{\eta}$ is invariant with respect to \hat{C} , i.e.*

$$(35) \quad \hat{\eta}(\hat{C}_{\mathcal{W}_1}(\mathcal{W}_2), \mathcal{W}_3) = \hat{\eta}(\mathcal{W}_2, \hat{C}_{\mathcal{W}_1}(\mathcal{W}_3)),$$

for any vector fields $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \in \mathcal{T}_{M^\infty}^{1,0}$.

Proof. From the definition of \hat{C} both sides of (35) are zero, when $\mathcal{W}_1 \in \text{Im}(T)$. When $\mathcal{W}_1 = \tau_{0,\alpha}$,

$$\hat{\eta}(\hat{C}_{\tau_{0,\alpha}} T^n(\tau_{0,\beta}), T^m(\tau_{0,\gamma})) = \delta_{nm} M_\alpha^\sigma [\eta(C_{\tau_{0,\sigma}}(\tau_{0,\beta}), \tau_{0,\gamma})]^\wedge$$

which is symmetric in the pairs (n, β) and (m, γ) , because η is symmetric and invariant with respect to C . \square

6. THE LIFTED tt^* -BUNDLES

In this section we consider various compatibility conditions (see Section 2.2) on the structures on the small phase space and we show that they are inherited by the lifted structures on the big phase space. We preserve the notation from the previous sections. Thus, the small phase space M has a metric η (the Poincaré pairing), a multiplication \bullet (with Higgs field C) and a real structure k which is compatible with η . Recall that k compatible with η means that $h := \eta(\cdot, k\cdot)$ is a pseudo-Hermitian metric. We denote by D the Chern connection of h . Since M is a Frobenius manifold, $(\nabla, \eta, C, R_0, R_\infty)$ is a Saito structure on $T^{1,0}M$, where ∇ is the Levi-Civita connection of η (which is flat and $\nabla(\tau_{0,\alpha}) = 0$, for any $1 \leq \alpha \leq N$), $R_0 = C_E$ is the multiplication by the Euler field and $R_\infty = \nabla E$.

The lifted structures were defined in the previous sections and will be denoted as before, $\hat{\eta}$, \hat{C} , \hat{k} and \hat{h} (with Chern connection \hat{D}). We shall also consider the natural lifts \hat{R}_0 and \hat{R}_∞ , which are endomorphisms on $T^{1,0}M^\infty$. Finally, we need to define a flat connection $\hat{\nabla}$ on the bundle $T^{1,0}M^\infty$. The connection $\hat{\nabla}$ will be part of a Saito structure on the big phase space and is defined as follows.

Definition 25. *The connection $\hat{\nabla}$ on $T^{1,0}M^\infty$ is defined by the condition that all vector fields $T^m(\tau_{0,\alpha})$, ($m \geq 0, \alpha = 1, \dots, N$) are $\hat{\nabla}$ -parallel.*

Note that the $\hat{\nabla}$ -covariant derivatives of vector fields $T^m(\hat{X})$, in directions from the image of T , vanish, i.e.

$$(36) \quad \hat{\nabla}_{T(\mathcal{W})} \left(T^m(\hat{X}) \right) = 0, \quad \forall \mathcal{W} \in \mathcal{T}_{M^\infty}, \quad X \in \mathcal{T}_M, \quad m \geq 0.$$

This follows from the above definition and Lemma 12. Note also that the two connections $\hat{\nabla}$ and ∇ (the latter defined in Lemma 5) are connected by the relation

$$\left(\hat{\nabla}_{T^n(\tau_{0,\alpha})} - \nabla_{T^n(\tau_{0,\alpha})} \right) (T^m(\tau_{0,\beta})) = \delta_{n,0} T^{m-1}(\tau_{0,\alpha} \circ \tau_{0,\beta}),$$

though we will not have to use this result.

Since the various notions from tt^* -geometry are built as an enrichment of the notion of Saito bundle, we begin by proving that the lifted data provides a Saito structure on the big phase space.

6.1. Lifted Saito bundles.

Theorem 26. *The data $(\hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty)$ is a Saito structure on $T^{1,0}M^\infty$.*

To prove Theorem 26, we need to check that $(\hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty)$ satisfies all defining conditions for Saito structures (see Definition 7). In particular, in Lemma 27 we show that the potentiality condition $d^{\hat{\nabla}} \hat{C} = 0$ holds and in Lemma 28 we show that the relation

$$\hat{\nabla}(\hat{R}_0) + \hat{C} = [\hat{C}, \hat{R}_\infty]$$

holds. The remaining conditions from Definition 7 can be checked easily, and we omit the proofs.

Lemma 27. *The equality $d^{\hat{\nabla}} \hat{C} = 0$ holds.*

Proof. From the various definitions it follows that

$$\begin{aligned} & (d^{\hat{\nabla}} \hat{C})_{T^p(\tau_{0,\alpha}), T^q(\tau_{0,\beta})} (T^m(\tau_{0,\gamma})) \\ &= \hat{\nabla}_{T^p(\tau_{0,\alpha})} \left(\hat{C}_{T^q(\tau_{0,\beta})} \right) (T^m(\tau_{0,\gamma})) - \hat{\nabla}_{T^q(\tau_{0,\beta})} \left(\hat{C}_{T^p(\tau_{0,\alpha})} \right) (T^m(\tau_{0,\gamma})) \\ & \quad - \hat{C}_{[T^p(\tau_{0,\alpha}), T^q(\tau_{0,\beta})]} (T^m(\tau_{0,\gamma})). \end{aligned}$$

When $p, q \geq 1$, both $\hat{C}_{T^p(\tau_{0,\gamma})}$ and $\hat{C}_{T^q(\tau_{0,\beta})}$ are trivial, and from Lemma 5, $[T^p(\tau_{0,\alpha}), T^q(\tau_{0,\beta})] = 0$. Hence

$$(d^{\hat{\nabla}} \hat{C})_{T^p(\tau_{0,\alpha}), T^q(\tau_{0,\beta})} (T^m(\tau_{0,\gamma})) = 0.$$

When $p = q = 0$ we obtain

$$\begin{aligned} & (d^{\hat{\nabla}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}} T^m(\tau_{0,\gamma}) \\ &= \hat{\nabla}_{\tau_{0,\alpha}} \left(\hat{C}_{\tau_{0,\beta}} T^m(\tau_{0,\gamma}) \right) - \hat{\nabla}_{\tau_{0,\beta}} \left(\hat{C}_{\tau_{0,\alpha}} T^m(\tau_{0,\gamma}) \right), \\ &= \hat{\nabla}_{\tau_{0,\alpha}} \left(M_{\beta}^{\nu} T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right) - \hat{\nabla}_{\tau_{0,\beta}} \left(M_{\alpha}^{\nu} T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right), \\ &= \frac{\partial M_{\beta}^{\nu}}{\partial t_0^{\alpha}} T^m \left([C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right) + M_{\beta}^{\nu} \hat{\nabla}_{\tau_{0,\alpha}} \left(T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right), \\ & \quad - \frac{\partial M_{\alpha}^{\nu}}{\partial t_0^{\beta}} T^m \left([C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right) - M_{\alpha}^{\nu} \hat{\nabla}_{\tau_{0,\beta}} \left(T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right), \end{aligned}$$

where we used the $\hat{\nabla}$ -flatness of $T^m(\tau_{0,\gamma})$ and the definition of \hat{C} . From

$$M_{\beta}^{\nu} = \eta^{\nu\mu} \langle\langle \tau_{0,1} \tau_{0,\mu} \tau_{0,\beta} \rangle\rangle_0$$

and relation (6),

$$(37) \quad \frac{\partial M_{\beta}^{\nu}}{\partial t_0^{\alpha}} = \eta^{\nu\mu} \langle\langle \tau_{0,1} \tau_{0,\mu} \tau_{0,\beta} \tau_{0,\alpha} \rangle\rangle_0,$$

which is symmetric in α and β . Thus,

$$(d^{\hat{\nabla}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}} (T^m(\tau_{0,\gamma})) = M_{\beta}^{\nu} \hat{\nabla}_{\tau_{0,\alpha}} \left(T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right) - M_{\alpha}^{\nu} \hat{\nabla}_{\tau_{0,\beta}} \left(T^m [C_{\tau_{0,\nu}}(\tau_{0,\gamma})]^{\wedge} \right).$$

Now, we write

$$(38) \quad C_{\tau_{0,\beta}}(\tau_{0,\sigma}) = c_{\beta\sigma}^{\mu} \tau_{0,\mu}$$

where $c_{\beta\sigma}^{\mu}$ are the structure constants of the Frobenius multiplication on M .

With this notation, the above relation becomes

$$(39) \quad (d^{\hat{\nabla}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}} (T^m(\tau_{0,\gamma})) = \left(M_{\beta}^{\mu} \tau_{0,\alpha} (\widehat{c_{\mu\gamma}^{\nu}}) - M_{\alpha}^{\mu} \tau_{0,\beta} (\widehat{c_{\mu\gamma}^{\nu}}) \right) T^m(\tau_{0,\nu}).$$

But from Lemma 12,

$$\tau_{0,\alpha} (\widehat{c_{\mu\gamma}^{\nu}}) = \left[\frac{\partial c_{\mu\gamma}^{\nu}}{\partial t_0^{\delta}} \right]^{\wedge} M_{\alpha}^{\delta}$$

and

$$(d^{\hat{\nabla}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}} T^m(\tau_{0,\gamma}) = \left[\frac{\partial c_{\mu\gamma}^{\nu}}{\partial t_0^{\delta}} \right]^{\wedge} \left(M_{\alpha}^{\delta} M_{\beta}^{\mu} - M_{\beta}^{\delta} M_{\alpha}^{\mu} \right) T^m(\tau_{0,\nu})$$

which is zero, because $\frac{\partial c_{\mu\gamma}^\nu}{\partial t_0^\delta}$ is symmetric in μ and δ (from $d^\nabla C = 0$) and $M_\alpha^\delta M_\beta^\mu - M_\beta^\delta M_\alpha^\mu$ is skew-symmetric in these indices (for any α, β fixed).

It remains to compute $(d^{\hat{\nabla}} \hat{C})_{\tau_0, \alpha, T^q(\tau_0, \beta)}(T^m(\tau_0, \gamma))$, when $q \geq 1$. Note that

$$\begin{aligned} & (d^{\hat{\nabla}} \hat{C})_{\tau_0, \alpha, T^q(\tau_0, \beta)}(T^m(\tau_0, \gamma)) \\ &= -\hat{\nabla}_{T^q(\tau_0, \beta)} \left(\hat{C}_{\tau_0, \alpha} T^m(\tau_0, \gamma) \right) - \hat{C}_{[\tau_0, \alpha, T^q(\tau_0, \beta)]} T^m(\tau_0, \gamma) \\ &= -\hat{\nabla}_{T^q(\tau_0, \beta)} \left(M_\alpha^\sigma T^m [C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge \right) + \hat{C}_{T^{q-1}(\tau_0, \alpha \circ \tau_0, \beta)}(T^m(\tau_0, \gamma)) \\ &= -T^q(\tau_0, \beta)(M_\alpha^\sigma) T^m \left([C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge \right) + \hat{C}_{T^{q-1}(\tau_0, \alpha \circ \tau_0, \beta)}(T^m(\tau_0, \gamma)), \end{aligned}$$

where we used $\hat{C}_{T^q(\tau_0, \beta)} = 0$, $[T^q(\tau_0, \beta), \tau_0, \alpha] = T^{q-1}(\tau_0, \alpha \circ \tau_0, \beta)$ and relation (36). Thus,

$$(40) \quad \begin{aligned} (d^{\hat{\nabla}} \hat{C})_{\tau_0, \alpha, T^q(\tau_0, \beta)}(T^m(\tau_0, \gamma)) &= -T^q(\tau_0, \beta)(M_\alpha^\sigma) T^m \left([C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge \right) \\ &\quad + \hat{C}_{T^{q-1}(\tau_0, \alpha \circ \tau_0, \beta)}(T^m(\tau_0, \gamma)). \end{aligned}$$

When $q \geq 2$, both terms from the right hand side of the above equality vanish (for the first term, see Lemma 18). If $q = 1$, then again from Lemma 18,

$$T(\tau_0, \beta)(M_\alpha^\sigma) = (\tau_0, \alpha \circ \tau_0, \beta)(\widehat{f^\sigma})$$

(recall that $f^\sigma(t_0^1, \dots, t_0^N) = t_0^\sigma$). We obtain:

$$\begin{aligned} & (d^{\hat{\nabla}} \hat{C})_{\tau_0, \alpha, T(\tau_0, \beta)}(T^m(\tau_0, \gamma)) \\ &= -(\tau_0, \alpha \circ \tau_0, \beta)(\widehat{f^\sigma}) T^m \left([C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge \right) + \hat{C}_{\tau_0, \alpha \circ \tau_0, \beta} T^m(\tau_0, \gamma). \end{aligned}$$

But the right hand side from the above expression is zero, because, for any primary field \mathcal{W} on M^∞ and $1 \leq \gamma \leq N$,

$$(41) \quad -\mathcal{W}(\widehat{f^\sigma}) T^m \left([C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge \right) + \hat{C}_{\mathcal{W}}(T^m(\tau_0, \gamma)) = 0.$$

(To prove (41) we may assume, without loss of generality, that $\mathcal{W} = \tau_0, \alpha$; then we use the definition of $\hat{C}_{\tau_0, \alpha}(T^m(\tau_0, \gamma))$ and $\tau_0, \alpha(\widehat{f^\sigma}) = M_\alpha^\sigma$, see Remark 17). Our claim follows. \square

Lemma 28. *The equality*

$$(42) \quad \hat{\nabla}_{\mathcal{W}}(\hat{R}_0) + \hat{C}_{\mathcal{W}} = [\hat{C}_{\mathcal{W}}, \hat{R}_\infty], \quad \forall \mathcal{W} \in \mathcal{T}_{M^\infty}$$

holds.

Proof. When $\mathcal{W} = T^m(\tau_0, \alpha)$ with $m \geq 1$, $\hat{C}_{\mathcal{W}} = 0$ and relation (42) is satisfied, because

$$\hat{\nabla}_{T^m(\tau_0, \alpha)}(\hat{R}_0)(T^n(\tau_0, \beta)) = \hat{\nabla}_{T^m(\tau_0, \alpha)}(T^n[R_0(\tau_0, \beta)]^\wedge) = 0,$$

where we used the definition of \hat{R}_0 and relation (36).

It remains to check that

$$(43) \quad \hat{\nabla}_{\tau_0, \alpha}(\hat{R}_0) + \hat{C}_{\tau_0, \alpha} = [\hat{C}_{\tau_0, \alpha}, \hat{R}_\infty].$$

For this, recall that

$$(44) \quad \hat{C}_{\tau_0, \alpha} (T^n(\tau_{0, \beta})) = M_\alpha^\sigma T^n \left([C_{\tau_0, \sigma}(\tau_{0, \beta})]^\wedge \right).$$

Also, it is straightforward that

$$(45) \quad [\hat{C}_{\tau_0, \alpha}, \hat{R}_\infty] (T^n(\tau_{0, \beta})) = M_\alpha^\sigma T^n ([C_{\tau_0, \sigma}, R_\infty](\tau_{0, \beta})^\wedge).$$

We now check that

$$(46) \quad \hat{\nabla}_{\tau_0, \alpha}(\hat{R}_0) (T^n(\tau_{0, \beta})) = M_\alpha^\sigma T^n \left([\nabla_{\tau_0, \sigma}(R_0)(\tau_{0, \beta})]^\wedge \right).$$

For this, we define functions $r_{\beta\mu}$ on M by $R_0(\tau_{0, \beta}) = r_{\beta\mu}\tau_{0, \mu}$. With this notation, relation (46) is proved as follows:

$$\begin{aligned} \hat{\nabla}_{\tau_0, \alpha}(\hat{R}_0)(T^n(\tau_{0, \beta})) &= \hat{\nabla}_{\tau_0, \alpha} (T^n [R_0(\tau_{0, \beta})]^\wedge) = \tau_{0, \alpha} (\widehat{r_{\beta\mu}}) T^n(\tau_{0, \mu}) \\ &= M_\alpha^\sigma \left[\frac{\partial r_{\beta\mu}}{\partial t_0^\sigma} \right]^\wedge T^n(\tau_{0, \mu}) = M_\alpha^\sigma T^n \left([\nabla_{\tau_0, \sigma}(R_0)(\tau_{0, \beta})]^\wedge \right). \end{aligned}$$

Combining the above relation with (44) we obtain:

$$\begin{aligned} \hat{\nabla}_{\tau_0, \alpha}(\hat{R}_0) (T^n(\tau_{0, \beta})) + \hat{C}_{\tau_0, \alpha} (T^n(\tau_{0, \beta})) \\ &= M_\alpha^\sigma T^n \left([\nabla_{\tau_0, \alpha}(R_0)(\tau_{0, \beta}) + C_{\tau_0, \sigma}(\tau_{0, \beta})]^\wedge \right) \\ &= M_\alpha^\sigma T^n ([C_{\tau_0, \sigma}, R_\infty](\tau_{0, \beta})^\wedge) = [\hat{C}_{\tau_0, \alpha}, \hat{R}_\infty] (T^n(\tau_{0, \beta})) \end{aligned}$$

where we used

$$\nabla(R_0) + C = [C, R_\infty],$$

because $(T^{1,0}M, \nabla, \eta, C, R_0, R_\infty)$ is a Saito bundle, and relation (45). We obtained (43), as required. \square

In the followings we introduce the real structure k into the picture and we study the tt^* -geometry on the big phase space.

6.2. Lifted harmonic Higgs bundles.

Theorem 29. *Assume that $(T^{1,0}M, C, h)$ is a harmonic Higgs bundle. Then $(T^{1,0}M^\infty, \hat{C}, \hat{h})$ is also a harmonic Higgs bundle.*

The proof follows by combining the Lemmas 30 and 31 (see below). These lemmas hold for h an arbitrary pseudo-Hermitian metric on M (not necessarily related to η by means of k), without any compatibility conditions between h and the Frobenius structure on M .

Lemma 30. *i) The following relation holds:*

$$(47) \quad \left(\partial^{\hat{D}} \hat{C} \right)_{\tau_0, \alpha, \tau_0, \beta} (T^n(\tau_{0, \gamma})) = M_\alpha^\nu M_\beta^\sigma T^n \left([(\partial^D C)_{\tau_0, \nu, \tau_0, \sigma}(\tau_{0, \gamma})]^\wedge \right),$$

for any $1 \leq \alpha, \beta, \gamma \leq N$ and $n \geq 0$. If m or p is bigger than zero, then

$$(48) \quad \left(\partial^{\hat{D}} \hat{C} \right)_{T^m(\tau_{0, \alpha}), T^p(\tau_{0, \beta})} (T^n(\tau_{0, \gamma})) = 0.$$

ii) In particular, if the first tt^* -equation $\partial^D C = 0$ for $(T^{1,0}M, C, h)$ holds, then the first tt^* -equation $\partial^{\hat{D}} \hat{C} = 0$ for $(T^{1,0}M^\infty, \hat{C}, \hat{h})$ holds as well.

Proof. From the definition of \hat{C} and

$$(49) \quad [T^m(\tau_{0,\alpha}), T^p(\tau_{0,\beta})] = 0, \quad \forall m, p \geq 1,$$

it is immediately clear that (48) is true when both $m, p \geq 1$. Also, note that $\hat{D}_{\mathcal{W}} = \hat{\nabla}_{\mathcal{W}}$, for any vector field \mathcal{W} in the image of T . Using this remark, together with the definition of \hat{C} again, we obtain:

$$(\partial^{\hat{D}} \hat{C})_{\tau_{0,\alpha}, T^p(\tau_{0,\beta})} (T^n(\tau_{0,\gamma})) = (d^{\hat{\nabla}} \hat{C})_{\tau_{0,\alpha}, T^p(\tau_{0,\beta})} (T^n(\tau_{0,\gamma})), \quad \forall p \geq 1,$$

which is zero from Lemma 27. It remains to prove (47). For this, we compute $\hat{D}_{\tau_{0,\alpha}} (\hat{C}_{\tau_{0,\beta}}) (T^n(\tau_{0,\gamma}))$ as follows:

$$\begin{aligned} \hat{D}_{\tau_{0,\alpha}} (\hat{C}_{\tau_{0,\beta}}) (T^n(\tau_{0,\gamma})) &= \hat{D}_{\tau_{0,\alpha}} (\hat{C}_{\tau_{0,\beta}} T^n(\tau_{0,\gamma})) - \hat{C}_{\tau_{0,\beta}} \hat{D}_{\tau_{0,\alpha}} (T^n(\tau_{0,\gamma})) \\ &= \hat{D}_{\tau_{0,\alpha}} (M_{\beta}^{\sigma} T^n [C_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) - M_{\alpha}^{\sigma} \hat{C}_{\tau_{0,\beta}} (T^n [D_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) \\ &= \eta^{\sigma\nu} \langle\langle \tau_{0,1} \tau_{0,\nu} \tau_{0,\beta} \tau_{0,\alpha} \rangle\rangle_0 T^n ([C_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) \\ &\quad + M_{\beta}^{\sigma} \hat{D}_{\tau_{0,\alpha}} (T^n [C_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) - M_{\alpha}^{\sigma} \hat{C}_{\tau_{0,\beta}} (T^n [D_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}), \end{aligned}$$

where in the third line we used

$$\tau_{0,\alpha}(M_{\beta}^{\sigma}) = \eta^{\sigma\nu} \langle\langle \tau_{0,1} \tau_{0,\nu} \tau_{0,\beta} \tau_{0,\alpha} \rangle\rangle_0.$$

To simplify notation, we define

$$\begin{aligned} E_1(\alpha, \beta, \gamma, n) &:= \eta^{\sigma\nu} \langle\langle \tau_{0,1} \tau_{0,\nu} \tau_{0,\beta} \tau_{0,\alpha} \rangle\rangle_0 T^n ([C_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) \\ E_2(\alpha, \beta, \gamma, n) &:= M_{\beta}^{\sigma} \hat{D}_{\tau_{0,\alpha}} (T^n [C_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}) = M_{\beta}^{\sigma} \hat{D}_{\tau_{0,\alpha}} (\widehat{c_{\sigma\gamma}^{\mu}} T^n(\tau_{0,\mu})) \\ E_3(\alpha, \beta, \gamma, n) &:= M_{\alpha}^{\sigma} \hat{C}_{\tau_{0,\beta}} (T^n [D_{\tau_{0,\sigma}}(\tau_{0,\gamma})]^{\wedge}), \end{aligned}$$

where $c_{\sigma\gamma}^{\mu}$ are the structure constants of the Frobenius multiplication on M , already defined in (38). With these

$$\hat{D}_{\tau_{0,\alpha}} (\hat{C}_{\tau_{0,\beta}}) (T^n(\tau_{0,\gamma})) = (E_1 + E_2 - E_3)(\alpha, \beta, \gamma, n).$$

Since

$$(\partial^{\hat{D}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}} (T^n(\tau_{0,\gamma})) = \hat{D}_{\tau_{0,\alpha}} (\hat{C}_{\tau_{0,\beta}}) (T^n(\tau_{0,\gamma})) - \hat{D}_{\tau_{0,\beta}} (\hat{C}_{\tau_{0,\alpha}}) (T^n(\tau_{0,\gamma})),$$

we need to compute the skew part (in α and β) of E_1 , E_2 and E_3 . It is clear that E_1 is symmetric in α and β . Also,

$$\begin{aligned} E_2(\alpha, \beta, \gamma, n) &= M_{\beta}^{\sigma} (\tau_{0,\alpha} (\widehat{c_{\sigma\gamma}^{\mu}}) T^n(\tau_{0,\mu}) + \widehat{c_{\sigma\gamma}^{\mu}} M_{\alpha}^{\nu} T^n [D_{\tau_{0,\nu}}(\tau_{0,\mu})]^{\wedge}) \\ &= M_{\beta}^{\sigma} \left(\left[\frac{\partial c_{\sigma\gamma}^{\mu}}{\partial t_0^{\nu}} \right]^{\wedge} M_{\alpha}^{\nu} T^n(\tau_{0,\mu}) + \widehat{c_{\sigma\gamma}^{\mu}} M_{\alpha}^{\nu} T^n [D_{\tau_{0,\nu}}(\tau_{0,\mu})]^{\wedge} \right) \end{aligned}$$

and a straightforward computation shows that

$$\begin{aligned} E_2(\alpha, \beta, \gamma, n) - E_2(\beta, \alpha, \gamma, n) &= M_{\beta}^{\sigma} M_{\alpha}^{\nu} \left[\frac{\partial c_{\sigma\gamma}^{\mu}}{\partial t_0^{\nu}} - \frac{\partial c_{\nu\gamma}^{\mu}}{\partial t_0^{\sigma}} \right]^{\wedge} T^n(\tau_{0,\mu}) \\ &\quad + M_{\beta}^{\sigma} M_{\alpha}^{\nu} T^n \left([c_{\sigma\gamma}^{\mu} D_{\tau_{0,\nu}}(\tau_{0,\mu}) - c_{\nu\gamma}^{\mu} D_{\tau_{0,\sigma}}(\tau_{0,\mu})]^{\wedge} \right). \end{aligned}$$

Since $\frac{\partial c_{\sigma\gamma}^\mu}{\partial t_0^\nu}$ is symmetric in σ and ν , the first term in the right hand side of the above relation vanishes and we obtain:

$$(50) \quad E_2(\alpha, \beta, \gamma, n) - E_2(\beta, \alpha, \gamma, n) = M_\beta^\sigma M_\alpha^\nu \left[c_{\sigma\gamma}^\mu f_{\nu\mu}^\delta - c_{\nu\gamma}^\mu f_{\sigma\mu}^\delta \right]^\wedge T^n(\tau_{0,\delta}),$$

where the functions $f_{\mu\gamma}^\delta$ are defined by $D_{\tau_{0,\mu}}(\tau_{0,\gamma}) = f_{\mu\gamma}^\delta \tau_{0,\delta}$. A similar computation shows that

$$(51) \quad E_3(\alpha, \beta, \gamma, n) - E_3(\beta, \alpha, \gamma, n) = M_\alpha^\nu M_\beta^\sigma \left[c_{\sigma\mu}^\delta f_{\nu\gamma}^\mu - c_{\nu\mu}^\delta f_{\sigma\gamma}^\mu \right]^\wedge T^n(\tau_{0,\delta}).$$

Hence we obtain

$$\begin{aligned} (\partial^{\hat{D}} \hat{C})_{\tau_{0,\alpha}, \tau_{0,\beta}}(T^n(\tau_{0,\gamma})) &= (E_2 - E_3)(\alpha, \beta, \gamma, n) - (E_2 - E_3)(\beta, \alpha, \gamma, n) \\ &= M_\alpha^\nu M_\beta^\sigma \left[c_{\sigma\gamma}^\mu f_{\nu\mu}^\delta - c_{\nu\gamma}^\mu f_{\sigma\mu}^\delta - c_{\sigma\mu}^\delta f_{\nu\gamma}^\mu + c_{\nu\mu}^\delta f_{\sigma\gamma}^\mu \right]^\wedge T^n(\tau_{0,\delta}). \end{aligned}$$

On the other hand, one may check that

$$(\partial^D C)_{\tau_{0,\nu}, \tau_{0,\sigma}}(\tau_{0,\gamma}) = \left(c_{\sigma\gamma}^\mu f_{\nu\mu}^\delta - c_{\nu\gamma}^\mu f_{\sigma\mu}^\delta - c_{\sigma\mu}^\delta f_{\nu\gamma}^\mu + c_{\nu\mu}^\delta f_{\sigma\gamma}^\mu \right) \tau_{0,\delta}.$$

Combining the above two relations, we obtain (47), as required. Our claim follows. \square

Lemma 31. *i) The following relation holds: for any $1 \leq \alpha, \beta, \gamma \leq N$ and $m \geq 0$,*

$$(52) \quad \begin{aligned} &\hat{D}R_{\tau_{0,\alpha}, \overline{\tau_{0,\beta}}} (T^m(\tau_{0,\gamma})) + [\hat{C}_{\tau_{0,\alpha}}, \hat{C}_{\overline{\tau_{0,\beta}}}^\dagger] (T^m(\tau_{0,\gamma})) \\ &= M_\alpha^\sigma \overline{M}_\beta^\nu T^m \left(\left[D R_{\tau_{0,\sigma}, \overline{\tau_{0,\nu}}}(\tau_{0,\gamma}) + [C_{\tau_{0,\sigma}}, C_{\overline{\tau_{0,\nu}}}^\dagger](\tau_{0,\gamma}) \right]^\wedge \right), \end{aligned}$$

where $C^\dagger \in \Omega^{0,1}(M, \text{End} T^{1,0} M)$ and $\hat{C}^\dagger \in \Omega^{0,1}(M^\infty, \text{End} T^{1,0} M^\infty)$ are, respectively, the h -adjoint of C and the \hat{h} -adjoint of \hat{C} .

ii) In particular, if the second tt^* -equation

$$D R + [C, C^\dagger] = 0$$

for $(T^{1,0} M, C, h)$ holds, then the second tt^* -equation

$$\hat{D} R + [\hat{C}, \hat{C}^\dagger] = 0$$

for $(T^{1,0} M^\infty, \hat{C}, \hat{h})$ holds as well.

Proof. We first compute the \hat{h} -adjoint $\hat{C}_{\overline{\tau_{0,\beta}}}^\dagger$ of $\hat{C}_{\tau_{0,\beta}}$, as follows:

$$\begin{aligned} \hat{h} \left(\hat{C}_{\tau_{0,\beta}} (T^n(\tau_{0,\alpha})), T^m(\tau_{0,\gamma}) \right) &= M_\beta^\sigma \hat{h} \left(T^n \left([C_{\tau_{0,\sigma}}(\tau_{0,\alpha})]^\wedge \right), T^m(\tau_{0,\gamma}) \right) \\ &= M_\beta^\sigma \delta_{nm} \left[h(C_{\tau_{0,\sigma}}(\tau_{0,\alpha}), \tau_{0,\gamma}) \right]^\wedge = M_\beta^\sigma \delta_{nm} \left[h(\tau_{0,\alpha}, C_{\overline{\tau_{0,\sigma}}}^\dagger(\tau_{0,\gamma})) \right]^\wedge \\ &= M_\beta^\sigma \hat{h} \left(T^n(\tau_{0,\alpha}), T^m \left([C_{\overline{\tau_{0,\sigma}}}^\dagger(\tau_{0,\gamma})]^\wedge \right) \right) \\ &= \hat{h} \left(T^n(\tau_{0,\alpha}), \overline{M}_\beta^\sigma T^m \left([C_{\overline{\tau_{0,\sigma}}}^\dagger(\tau_{0,\gamma})]^\wedge \right) \right). \end{aligned}$$

We obtain:

$$\hat{C}_{\overline{\tau_{0,\beta}}}^\dagger (T^m(\tau_{0,\gamma})) = \overline{M}_\beta^\sigma T^m \left([C_{\overline{\tau_{0,\sigma}}}^\dagger(\tau_{0,\gamma})]^\wedge \right),$$

which, combined with

$$\hat{C}_{\tau_0, \alpha} (T^m(\tau_0, \gamma)) = M_\alpha^\sigma T^m ([C_{\tau_0, \sigma}(\tau_0, \gamma)]^\wedge)$$

gives

$$[\hat{C}_{\tau_0, \alpha}, \hat{C}_{\tau_0, \beta}^\dagger] (T^m(\tau_0, \gamma)) = M_\alpha^\sigma \overline{M_\beta^\nu} T^m ([C_{\tau_0, \sigma}, C_{\tau_0, \nu}^\dagger](\tau_0, \gamma)^\wedge).$$

The above relation, together with

$$\hat{D}R_{\tau_0, \alpha, \overline{\tau_0, \beta}} (T^m(\tau_0, \gamma)) = M_\alpha^\sigma \overline{M_\beta^\nu} T^m ([{}^D R_{\tau_0, \sigma, \overline{\tau_0, \nu}}(\tau_0, \gamma)]^\wedge)$$

(see Lemma 20), implies (52). This proves *i*). Claim *ii*) follows from *i*) and

$$\hat{D}R_{\mathcal{W}_1, \mathcal{W}_2} = 0, \quad [\hat{C}_{\mathcal{W}_1}, \hat{C}_{\mathcal{W}_2}^\dagger] = 0$$

which hold when \mathcal{W}_1 or \mathcal{W}_2 belongs to the image of T . □

6.3. Lifted harmonic real Saito bundles.

Theorem 32. *Assume that $(T^{1,0}M, \nabla, \eta, R_0, R_\infty, k)$ is a real Saito bundle (respectively, a harmonic real Saito bundle). Then $(T^{1,0}M^\infty, \hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty, \hat{k})$ is also a real Saito bundle (respectively, a harmonic real Saito bundle).*

The above theorem is a consequence of Theorems 26 and 29 and the following lemma.

Lemma 33. *If h is compatible with η , then \hat{h} is compatible with $\hat{\eta}$.*

Proof. Recall that h is compatible with η if the Chern connection D of h preserves η , i.e. $D(\eta) = 0$ and the compatibility of \hat{h} and $\hat{\eta}$ is defined in a similar way. In order to prove our claim, we will show that

$$(53) \quad \hat{D}_{\mathcal{W}}(\hat{\eta}) = 0, \quad \forall \mathcal{W} \in \text{Im}(T)$$

and

$$(54) \quad \hat{D}_{\tau_0, \gamma}(\hat{\eta}) (T^n(\tau_0, \alpha), T^m(\tau_0, \beta)) = M_\gamma^\sigma \delta_{nm} [D_{\tau_0, \sigma}(\eta)(\tau_0, \alpha, \tau_0, \beta)]^\wedge.$$

To prove these two relations, we notice that for any vector field $\mathcal{V} \in \mathcal{T}_{M^\infty}^{1,0}$ and $n, m \geq 0$, $1 \leq \alpha, \beta \leq N$,

$$(55) \quad \begin{aligned} \hat{D}_{\mathcal{V}}(\hat{\eta}) (T^n(\tau_0, \alpha), T^m(\tau_0, \beta)) &= -\hat{\eta} \left(\hat{D}_{\mathcal{V}} (T^n(\tau_0, \alpha)), T^m(\tau_0, \beta) \right) \\ &\quad - \hat{\eta} \left(T^n(\tau_0, \alpha), \hat{D}_{\mathcal{V}} (T^m(\tau_0, \beta)) \right), \end{aligned}$$

because $\hat{\eta} (T^n(\tau_0, \alpha), T^m(\tau_0, \beta))$ is constant. This relation, together with the expression of \hat{D} from Lemma 19, implies (53). Letting $\mathcal{V} := \tau_0, \gamma$ in (55) we obtain:

$$\begin{aligned} \hat{D}_{\tau_0, \gamma}(\hat{\eta}) (T^n(\tau_0, \alpha), T^m(\tau_0, \beta)) &= \\ &= -M_\gamma^\sigma \hat{\eta} \left(T^n \left([D_{\tau_0, \sigma}(\tau_0, \alpha)]^\wedge \right), T^m(\tau_0, \beta) \right) - M_\gamma^\sigma \hat{\eta} \left(T^n(\tau_0, \alpha), T^m \left([D_{\tau_0, \sigma}(\tau_0, \beta)]^\wedge \right) \right) \\ &= -M_\gamma^\sigma \delta_{mn} [\eta (D_{\tau_0, \sigma}(\tau_0, \alpha), \tau_0, \beta)]^\wedge - M_\gamma^\sigma \delta_{mn} [\eta (\tau_0, \alpha, D_{\tau_0, \sigma}(\tau_0, \beta))]^\wedge \\ &= M_\gamma^\sigma \delta_{nm} [D_{\tau_0, \sigma}(\eta)(\tau_0, \alpha, \tau_0, \beta)]^\wedge, \end{aligned}$$

i.e. relation (54) holds as well. \square

Remark 34. The above lemma, combined with Theorem 29, imply that if $(T^{1,0}M, D, C, h, k)$ is a $DChk$ -bundle, then $(T^{1,0}M^\infty, \hat{D}, \hat{C}, \hat{h}, \hat{k})$ is a $\hat{D}\hat{C}\hat{h}\hat{k}$ -bundle (the definition of $DChk$ -bundles was recalled in Remark 10).

6.4. Lifted harmonic potential real Saito bundles.

Theorem 35. *Suppose that (M, \bullet, e, η, E) is a harmonic Frobenius manifold, with real structure k and potential A . Then $T^{1,0}M^\infty$, with the data $(\hat{\nabla}, \hat{\eta}, \hat{C}, \hat{R}_0, \hat{R}_\infty, \hat{k}, \hat{A})$, is a harmonic potential real Saito bundle.*

Proof. It is easy to check that taking Hermitian adjoints of endomorphisms, with respect to h and \hat{h} , commutes with taking natural lifts, so that $(\hat{A})^\dagger = \widehat{A^\dagger}$ (we use the same symbol \dagger to denote h and \hat{h} -adjoints). There is a similar commutativity property when the metrics η and $\hat{\eta}$ are used instead of h and \hat{h} . Thus, since A is η -self adjoint, \hat{A} is $\hat{\eta}$ -self adjoint. Similarly,

$$\hat{R}_\infty + [(\hat{A})^\dagger, \hat{R}_0] = \left(R_\infty + [A^\dagger, R_0] \right)^\wedge$$

is \hat{h} -self adjoint, because $R_\infty + [A^\dagger, R_0]$ is h -self adjoint. From Theorem 32, we only have to check that

$$(56) \quad \hat{D}^{(1,0)} = \hat{\nabla} - [\widehat{A^\dagger}, \hat{C}], \quad \hat{D}^{(1,0)}\hat{A} = \hat{C}.$$

Using that $T^m(\tau_{0,\beta})$ are $\hat{\nabla}$ -parallel, the first equality (56) is equivalent to

$$(57) \quad \hat{D}_{T^n(\tau_{0,\alpha})}(T^m(\tau_{0,\beta})) + [\widehat{A^\dagger}, \hat{C}_{T^n(\tau_{0,\alpha})}](T^m(\tau_{0,\beta})) = 0.$$

When $n > 0$ both terms of (57) are zero. We now show that (57) holds also for $n = 0$. For this, we use

$$(58) \quad \hat{D}_{\tau_0,\alpha}(T^m(\tau_{0,\beta})) = M_\alpha^\sigma T^m \left([D_{\tau_0,\sigma}(\tau_{0,\beta})]^\wedge \right)$$

and

$$(59) \quad [\widehat{A^\dagger}, \hat{C}_{\tau_0,\alpha}](T^m(\tau_{0,\beta})) = M_\alpha^\sigma T^m \left([A^\dagger, C_{\tau_0,\sigma}](\tau_{0,\beta})^\wedge \right)$$

(easy check). On the other hand, $\nabla = D + [A^\dagger, C]$ (M with the given data is a harmonic Frobenius manifold, with potential A) and, since $\nabla(\tau_{0,\beta}) = 0$,

$$(60) \quad D_{\tau_0,\sigma}(\tau_{0,\beta}) + [A^\dagger, C_{\tau_0,\sigma}](\tau_{0,\beta}) = 0.$$

Combining (58), (59) and (60) we obtain

$$\hat{D}_{\tau_0,\alpha}(T^m(\tau_{0,\beta})) + [\widehat{A^\dagger}, \hat{C}_{\tau_0,\alpha}](T^m(\tau_{0,\beta})) = 0.$$

The first equality (56) is proved. The second equality (56) can be proved equally easy, by using

$$(61) \quad \begin{aligned} \hat{D}_{\tau_0,\alpha}(\hat{A})(T^m(\tau_{0,\beta})) &= M_\alpha^\sigma T^m \left([D_{\tau_0,\sigma}(A)(\tau_{0,\beta})]^\wedge \right) \\ \hat{D}_{T^n(\tau_{0,\alpha})}(\hat{A})(T^m(\tau_{0,\beta})) &= 0, \end{aligned}$$

the definition of \hat{C} and the condition $D^{1,0}A = C$.

\square

Remark 36. Relations (59) and (61) remain true when A^\dagger , respectively A , are replaced by any endomorphism of $T^{1,0}M$. Combining these facts with Remark 34, it is easy to see that if $(T^{1,0}M, C, h, k, \mathcal{U}, \mathcal{Q})$ is a CV-bundle (see Remark 10), then $(T^{1,0}M^\infty, \hat{C}, \hat{h}, \hat{k}, \hat{\mathcal{U}}, \hat{\mathcal{Q}})$ is also a CV-bundle.

7. DISCUSSION

The construction of these Hermitian structures on the big phase space rests on the pre-existence of two different structures:

- the lifting map $u^\alpha = \eta^{\alpha\beta} \ll \tau_{0,1}\tau_{0,\beta} \gg_0$;
- the Hermitian structures on the small phase space,

and one should comment separably on the tractability of each of these two points.

The first point rests on the work of Dijkgraaf and Witten [2]. We repeat verbatim their construction, using their normalization of the topological recursion relation which differs by a factor to the one used in [14, 15] and in the above. The string equation gives (where $u_\alpha = \eta_{\alpha\beta}u^\beta$ and $t_{i,\alpha} = \eta_{\alpha\beta}t_i^\beta$)

$$u_\alpha = t_{0,\alpha} + \sum_{i=0}^{\infty} (i+1)t_{i+1,\beta} \ll \tau_i(\gamma_\beta)\gamma_\alpha \gg_0$$

and small phase space calculations give the constitutive relations

$$\ll \tau_i(\gamma_\beta)\gamma_\alpha \gg_0 = R_{\alpha,\beta,i}(u_1, \dots, u_N).$$

Combining these gives

$$u_\alpha = t_{0,\alpha} + \sum_{i=0}^{\infty} (i+1)t_{i+1,\beta} R_{\alpha,\beta,i}(u_1, \dots, u_N).$$

Inverting these equations gives $u_\alpha = u_\alpha(t_\beta^n)$, and hence the two-point correlation functions $\ll \tau_{0,1}\tau_{0,\beta} \gg_0$ as functions of the big phase space variables, as required in order to perform the canonical lift described in Section 5. If $\dim(M) = 1$ this reduces to the single equation

$$(62) \quad u = t_0 + \sum_{i=1}^{\infty} t_i u^i.$$

Thus the extension to the big phase space is entirely tractable, up to the inversion of these equations.

Hermitian structures in one dimension are trivial. One just has $h(\partial_0, \partial_0) = |a(t)|$ for some non-vanishing function $a(t)$, and the corresponding anti-holomorphic involution is then given by $k(\partial_0) = a^{-1}|a|\partial_0$. In fact this gives a positive CDV-structure [21]. Thus combining this with the solution to (62) gives Hermitian structures on the big phase space to 2 dimensional gravity [22].

The existence of tt^* -geometries in higher dimensions is harder. The Hermitian structures involve the construction of solutions to Toda and harmonic map type equations [1, 9], often with very specific boundary conditions. Even in dimension two this involves specific solutions to the Painlevé III equation. While these equations are integrable, one is rapidly drawn into

very sophisticated isomonodromy problems. For details of these constructions see, for examples originating in quantum cohomology, [12], and for examples originating in singularity theory, [11, 19].

Ultimately these constructions rely on the integrability properties of the tt^* -equations [1, 6]. The tt^* -equations are the compatibility conditions, or zero-curvature equations, for the deformed connections

$$\begin{aligned} {}^{(\lambda)}D_X Y &= (D_X - \lambda C_X)Y, \\ {}^{(\lambda)}D_{\bar{X}} Y &= (D_{\bar{X}} - \lambda^{-1} C_X^\dagger)Y. \end{aligned}$$

It is easy to show that the $\lambda^{\pm 2}$ -terms in the curvature of this connection vanish from the properties of the Higgs field, the $\lambda^{\pm 1}$ -terms vanish from the first tt^* -equation $\partial^D C = 0$, and the λ^0 -term vanishes from the second tt^* -equation ${}^D R + [C, C^\dagger] = 0$. They thus provide a Lax pair for, and hence the integrability of, the tt^* -equations.

The results in this paper show that there exists a solution of the analogous Lax-pair on the big phase space M^∞ . However, while one may speculate that they define an integrable system on M^∞ all that has been shown is that any solution on M may be naturally lifted to a solution on M^∞ , not that all solutions arise in this way. The integrability aspects of the tt^* -equations on the big phase space deserves a separate study.

One geometric structure that plays a prominent role in quantum cohomology is the Euler field, which on the big phase space takes the form

$$\chi = - \sum_{m,\alpha} (m + b_\alpha - b_1 - 1) \tilde{t}_m^\alpha \tau_m(\gamma_\alpha) - \sum_{m,\alpha,\beta} C_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1}(\gamma_\beta),$$

with the associated quasihomogeneity equation

$$\langle\langle \chi \rangle\rangle_g = 2(b_1 + 1)(1 - g)\mathcal{F}_g + \frac{1}{2}\delta_{g,0} \sum_{\alpha,\beta} C_{\alpha\beta} t_0^\alpha t_0^\beta - \frac{1}{24} \int_V c_1(V) \cup c_{d-1}(V)$$

(for a precise definition of the various constants, see [14, 15]). To develop further these ideas one should study the homogeneity properties of the lifted objects on the big phase space, starting with their homogeneity properties on the small phase space. Another direction of research would be to reformulate these constructions in the semi-simple case in terms of idempotents and canonical coordinates, following [17]. However, a more geometric problem is to seek a description of these Hermitian structures in terms of Givental's Lagrangian cones [8]. Recall that Givental showed that the function \mathcal{F}_0 satisfies the Topological Recursion Relations, the String Equation and the Dilaton Equation if and only if a corresponding Lagrangian submanifold has certain natural geometric properties. This gives a beautiful interpretation of quantum cohomology and leads naturally to the quantization of these objects. Understanding how the (compatible) Hermitian structures introduced in this paper can be interpreted within this framework would be of great interest. In a sense the results of this paper could be seen as a 'pre-Givental' approach. Understanding the symmetries and quantization of the tt^* -equations, following Givental, would provide an elegant solution to the problems addressed in this paper. We hope to address such problems in subsequent work.

REFERENCES

- [1] Cecotti, S., Vafa, C.: *Topological-antitopological fusion*, Nuclear Physics **B 367** (1991), p. 359-461.
- [2] Dijkgraaf, R., Witten, E.: *Mean field-theory, topological field-theory, and multimatrix models*, Nuclear Physics **B 342:3** (1990) p. 486-522.
- [3] Donagi, R., Wendland, K.: eds., *From Hodge Theory to Integrability and TQFT: tt^* geometry*, Proceedings of Symposia in Pure Mathematics, vol. 78, American Mathematical Society (2008)
- [4] Dubrovin, B., *Geometry of 2D topological field theories in Integrable Systems and Quantum Groups*, ed. Francaviglia, M. and Greco, S.. Springer lecture notes in mathematics, **1620**, 120-348.
- [5] Dubrovin, B.: *Integrable systems in topological field theory*, Nuclear Physics **B 379** (1992) 627-689.
- [6] Dubrovin, B.: *Geometry and integrability of topological-antitopological fusion*, Communications in Mathematical Physics, **152:3** (1993), p. 539-564.
- [7] Getzler, E.: *The jet-space of a Frobenius manifold and higher-genus Gromov-Witten invariants in Frobenius manifolds: Quantum Cohomology and Singularities*, ed. Hertling, C and Marcolli, M.. Aspects Math., E36, Vieweg, Wiesbaden, 2004.
- [8] Givental, A.B.: *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001) 551-568.
- [9] Guest, M.A. and Lin, C-S.: *Nonlinear PDE aspects of the tt^* -equations of Cecotti and Vafa* arXiv:1010.1889v1
- [10] Hertling, C., Manin, Yuri I.: *Weak Frobenius Manifolds*, Internat. Math. Res. Notices, no. 6 (1999), p. 277-286.
- [11] Hertling, C.: *tt^* -geometry, Frobenius manifolds, their connections, and the construction for singularities*, J. Reine Angew. Math. 555 (2003), p. 77-161.
- [12] Iritani, H.: *tt^* -geometry in quantum cohomology*, arXiv:0906.1307
- [13] Kontsevich, M.: *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), 1-23.
- [14] Liu, X. and Tian, G.: *Virasoro Constraints for Quantum Cohomology*, J. Differential Geometry, **50** (1998) 537-591.
- [15] Liu, X.: *Quantum Product on the Big Phase Space and the Virasoro Conjecture*, Advances in Mathematics, **169** (2002) 313-375.
- [16] Liu, X.: *Gromov-Witten invariants and moduli spaces of curves*, Proceedings International Congress of Mathematicians, 2006, vol. II, p. 791-812.
- [17] Liu, X.: *Idempotents on the big phase space*, Contemporary Math., vol. 403 (2006), p. 43-66.
- [18] Sabbah, C.: *Isomonodromic deformations and Frobenius manifolds - An Introduction*, Springer and EDP Sciences (2007).
- [19] Sabbah, C.: *Universal unfolding of Laurent polynomials and tt^* structures*, From Hodge theory to integrability and TQFT: tt^* -geometry, Proc. Symposia in Pure Math., vol. 78 (2008), p. 1-29.
- [20] Saito, K.: *The higher residue pairings $K_F^{(k)}$ for a family of hypersurfaces singular points*, Singularities, Proc. of Symposia in Pure Math., vol. 40, American Mathematical Society (1983), p. 441-463.
- [21] Takahashi, A.: *tt^* geometry of rank two*, Int. Math. Res. Notices, (2004) 1099-1114
- [22] Witten, E.: *On the structure of the topological phase of two dimensional gravity*, Nuclear Phys. **B 340** (1990), 281-332,
Two-dimensional gravity and intersection theory on moduli space, Surveys in Differential Geometry **1** (1991), 243-310.

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