NOVIKOV ALGEBRAS AND A CLASSIFICATION OF MULTICOMPONENT CAMASSA-HOLM EQUATIONS

IAN A.B. STRACHAN AND BŁAŻEJ M. SZABLIKOWSKI

Abstract. A class of multi-component integrable systems associated to Novikov algebras, which interpolate between KdV and Camassa-Holm type equations, is obtained. The construction is based on the classification of low-dimensional Novikov algebras by Bai and Meng. These multi-component bi-Hamiltonian systems obtained by this construction may be interpreted as Euler equations on the centrally extended Lie algebras associated to the Novikov algebras. The related bilinear forms generating cocycles of first, second and third order are classified. Several examples, including known integrable equations, are presented.

Contents

Date: January 24, 2014.
Key words and phrases. Novikov algebras, bi-Hamiltonian structures, Camassa-Holm equations.

1. Introduction

The Camassa-Holm equation [10]

$$v_t - v_{xxt} = \alpha v_x - 3vv_x + 2v_x v_{xx} + vv_{xxx}\,, \tag{1.1}$$

an example of a (1+1)-dimensional integrable system, has many intriguing mathematical properties, amongst them being:

- the existence of multi 'peakon' solutions;
- the non-existence of a τ function or functions.

Its other properties have more in common with equations such as the KdV equation: the existence of a Lax pair, solvability via the inverse scattering transform, and a bi-Hamiltonian structure. In fact, the bi-Hamiltonian structure, and hence the Camassa-Holm itself, may be found by exploiting the tri-Hamiltonian structure of the KdV hierarchy [15, 16, 34]. The component parts of the bi-Hamiltonian pair for the KdV equation, namely

$$\frac{d}{dx}\,, \qquad \frac{d^3}{dx^3}\,, \qquad u\frac{d}{dx} + \frac{1}{2}u_x \tag{1.2}$$

are pair-wise compatible and hence may be recombined to form the bi-Hamiltonian structures

$$\begin{aligned} \mathcal{P}_1 &= \frac{d}{dx} - \frac{d^3}{dx^3}\,, \\ \mathcal{P}_2 &= u\frac{d}{dx} + \frac{1}{2}u_x \end{aligned}$$

and applying the Lenard-Magri recursion scheme results in the Camassa-Holm equation (1.1), where $u = v - v_{xx}$. Another common feature between Camassa-Holm and KdV equations, closely related to the above bi-Hamiltonian structure, is that both systems may be written as Euler equations on the Virasoro algebra (see [26] and references therein).

The class of Hamiltonian operators such that this tri-Hamiltonian duality may be most easily applied was first derived by Balinskii and Novikov [7] as a special case of the Dubrovin-Novikov operators of hydrodynamic type [13]. The conditions for the operator

$$\mathcal{P}^{ij} = c^{ij}_k u^k \frac{d}{dx} + b^{ij}_k u^k_x\,, \qquad c^{ij}_k = b^{ij}_k + b^{ji}_k\,,$$

to be a Hamiltonian places purely algebraic conditions on the constants b^{ij}_k, and the corresponding algebraic structure - on regarding these constants as structure constants - is now known as a Novikov algebra. With each Novikov algebra there is associated a translationally invariant Lie algebra, which can be centrally extended. Condition for the existence of cocycles (either first-order or third-order Gelfand-Fuks cocycles) is equivalent to the existence of symmetric bilinear forms g and h on the Novikov algebra which satisfy certain (quasi-Frobenius and Frobenius, respectively) compatibility conditions. Second order cocycles result in antisymmetric bilinear forms which again satisfy certain algebraic relations. This construction is outlined in Section 2. Thus one obtains a multi-component tri-Hamiltonian structure

$$g^{ij}\frac{d}{dx}\,, \qquad h^{ij}\frac{d^3}{dx^3}\,, \qquad \mathcal{P}^{ij}$$

in direct analogue to (1.2), defined algebraically in terms of Novikov algebras and compatible symmetric bilinear forms on the algebra. Using these ideas, the centrally extended translationally invariant Lie algebras associated to Novikov algebras can be considered as multi-component linear generalization of the Virasoro algebra.

A two-component integrable generalization of the Camassa-Holm equation, called CH2, has been proposed in [30, 11]. This new system admits a Lax pair [11], of the same type as the original

Camassa-Holm equation, which is connected to the energy-dependent Schrödinger spectral problem, see [1, 2]. An alternative Lie algebraic approach for the derivation of CH2, which is closer to the one considered in this article, was presented in [14]. Further, taking the advantage of the energy-dependent Schrödinger spectral problems the CH2 equation is generalized in [21] to produce an integrable multi-component family CH(n,k) of equations with n components and k velocities. The direct relationships between our results with these from [21] remains to be clarified (see Section 4.3 and Example 6.4 for some specific examples).

The purpose of this paper is two-fold. By mirroring the construction of the Camassa-Holm equation we will obtain multi-component versions of this equation (in fact, by splitting the various structures more finely one can obtain equations which interpolate between KdV and Camassa-Holm equations). The nonlinear terms in these equations are controlled by the properties of the Novikov algebra and associated bilinear forms. Secondly, using the (low-dimensional) classification of Novikov algebras obtained by Bai and Meng [4] - extended to classify the second and third order Gelfand-Fuks cocycles - one can obtain a classification scheme for equations in this class purely in terms of an underlying algebraic structure. It should be noted that these algebras are classified over $\mathbb{C}$; in some cases a finer decomposition exists over $\mathbb{R}$: see, for example [9]. It turns out that many Novikov algebras result in highly degenerate systems of integrable equations, so the classification scheme has to be extended to rule out these degenerate cases.

2. Novikov algebras

We begin by briefly reviewing some of the basic concepts of linear Poisson tensors and brackets that will be needed elsewhere in the paper. The material may all be found in the original papers [7, 13] or in expositions such as [12]. Other basic results may be found in the Appendix.

2.1. **Linear Poisson tensors of hydrodynamic type.** Following [7] consider a homogeneous first order $n\times n$ operator[1]

$$\mathcal{P}^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k u^k_x\,, \qquad x\in\mathbb{S}^1, \tag{2.1}$$

depending on fields $u^1(x),\dots,u^n(x)$. Here, $g^{ij}(u)=c^{ij}_k u^k$ is symmetric and b^{ij}_k, c^{ij}_k are constants. The operator is a Poisson operator if and only if

- $c^{ij}_k = b^{ij}_k + b^{ji}_k$;
- b^{ij}_k is the set of structure constants of an algebra $\mathbb{A}$, that is $e^i\cdot e^j = b^{ij}_k\, e^k$ where $e^1,\dots,e^n$ are basis vectors, such that

$$(a\cdot b)\cdot c = (a\cdot c)\cdot b\,, \tag{2.2a}$$

$$(a\cdot b)\cdot c - a\cdot(b\cdot c) = (b\cdot a)\cdot c - b\cdot(a\cdot c)\,. \tag{2.2b}$$

This structure $\mathbb{A}$ is called a Novikov algebra.[2] The relations (2.2) may be written in a simple way by defining left and right multiplications by $L_a b = R_b a = a\cdot b$. With these the identities (2.2) are equivalent to

$$[R_a, R_b] = 0\,, \qquad [L_a, L_b] = L_{[a,b]}\,,$$

respectively. Here $[a,b] = a\cdot b - b\cdot a$. Novikov Lie algebras $\mathbb{A}$ are Lie admissible, that is the commutator defines structure of a Lie algebra on the underlying vector spaces $\mathbb{A}$. Note that if the multiplication is commutative then the Novikov conditions (2.2) reduce to associativity conditions.

[1] The summation convention is used throughout this article.

[2] All structures are considered over the field of complex numbers $\mathbb{C}$.

Let $\mathbb{A}^*$ be the dual algebra with respect to the standard pairing $(\,,\,) : \mathbb{A}^* \times \mathbb{A} \to \mathbb{C}$. Then $(L_a^* u, b) = (R_b^* u, a) := (u, a \cdot b)$, where $a, b \in \mathbb{A}$ and $u \in \mathbb{A}^*$. The following relation, which will be needed later, is equivalent to (2.2a):

$$R_b L_a c = L_{R_b a} c \quad \Longleftrightarrow \quad L_a^* R_b^* u = L_{R_b a}^* u. \tag{2.3}$$

2.2. **Lie-Poisson brackets.** Consider the infinite-dimensional Lie algebra $\mathscr{L}_\mathbb{A}$ on the space of $\mathbb{A}$-valued functions of $x \in \mathbb{S}^1$, with a Lie bracket of the form

$$[\![a, b]\!] := a_x \cdot b - b_x \cdot a\,, \qquad a_x \equiv \frac{da}{dx}. \tag{2.4}$$

In fact, (2.4) defines a Lie bracket if and only if the multiplication $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$ satisfies the conditions (2.2) and hence the algebra $\mathbb{A}$ is a Novikov algebra.

The Poisson bracket associated to the Poisson operator (2.1) is a Lie-Poisson bracket associated to the Lie algebra $\mathscr{L}_\mathbb{A}$ (see Appendix A.1):

$$\{\mathcal{H}, \mathcal{F}\}[u] := \int_{\mathbb{S}^1} \frac{\delta \mathcal{F}}{\delta u^i} \mathcal{P}^{ij} \frac{\delta \mathcal{H}}{\delta u^j}\, dx \equiv \langle u, [\![\delta_u \mathcal{F}, \delta_u \mathcal{H}]\!]\rangle\,, \qquad u \in \mathscr{L}_\mathbb{A}^*, \tag{2.5}$$

where $\mathcal{H}, \mathcal{F} \in \mathscr{F}(\mathscr{L}_\mathbb{A}^*)$ are functionals on the (regular) dual space to $\mathscr{L}_\mathbb{A}$. Here, the duality pairing $\mathscr{L}_\mathbb{A}^* \times \mathscr{L}_\mathbb{A} \to \mathbb{C}$ is given by

$$\langle u, a\rangle := \int_{\mathbb{S}^1} (u, a)\, dx\,,$$

where $u \in \mathscr{L}_\mathbb{A}^*$ and $a \in \mathscr{L}_\mathbb{A}$. Functionals from $\mathscr{F}(\mathscr{L}_\mathbb{A}^*)$ have the form

$$\mathcal{H}[u] = \int_{\mathbb{S}^1} H(u, u_x, u_{xx}, \ldots)\, dx\,,$$

their (variational) differentials $\delta_u \mathcal{H} = \frac{\delta \mathcal{H}}{\delta u^i} e^i$ belong to $\mathscr{L}_\mathbb{A}$, which follows from the relation (A.2). The coadjoint action, such that (A.1), is

$$\mathrm{ad}_a^* u = -(R_a^* u)_x - L_{a_x}^* u.$$

2.3. **Central extensions.** Centrally extended Poisson bracket (2.5) is given by

$$\{\mathcal{H}, \mathcal{F}\}[u] := \langle u, [\![\delta_u \mathcal{H}, \delta_u \mathcal{F}]\!]\rangle + \omega(\delta_u \mathcal{H}, \delta_u \mathcal{F})\,, \tag{2.6}$$

where the bilinear map ω on $\mathscr{L}_\mathbb{A}$ is a 2-cocycle, see Appendix A.2. Let ω be defined by means of a 1-cocycle $\phi : \mathscr{L}_\mathbb{A} \to \mathscr{L}_\mathbb{A}^*$, i.e. the relation (A.5) is valid. Then, the Poisson operator $\mathcal{P}$ of (2.6), such that $\{\mathcal{H}, \mathcal{F}\} := \langle \mathcal{P}\delta_u \mathcal{H}, \delta_u \mathcal{F}\rangle$, has the form

$$\mathcal{P}\gamma = -\,\mathrm{ad}_\gamma^* u + \phi(\gamma)\,, \qquad \gamma \in \mathscr{L}_\mathbb{A}. \tag{2.7}$$

We will consider now some differential 1-cocycles, yielding central extensions of Lie algebras associated with Novikov algebras. They are generated by appropriate bilinear forms satisfying various algebraic conditions and were originally derived in [7].

- <u>Order-one cocycles:</u> A symmetric bilinear form g on $\mathbb{A}$, $g(a, b) \equiv (\tilde{g}(a), b)$, defines 1-cocycle of order one $\phi = \tilde{g}\frac{d}{dx}$, that is the associated 2-cocycle is

  $$\omega_1(a, b) = \int_{\mathbb{S}^1} g(a_x, b)\, dx\,,$$

  if and only if the quasi-Frobenius condition

$$g(a \cdot b, c) = g(a, c \cdot b) \tag{2.8}$$

holds for all $a, b, c \in \mathbb{A}$. The condition (2.8) is equivalent either to

$$\tilde{g}(R_b a) = R_b^* \tilde{g}(a), \tag{2.9}$$

or

$$L_a^* \tilde{g}(b) = L_b^* \tilde{g}(a). \tag{2.10}$$

From the more general formalism of Poisson tensors of hydrodynamic type [13] we know that the bilinear form $g^{ij}(u) = c_k^{ij} u^k + g_0^{ij}$, where g_0 is generating the above first order cocycle, if $\det g^{ij}(u) \neq 0$, can be interpreted as a contravariant flat metric. Thus this bilinear form may be thought about in two different ways: as an inhomogeneous flat metric, or as a homogeneous flat metric with the addition of a first-order cocycle term. Even if $\det(c^{ij}_{\ k} u^k) = 0$ it still defines a Poisson tensor [17].

- Order-two cocycles: A skew-symmetric bilinear form f on $\mathbb{A}$, $f(a, b) \equiv (\tilde{f}(a), b)$, defines 1-cocycle of order two $\phi = \tilde{f} \frac{d^2}{dx^2}$, with the associated 2-cocycle

$$\omega_2(a, b) = \int_{\mathbb{S}^1} f(a_{xx}, b)\, dx\,,$$

if and only if

$$f(a \cdot b, c) = f(a, c \cdot b) \tag{2.11}$$

and

$$f(a \cdot b, c) + f(b \cdot c, a) + f(c \cdot a, b) = 0\,, \tag{2.12}$$

that is the quasi-Frobenius and cyclic conditions are satisfied. The condition (2.11) may be rewritten in two equivalent ways: as

$$\tilde{f}(R_b a) = R_b^* \tilde{f}(a)\,, \tag{2.13}$$

or as

$$L_a^* \tilde{f}(b) = -L_b^* \tilde{f}(a). \tag{2.14}$$

On the other hand (2.12) is equivalent to

$$L_b^* \tilde{f}(a) = \tilde{f}(a \cdot b - b \cdot a).$$

- Order-three cocycles: A symmetric bilinear form h on $\mathbb{A}$, $\tilde{h}(a, b) \equiv (\tilde{h}(a), b)$, defines 1-cocycle of third order $\phi = \tilde{h} \frac{d^3}{dx^3}$, with the associated 2-cocycle

$$\omega_3(a, b) = \int_{\mathbb{S}^1} h(a_{xxx}, b)\, dx\,,$$

if and only if the tensor $h(a \cdot b, c)$ is totally symmetric, which reduces to two conditions:

$$h(a \cdot b, c) = h(a, c \cdot b), \qquad h(a, b \cdot c) = h(a, c \cdot b). \tag{2.15}$$

The conditions (2.15) are equivalent to

$$L_a^* \tilde{h}(b) = R_a^* \tilde{h}(b) = \tilde{h}(a \cdot b) = \tilde{h}(b \cdot a). \tag{2.16}$$

There are no higher order cocycles with constants coefficients. Let us denote by $\mathscr{C}_{\mathbb{A}}^i$ the linear space spanned by bilinear forms generating cocycles of i-th order and associated with a Novikov algebra $\mathbb{A}$.

3. Multicomponent bi-Hamiltonian Camassa-Holm hierarchies

Consider the following pair of compatible Poisson operators $\mathcal{P}_0$ and $\mathcal{P}_1$ on $\mathscr{L}_{\mathbb{A}}$, associated with some Novikov algebra $\mathbb{A}$:

$$\mathcal{P}_1\gamma = (R^*_\gamma u)_x + L^*_{\gamma_x}u + \tilde{g}_1\gamma_x + \tilde{f}_1\gamma_{xx} + \tilde{h}_1\gamma_{xxx} \tag{3.1a}$$

and

$$\mathcal{P}_0\gamma = \tilde{g}_0\gamma_x + \tilde{f}_0\gamma_{xx} + \tilde{h}_0\gamma_{xxx}, \tag{3.1b}$$

where $u \in \mathscr{L}^*_{\mathbb{A}}$ and $\gamma \in \mathscr{L}_{\mathbb{A}}$. Here g_0 and g_1 are symmetric bilinear forms on $\mathbb{A}$ generating first order 1-cocycles, f_0 and f_1 are skew-symmetric bilinear forms generating 1-cocycles of order two, while h_0 and h_1 are symmetric bilinear forms generating third order 1-cocycles. The operator $\mathcal{P}_1$ has the form of a centrally extended Poisson operator (2.7). The compatibility of $\mathcal{P}_0$ and $\mathcal{P}_1$ is a consequence of the fact that any linear combination of 1-cocycles is itself a 1-cocycle. In many research papers the second Poisson structure is obtained by freezing the first one, which often yields restricted class of compatible Poisson operators. In the case of the linear Poisson operator $\mathcal{P}_1$ our approach gives the most general compatible with it Poisson operator $\mathcal{P}_2$ with constant coefficients.

We can formulate the bi-Hamiltonian chain associated to the above Poisson pair in the form

$$\begin{aligned} u_{t_0} &= \mathcal{P}_0\delta_u\mathcal{H}_0 \equiv 0\,,\\ u_{t_1} &= \mathcal{P}_1\delta_u\mathcal{H}_0 = \mathcal{P}_0\delta_u\mathcal{H}_1\,,\\ u_{t_2} &= \mathcal{P}_1\delta_u\mathcal{H}_1 = \mathcal{P}_0\delta_u\mathcal{H}_2\,,\\ &\ \ \vdots, \end{aligned} \tag{3.2}$$

where $\mathcal{H}_i \in \mathscr{F}(\mathscr{L}^*_{\mathbb{A}})$ and $\mathcal{H}_0$ is a Casimir of $\mathcal{P}_0$, t_i are evolution parameters (times) of respective flows associated to vector fields u_{t_i} on $\mathscr{L}^*_{\mathbb{A}}$.

Consider the operator $\Lambda : \mathscr{L}_{\mathbb{A}} \to \mathscr{L}^*_{\mathbb{A}}$ defined by

$$\Lambda\gamma := \tilde{g}_0\gamma + \tilde{f}_0\gamma_x + \tilde{h}_0\gamma_{xx}, \tag{3.3}$$

such that $\mathcal{P}_0\gamma \equiv \Lambda\gamma_x$. The operator Λ is self-adjoint, i.e. $\Lambda^\dagger = \Lambda$, and when Λ is invertible it can be interpreted as an inertia operator, see Remark 3.2. We assume that Λ as a map is a diffeomorphism, and hence we impose its invertibility by the condition that the bilinear form g_0 is nondegenerate (see Remark 3.3). As we will see in the following theorem, to obtain local forms of evolution equations one must perform the change of coordinates

$$v := \Lambda^{-1}u, \tag{3.4}$$

which is of (linear) Miura-type. Thus the hierarchy (3.2) transforms into

$$\begin{aligned} v_{t_0} &= \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_0 \equiv 0\,,\\ v_{t_1} &= \tilde{\mathcal{P}}_1\delta_v\mathcal{H}_0 = \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_1\,,\\ v_{t_2} &= \tilde{\mathcal{P}}_1\delta_v\mathcal{H}_1 = \tilde{\mathcal{P}}_0\delta_v\mathcal{H}_2\,,\\ &\ \ \vdots, \end{aligned} \tag{3.5}$$

where $\tilde{\mathcal{P}}_i = \Lambda^{-1}\mathcal{P}_i(\Lambda)^{-1}$ and $\delta_v\mathcal{H}_j = \Lambda\delta_u\mathcal{H}_j$.

Theorem 3.1. *The first two evolution equations from the hierarchy (3.5) are*

$$\begin{aligned} v_{t_1} &= v_x \cdot c\,, \\ \tilde g_0(v_{t_2}) + \tilde f_0(v_{xt_2}) + \tilde h_0(v_{xxt_2}) &= \tilde g_0(v_x\cdot(v\cdot c)) + \tilde g_0(v\cdot(v_x\cdot c)) + L^*_{v\cdot c}\tilde g_0(v_x) \\ &\quad + \tilde f_0(v_x\cdot(v_x\cdot c)) + \tilde f_0(v_{xx}\cdot(v\cdot c)) \\ &\quad + 2\tilde h_0((v_x\cdot c)\cdot v_{xx}) + \tilde h_0((v\cdot c)\cdot v_{xxx}) \\ &\quad + \tilde g_1(v_x\cdot c) + \tilde f_1(v_{xx}\cdot c) + \tilde h_1(v_{xxx}\cdot c)\,, \end{aligned}$$

where $c = \mathrm{const} \in \mathscr{L}_{\mathbb{A}}$ *and* $\delta_u \mathcal{H}_0 = c$*. The densities of the first three conserved quantities (Hamiltonians), such that*

$$\mathcal{H}_i[v] = \int_{\mathbb{S}^1} H_i(v, v_x, v_{xx}, \dots)\, dx\,, \qquad i = 0, 1, \dots$$

are

$$\begin{aligned} H_0 &= g_0(c, v)\,, \\ H_1 &= \frac12\, g_0(v, v\cdot c) + \frac12\, f_0(v_x, v\cdot c) + \frac12\, h_0(v_{xx}, v\cdot c)\,, \\ H_2 &= \frac13\, g_0(v, v\cdot(v\cdot c)) + \frac13\, f_0(v_x, v\cdot(v\cdot c)) + \frac13\, h_0(v\cdot c, v\cdot v_{xx}) \\ &\quad + \frac16\, g_0(v\cdot c, v\cdot v) + \frac16\, h_0(v_x\cdot c, v_x\cdot v) \\ &\quad + \frac12\, g_1(v, v\cdot c) + \frac12\, f_1(v_x, v\cdot c) + \frac12\, h_1(v_{xx}, v\cdot c)\,. \end{aligned}$$

Proof. Recall that by Poincaré lemma a closed 1-form (co-vector) γ in a star-shape (local) coordinate system $\{u\}$ is exact, that is $\gamma = \delta_u \mathcal{H}$. The functional $\mathcal{H}$ can be obtained using the homotopy formula [33]:

$$\mathcal{H}[u] = \int_0^1 \langle u, \gamma(\lambda u)\rangle d\lambda\,.$$

A useful relation, which follows from (2.9), (2.13) and (2.16), is

$$R^*_a \Lambda = \Lambda R_a \qquad \Longleftrightarrow \qquad \Lambda^{-1} R^*_a = R_a \Lambda^{-1}\,, \qquad a \in \mathbb{A}\,,$$

which will be used below.

Casimirs of $\mathcal{P}_0$ are constants $c \in \mathbb{A}$. Hence, taking $\delta_u \mathcal{H}_0 = c$ we have

$$\mathcal{H}_0 = \langle u, c\rangle = \langle \Lambda v, c\rangle.$$

Thus,

$$u_{t_1} = \mathcal{P}_1 \delta_u \mathcal{H}_0 = R^*_c u_x$$

and from (3.2) one finds that $\delta_u \mathcal{H}_1 = \Lambda^{-1} R^*_c u$. Using the homotopy formula one finds that

$$\mathcal{H}_1 = \frac12 \langle \Lambda^{-1} u, R^*_c u\rangle = \frac12 \langle \Lambda v, v\cdot c\rangle.$$

The second flow is

$$\begin{aligned} u_{t_2} &= \mathcal{P}_1 \delta_u \mathcal{H}_1 \\ &= (R^*_{R_c\Lambda^{-1}u} u)_x + L^*_{\Lambda^{-1}u_x} R^*_c u + \tilde g_1 R_c \Lambda^{-1} u_x + \tilde f_1 R_c \Lambda^{-1} u_{xx} + \tilde h_1 R_c \Lambda^{-1} u_{xxx} \end{aligned} \tag{3.6}$$

or equivalently

$$\Lambda v_{t_2} = (R^*_{R_c v} \Lambda v)_x + L^*_{v_x} \Lambda R_c v + \tilde g_1 R_c v_x + \tilde f_1 R_c v_{xx} + \tilde h_1 R_c v_{xxx}.$$

Note that

$$L^*_{v_x}\tilde{f}_0 R_c v_x = 0, \tag{3.7}$$

which is a consequence of relations (2.3), (2.13) and (2.14). Using the properties of the symmetric forms $\tilde{g}_0$, $\tilde{h}_0$ and (3.7) one can show that

$$L^*_{v_x}\Lambda R_c v = \frac{1}{2}\Big(L^*_v\tilde{g}_0 R_c v + L^*_{v_x}\tilde{h}_0 R_c v_x\Big)_x .$$

Hence,

$$\delta_v\mathcal{H}_2 \equiv \Lambda\delta_u\mathcal{H}_2 = R^*_{R_c v}\Lambda v + \frac{1}{2}L^*_v\tilde{g}_0 R_c v + \frac{1}{2}L^*_{v_x}\tilde{h}_0 R_c v_x\,, + \tilde{g}_1 R_c v + \tilde{f}_1 R_c v_x + \tilde{h}_1 R_c v_{xx}$$

and finally, using the homotopy formula, one obtains

$$\begin{aligned}\mathcal{H}_2 = &\frac{1}{3}\langle\Lambda v, R_{R_c v}v\rangle + \frac{1}{6}\langle\tilde{g}_0 R_c v, L_v v\rangle + \frac{1}{6}\langle\tilde{h}_0 R_c v_x, L_{v_x}v\rangle\\ &+\frac{1}{2}\langle\tilde{g}_1 R_c v, v\rangle + \frac{1}{2}\langle\tilde{f}_1 R_c v_x, v\rangle + \frac{1}{2}\langle\tilde{h}_1 R_c v_{xx}, v\rangle,\end{aligned}$$

which finishes the proof. □

For non-nilpotent Novikov algebras the invertibility of Λ guarantees the existence of the (hereditary) recursion operator $\mathcal{R} = \mathcal{P}_1\mathcal{P}_0^{-1}$, such that $u_{t_i} = \mathcal{R}^i u_{t_0}$, and hence the existence of the infinite hierarchies of commuting evolution equations and conserved quantitates. Thus the equations from the hierarchy (3.2) (or equivalently (3.5)) are integrable, see [33].

Remark 3.2. Observe that the Hamiltonian flow (3.6) on the dual space $\mathscr{L}^*_{\mathbb{A}}$ is the Euler equation (see Appendix A.3), corresponding to the centrally extended Lie algebra $\mathscr{L}_{\mathbb{A}}\oplus\mathscr{C}^1_{\mathbb{A}}\oplus\mathscr{C}^2_{\mathbb{A}}\oplus\mathscr{C}^3_{\mathbb{A}}$, with the quadratic Hamiltonian

$$\mathcal{H}_1 = \frac{1}{2}\langle u, \Lambda^{-1}R^*_c u\rangle.$$

This Euler equation transformed through (3.4) to $\mathscr{L}_{\mathbb{A}}$ is the second flow from Theorem 3.1.

These formulae simplify if the Novikov algebra $\mathbb{A}$ has a right-unity e, i.e. $u\cdot e = u$ for all $u\in\mathbb{A}$ (recall that if the algebra has a left-unity then it is automatically commutative and associative). Then, taking $c = e$ the evolution equations and Hamiltonians from Theorem 3.1 simplify to

$$v_{t_1} = v_x\,,$$

$$\begin{aligned}\tilde{g}_0(v_{t_2}) + \tilde{f}_0(v_{xt_2}) + \tilde{h}_0(v_{xxt_2}) = \Big(&\tilde{g}_0(v\cdot v) + \frac{1}{2}L^*_v\,\tilde{g}_0\,v + \tilde{f}_0(v_x\cdot v) + \frac{1}{2}\tilde{h}_0(v_x\cdot v_x)\\ &+\tilde{h}_0(v\cdot v_{xx}) + \tilde{g}_1 v + \tilde{f}_1 v_x + \tilde{h}_1 v_{xx}\Big)_x\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}H_0 &= g_0(e,v)\,,\\ H_1 &= \frac{1}{2}\,g_0(v,v) + \frac{1}{2}\,f_0(v_x,v) + \frac{1}{2}\,h_0(v_{xx},v)\,,\\ H_2 &= \frac{1}{2}\,g_0(v,v\cdot v) + \frac{1}{3}\,f_0(v_x,v\cdot v) + \frac{1}{3}\,h_0(v,v\cdot v_{xx}) + \frac{1}{6}\,h_0(v,v_x\cdot v_x)\\ &\quad+\frac{1}{2}\,g_1(v,v) + \frac{1}{2}\,f_1(v_x,v) + \frac{1}{2}\,h_1(v_{xx},v)\,.\end{aligned}$$

Remark 3.3. Note that the invertibility of the inertia operator (3.3), that is $\Lambda = \tilde{g} + \tilde{f}\partial_x + \tilde{h}\partial_x^2$ (for simplicity the subscript is omitted here), is understood in the sense of invertibility of a pseudo-differential operator [33].[3] That is, if $\det \tilde{g} \neq 0$, the inverse is given by a formal differential series

$$\Lambda^{-1} = \tilde{g}^{-1} - \tilde{g}^{-1}\tilde{f}\tilde{g}^{-1}\partial_x + \tilde{g}^{-1}\left(\tilde{f}\tilde{g}^{-1}\tilde{f} - \tilde{h}\right)\tilde{g}^{-1}\partial_x^2 + \dots \; .$$

There are two important special cases which will occur in the next section, and for which Theorem 3.1 still holds. The first one is when $\tilde{g} = 0$ and $\det \tilde{f} \neq 0$. In this case the inverse is

$$\Lambda^{-1} = \tilde{f}^{-1}\partial_x^{-1} - \tilde{f}^{-1}\tilde{h}\tilde{f}^{-1} + \tilde{f}^{-1}\tilde{h}\tilde{f}^{-1}\tilde{h}\tilde{f}^{-1}\partial_x + \dots \; .$$

The second is when $\tilde{g} = \tilde{f} = 0$ and $\det \tilde{h} \neq 0$. In this case the inverse is $\Lambda^{-1} = \tilde{h}^{-1}\partial_x^{-2}$.

4. A classification of integrable systems on Novikov algebras

In [4] Novikov algebras $\mathbb{A}$ of dimension up to $n = 3$ were classified over $\mathbb{C}$ and in [6] this work was extended to classify four-dimensional transitive Novikov algebras[4]. In [9] a different approach to the classification of low-dimensional Novikov algebras $\mathbb{A}$ was presented. This is based on the classification of the associated Lie algebras $(\mathbb{A}, [\,,\,])$,[5] defined by means of a commutator $[a, b] = a \cdot b - b \cdot a$. As result the authors classified four-dimensional Novikov algebras for which associated Lie algebras are nilpotent. However, the full classification of Novikov algebras of dimension four is far from being complete.

To be able to compute the associated evolution equations, using Theorem 3.1, one must additionally classify the bilinear forms generating central extensions of the Lie algebras $\mathscr{L}_{\mathbb{A}}$. This classification is presented in Appendix B. The various defining relations for these bilinear forms are just linear equation and hence may be solved using, for example, Mathematica. However, not all of these Novikov algebras will lead to the construction of 'proper' evolution equations:

- All Novikov algebras that are direct sums of lower dimensional algebras can be omitted as they lead to evolution equations consisting of decoupled systems associated to the lower dimensional Novikov algebras.
- We are interested in the construction of evolution equations for which the coefficients of associated (infinite-dimensional) vector fields will depend on all fields. This leads to the exclusion of Novikov algebras for which the generic right multiplication (or equivalently left multiplication) has rank lower then the algebra dimension.[6] This occurs for all transitive Novikov algebras appearing in the classification schemes in [4, 6, 9].
- The invertibility of the inertia operator Λ impose nondegeneracy of the bilinear form g_0. Hence, the Novikov algebras for which 'generic' bilinear forms g, generating central extension of first order, are degenerate can also be omitted.

As result, the only relevant Novikov algebras are in dimension one: the field of complex numbers $\mathbb{C}$; in dimension two: $(N3)$–$(N6)$; in dimension three: $(C6)$, $(C8)$, $(C9)$, $(C16)$, $(C19)$, $(D2)$–$(D5)$; in dimension four (within the Novikov algebras from [6, 9] and this paper): $\widetilde{A}_{3,3}$, $\widetilde{A}_{3,4}$, $N_{22}^{\mathfrak{h}_1}$, $N_{23}^{\mathfrak{h}_1}$, $N_{24}^{\mathfrak{h}_1}$, $N_{27}^{\mathfrak{h}_2}$ and $\mathbb{A}_4$ (see Section 6). We use the symbols adopted in [4] and [9].

[3]There is also possible other approach to the invertibility of the inertia operator such as the analytic approach presented in [25].

[4]A Novikov algebra is called transitive (or right-nilpotent) if every R_a is nilpotent.

[5]Naturally, these Lie algebras should not be confused with the translationally invariant Lie algebras $\mathscr{L}_{\mathbb{A}}$.

[6]One could claim that some of this cases can lead to proper reduced systems. However, careful inspection shows that for such low-dimensional Novikov algebras this cannot be the case.

Most of the relevant Novikov algebras under consideration in dimension two, three and four lead to the construction of evolution equations from Theorem 3.1 in a triangular form.[7] The only non-triangular systems are associated to the algebras $(N4)$, $(C8)$ and $\mathbb{A}_4$.

Contrary to Section 2 we will use here the 'contravariant' convention for Novikov algebras (so the formulae agree with the prior work [4]). Therefore the structure constants of a Novikov algebra $\mathbb{A}$, with basis vectors $e_1, \dots, e_n$, are given by b^i_{jk} such that

$$(a \cdot b)^i := b^i_{jk} a^j b^k \qquad \Longleftrightarrow \qquad e_i \cdot e_j = b^k_{ij}\, e_k, \tag{4.1}$$

where $a, b \in \mathbb{A}$. The related characteristic matrix is given by $\mathcal{B} = (b_{ij})$, where $b_{ij} := b^k_{ij}\, e_k$.

4.1. **Dimension one.** The only relevant one-dimensional Novikov algebra $\mathbb{A}$ is generated by the rule $u \cdot v := uv$. This algebra is obviously isomorphic to $\mathbb{C}$.

Let $\tilde{g}_0 = g$, $\tilde{g}_1 = \alpha$, $\tilde{h}_0 = h$ and $\tilde{h}_1 = \beta$. There are no cocycles of second order, thus $\tilde{f}_1 = \tilde{f}_2 = 0$. For $c = 1$ the evolution equation (3.8) becomes

$$g v_t + h v_{xxt} = \alpha v_x + 3 g v v_x + 2 h v_x v_{xx} + h v v_{xxx} + \beta v_{xxx}\,, \tag{4.2}$$

here $v \in \mathscr{L}_{\mathbb{C}}$. The first order linear term involving the constant α can always be eliminated by a linear change of independent coordinates. The evolution equation (4.2) (with $\alpha = 0$) was obtained in [26]. In particular for $g = 1$ and $h = \alpha = 0$ we obtain the Korteweg–de Vries (KdV) equation

$$v_t = 3 v v_x + \beta v_{xxx}\,,$$

for $g = 1$ and $h = -1$ we have the Camassa-Holm equation [10]

$$v_t - v_{xxt} = \alpha v_x + 3 v v_x - 2 v_x v_{xx} - v v_{xxx} + \beta v_{xxx}\,.$$

Finally, if $g = \beta = \alpha = 0$ and $h = 1$ then (4.2) becomes the Hunter-Saxton equation [22]

$$v_{xxt} = 2 v_x v_{xx} + v v_{xxx}\,.$$

4.2. **Two-dimensional algebra** $(N4)$**.** This Novikov algebra is non-abelian and associative, its structure matrix is

$$B = \begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}.$$

Consider the most general form of equation (3.8) with the right unity $c = (0,\, 1)^{\mathrm{T}}$. Thus we take the following bilinear forms generating associated cocycles (see Table 1)

$$\tilde{g}_0 = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \quad \tilde{f}_0 = \begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}, \quad \tilde{h}_0 = \begin{pmatrix} 0 & 0 \\ 0 & h \end{pmatrix},$$

and

$$\tilde{g}_1 = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix}, \quad \tilde{f}_1 = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \quad \tilde{h}_1 = \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}.$$

Let $(u,\, v)^{\mathrm{T}} \in \mathscr{L}_{\mathbb{A}}$. Then, the equation (3.8) has the form

$$\begin{aligned} g_{11} u_t + g_{12} v_t + f v_{xt} &= \big(\alpha_{11} u + g_{11} u v + \alpha_{12} v + g_{12} v^2 + f v v_x + \gamma v_x\big)_x\,, \\ g_{12} u_t + g_{22} v_t - f u_{xt} + h v_{xxt} &= \Big(\alpha_{12} u + \frac{1}{2} g_{11} u^2 + 2 g_{12} u v + \alpha_{22} v + \frac{3}{2} g_{22} v^2 - f u_x v \\ &\qquad - \gamma u_x + \frac{1}{2} h v_x^2 + h v v_{xx} + \beta v_{xx}\Big)_x. \end{aligned} \tag{4.3}$$

[7] By triangular evolution system we mean such that can be represented in the form: $(u_1)_t = K_1(u_1)$, $(u_2)_t = K_2(u_1, u_2)$, $(u_3)_t = K_3(u_1, u_2, u_3)$, $\dots$.

We now show how various examples of 2-component Camassa-Holm equations that have appeared already in the literature fall into this scheme by identifying the underlying Novikov algebras and bilinear forms.

Example 4.1. For the Novikov algebra $(N4)$ with the particular choice

$$\tilde{g}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{g}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

one obtains the system

$$\begin{aligned} u_t + f v_{xt} &= \left(uv + f v v_x + \gamma v_x\right)_x \, , \\ v_t - f u_{xt} + h v_{xxt} &= \left(\frac{1}{2}u^2 + \frac{3}{2}v^2 - f u_x v - \gamma u_x + \frac{1}{2}h v_x^2 + h v v_{xx} + \beta v_{xx}\right)_x \, , \end{aligned}$$

which, for $f = \gamma = 0$, reduces to

$$\begin{aligned} u_t &= (uv)_x \, , \\ v_t + h v_{xxt} &= \left(\frac{1}{2}u^2 + \frac{3}{2}v^2 + \frac{1}{2}h v_x^2 + h v v_{xx} + \beta v_{xx}\right)_x \, . \end{aligned}$$

This system, when $\beta = 0$, was obtained in [19] and it is an extension of the Ito equation [23] $(h = 0)$.

Example 4.2. For the Novikov algebra $(N4)$ with the particular choice

$$\tilde{g}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{g}_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

one obtains the system

$$\begin{aligned} v_t + f v_{xt} &= \left(\alpha_1 u + v^2 + f v v_x + \gamma v_x\right)_x \, , \\ u_t - f u_{xt} + h v_{xxt} &= \left(2uv + \alpha_2 v - f u_x v - \gamma u_x + \frac{1}{2}h v_x^2 + h v v_{xx} + \beta v_{xx}\right)_x \, , \end{aligned}$$

and in the case $f = h = 0$ this reduces to

$$\begin{aligned} v_t &= \left(\alpha_1 u + v^2 + \gamma v_x\right)_x \, , \\ u_t &= (2uv + \alpha_2 v - \gamma u_x + \beta v_{xx})_x \, . \end{aligned}$$

For $\alpha_2 = 0$ this system is the dispersive water waves (DWW) equation considered in [28]. If $\beta = \alpha_2 = 0$ and $\gamma = 1$ this system is equivalent to the Kaup-Broer system [24, 8], which under the constraint $u = 0$ reduces to the Burgers equation.

Example 4.3. For the Novikov algebra $(N4)$ with the choice

$$\tilde{g}_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \tilde{g}_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

we get the system

$$\begin{aligned} u_t - f v_{xt} &= (uv - f v v_x - \alpha_1 u - \gamma v_x)_x \, , \\ v_t - f u_{xt} + h v_{xxt} &= \left(-\frac{1}{2}u^2 + \frac{3}{2}v^2 + \alpha_2 v - f u_x v - \gamma u_x + \frac{1}{2}h v_x^2 + h v v_{xx} + \beta v_{xx}\right)_x \, , \end{aligned}$$

which for $h = -1$ and $f = \beta = \alpha_1 = 0$ is the 2-component Camassa-Holm equation (CH2) derived in [30, 11], see also [14]. If $f = h = 0$ the above system reduces to another water wave type equation

$$\begin{aligned} u_t &= (uv - \alpha_1 u - \gamma v_x)_x \,, \\ v_t &= \left(-\frac{1}{2}u^2 + \frac{3}{2}v^2 + \alpha_2 v - \gamma u_x + \beta v_{xx}\right)_x . \end{aligned}$$

Example 4.4. The algebra $(N4)$ is the only Novikov algebra of dimension less or equal to 3 with non-degenerate bilinear form generating second order central extension. Thus with the particular choice

$$\tilde{g}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{f}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(allowed since $\tilde{f}_0$ is non-degenerate) one obtains the system

$$\begin{aligned} v_{xt} &= (\alpha_{11} u + \alpha_{12} v + v v_x + \gamma v_x)_x \,, \\ -u_{xt} + h v_{xxt} &= \left(\alpha_{12} u + \alpha_{22} v - u_x v - \gamma u_x + \frac{1}{2} h v_x^2 + h v v_{xx} + \beta v_{xx}\right)_x . \end{aligned}$$

4.3. **Three-dimensional Novikov algebra** $(C8)$**.** This Novikov algebra is a straightforward three-dimensional generalisation of the two-dimensional algebra $(N4)$. In fact, this algebra admits a straightforward generalization to arbitrary dimensions (see Section 6).

Here we will consider only one particular case with

$$\tilde{g}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{f}_0 = \begin{pmatrix} 0 & 0 & f_1 \\ 0 & 0 & f_2 \\ -f_1 & -f_2 & 0 \end{pmatrix}, \quad \tilde{h}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h \end{pmatrix}$$

and

$$\tilde{g}_1 = \begin{pmatrix} 0 & \alpha_1 & 0 \\ \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}, \quad \tilde{f}_1 = \begin{pmatrix} 0 & 0 & \gamma_1 \\ 0 & 0 & \gamma_2 \\ -\gamma_1 & -\gamma_2 & 0 \end{pmatrix}, \quad \tilde{h}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

Then, for $(u, v, w)^{\mathrm{T}} \in \mathscr{L}_{\mathbb{A}}$ and $c = (0, 0, 1)^{\mathrm{T}}$ the system (3.8) takes the form

$$\begin{aligned} u_t + f_1 w_{xt} &= (\alpha_1 v + uw + f_1 w w_x + \gamma_1 w_x)_x \,, \\ v_t + f_2 w_{xt} &= (\alpha_1 u + vw + f_2 w w_x + \gamma_2 w_x)_x \,, \\ w_t - f_1 u_{xt} - f_2 v_{xt} + h w_{xxt} &= \bigg(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{3}{2}w^2 + \alpha_2 w - f_1 u_x w - f_2 v_x w - \gamma_1 u_x - \gamma_2 v_x \\ &\qquad + \frac{1}{2} h w_x^2 + h w w_{xx} + \beta w_{xx}\bigg)_x . \end{aligned}$$

For $\alpha_{1,2} = f_{1,2} = \gamma_{1,2} = 0$ we obtain:

$$\begin{aligned} u_t &= (uw)_x \,, \\ v_t &= (vw)_x \,, \\ w_t + h w_{xxt} &= \left(\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{3}{2}w^2 + \frac{1}{2} h w_x^2 + h w w_{xx} + \beta w_{xx}\right)_x . \end{aligned}$$

Remark 4.5. This system for $\beta = 0$, after the change of dependent variables: $q_1 = w + h w_{xx}$, $q_2 = u^2$ and $q_3 = v^2$, is equivalent to the system obtained from CH$(3, 1)$ [21] by the scaling $q_3 \mapsto \mu q_3$ followed by the limit $\mu \mapsto 0$.[8]

[8]We would like to thank the referee for this observation.

5. An n-dimensional abelian and associative Novikov algebra $\mathbb{T}_n$

It turns out that many Novikov algebras with nontrivial algebraic properties result in systems of evolution equations which are degenerate, for example, not fully nonlinear in all of the variables or, as in the example below, triangular. Consider an n-dimensional abelian $\mathbb{T}_n$ algebra defined by the multiplication rule

$$(a\cdot b)^i=\sum_{k=1}^{i}a^k\,b^{i-k+1}\qquad\Longleftrightarrow\qquad e_i\cdot e_j=\delta^k_{i+j-1}\,e_k\,,$$

where $a,b\in\mathbb{T}_n$. The related structure constants are $b^k_{ij}=\delta^k_{i+j-1}$. For dimensions $n=1,2$ and 4 the algebra $\mathbb{T}_n$ coincides with Novikov algebras $\mathbb{C}$, $(N3)$ and $\widetilde{A}_{3,3}$, respectively.

Proposition 5.1. *For any dimension n the algebra $\mathbb{T}_n$ is an associative Novikov algebra. Moreover:*

- *an arbitrary symmetric bilinear form $\tilde g=(g_{ij})$ on $\mathbb{T}_n$, satisfies the quasi-Frobenius condition (2.8) if and only if*
$$g_{ij}=\begin{cases}g_{1,i+j-1} & \text{for}\quad 1\leqslant i+j-1\leqslant n\\ 0 & \text{otherwise}\end{cases},$$
such bilinear form is non-degenerate if $g_{1n}\neq 0$;
- *there is no nonzero anti-symmetric bilinear form $\tilde f=(f_{ij})$ on $\mathbb{T}_n$ that satisfies the conditions (2.11) and (2.12).*

The proof is straightforward. Since the algebra $\mathbb{T}_n$ is abelian the only condition for bilinear forms generating the central extensions of the Lie algebra $\mathscr{L}_{\mathbb{T}_n}$ is the quasi-Frobenius condition (2.8).

As consequence of the above proposition the most general bilinear forms generating the Poisson pair (3.1) are given by

$$(\tilde g_0)_{ij}=g_{i+j-1},\quad (\tilde h_0)_{ij}=h_{i+j-1},\quad (\tilde g_1)_{ij}=\alpha_{i+j-1},\quad (\tilde h_1)_{ij}=\beta_{i+j-1}$$

for $1\leqslant i+j-1\leqslant n$ and 0 otherwise. Moreover $(\tilde f_0)_{ij}=(\tilde f_1)_{ij}=0$.

The element $c=(\delta^i_1)\in\mathbb{T}_n$ is a unity. For $v=(v^i)\in\mathscr{L}_{\mathbb{T}_n}$ the equation (3.8) has the following system

$$\begin{aligned}g_{i+j-1}v^j_t+h_{i+j-1}v^j_{xxt}=\Bigg(&\sum_{k=1}^{j}\Big(\frac{3}{2}g_{i+j-1}v^kv^{j-k+1}+\frac{1}{2}h_{i+j-1}v^k_x\,v^{j-k+1}_x+h_{i+j-1}v^kv^{j-k+1}_{xx}\Big)\\&+\alpha_{i+j-1}v^j+\beta_{i+j-1}v^j_{xx}\Bigg)_x,\end{aligned}\tag{5.1}$$

where $1\leqslant i\leqslant n$ and the summation for j is over $1\leqslant j\leqslant n-i+1$. The bilinear form $\tilde h$ is non-degenerate if $h_n\neq 0$. In this case it is allowed to take $\tilde g_0=0$.

Consider the special case given by

$$(\tilde g_0)_{ij}=g\delta^n_{i+j-1},\quad (\tilde h_0)_{ij}=h\delta^n_{i+j-1},\quad (\tilde g_1)_{ij}=\alpha\delta^n_{i+j-1},\quad (\tilde h_1)_{ij}=\beta\delta^n_{i+j-1}.$$

Then, the above system takes the form

$$gv^i_t+hv^i_{xxt}=\Bigg(\sum_{k=1}^{i}\Big(\frac{3}{2}gv^kv^{i-k+1}+\frac{1}{2}hv^k_x\,v^{i-k+1}_x+hv^kv^{i-k+1}_{xx}\Big)+\alpha v^i+\beta v^i_{xx}\Bigg)_x,\tag{5.2}$$

where $1\leqslant i\leqslant n$.

The above systems (5.1) and (5.2) are triangular: with the only one genuinely nonlinear equation for $i=n$ and $i=1$, respectively, the remaining equations are sequentially linear. In fact all triangular systems associated with low-dimensional algebras posses similar property.

6. An n-dimensional Novikov algebra $\mathbb{A}_n$

Consider the following n-dimensional algebra $\mathbb{A}_n$, this being a generalisation of the Novikov algebras $(N4)$ and $(C8)$, defined by the rule

$$(a\cdot b)^i := a^i b^n \qquad \Longleftrightarrow \qquad e_i\cdot e_j = \delta_i^k\delta_j^n\, e_k,$$

so the structure constants are given by $b_{ij}^k = \delta_i^k\delta_j^n$.

Proposition 6.1. *For any dimension n the algebra $\mathbb{A}_n$ is an associative Novikov algebra. If $n\geqslant 2$ it is non-abelian. The associated Lie algebra structure on $\mathbb{A}_n$ is non-nilpotent. Moreover:*

- *an arbitrary symmetric bilinear form $\tilde g$ on $\mathbb{A}_n$ satisfies the quasi-Frobenius condition (2.8);*
- *an anti-symmetric bilinear form $\tilde f = (f_{ij})$ on $\mathbb{A}_n$ satisfies the conditions (2.11) and (2.12) if and only if*

$$f_{ij} = 0 \quad \text{for} \quad i\neq n \text{ and } j\neq n\,;$$

- *a symmetric bilinear form $\tilde h = (h_{ij})$ on $\mathbb{A}_n$ satisfies the conditions (2.12) if and only if*

$$h_{ij} = 0 \quad \text{for} \quad i\neq n \text{ or } j\neq n\,.$$

The proof is a straightforward calculation and will not be presented.

Remark 6.2. Let $\mathcal{C}^\infty(\mathbb{S}^1)$ be the space of smooth functions on the circle $\mathbb{S}^1$ and $\mathrm{Vect}(\mathbb{S}^1)$ be the Lie algebra of vector fields $u(x)\partial_x$ on $\mathbb{S}^1$. The Lie bracket in $\mathrm{Vect}(\mathbb{S}^1)$ is given by the formula

$$[u\,\partial_x, v\,\partial_x] = (uv_x - u_x v)\partial_x\,, \qquad x\in\mathbb{S}^1.$$

Consider the semidirect product $\mathscr{G}_n(\mathbb{S}^1) := \mathrm{Vect}(\mathbb{S}^1)\ltimes\mathcal{C}^\infty(\mathbb{S}^1)^{\oplus n}$ of $\mathrm{Vect}(\mathbb{S}^1)$ with n copies of $\mathcal{C}^\infty(\mathbb{S}^1)$. The Lie bracket in $\mathscr{G}_n(\mathbb{S}^1)$ is given by

$$[(u\partial_x, \boldsymbol{f}), (v\partial_x, \boldsymbol{g})] = ((uv_x - u_x v)\partial_x, u\,\boldsymbol{g}_x - v\,\boldsymbol{f}_x)\,,$$

where $(u\partial_x, \boldsymbol{f}), (v\partial_x, \boldsymbol{g})\in\mathscr{G}_n(\mathbb{S}^1)$. The Lie algebra $\mathscr{L}_{\mathbb{A}_n}$ associated with the n-dimensional Novikov algebra $\mathbb{A}_n$ is isomorphic to $\mathscr{G}_{n-1}(\mathbb{S}^1)$. The isomorphism $\mathscr{G}_{n-1}(\mathbb{S}^1)\to\mathscr{L}_{\mathbb{A}_n}$ is given by $(u\partial_x, \boldsymbol{f})\mapsto -(\boldsymbol{f}, u)$. The particular algebra $\mathscr{G}_1(\mathbb{S}^1)$, with related integrable systems, was extensively studied, see [32, 31, 18, 19] and references therein.

Proposition 6.1 provides conditions on the bilinear forms generating the central extension of the Lie algebra $\mathscr{L}_{\mathbb{A}_n}$. Consequently the most general bilinear forms generating the Poisson pair (3.1) are given by

$$(\tilde g_0)_{ij} = g_{ij}, \qquad (\tilde f_0)_{ij} = \delta_j^n f_i - \delta_i^n f_j, \qquad (\tilde h_0)_{ij} = \delta_i^n\delta_j^n h$$

and

$$(\tilde g_1)_{ij} = \alpha_{ij}, \qquad (\tilde f_1)_{ij} = \delta_j^n\gamma_i - \delta_i^n\gamma_j, \qquad (\tilde h_1)_{ij} = \delta_i^n\delta_j^n\beta.$$

The element $c = (c^i)\in\mathbb{A}_n$ is the right unity iff $c^n = 0$. Hence it is not unique. For $v = (v^i)\in\mathscr{L}_{\mathbb{A}_n}$ the equation (3.8) has the following form:

$$\begin{aligned} i\neq n: &\qquad g_{ij}v^j_t + f_i v^n_{xt} = \left(g_{ij}v^jv^n + f_iv^nv^n_x + \alpha_{ij}v^j + \gamma_i v^n_x\right)_x\,,\\ i = n: &\qquad g_{nj}v^j_t - f_jv^j_{xt} + hv^n_{xxt} = \Big(g_{nj}v^jv^n + \frac{1}{2}g_{jk}v^jv^k - f_jv^j_xv^n + \frac{1}{2}h(v^n_x)^2\\ &\qquad\qquad + hv^nv^n_{xx} + \alpha_{nj}v^j - \gamma_jv^j + \beta v^n_{xx}\Big)_x\,. \end{aligned} \tag{6.1}$$

For $n = 2$ this system is equivalent to (4.3).

Example 6.3. Consider the particular choice:[9]

$$\tilde{g}_0 = \begin{pmatrix} \mathrm{Id}_{\boldsymbol{n-2}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{g}_1 = \begin{pmatrix} \boldsymbol{\alpha} & 0 & 0 \\ 0 & \alpha_{n-1} & 0 \\ 0 & 0 & \alpha_n \end{pmatrix},$$

where $\mathrm{Id}_{n-2} = \mathrm{diag}(1,\dots,1)$ and $\boldsymbol{\alpha} = \mathrm{diag}(\alpha_1,\dots,\alpha_{n-2})$. Let $\boldsymbol{f} = (f_1,\dots,f_{n-2})^{\mathrm{T}}$ and $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_{n-2})^{\mathrm{T}}$. Then, the system (6.1) for $v = (\boldsymbol{q}, w, u)^{\mathrm{T}} \in \mathscr{L}_{\mathbb{A}_n}$ takes the form

$$\begin{aligned}
\boldsymbol{q}_t + u_{xt}\boldsymbol{f} &= (u\boldsymbol{q} + uu_x\boldsymbol{f} + \boldsymbol{\alpha}\boldsymbol{q} + u_x\boldsymbol{\gamma})_x\,, \\
u_t + f_{n-1}u_{xt} &= \left(u^2 + f_{n-1}uu_x + \alpha_{n-1}w + \gamma_{n-1}u_x\right)_x\,, \\
w_t - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{q}_{xt} - f_{n-1}w_{xt} + hu_{xxt} &= \Big(2uw + \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} - u\boldsymbol{f}^{\mathrm{T}}\boldsymbol{q}_x - f_{n-1}uw_x + \frac{1}{2}hu_x^2 \\
&\qquad + huu_{xx} + \alpha_n u - \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{q}_x - \gamma_{n-1}w_x + \beta u_{xx}\Big)_x\,.
\end{aligned}$$

For $\boldsymbol{f} = \boldsymbol{0}$ and $f_{n-1} = h = 0$ it reduces to the system

$$\begin{aligned}
\boldsymbol{q}_t &= (u\boldsymbol{q} + \boldsymbol{\alpha}\boldsymbol{q} + u_x\boldsymbol{\gamma})_x\,, \\
u_t &= \left(u^2 + \alpha_{n-1}w + \gamma_{n-1}u_x\right)_x\,, \\
w_t &= \left(2uw + \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} + \alpha_n u - \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{q}_x - \gamma_{n-1}w_x + \beta u_{xx}\right)_x,
\end{aligned}$$

which for $\boldsymbol{\alpha} = \boldsymbol{0}$, $\boldsymbol{\gamma} = 0$, $\alpha_n = \beta = 0$ and $\alpha_{n-1} = 2$, $\gamma_{n-1} = -1$ is the multicomponent dispersive water wave equation considered in [29].

Example 6.4. Finally, consider the case of $\tilde{g}_0 = \mathrm{diag}(1,\dots,1)$ and $\tilde{g}_1 = 0$. Let $\boldsymbol{f} = (f_1,\dots,f_{n-1})^{\mathrm{T}}$ and $\boldsymbol{\gamma} = (\gamma_1,\dots,\gamma_{n-1})^{\mathrm{T}}$. Then, taking $v = (\boldsymbol{u}, w)^{\mathrm{T}} \in \mathscr{L}_{\mathbb{A}_n}$ we get the following multicomponent system

$$\begin{aligned}
\boldsymbol{u}_t + w_{xt}\boldsymbol{f} &= (w\boldsymbol{u} + ww_x\boldsymbol{f} + w_x\boldsymbol{\gamma})_x\,, \\
w_t - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{u}_{xt} + hw_{xxt} &= \left(\frac{3}{2}w^2 + \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} - w\boldsymbol{f}^{\mathrm{T}}\boldsymbol{u}_x + \frac{1}{2}hw_x^2 + hww_{xx} - \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{u}_x + \beta w_{xx}\right)_x,
\end{aligned}$$

which for $\boldsymbol{f} = \boldsymbol{\gamma} = 0$ reduces to

$$\begin{aligned}
\boldsymbol{u}_t &= (w\boldsymbol{u})_x\,, \\
w_t + hw_{xxt} &= \left(\frac{3}{2}w^2 + \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} + \frac{1}{2}hw_x^2 + hww_{xx} + \beta w_{xx}\right)_x.
\end{aligned}$$

Similarly to Section 4.3 and Remark 4.5, this system for $\beta = 0$, after the change of dependent variables: $q_1 = w + hw_{xx}$, $q_i = \sum_{j=1}^{n-i+1} u_j^2$, where $i = 2,\dots,n$, is equivalent to the system that one can obtain from CH$(n,1)$ [21] by scalings $q_{n-i+1} \mapsto \mu_i q_{n-i+1}$, for $i = 1,\dots,n-2$, followed by the limits $\mu_i \mapsto 0$. There is much scope for the investigation of the links between seemingly different systems via such nonlinear changes of variables and scalings.

7. Equations of Hydrodynamic type on Novikov algebras

Taking the dispersionless limit of the evolution equations obtained in Theorem 3.1 gives, in the coordinates u, the following equations of hydrodynamic type:

$$\begin{aligned}
u_{t_1} &= R_c^* u_x\,, \\
u_{t_2} &= (R^*_{R_c\Lambda^{-1}u}u)_x + L^*_{\Lambda^{-1}u_x}R_c^* u + \tilde{g}_1 R_c \Lambda^{-1}u_x\,,
\end{aligned} \tag{7.1}$$

[9]Here the matrices are of dimension $\{n-2,1,1\} \times \{n-2,1,1\}$.

where the inertia operator is given by $\Lambda \equiv \tilde{g}_0$. These are the first two equations from the bi-Hamiltonian hierarchy (3.2) generated by means of the following compatible Poisson operators:

$$\mathcal{P}_1\gamma = (R^*_\gamma u)_x + L^*_{\gamma_x}u + \tilde{g}_1\gamma_x, \qquad \mathcal{P}_0\gamma = \tilde{g}_0\gamma_x,$$

where $\gamma \in \mathscr{L}_{\mathbb{A}}$. The first three densities of Hamiltonian functionals are

$$\begin{aligned} H_0 &= g_0(c, v),\\ H_1 &= \frac{1}{2}\, g_0(v, v\cdot c),\\ H_2 &= \frac{1}{3}\, g_0(v, v\cdot(v\cdot c)) + \frac{1}{6}\, g_0(v\cdot c, v\cdot v) + \frac{1}{2}\, g_1(v, v\cdot c), \end{aligned}$$

where $v = \tilde{g}_0^{-1}u$. These simplify further when the algebra has a right unity e and one takes $c = e$, or if the algebra has a left unity, in which the algebra is automatically associative. For all the explicit examples of Novikov algebras and bilinear forms (with $\det g_0 \neq 0$) discussed in this paper the associated Haantjes tensor vanishes. But only those systems associated to the algebras $(N4)$, $(C8)$ and $\mathbb{A}_n$ are hyperbolic and thus are diagonalisable (see Appendix A.4).

Let us consider the dispersionless limit of the n-component equation (6.1). Thus, for $u = (u^i) \in \mathscr{L}^*_{\mathbb{A}_n}$ such that $(u^i) = (g_{ij}v^j)$ we obtain[10]

$$u^i_t = \left(\eta_{nj}u^ju^i + \alpha^{ij}\eta_{jk}u^k\right)_x, \qquad i = 1,\dots,n\,, \tag{7.2}$$

where $\eta = g^{-1}$, such that $(\eta_{ij}) = (g^{ij})$, and $(\alpha^{ij}) \equiv (\alpha_{ij})$.

Proposition 7.1. *The Haantjes tensor (A.8) of the equations of hydrodynamic type (7.2) is identically zero.*

Proof. We have

$$A^i_k = \eta_{nj}u^j\delta^i_k + u^i\eta_{nk} + \alpha^{ij}\eta_{jk} \quad\Longrightarrow\quad \frac{\partial A^i_k}{\partial u^j} = \eta_{nj}\delta^i_k + \eta_{nk}\delta^i_j$$

such that (A.7). Hence, the Nijenhuis tensor is given by

$$N^i_{jk} = \alpha^{ir}\eta_{rj}\eta_{nk} + \eta_{nr}\alpha^{rs}\eta_{sj}\delta^i_k + \eta_{nj}\eta_{nr}u^r\delta^i_k - \{j \leftrightarrow k\}.$$

Then, after straightforward, but slightly tedious, calculations one can show that the Haantjes tensor (A.8) vanishes. □

It turns out that the eigenvalues/characteristic speeds of the system (7.2) are, when $n = 2\,, 3$ or 4, distinct. We conjecture that this is true for arbitrary sized systems.

In fact, as remarked above, all the dispersionless systems constructed *from the explicit Novikov algebras* in this paper have vanishing Haantjes tensor. This leads us to the following:

Conjecture 7.2. *The equations of hydrodynamic type (7.1) constructed from an arbitrary Novikov algebra have vanishing Haantjes tensor.*

In the case of a commutative Novikov algebra (which are automatically associative) the conjecture is true: in this case the Nijenhuis tensor vanishes automatically. We have not been able to prove this conjecture in general, even for those Novikov algebras which arise from a derivation ∂ on a commutative, associative algebra (i.e. $a\cdot b = a\partial b$). We end by noting that the *explicit* construction of flat coordinates for the (inverse) flat metric $g^{ij}(u) = c^{ij}_{\ \ k}u^k + g^{ij}_0$ with $\det(g^{ij}(u)) \neq 0$ for a non-commutative Novikov algebra, a problem stated in Balinskii and Novikov [7] nearly 30 years ago, remains unsolved.

[10]We slightly abuse here the convention previously adopted.

Acknowledgement

The authors wishes to thank the anonymous referee for providing to our attention the reference [21] and some important comments on the connections between various Camassa-Holm type systems via nonlinear changes of variables.

Appendix A.

Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra, $\mathfrak{g}^*$ its (regular) dual space and $\langle\,,\,\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{C}$ the standard duality pairing. The coadjoint action $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is defined through the formula

$$\langle \mathrm{ad}^*_a u, b\rangle := -\langle u, [a, b]\rangle, \tag{A.1}$$

where $a, b \in \mathfrak{g}$ and $u \in \mathfrak{g}^*$.

A.1. **Lie-Poisson structure.** The Lie-Poisson (or Kirillov-Kostant) bracket in the space of smooth functions $\mathcal{C}^\infty(\mathfrak{g}^*)$ on the dual Lie algebra $\mathfrak{g}^*$ is given by

$$\{H, F\}(u) := \big\langle u, [dF, dH]\big\rangle, \qquad u \in \mathfrak{g}^*,$$

where $H, F \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and theirs differentials $dF, dG \in \mathfrak{g}$. The differential of $F \in \mathcal{C}^\infty(\mathfrak{g}^*)$ is defined by the relation

$$\langle v, dF\rangle := \frac{d}{d\varepsilon} F[u + \varepsilon v]\bigg|_{\varepsilon=0}, \tag{A.2}$$

where $v \in \mathfrak{g}^*$. The Hamiltonian equation corresponding to a function H on $\mathfrak{g}^*$ with respect to the Poisson-Lie structure has the form

$$u_t = \mathcal{P} dH \equiv -\,\mathrm{ad}^*_{dH} u\,, \qquad u \in \mathfrak{g}^*,$$

where $\mathcal{P}$ is the Poisson operator such that $\{H, F\} = \big\langle \mathcal{P} dH, dF\big\rangle$.

A.2. **Central extension.** A 2-cocycle is a bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \mapsto \mathbb{C}$, which is skew-symmetric:

$$\omega(a, b) = -\omega(b, a)\,, \tag{A.3}$$

and it satisfies the cyclic-condition:

$$\omega([a, b], c) + \omega([c, a], b) + \omega([b, c], a) = 0, \tag{A.4}$$

where $a, b, c \in \mathfrak{g}$. Then, the centrally extended algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ is defined via the Lie bracket

$$[(a, \alpha), (b, \beta)] = ([a, b], \omega(a, b))\,,$$

where $a, b \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{C}$.

Let the 2-cocycle ω be defined by means of a 1-cocycle $\phi : \mathfrak{g} \to \mathfrak{g}^*$, that is

$$\omega(a, b) := \langle \phi(a), b\rangle. \tag{A.5}$$

Then, the skew-symmetry (A.3) is equivalent to skew-adjoint's of ϕ, that is $\phi^\dagger = -\phi$, where $\langle \phi^\dagger(a), b\rangle := \langle \phi(b), a\rangle$. The cyclic-condition (A.4) impose the following relation on ϕ:

$$\phi([a, b]) = \mathrm{ad}^*_a \phi(b) - \mathrm{ad}^*_b \phi(a),$$

where $a, b \in \mathfrak{g}$.

The coadjoint action on the centrally extended Lie algebra $\tilde{\mathfrak{g}}$ restricted to $\mathfrak{g}$ is given by

$$\widetilde{\mathrm{ad}}^*_a u = \mathrm{ad}^*_a u - \delta\phi(a),$$

where $\delta \in \mathbb{K}$ is a (fixed) charge. For our purposes it can be taken as $\delta = 1$. Important is fact that more general cocycles can be obtained as linear compositions of respectively one or two-cocycles.

A.3. **Euler equation.** An invertible self-adjoint operator $\Lambda : \mathfrak{g} \to \mathfrak{g}^*$ defining the quadratic Hamiltonian

$$H = \frac{1}{2}\langle u, \Lambda^{-1}u\rangle \tag{A.6}$$

is called an inertia operator on $\mathfrak{g}$. The corresponding Hamiltonian equation on $\mathfrak{g}^*$ is

$$u_t = -\operatorname{ad}^*_{\Lambda^{-1}u} u\,, \qquad u \in \mathfrak{g}^*,$$

since $dH = \Lambda^{-1}u$. This equation is the Euler equation on the dual space $\mathfrak{g}^*$ corresponding to geodesic flows on a Lie Group G associated with $\mathfrak{g}$. In fact the inertia operator defines the metric $(\ ,\ )_e$ at the identity $e \in G$ such that $(v, w)_e = \langle \Lambda v, w\rangle$, where $v, w \in \mathfrak{g} \equiv T_eG$. For more information on the subject we refer the reader to [3, 27].

The Euler equation on the centrally extended Lie algebra $\tilde{\mathfrak{g}}$ for the quadratic Hamiltonian (A.6) takes the form

$$\begin{aligned} u_t &= -\operatorname{ad}^*_{\Lambda^{-1}u} u + \delta\phi\big(\Lambda^{-1}u\big)\,,\\ \delta_t &= 0, \end{aligned}$$

where $(u, \delta) \in \tilde{\mathfrak{g}}^*$. The reduction of this Euler equation to $\mathfrak{g}^*$ is natural.

A.4. **Diagonalizability of hydrodynamic systems.** An evolution equation of hydrodynamic type:

$$u^i_t = A^i_k(u)u^k_x\,, \qquad i = 1, \ldots, n. \tag{A.7}$$

is diagonalizable if there are coordinates, so-called Riemann invariants, in which it can be presented in the form

$$R^i_t = \Lambda_i(R)R^i_x \qquad \textit{(no summation)}.$$

Then, if the characteristic speeds Λ_i are mutually distinct and satisfy the semi-Hamiltonian condition, which is automatic for Hamiltonian hydrodynamic systems with nondegenerate metric, (A.7) is integrable by means of the so-called generalised hodograph method [35]. We say that the system of hydrodynamic type (A.7) is hyperbolic if the tensor $A = (A^i_k(u))$ has n linearly independent eigenvectors. The diagonalizability criterion for hyperbolic systems (A.7) is provided by vanishing of the Haantjes tensor [20]:

$$H_A(X, Y) = N_A(AX, AY) - AN_A(X, AY) - AN_A(AX, Y) + A^2N_A(X, Y) \tag{A.8}$$

which is defined via the Nijenhuis tensor:

$$N_A(X, Y) = [AX, AY] - A[X, AY] - A[AX, Y] + A^2[X, Y]\,.$$

More precisely, a hyperbolic system of hydrodynamic type (A.7) is diagonalizable if and only if the corresponding Haantjes tensor (A.8) vanishes.

Appendix B.

Here we present a classification of bilinear forms generating central extensions of the Lie algebras $\mathscr{L}_{\mathbb{A}}$ (see Section 2.3). This classification is based on the classification of Novikov algebras in dimension $\leqslant 4$ presented in [4] and [9]. We use the same names for the different types of Novikov algebras as used in [4] and [9]. The most general form of the bilinear forms generating cocycles of order one, two and three, are denoted by g, f and h respectively. Our classification extends that of [5], where the symmetric bilinear forms satisfying the quasi-Frobenius condition were presented. The characteristic matrix of a Novikov algebra $\mathbb{A}$ is $\mathcal{B} = (b_{ij})$ defined by $b_{ij} := e_i \cdot e_j = b_{ij}^k e_k$ according to (4.1). From the classification we exclude the Novikov algebras that are trivial, i.e. $b_{ij} = 0$, or can be represented as a direct sum of lower dimensional Novikov algebras. In dimension four we only consider non-transitive Novikov algebras. The classification is presented in Tables 1–6.[11]

Table 1. Classification of bilinear forms associated with one and two-dimensional Novikov algebras.

type	charact. matrix	g	f	h	comments
$\mathbb{C}$	e_1	g_{11}	0	h_{11}	
$(T2)$	$\begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & 0 \end{pmatrix}$	transitive
$(T3)$	$\begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$	transitive
$(N3)$	$\begin{pmatrix} e_1 & e_2 \\ e_2 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} \\ h_{12} & 0 \end{pmatrix}$	
$(N4)$	$\begin{pmatrix} 0 & e_1 \\ 0 & e_2 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & f_{12} \\ -f_{12} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$	$\det f \neq 0$
$(N5)$	$\begin{pmatrix} 0 & e_1 \\ 0 & e_1 + e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$	
$(N6)$	$\begin{pmatrix} 0 & e_1 \\ \kappa e_1 & e_2 \end{pmatrix}$	$\begin{pmatrix} 0 & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & h_{22} \end{pmatrix}$	$\kappa \neq 0, 1$

[11] The special cases $(A6)'$, $(C10)'$ were overlooked in [5]. The case $(D6)'$ must be distinguished because of the form f. In [9] there is a misprint in the multiplication table of the algebra $N_{20}^{\mathfrak{h}_1}$.

TABLE 2. Classification of bilinear forms associated with three-dimensional Novikov algebras of type A.

type	charact. matrix	g	f	h	comments
$(A3)$	$\begin{pmatrix}0&0&0\\0&e_1&0\\0&0&e_1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&g_{22}&g_{23}\\0&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&f_{23}\\0&-f_{23}&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&h_{22}&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	transitive $\det g=0$
$(A4)$	$\begin{pmatrix}0&0&0\\0&0&e_1\\0&e_1&e_2\end{pmatrix}$	$\begin{pmatrix}0&0&g_{22}\\0&g_{22}&g_{23}\\g_{22}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&h_{22}\\0&h_{22}&h_{23}\\h_{22}&h_{23}&h_{33}\end{pmatrix}$	transitive
$(A5)$	$\begin{pmatrix}0&0&0\\0&0&e_1\\0&-e_1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&g_{22}&g_{23}\\0&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&f_{23}\\0&-f_{23}&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&h_{22}&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	transitive $\det g=0$
$(A6)$	$\begin{pmatrix}0&0&0\\0&e_1&e_1\\0&-e_1&\kappa e_1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&g_{22}&g_{23}\\0&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&f_{23}\\0&-f_{23}&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&h_{22}&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	$\kappa\neq-1$ transitive $\det g=0$
$(A6)'$	$\begin{pmatrix}0&0&0\\0&e_1&e_1\\0&-e_1&\kappa e_1\end{pmatrix}$	$\begin{pmatrix}0&-g_{13}&g_{13}\\-g_{13}&g_{22}&g_{23}\\g_{13}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&f_{23}\\0&-f_{23}&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&h_{22}&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	$\kappa=-1$ transitive
$(A7)$	$\begin{pmatrix}0&0&0\\0&0&e_1\\0&\kappa e_1&e_2\end{pmatrix}$	$\begin{pmatrix}0&0&g_{22}\\0&g_{22}&g_{23}\\g_{22}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	$\kappa\neq1$ transitive
$(A8)$	$\begin{pmatrix}0&0&0\\0&0&0\\0&e_1&e_2\end{pmatrix}$	$\begin{pmatrix}0&0&g_{13}\\0&0&g_{23}\\g_{13}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&h_{23}\\0&h_{23}&h_{33}\end{pmatrix}$	transitive $\det g=0$
$(A10)$	$\begin{pmatrix}0&0&0\\0&0&0\\0&e_2&e_1\end{pmatrix}$	$\begin{pmatrix}0&0&g_{13}\\0&0&g_{23}\\g_{13}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&h_{13}\\0&0&0\\h_{13}&0&h_{33}\end{pmatrix}$	transitive $\det g=0$
$(A11)$	$\begin{pmatrix}0&0&0\\0&0&0\\e_1&\kappa e_2&0\end{pmatrix}$	$\begin{pmatrix}0&0&g_{13}\\0&0&g_{23}\\g_{13}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&h_{33}\end{pmatrix}$	$\lvert\kappa\rvert\leqslant1,\ \kappa\neq0$ transitive $\det g=0$
$(A12)$	$\begin{pmatrix}0&0&0\\0&0&0\\e_1&e_1+e_2&0\end{pmatrix}$	$\begin{pmatrix}0&0&g_{13}\\0&0&g_{23}\\g_{13}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&h_{33}\end{pmatrix}$	transitive $\det g=0$
$(A13)$	$\begin{pmatrix}0&0&0\\0&e_1&0\\e_1&\frac{e_2}{2}&0\end{pmatrix}$	$\begin{pmatrix}0&0&\frac{1}{2}g_{22}\\0&g_{22}&g_{23}\\\frac{1}{2}g_{22}&g_{23}&g_{33}\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&0&0\\0&0&h_{33}\end{pmatrix}$	transitive

TABLE 3. Classification of bilinear forms associated with three-dimensional Novikov algebras of type C.

type	charact. matrix	g	f	h	comments
$(C6)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f_{23} \\ 0 & -f_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & 0 \\ h_{13} & 0 & h_{33} \end{pmatrix}$	
$(C7)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_2+e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & 0 \\ h_{13} & 0 & h_{33} \end{pmatrix}$	$\det g = 0$
$(C8)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{12} & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ -f_{13} & -f_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	
$(C9)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f_{23} \\ 0 & -f_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa \neq 0, 1$
$(C10)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & 0 & e_2+e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa \neq 0, 1$ $\det g = 0$
$(C10)'$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & 0 & e_2+e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & 0 \\ -f_{13} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa = 0$
$(C11)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix}$	$\det g = 0$
$(C12)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & \kappa e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & 0 \\ h_{13} & 0 & h_{33} \end{pmatrix}$	$\kappa \neq 0, 1$ $\det g = 0$
$(C13)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ \gamma e_1 & \kappa e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\gamma, \kappa \neq 0, 1$ $\det g = 0$
$(C14)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_1+e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h_{23} \\ 0 & h_{23} & h_{33} \end{pmatrix}$	$\det g = 0$
$(C15)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & e_1+\kappa e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa \neq 0, 1$ $\det g = 0$
$(C16)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & g_{22} & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & f_{23} \\ 0 & -f_{23} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	
$(C17)$	$\begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2 \\ 0 & e_1 & e_2+e_3 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\det g = 0$
$(C18)$	$\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & 0 & 0 \\ g_{13} & 0 & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & 0 \\ -f_{13} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\det g = 0$
$(C19)$	$\begin{pmatrix} 0 & 0 & e_1+e_2 \\ 0 & 0 & e_2 \\ 0 & -e_2 & e_1+e_3 \end{pmatrix}$	$\begin{pmatrix} g_{23} & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	

TABLE 4. Classification of bilinear forms associated with three-dimensional Novikov algebras of type D.

type	charact. matrix	g	f	h	comments
$(D2)$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & 0 & g_{11} \\ g_{13} & g_{11} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & 0 & h_{13} \\ 0 & 0 & h_{11} \\ h_{13} & h_{11} & h_{33} \end{pmatrix}$	$\det h \neq 0$
$(D3)$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ e_1+e_2 & e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{23} & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & h_{13} \\ 0 & 0 & 0 \\ h_{13} & 0 & h_{33} \end{pmatrix}$	
$(D4)$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{e_1}{2} & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} 2g_{23} & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	
$(D5)$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \frac{e_1}{2} & 0 & e_2+e_3 \end{pmatrix}$	$\begin{pmatrix} 2g_{23} & 0 & g_{13} \\ 0 & 0 & g_{23} \\ g_{13} & g_{23} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	
$(D6)$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & (2\kappa-1)e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & 0 & \kappa g_{11} \\ g_{13} & \kappa g_{11} & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa \neq 0, \frac{1}{2}, 1$
$(D6)'$	$\begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & e_2 \\ \kappa e_1 & (2\kappa-1)e_2 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & 0 & g_{13} \\ 0 & 0 & 0 \\ g_{13} & 0 & g_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & 0 \\ -f_{13} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & h_{33} \end{pmatrix}$	$\kappa = 0$ $\det g = 0$

TABLE 5. Classification of bilinear forms associated with all four-dimensional abelian nontransitive Novikov algebras and additionally the Novikov algebra $\mathbb{A}_4$.

type	charact. matrix	g	f	h	comments
$\widetilde{A}_{3,1}$	$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & 0 & 0 & 0 \\ g_{13} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & 0 & 0 & 0 \\ h_{13} & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$\widetilde{A}_{3,2}$	$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & e_3 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{22} & g_{14} \\ g_{12} & g_{22} & 0 & 0 \\ g_{22} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & h_{22} & h_{14} \\ h_{12} & h_{22} & 0 & 0 \\ h_{22} & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$\widetilde{A}_{3,3}$	$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & e_3 & e_4 & 0 \\ e_3 & e_4 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{22} & g_{23} \\ g_{12} & g_{22} & g_{23} & 0 \\ g_{22} & g_{23} & 0 & 0 \\ g_{23} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & h_{22} & h_{23} \\ h_{12} & h_{22} & h_{23} & 0 \\ h_{22} & h_{23} & 0 & 0 \\ h_{23} & 0 & 0 & 0 \end{pmatrix}$	$\det h \neq 0$
$\widetilde{A}_{3,4}$	$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ e_2 & e_4 & e_4 & 0 \\ e_3 & e_4 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{23} \\ g_{12} & g_{23} & g_{23} & 0 \\ g_{13} & g_{23} & 0 & 0 \\ g_{23} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{23} \\ h_{12} & h_{23} & h_{23} & 0 \\ h_{13} & h_{23} & 0 & 0 \\ h_{23} & 0 & 0 & 0 \end{pmatrix}$	$\det h \neq 0$
$\mathbb{A}_4$	$\begin{pmatrix} 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 \\ 0 & 0 & 0 & e_3 \\ 0 & 0 & 0 & e_4 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & g_{22} & g_{23} & g_{24} \\ g_{13} & g_{23} & g_{33} & g_{34} \\ g_{14} & g_{24} & g_{34} & g_{44} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & f_{14} \\ 0 & 0 & 0 & f_{24} \\ 0 & 0 & 0 & f_{34} \\ -f_{14} & -f_{24} & -f_{34} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{44} \end{pmatrix}$	non-abelian

TABLE 6. Classification of bilinear forms associated with all four-dimensional non-transitive Novikov algebras for which the associated Lie algebra is non-abelian and nilpotent.

type	charact. matrix	g	f	h	comments
$N_{19}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & 0 & 0 & 0 \\ g_{13} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{20}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & e_3 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{22} & g_{14} \\ g_{12} & g_{22} & 0 & 0 \\ g_{22} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{21}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & e_4 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{22} \\ g_{12} & g_{22} & 0 & 0 \\ g_{13} & 0 & 0 & 0 \\ g_{22} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{22} \\ h_{12} & h_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{22} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{22}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & 0 & 0 & e_3 \\ e_3 & 0 & 0 & 0 \\ e_4 & e_3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{24} & g_{14} \\ g_{12} & 0 & 0 & g_{24} \\ g_{24} & 0 & 0 & 0 \\ g_{14} & g_{24} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	
$N_{23}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & e_4 & 0 & e_3 \\ e_3 & 0 & 0 & 0 \\ e_4 & e_3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{24} & g_{22} \\ g_{12} & g_{22} & 0 & g_{24} \\ g_{24} & 0 & 0 & 0 \\ g_{22} & g_{24} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{22} \\ h_{12} & h_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{22} & 0 & 0 & 0 \end{pmatrix}$	
$N_{24}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & e_3 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{44} & g_{14} \\ g_{12} & g_{44} & 0 & 0 \\ g_{44} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & g_{44} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	
$N_{25}^{\mathfrak{h}_1}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3 & e_4 \\ e_2 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & e_3 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{44} & g_{14} \\ g_{12} & 0 & 0 & 0 \\ g_{44} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & g_{44} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & h_{14} \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ h_{14} & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{15}^{\mathfrak{h}_2}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3+e_4 & e_4 \\ e_2 & 0 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{12} & 0 & 0 & 0 \\ g_{13} & 0 & 0 & 0 \\ g_{14} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{16}^{\mathfrak{h}_2}$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3+e_4 & e_4 \\ e_2 & e_4 & 0 & 0 \\ e_3 & 0 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{22} \\ g_{12} & g_{22} & 0 & 0 \\ g_{13} & 0 & 0 & 0 \\ g_{22} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\det g = 0$
$N_{17}^{\mathfrak{h}_2}(\kappa)$	$\begin{pmatrix} e_1 & e_2+e_3 & e_3+e_4 & e_4 \\ e_2 & 2e_3+\kappa e_4 & e_4 & 0 \\ e_3 & e_4 & 0 & 0 \\ e_4 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{23} \\ g_{12} & g_{22} & g_{23} & 0 \\ g_{13} & g_{23} & 0 & 0 \\ g_{23} & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$g_{13} = \frac{1}{2}g_{22} + \frac{1-\kappa}{2}g_{23}$

References

[1] Antonowicz, M., Fordy, A.P., *Factorisation of energy dependent Schrdinger operators: Miura maps and modified systems*, Comm. Math. Phys. **124** (1989) 465–486

[2] Antonowicz, M., Fordy, A.P., *Hamiltonian structure of nonlinear evolution equations*, in A.P. Fordy, editor, Soliton theory: a survey of results, pages 273–312, MUP, Manchester, 1990

[3] Arnold, V.I., Khesin, B.A., *Topological methods in hydrodynamics*, Springer-Verlag, New York, 1998

[4] Bai, C., Meng, D., *The classification of Novikov algebras in low dimensions*, J. Phys. A **34** (2001) 1581–1594

[5] Bai, C., Meng, D., *Addendum: invariant bilinear forms*, J. Phys. A **34** (2001) 8193–8197

[6] Bai, C. , Meng, D., *Transitive Novikov algebras on four-dimensional nilpotent Lie algebras*, Internat. J. Theoret. Phys. **40** (2001) 1761–1768

[7] Balinskii, A.A., Novikov, S.P., *Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras*, Soviet Math. Dokl. **32** (1985) 228–231.

[8] Broer, L.J.F., *Approximate equations for long water waves*, Appl. Sci. Res. **31** (1975) 377–395

[9] Burde, D., de Graaf, W., *Classification of Novikov algebras*, Appl. Algebra Engrg. Comm. Comput. **24** (2013) 1–15

[10] Camassa, R., Holm, D.D., *An integrable shallow water wave equation with peaked solitons*, Phys. Rev. Lett. **71** (1993) 1661–1664.

[11] Chen, M., Liu Si-Qi, Zhang, Y., *A two-component generalization of the Camassa-Holm equation and its solutions*, Lett. Math. Phys. **75** (2006) 1–15

[12] Dorfman, I. *Dirac Structures and Integrability of Nonlinear Evolution Equations*, (Wiley, Chichester U.K., 1993).

[13] Dubrovin, B.A., Novikov, S.P., *On Poisson brackets of hydrodynamic type*, Soviet Math. Dokl. **30** (1984) 651–654.

[14] Falqui, G., *On a Camassa-Holm type equation with two dependent variables*, J. Phys. A: Math. Gen. **39** (2006) 327–342

[15] Fokas, A.S., Fuchssteiner, B., *Sympletic structures, their Bäcklund transformations and hereditary symmetries*, Physica D **4** (1981) 47–66.

[16] Fuchssteiner, B. *Some tricks from the symmetry-toolbox for nonlinear equations: Generalizations of the Camassa-Holm equation*, Physica D **95** (1996) 229-243.

[17] Grinberg, N.I., *On Poisson brackets of hydrodynamic type with a degenerate metric*, Russian Math. Surveys **40:4** (1985) 231-232.

[18] Guha, P. *Geodesic flows, bi-Hamiltonian structure and coupled KdV type systems*, J. Math. Anal. Appl. **310** (2005) 45–56

[19] Guha, P., Olver, P. J., *Geodesic flow and two (super) component analog of the Camassa-Holm equation*, SIGMA **2** (2006) 054

[20] Haantjes, J., *On X_m-forming sets of eigenvectors*, Indag. Math. **17** (1955) 158–162

[21] Holm, D.D., Ivanov, R.I., *Multi-component generalizations of the CH equation: geometrical aspects, peakons and numerical examples*, J. Phys. A: Math. Theor. **43** (2010) 492001

[22] Hunter, J.K., Saxton, R., *Dynamics of director fields*, SIAM J. Appl. Math. **51** (1991) 1498–1521

[23] Ito M., *Symmetries and conservation laws of a coupled nonlinear wave equation*, Phys. Lett. A **91** (1982) 335–338

[24] Kaup, D.J., *A higher-order water-wave equation and the method for solving it*, Progr. Theoret. Phys. **54** (1975) 396–408

[25] Khesin, B., Lenells, J., Misiołek, G., *Generalized Hunter-Saxton equation and the geometry of the group of circle diffeomorphisms*, Math. Ann. **342** (2008) 617–656

[26] Khesin, B., Misiołek, G., *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math. **176** (2003) 116–144

[27] Khesin, B., Wendt, R.,*The geometry of infinite-dimensional groups*, Springer-Verlag, Berlin, 2009

[28] Kupershmidt, B.A., *Mathematics of dispersive water waves*, Comm. Math. Phys. **99** (1985) 51–73

[29] Kupershmidt, B.A., *A multicomponent water wave equation*, J. Phys. A **18** (1985) L1119–L1122

[30] Liu, Si-Qi, Zhang, Y., *Deformations of semisimple bihamiltonian structures of hydrodynamic type*, J. Geom. Phys. **54** (2005) 427–453

[31] Marcel, P., Ovsienko, V., Roger, C., *Extension of the Virasoro and Neveu-Schwarz algebras and generalized Sturm-Liouville operators*, Lett. Math. Phys. **40** (1997) 31–39

[32] Marshall, I., *A Lie algebraic setting for Miura maps related to an energy dependent linear problem*, Comm. Math. Phys. **133** (1990) 509–520

[33] Olver, P.J., *Applications of Lie groups to differential equations*, second edition, Springer-Verlag, New York, 1993
[34] Olver, P.J. and Rosenau, P., *Tri-Hamiltonian soliton-compacton duality*, Phys. Rev. E **53** (1996) 1900–1906.
[35] Tsarëv, S.P., *Poisson brackets and one-dimensional Hamiltonian systems of hydrodynamic type*, Dokl. Akad. Nauk SSSR **282** (1985) 534–537

School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QQ, U.K.
E-mail address: ian.strachan@glasgow.ac.uk

Faculty of Physics, Adam Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland
E-mail address: bszablik@amu.edu.pl