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Quantum Optical State Comparison Amplifier

Electra Eleftheriadou[1], Stephen M. Barnett[2] and John Jeffers[1*]

[1] *Department of Physics, University of Strathclyde,*
John Anderson Building,
107 Rottenrow, Glasgow G4 0NG, U.K.
[2] *School of Physics and Astronomy,*
University of GLasgow, Kelvin Building,
University Avenue, Glasgow, G12 8QQ, U.K.
[*]john.jeffers@strath.ac.uk
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It is a fundamental principle of quantum theory that an unknown state cannot be copied or, as a consequence, an unknown optical signal cannot be amplified deterministically and perfectly. Here we describe a protocol that provides nondeterministic quantum optical amplification in the coherent state basis with high gain, high fidelity and which does not use quantum resources. The scheme is based on two mature quantum optical technologies, coherent state comparison and photon subtraction. The method compares favourably with all previous nondeterministic amplifiers in terms of fidelity and success probability.

Signal amplification is a simple concept in classical physics. In electromagnetism there is no theoretical impediment to amplifying a time-varying electric field E to form a perfectly-copied larger signal gE, where $g(> 1)$ is the gain factor. There are, of course, practical limitations: the gain is typically saturated because only a finite amount of energy is available for the amplifier. Furthermore the utility of the device is limited in practice by the fact that the amplifier normally adds noise to the signal.

A perfect quantum optical amplifier would increase the mean photon number of a state multiplicatively without affecting the other intrinsic properties. For example, it would transform the coherent state $|\alpha\rangle$, the nearest quantum equivalent to a classical stable wave, as follows

$$|\alpha\rangle \rightarrow |g\alpha\rangle. \qquad (1)$$

However, it has been known for some time that quantum-level linear optical amplifiers have more stringent limitations on their operation. It is impossible to amplify an unknown quantum optical signal without adding noise [1], the minimum value of which is a consequence of the uncertainty principle [2]. This extra required added noise swamps the quantum properties of a signal. Were it otherwise it would be possible to violate the no-cloning theorem [3] and achieve superluminal communication [4].

Ralph and Lund [5] suggested that this noise limit could be beaten by nondeterministic amplifiers: ones that work only in postselection. Such amplifiers transform the coherent state $|\alpha\rangle \rightarrow c|g\alpha\rangle$, where c is a constant satisfying $|c| \leq \frac{1}{g}$. They proposed an amplifier based on the quantum scissors device [6, 7], and this was later realised experimentally [8, 9]. The scheme has been extended to amplification of photonic polarization qubits using two such amplifiers [10, 11].

There are technical limitations to such a device, however, in that the amplifier requires single photons as a resource, its success probability is only a few percent and

it only works in the $|0\rangle$ and $|1\rangle$ basis, so the condition on the output is $|g\alpha| \ll 1$. This last restriction could be circumvented somewhat by operating several amplifiers in parallel [5], or using a two-photon version of the quantum scissors device [12], but at the practical cost of a very low count rate for a pair of coincident, separately-heralded photons. As a consequence the effective probability of amplification is tiny. Photon addition [13] and subtraction [14] can also be used to form a nondeterministic amplifier with $g = \sqrt{2}$ [15, 16], but with the same type of limitations as the scissors-based device. A further amplification protocol employs weak measurements based on a cross-Kerr effect [17]. Although the nominal success probability is high in this scheme, the fact that it relies on optical nonlinearity means that it also has a low effective success probability.

A scheme for state amplification without the use of quantum resources was suggested by Marek and Filip [18]. Surprisingly, thermal noise is added to the signal. The state is then bose-conditioned by a photon subtraction performed with a weakly-reflecting beam splitter and a detector. This performs the effective amplification. The amplification is not perfect, but in experiment it has been shown to produce larger amplitude output states with phase variances considerably smaller than that of the input [19]. The scheme can be improved slightly if a standard optical amplifier is used to add the initial noise [20, 21]. The success probability in all cases is limited by the probability of photon subtraction, which is of the order of a few percent for such low amplitude states.

In this letter we describe a remarkably simple method for amplifying coherent states based on comparing the input state with a known coherent state [22]. This type of comparison has already been used in an experimental realisation of a quantum digital signature scheme [23]. It is also used in various binary quantum receivers [24–26]. We shall see that as an amplifier it has several advantages
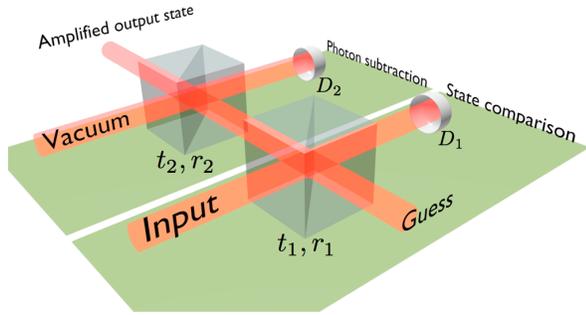
FIG. 1. (Color online) The state comparison amplifier. Bob attempts to null Alice's input with his guess state at the first beam splitter. The second beam splitter and detector are used for photon subtraction. The output state is accepted if the first detector does not fire and the second one does.

over earlier methods.

We assume that Alice sends coherent states selected at random from a known set to Bob. Bob's task is to amplify them, e.g. for later splitting and distribution of identical copies to multiple users on a network, or to determine the phase of the coherent state accurately. Higher amplitude coherent states have a smaller phase variance, so this latter application amounts to the sharing of a reference frame - an important task in quantum communication [27]. There are many other possibilities.

Bob performs the amplification using the device shown in Fig. 1. He mixes the unknown input state with another coherent state (the guess state) at a beam splitter and compares them. The beam splitter has transmission and reflection coefficients t_1 and r_1 which we take to be real, and there is a phase change of π on reflection from the lower arm. One output arm of the beam splitter falls upon a photodetector, and the other output is postselected based on no photocounts being recorded. This system therefore performs a state comparison between two coherent states, the reflected part of our guess state and the transmitted part of the input state. If Alice's input state is $|\alpha\rangle$ and Bob chooses his guess state to be $|\beta\rangle = |t_1\alpha/r_1\rangle$ (correctly) the transmitted part of the input state interferes destructively with the reflected part of the guess state and the detector cannot fire. The output in the upper arm is then a coherent state of amplitude α/r_1, which is larger than α. This is the gain mechanism for the device. If Bob chooses his guess state incorrectly, coherent light leaks into the detector arm. This can also sometimes cause no counts at the detector. Thus the output state, *conditioned* on the detector not firing, is in general a mixed state, weighted by the probability that no counts are recorded. The probability is maximised when destructive interference occurs in the

detector arm, for which the output coherent amplitude also reaches its maximum.

Bob can improve the quality of his output at small cost to the success probability if he performs a photon subtraction on the output. Coherent states are eigenstates of the annihilation operator, and so this process has no effect on them [28], but for a mixture of coherent states with different mean photon numbers the probabilities in the mixture are adjusted. When the detector fires, a subtraction occurs and it is more likely to have been due to a high-amplitude coherent state rather than a low amplitude one. Thus a subtraction is more likely to occur when Bob has chosen his guess state well. If the photon subtraction is performed with a beam splitter of transmission coefficient t_2, then $g = t_2/r_1$ is the nominal gain of the composite system.

For input and guess states $|\alpha\rangle$ and $|\beta\rangle$ the coherent amplitude in the nominal vacuum output is $t_1\alpha - r_1\beta$, and the other beam splitter output passes to the photon subtraction stage. The amplitude in the subtraction arm is therefore $-r_2(t_1\beta + r_1\alpha)$, and the output amplitude is $t_2(t_1\beta + r_1\alpha)$. We assume that the input and guess states are chosen from probability distributions over the coherent states

$$\hat{\rho}_{\text{in}} = \int d^2\bar{\alpha} P(\bar{\alpha})|\bar{\alpha}\rangle\langle\bar{\alpha}|,$$
$$\hat{\rho}_{\text{g}} = \int d^2\bar{\beta} Q(\bar{\beta})|\bar{\beta}\rangle\langle\bar{\beta}|, \qquad (2)$$

and can calculate the output state and the fidelity based on these and the properties of the device. The fidelity is

$$F = \int d^2\alpha P(\alpha)\langle g\alpha|\hat{\rho}_{\text{out}}|g\alpha\rangle, \qquad (3)$$

where $\hat{\rho}_{\text{out}}$ is the output state conditioned both on the input state distributions from eq.(2) and on the successful operation of the device, i.e. the comparison detector does not fire and the subtraction detector does. This is the probability that the output state passes a measurement test comparing it to the amplified version of the input state, and can be written

$$F = P(T|S) = \frac{P(T,S)}{P(S)}$$
$$= \frac{\int d^2\bar{\alpha} \int d^2\bar{\beta} P(T|S,\bar{\alpha},\bar{\beta})P(S|\bar{\alpha},\bar{\beta})P(\bar{\alpha})Q(\bar{\beta})}{\int d^2\bar{\alpha} \int d^2\bar{\beta} P(S|\bar{\alpha},\bar{\beta})P(\bar{\alpha})Q(\bar{\beta})} \quad (4)$$

where $P(T|S)$ is the probability that the output state will pass the fidelity test given that the device operates succesfully.

So far we have made no assumptions about the forms of the input and guess probability distributions, but it is instructive to consider two cases in order to comply with the requirements of a realistic communication system. The first scenario is one in which the set of states to be amplified is restricted to the binary alphabet $\{|\alpha\rangle, |-\alpha\rangle\}$ and in the second each state in the set has the same mean
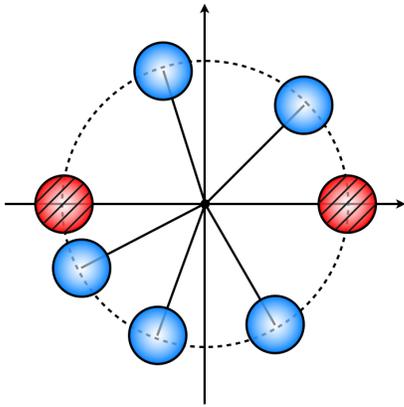
FIG. 2. (Color online) The sets of input states considered, binary (red, hatched), phase-covariant (blue shaded).

photon number, but with completely uncertain phase (Fig. 2). Here the mean photon number could either be agreed in advance, or determined by simply measuring the first few states. We will mostly leave discussion of the phase covariant amplifier to the supplementary material. Without loss of generality we will assume from here on that α is real and positive.

Suppose that Alice chooses randomly from a binary set of states so that

$$P(\bar{\alpha}) = \frac{1}{2}\left[\delta^2(\bar{\alpha} - \alpha) + \delta^2(\bar{\alpha} + \alpha)\right]. \qquad (5)$$

The best choice for Bob's guess state is to pick at random from the set $\{|\pm t_1\alpha/r_1\rangle\}$. He knows which state he chooses, so we suppose that he chooses $+$ without loss of generality and so $Q(\bar{\beta}) = \delta^2(\bar{\beta} - t_1\alpha/r_1)$. The probabilities which form the fidelity and success probability in eq. (4) are calculated using the Kelley-Kleiner formula [29], and we leave the details for the supplementary material.

Fig. 3 shows the fidelity as a function of gain. The fidelity drops as the gain is increased initially, rises to unity at intermediate gain, and then decays to a lower value at high gain. The increase at intermediate gain is a combined effect of the photon subtraction and comparison. If the first beam splitter is 50/50, then its conditioned output that falls upon the subtraction beam splitter has a coherent amplitude of either $\sqrt{2}\alpha$ (if Bob has chosen his state correctly) or zero (incorrectly). If it is zero then no photon subtraction can take place. Therefore, if there is a photon subtraction Bob must have chosen correctly and a perfect amplified copy of the state is made. At high gain for perfect detector efficiency the fidelity decays to $1/(1 + \exp[-4\alpha^2])$, which approaches unity for large α. Fidelity is not the only possible measure of output quality, and as a comparison in the supplementary material we show an output signal to noise ratio which increases with gain.
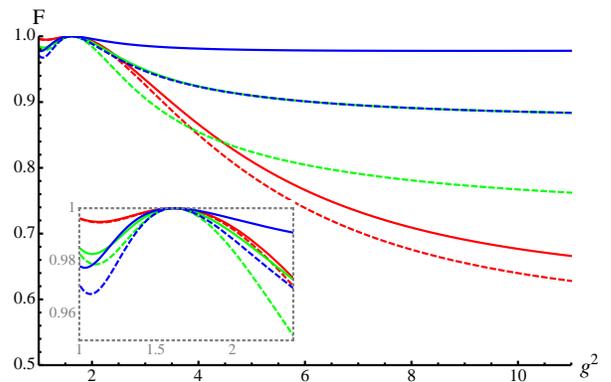

FIG. 3. (Color online) Fidelity as a function of intensity gain for the binary system. The red (bottom two) , green and blue (top two) plots correspond to $\alpha^2 = 0.1, 0.5, 1$. Full (dashed) curves are for detector quantum efficiencies of $\eta = 1(0.5)$. The intensity reflection coefficient of the subtraction beam splitter is 0.1. The inset shows an expanded view of the low gain region. The fidelity for $\alpha^2 = 1, \eta = 0.5$ is very close to that for $\alpha^2 = 0.5, \eta = 1$ for gains greater than about 3.
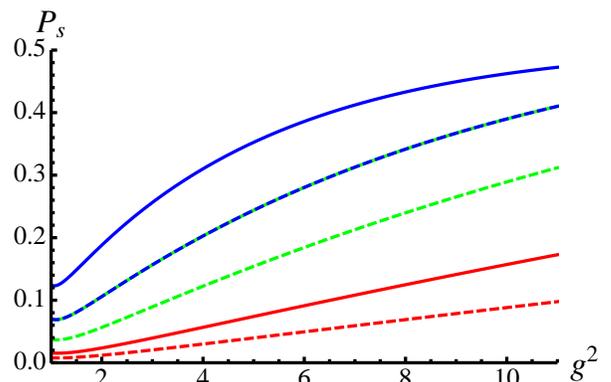

FIG. 4. (Color online) Success probability as a function of intensity gain for the binary system. Parameters as for Fig. 3.

The success probability (Fig. 4) is dominated at low gain by the photon subtraction probability, but for higher gains the subtraction probability approaches one and the success probability becomes the probability of obtaining no counts at the comparison detector.

It is important to note that the performance of our device is relatively insensitive to experimental detector imperfections. The fidelity is reasonably robust to nonunit quantum efficiency, as shown in Fig. 3. At high gain a nonunit quantum efficiency reduces the fidelity to that which would be obtained by operating the device with reduced coherent amplitude input $\alpha\sqrt{\eta}$. The effects on success probability shown in Fig. 4 depend on two competing factors. There is an increased probability of obtaining no counts at the comparison detector, but a decreased probability of photon subtraction. In a realistic experimental scenario the ideal is to keep dark count rates low
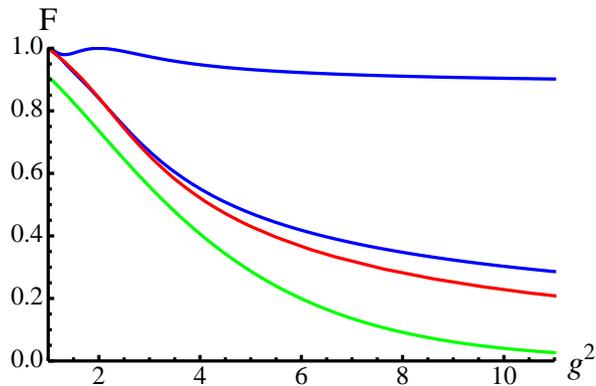
FIG. 5. (Color online) Fidelity as a function of intensity gain for the state comparison amplifier. Curves are from top to bottom: state comparison binary system (blue upper), state comparison phase-covariant (blue lower), noise addition (red) and scissors-based (green). Detector efficiencies and photon subtractions are assumed perfect in all cases.

enough that they do not affect the results. For example, in the state comparison experiment of reference [23] the photon flux at the detectors is of the order of $10^7 s^{-1}$, and the dark count rates for the SPAD detectors are $320 s^{-1}$, rendering the effects of dark counts insignificant.

If Bob and Alice do not initially share a phase reference, and all that Bob knows is the mean photon number of Alice's input then

$$P(\bar{\alpha}) = \frac{1}{2\pi\alpha}\delta(|\bar{\alpha}| - \alpha), \qquad (6)$$

with $Q(\bar{\beta})$ as before. The relevant probabilities are easily calculated, but again we leave the details for the supplementary material. The plots of the fidelities, and success probabilities are also to be found there.

It is instructive to compare the fidelity with that obtained using other methods of postselecting amplification. In Fig. 5 we show the fidelity obtained using the state-comparison amplifier, using the quantum scissors-based amplifier [5, 8, 9] and using the noise addition amplifier [18, 19] for a mean input photon number of 0.5. Overall the advantages of the state-comparison amplifier are obvious. We can see that for the binary alphabet the state comparison amplifier significantly outperforms the other systems. The effect of the photon subtraction ensures perfect amplification for a twofold gain, and no other amplifier can reach this whilst still retaining a high success probability. The other amplifiers do not use the extra knowledge of the state efficiently.

For the phase covariant state set, again the state comparison amplifier outperforms the other systems, although for low gain its edge over the noise addition amplifier is minimal in terms of fidelity. However, it does have another advantage over this system. When it works the state comparison amplifier provides knowledge of both

the state to be amplified and the amplified state, knowledge which is not available in the noise addition amplifier.

The lower fidelity associated with the scissors-based amplifier is largely due to the fact that it can only produce a superposition of zero and one photon, and this is not useful for amplifying a coherent state with a mean photon number of 0.5. For much lower mean input photon numbers the scissors-based amplifier has a higher fidelity than the other methods provided that the gain is low enough.

It is clear that to be useful in future quantum communications systems postselecting amplifiers must approach the ultimate limits of performance [30]. Here we have described a nondeterministic amplifier based on the comparison of coherent states that outperforms other schemes over a wide range of input amplitudes and amplifier gains. It does not require quantum resources, and operates with high fidelity and high success probability. It uses two already demonstrated experimental techniques, and will be relatively straightforward to implement. The gain of the amplifier can be chosen by adjusting the reflectivity of a beam splitter, making the system ideal for demonstration via polarization.

The main reason why this amplifier works well is that it uses the available information about the input states in an effective manner. Knowledge of the input basis is knowledge that can be used in the amplifier. Amplification is *normally* performed by dumping energy into the system - an optical mode. This works best if we do it in the appropriate basis, so if we want to amplify coherent states we should place the energy in the coherent state basis.

The amplifier is robust to realistic values of detector imperfections. It still works well for nonunit detector efficiencies, and dark count rates ought to be low enough to render them unimportant. Losses within the other components, such as at the beam splitters or in any connecting fibers, will be small and will normally reduce fidelity and/or gain by a commensurate amount.

The main limitation of the device is that it works best in a limited state space. This is a feature common to all nondeterministic amplifiers, although the limits are different for each one. Both the scissors-based and photon addition/subtraction amplifiers have an output which cannot contain more than one photon, and for the noise addition amplifier the amount of added noise, and therefore the gain, is tailored to the input amplitude in order to maximise phase concentration. As implied above our amplifer turns the state space limitation to an advantage, in that the amplifier can be tailored to work in the limited basis that is used in a particular communication system. In any case the state space that the device works on can always be widened at a cost to the success probability. Whether or not there is a cost to the fidelity depends on the gain chosen and the states added. We also remark that the amplifier described here provides a gain which

is dependent on the input state and this is the case for all postselecting amplifiers so far. It renders them effectively nonlinear, but it does not bring into question their status as amplifiers [30].

None of the earlier schemes can amplify superpositions of coherent states, and the same is true of the state comparison amplifier as proposed here. In principle this device could amplify a limited set of superpositions of coherent states, but to do so it would require as inputs guess states that were themselves superpositions. In any case superpositions are of limited use in a communications system, as propagation would quickly destroy coherence.

The most striking results of the state comparison amplifier are for an input chosen from a binary set of coherent states, where for a gain of just less than twofold perfect amplification can be achieved. This suggests an application of the device as an ideal quantum optical repeater, stationed every few km in a low-loss optical fibre communication system - the quantum equivalent of erbium doping.

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