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Ultra-relativistic electrostatic Bernstein waves

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Abstract

A new general form of the dispersion relation for electrostatic Bernstein waves in ultra-relativistic pair plasmas, characterized by $a^{-1} = k_B T / (m_e c^2) \gg 1$, is derived in this paper. The parameter $S_p = a \Omega_0 / \omega_p$, where Ω_0 is the rest cyclotron frequency for electrons or positrons and ω_p is the electron (or positron) plasma frequency, plays a crucial role in characterizing these waves. In particular, S_p has a restricted range for permitted wave solutions; this range is effectively unlimited for classical plasmas, but is significant for the ultra-relativistic case. The characterization of these waves is applied in particular to the presence of such plasmas in pulsar atmospheres.

(Some figures may appear in colour only in the online journal)

1. Introduction

A uniform, magnetized plasma when described by classical (that is, non relativistic) kinetic theory can support a spectrum of electrostatic waves that propagate undamped in the direction perpendicular to the equilibrium magnetic field orientation [1], known as Bernstein modes. These are in addition to the complex electromagnetic waves supported by such plasmas; much attention was focused on these wave descriptions in the 1950s and 1960s, particularly in the context of electromagnetic losses in energetic (fusion) plasmas, and considerable efforts were made to construct comprehensive dispersion relations for such radiation scenarios [2–4], though such models were restricted to stationary ion plasmas. More recent analysis reflects the significance of cyclotron heating in tokamaks, and so there has been a resurgence of activity addressing the mildly relativistic form of the dispersion relations near the fundamental cyclotron harmonic (for example, [5–7]).

Classical Bernstein modes appear near harmonics of the cyclotron frequency, except in the vicinity of the hybrid frequency, at which point the topology of the dispersion curves changes and band-gaps occur for higher frequencies. As the plasma becomes more energetic, this characteristic spectrum changes significantly, not just because the cyclotron modes are momentum-dependent, but also because the equilibrium distribution function can no longer be taken as the classical Maxwellian. In an electron–positron plasma, there is a further

complication, in that the positive ion mobility is identical to the electron, and therefore the dynamics of both species are required.

In previous papers [8, 9], we obtained dispersion curves for Bernstein modes in a weakly relativistic electron–positron plasma, in which the equilibrium distribution function was taken to be the classical Maxwellian, but the full relativistic correction for the mass-dependent cyclotron frequency was included. A critical parameter in this calculation was the non-dimensional reciprocal relativistic temperature $a = m_e c^2 / (k_B T)$. The limitation of describing the equilibrium electron–positron plasma by a classical Maxwellian distribution lay in the restricted range of values of a which resulted in feasible calculations of the dispersion curves. Thus, in our previous paper [8, 9], we presented results for $a = 10, 20$ and 50 , where $a = 50$ is typical for tokamak experimental devices in fusion research.

We are interested in astrophysical electron–positron plasmas, which are expected to be highly relativistic, and thus requiring $a \ll 1$. The dynamics of highly relativistic electron–positron plasmas in pulsar atmospheres has been a topic of discussion in the recent literature, from compressive shock acceleration [10], to streaming flows [11, 12], electromagnetic waves [13, 14] and electrostatic modes [15]. In each case, the relativistic Maxwell–Boltzmann–Jüttner distribution function [16] plays a key role in characterizing the behaviour, with the central parameter a taking very small values. In this paper we revisit the electrostatic Bernstein modes in the ultra-relativistic



limit $10^{-6} \leq a \leq 0.1$ in order to construct a clearer picture of the possible role of such waves in energetic pair plasmas, and in particular, pulsar magnetospheres.

We state without proof the relativistic Bernstein wave dispersion relation for (electrostatic) waves of frequency ω and wavenumber k derived previously [8]:

$$1 = -4\pi\omega_p^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dp_{\parallel} \int_0^{\infty} dp_{\perp} \frac{\partial f_0}{\partial p_{\perp}} \frac{p_{\perp}^2 n^2}{\Delta_n \zeta^2} J_n^2(\zeta), \quad (1)$$

where

$$f_0 = (4\pi m_e^3 c^3)^{-1} \frac{a}{K_2(a)} e^{-a\gamma} \quad (2)$$

is the Maxwell–Boltzmann–Jüttner distribution [16] (the relativistic Maxwellian distribution), n is an integer, J_n is the Bessel function of the first kind, K_2 is the modified Bessel function of the second kind, and the following notation holds:

$$a = \frac{m_e c^2}{k_B T} \quad (3)$$

$$\gamma = [1 + (p_{\parallel}^2 + p_{\perp}^2)/(m_e^2 c^2)]^{1/2} \quad (4)$$

$$\zeta = \frac{k_{\perp} p_{\perp}}{\Omega_0 m_e} \quad (5)$$

$$\Delta_n = \omega^2 - n^2 \Omega_0^2 / \gamma^2 \quad (6)$$

$$\Omega_0 = eB_0/m_e \quad (7)$$

in which the subscripts \parallel and \perp refer to directions parallel, and perpendicular, to the homogeneous equilibrium magnetic field B_0 .

The mathematics is simplified by introducing the following non-dimensional quantities:

$$\hat{\omega} = \omega / \Omega_0 \quad (8)$$

$$\hat{\omega}_p = \omega_p / \Omega_0 \quad (9)$$

$$\hat{p}_{\parallel} = p_{\parallel} / (m_e c) \quad (10)$$

$$\hat{p}_{\perp} = p_{\perp} / (m_e c) \quad (11)$$

$$\hat{k}_{\perp} = k_{\perp} c / \Omega_0. \quad (12)$$

We are interested in this paper in the ultra-relativistic limit, characterized by $a \ll 1$; in this limit, it can be shown that the equilibrium distribution function f_0 is given by the Maxwell–Boltzmann–Jüttner form [16]:

$$f_0 \simeq \frac{a^3}{8\pi m_e^3 c^3} e^{-a\gamma} = f_{\text{MBJ}} \quad (13)$$

$$\frac{\partial f_0}{\partial p_{\perp}} \simeq -\frac{a^4 \hat{p}_{\perp}}{8\pi \gamma m_e^4 c^4} e^{-a\gamma} \quad (14)$$

using $K_2(a) \sim 2/a^2$.

After some rearrangement, the dispersion relation becomes

$$\hat{\omega}^2 = \frac{\hat{\omega}_p^2 a^4}{\hat{k}_{\perp}^2} \sum_{n=1}^{\infty} n^2 \int_0^{\infty} d\hat{p}_{\perp} \hat{p}_{\perp} J_n^2(\hat{k}_{\perp} \hat{p}_{\perp}) \times \int_{-\infty}^{\infty} d\hat{p}_{\parallel} \frac{\gamma e^{-a\gamma}}{\gamma^2 - n^2 S^2}. \quad (15)$$

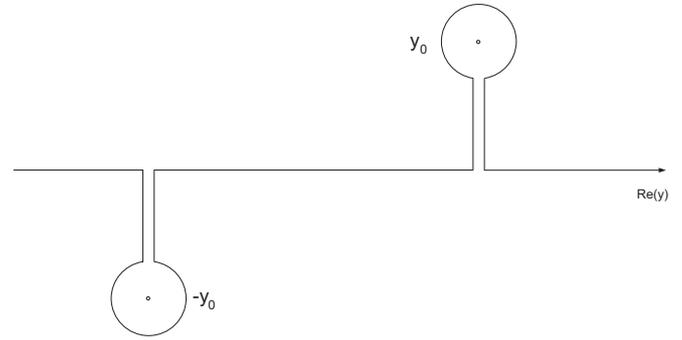


Figure 1. Showing the integration contour for the Landau prescription.

It is useful to introduce a new set of dimensionless terms, namely

$$x = a \hat{p}_{\perp} \quad (16)$$

$$y = a \hat{p}_{\parallel} \quad (17)$$

$$S = a / \hat{\omega} \quad (18)$$

$$S_p = a / \hat{\omega}_p \quad (19)$$

$$k = \hat{k}_{\perp} / a \quad (20)$$

$$\Gamma = a\gamma = (x^2 + y^2 + a^2)^{1/2}. \quad (21)$$

With these new variables, the dispersion relation takes on a simpler form:

$$S_p^2 = (S^2/k^2) \sum_{n=1}^{\infty} n^2 \int_0^{\infty} dx x J_n^2(kx) \int_{-\infty}^{\infty} dy \frac{\Gamma e^{-\Gamma}}{\Gamma^2 - n^2 S^2}. \quad (22)$$

Note that the relativistic parameter a now appears only in Γ . Since our interest lies in the ultra-relativistic limit for which $a \ll 1$, to a good approximation $\Gamma \simeq (x^2 + y^2)^{1/2}$. Under this approximation it follows then that a is explicitly scaled out of the dispersion relation, and the only variable parameter is S_p , which is proportional to $an_p^{-1/2}$, where n_p is the number density of electrons or positrons. Having obtained a solution of the dispersion relation for a given S_p in terms of k and S , we recover the physical quantities via the relations $\omega_p = a\Omega_0/S_p$, $\omega = a\Omega_0/S$, $k_{\perp} = a\Omega_0 k/c$. Only at this point does the relativistic parameter a appear in the solution. Note that the phase velocity $v_p = \omega/k_{\perp}$ of the Bernstein waves is independent of a . Note also that S , and therefore ω , may be a complex number, implying damping of the Bernstein waves.

2. Analysis of the dispersion relation

The y -integral has poles, and therefore must be treated following the Landau description, that is, y must be considered to be a complex variable, with the integration to be taken along a certain path \mathcal{C} in the complex plane. It is well known that a full solution of the dispersion relation will yield information on both the frequency and damping of the waves.

Following Landau, we choose the path of integration as in figure 1, noting that the poles occur if $\Gamma^2 = n^2 S^2$, that is, if $x^2 + y^2 = n^2 S^2$. Note that if $x < nS$ and S is real, then

there is a finite value of y for which a singularity occurs; if $x > ns$ then there are no singularities present in the real y integration. However, the whole range of integration in the x, y plane is always of interest, and so there will always be the possibility of a complex contribution to the integration arising from poles encountered in the integration contour. Consequently we must express S as $S = s + iv$, where v is related to the damping coefficient. The poles occur at $\pm y_0$, where $y_0 = [n^2(s + iv)^2 - x^2]^{1/2}$. The term R arising from the residue at the poles is given by

$$\begin{aligned} R &= -2\pi i \frac{n(s + iv)}{[n^2(s + iv)^2 - x^2]^{1/2}} e^{-n(s+iv)} \\ &= -2\pi i \frac{n(s + iv)e^{-ns}}{[n^2(s^2 - v^2) - x^2 + i2n^2sv]^{1/2}} \\ &\quad \times [\cos(nv) - i \sin(nv)] \\ &= -2\pi in(s + iv)[\cos(nv) - i \sin(nv)] \\ &\quad \times [(d + b)^{1/2} - i(d - b)^{1/2}] e^{-ns} / (\sqrt{2}d) \end{aligned} \quad (23)$$

where

$$b = n^2(s^2 - v^2) - x^2 \quad (24)$$

$$d^2 = [n^2(s^2 - v^2) - x^2]^2 + 4n^4s^2v^2. \quad (25)$$

The integral along the real y -axis is independent of whether or not $x < ns$, since now that S is complex, there are no singularities present on the real y -axis. In this case, the whole integral is given by

$$\begin{aligned} &2 \frac{(s + iv)^2}{k^2} \sum_{n=1}^{\infty} n^2 \int_0^{\infty} dx x J_n^2(kx) \int_0^{\infty} dy \frac{\Gamma e^{-\Gamma}}{\Gamma^2 - n^2(s + iv)^2} \\ &= \frac{2}{k^2} \sum_{n=1}^{\infty} n^2 \int_0^{\infty} dx x J_n^2(kx) \int_0^{\infty} dy \Gamma e^{-\Gamma} (s + iv)^2 \\ &\quad \times \frac{\Gamma^2 - n^2(s^2 - v^2) + i2n^2sv}{[\Gamma^2 - n^2(s^2 - v^2)]^2 + 4n^4s^2v^2}. \end{aligned} \quad (26)$$

The Landau prescription now requires us to obtain both the real and imaginary parts of the dispersion relation, yielding a pair of simultaneous equations for s and v , given S_p^2 . Before stating these two equations, we apply a change of variables as follows:

$$\begin{aligned} x &= nsu \\ y &= nsw. \end{aligned} \quad (27)$$

Then $\Gamma^2 = x^2 + y^2 = n^2s^2(u^2 + w^2)$ and, after considerable algebra, we arrive at the following expressions for the real part A and imaginary part B of the dispersion relation:

$$\begin{aligned} S_p^2 &= 2 \frac{s^4}{k^2} \sum_{n=1}^{\infty} n^4 \int_0^{\infty} du u J_n^2(knsu) \\ &\quad \times \int_0^{\infty} dw \beta e^{-ns\beta} \frac{\beta^2(1 - v^2/s^2) - (1 + v^2/s^2)^2}{(\beta^2 - 1 + v^2/s^2)^2 + 4v^2/s^2} \\ &\quad - \sqrt{2}\pi \frac{s^4}{k^2} \sum_{n=1}^{\infty} n^4 e^{-ns} \int_0^{(1-v^2/s^2)^{1/2}} du \frac{u}{\delta} J_n^2(knsu) \\ &\quad \times \{ (1 - 3v^2/s^2)[(\delta - \alpha)^{1/2} \cos(nv) \\ &\quad + (\delta + \alpha)^{1/2} \sin(nv)] - (v/s)(3 - v^2/s^2) \\ &\quad \times [(\delta + \alpha)^{1/2} \cos(nv) - (\delta - \alpha)^{1/2} \sin(nv)] \} \dots (A), \end{aligned} \quad (28)$$

where

$$\alpha = 1 - v^2/s^2 - u^2 \quad (29)$$

$$\beta = (u^2 + w^2)^{1/2} \quad (30)$$

$$\delta = [(1 - v^2/s^2 - u^2)^2 + 4v^2/s^2]^{1/2} \quad (31)$$

and where the first term (containing the double integral) is the real contribution arising from the integration along the real y -axis, and the remaining terms are the real contribution arising from the residue term. Note that the integration limits on the residue term arise from the restricted range of x for which the pole appears, since there are no singularities if $x^2 > n^2(s^2 - v^2)$, that is, if $u^2 > 1 - v^2/s^2$.

The imaginary term of the dispersion relation can be written as follows:

$$\begin{aligned} 0 &= 4 \frac{s^3v}{k^2} \sum_{n=1}^{\infty} n^4 \int_0^{\infty} du u J_n^2(knsu) \\ &\quad \times \int_0^{\infty} dw \frac{\beta^3 e^{-ns\beta}}{(\beta^2 - 1 + v^2/s^2)^2 + 4v^2/s^2} \\ &\quad - \sqrt{2}\pi \frac{s^4}{k^2} \sum_{n=1}^{\infty} n^4 e^{-ns} \int_0^{(1-v^2/s^2)^{1/2}} du \frac{u}{\delta} J_n^2(knsu) \\ &\quad \times \{ (1 - 3v^2/s^2)[(\delta + \alpha)^{1/2} \cos(nv) - (\delta - \alpha)^{1/2} \sin(nv)] \\ &\quad + (v/s)(3 - v^2/s^2) \\ &\quad [(\delta - \alpha)^{1/2} \cos(nv) + (\delta + \alpha)^{1/2} \sin(nv)] \} \dots (B). \end{aligned} \quad (32)$$

These real and imaginary parts of the dispersion relation have been derived under the assumption that $a \ll 1$; a more precise derivation of these expressions requires the use of the full expression for Γ :

$$\Gamma^2 = x^2 + y^2 + a^2 \quad (33)$$

leading to the generalization of the notation as follows:

$$\alpha_n = 1 - u^2 - v^2/s^2 - a^2/(n^2s^2) \quad (34)$$

$$\beta_n = [u^2 + w^2 + a^2/(n^2s^2)]^{1/2} \quad (35)$$

$$\delta_n = (\alpha_n^2 + 4v^2/s^2)^{1/2} \quad (36)$$

Expressions equations (28) and (32) follow identically with α_n replacing α , and so on, with the further modification that the integration limits in the u integration (arising from the residue term) are no longer 0 to $(1 - v^2/s^2)^{1/2}$ but instead 0 to u_n , where $u_n^2 = 1 - v^2/s^2 - a^2/(n^2s^2)$. Note that u_n is always real, because there is no singularity in the integrand if $a > ns$, since then $\beta_n > 1$ and therefore no residue contribution is required.

The interpretation of s and v in terms of a normalized wave frequency $\hat{\omega}_r$ and associated damping rate $\hat{\omega}_i$ is straightforward. Previously we had $S = a/\hat{\omega}$ when S was purely real. The generalization to complex S yields

$$s + iv = \frac{a}{\hat{\omega}_r - i\hat{\omega}_i} \quad (37)$$

yielding in turn,

$$\hat{\omega}_r = \frac{as}{s^2 + v^2}, \quad \hat{\omega}_i = \frac{av}{s^2 + v^2}. \quad (38)$$

Hence the damping rate expressed in terms of the frequency of the wave is given by

$$\frac{\hat{\omega}_i}{\hat{\omega}_r} = \frac{v}{s}. \quad (39)$$

3. Intersection of dispersion curves with the S -axis: $k = 0$

It is of particular interest to determine the behaviour of the plasma waves in the limit of small wave number, i.e. where the dispersion relation intersects the s -axis. It is clear that in the limit $k \rightarrow 0$, all terms with $n > 1$ will disappear because of the Bessel function properties. The only remaining term is the one in which $n = 1$, and for which $J_1^2(ksu) \approx k^2 s^2 u^2 / 4$. After cancellation of k^2 throughout, we obtain two equations for s and v , from the dispersion relation parts A and B :

$$S_p^2 = \frac{s^6}{2} \int_0^\infty du u^3 \int_0^\infty d\omega \beta e^{-s\beta} \frac{\beta^2(1-v^2/s^2) - (1+v^2/s^2)^2}{(\beta^2 - 1 + v^2/s^2)^2 + 4v^2/s^2} \\ - \sqrt{2\pi} \frac{s^6}{4} e^{-s} \int_0^{(1-v^2/s^2)^{1/2}} du \frac{u^3}{\delta} \\ \times \{(1 - 3v^2/s^2)[(\delta - \alpha)^{1/2} \cos(v) + (\delta + \alpha)^{1/2} \sin(v)] \\ - (v/s)(3 - v^2/s^2) \\ \times [(\delta + \alpha)^{1/2} \cos(v) - (\delta - \alpha)^{1/2} \sin(v)]\} \quad (40)$$

for the real part, and for the imaginary part:

$$0 = 4v \int_0^\infty du u^3 \int_0^\infty d\omega \frac{\beta^3 e^{-s\beta}}{(\beta^2 - 1 + v^2/s^2)^2 + 4v^2/s^2} \\ - \sqrt{2\pi} s e^{-s} \int_0^{(1-v^2/s^2)^{1/2}} du \frac{u^3}{\delta} \\ \times \{(1 - 3v^2/s^2)[(\delta + \alpha)^{1/2} \cos(v) - (\delta - \alpha)^{1/2} \sin(v)] \\ + (v/s)(3 - v^2/s^2)[(\delta - \alpha)^{1/2} \cos(v) + (\delta + \alpha)^{1/2} \sin(v)]\}. \quad (41)$$

The solution strategy here is to use equation (41) to determine a relationship between s and v , and then evaluate S_p^2 using equation (40) with these same values of s and v . In other words, equation (41) is used to identify sets of (s, v) pairs that ensure the imaginary part of the dispersion relation vanishes; these (s, v) pairs are then substituted into equation (40) to discover the corresponding values of S_p^2 . Typically, $v < s$, so that the damping is small: an example of v as a function of s for the case $k = 0$ is plotted in figure 2.

It is possible to identify certain general trends from examining the character of the real and imaginary parts of the dispersion relation. Firstly, since S_p is real, then a physical solution for S_p is only possible if the right-hand side of equation (40) is positive, that is, if the two terms produce a positive definite sum. Secondly, the requirement that the imaginary part of the dispersion relation vanishes requires a negative-definite contribution from the second term (the residue contribution) in equation (41), since the first integral is positive definite.

From these two broad observations, it is evident that as $s \rightarrow 0$, $S_p^2 \rightarrow 0$ from equation (40), irrespective of the value of v , since the s^6 common factor dominates the behaviour. If $v \rightarrow 0$ as $s \rightarrow 0$, then the imaginary part of the dispersion relation is also satisfied, since the principal integral term has v as an overall factor, and the residue contribution is multiplied by s . Hence there is a consistent solution to the dispersion relation in which S_p and $v \rightarrow 0$ as $s \rightarrow 0$, with v approaching zero at least as fast as s .

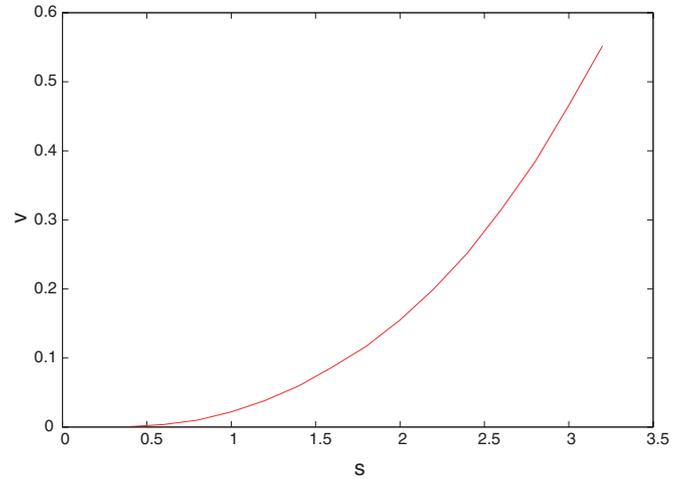


Figure 2. A plot of the normalized damping term v as a function of the normalized real frequency s (horizontal axis), for the case $k = 0$.

Note that the case where $s \rightarrow 0$ but v remains finite is precluded by the range of integration in the u integral, since there is no singularity if u is real, and if there is a contribution from the residue term, then $u^2 < 1 - v^2/s^2$. Hence we cannot have a damped solution as $k \rightarrow 0$ in which the damping does not tend to zero at least as fast as s .

It is possible to consider the limit $s \rightarrow 0$, for which the imaginary term is very small. The principal integral part of equation (40) in the limit of negligible damping becomes

$$\frac{1}{2} s^6 \psi(s) \quad (42)$$

$$\psi(s) = \int_0^\infty du u^3 \mathcal{P} \int_0^\infty d\omega \frac{\beta \exp(-s\beta)}{\beta^2 - 1}, \quad (43)$$

where \mathcal{P} denotes the principal integral. We can differentiate under the integral sign to form the following inhomogeneous differential equation for ψ :

$$\frac{d^2 \psi}{ds^2} - \psi = \int_0^\infty du u^3 \int_0^\infty d\omega \beta \exp(-s\beta). \quad (44)$$

The w -integration can be performed in closed form, in terms of the Bessel functions $K_0(z)$ and $K_1(z)$:

$$\int_0^\infty d\omega \beta \exp(-s\beta) = u^2 K_0(su) + (u/s) K_1(su) \quad (45)$$

which, after integration with respect to u , leads to

$$\frac{d^2 \psi}{ds^2} - \psi = 80/s^6 \quad (46)$$

Assuming that ψ is negligible compared with its second derivative, equation (46) can be approximated by

$$\frac{d^2 \psi}{ds^2} \approx 80/s^6, \quad (47)$$

which has solution $\psi = 4/s^4$. The next term in the series expansion comes from substituting $\psi(s) = \psi_1(s) + 4/s^4$ into

equation (46), which yields an equation for ψ_1 , and so on. Continuing in this way, the small- s expansion for ψ is

$$\psi(s) \approx 4/s^4 + 2/(3s^2) - (2/3) \ln(s) \dots \quad (48)$$

The dispersion relation is then given by $s^6\psi/2$, leading to the dominant form of the dispersion relation for small s :

$$S_p^2 \approx 2s^2 \quad (49)$$

which can be written in the original plasma variables as

$$\omega^2 = 2\omega_p^2. \quad (50)$$

Given that $s \rightarrow 0$ represents the high-frequency limit of electrostatic waves (as seen from equation (38)), then it is appropriate that equation (49) yields the pair-plasma oscillation in that limit. Note that the traditional classical case had solutions for $k = 0$ that corresponded to the cyclotron harmonics, plus the hybrid oscillation. The weakly relativistic case had only one solution for $k = 0$, namely the hybrid frequency; here, because we are assuming a small, the hybrid frequency is dominated by the plasma frequency, and so equation (49) recovers that solution. However, we do have another, lower-frequency solution that comes from finite s , but the value of this low-frequency limit is governed by the behaviour of s as a function of S_p .

Consider now the case s becomes large. In equation (40), the $\beta \exp(-\beta s)$ factor in the principal integral term means that the contribution to the integrand from this factor peaks near $\beta \lesssim 1/s$, and is significantly attenuated for β values that exceed this; hence most of the contribution to this integral comes from close to this value. The second factor in the integrand controls its sign: if $\beta^2 < \frac{(1+\epsilon^2)^2}{1-\epsilon^2}$ where $\epsilon = v/s$ then the integrand becomes negative definite, meaning that there is a possibility that the integral itself is less than zero. Given that the residue contribution is likely to be mainly negative definite (assuming that ϵ is small), then there is a maximum value of s for which S_p^2 can be non-negative, simply from the character of the integral. Hence we expect that, for S_p^2 to be a continuous function of s , there must be a maximum value of S_p^2 over a finite s range.

There is also a physical reason why we expect there to be a finite range of S_p over which relativistic Bernstein modes can be found. The validity of the kinetic theory model for plasmas depends on phenomena occurring on scale lengths larger than the Debye length, to ensure that there are no anomalous small-scale transport properties that are dominated by unbalanced nearest-neighbour electromagnetic interactions (for example, see the discussions of anomalous transport that may result if this condition is violated, given in [17–19]). Hence we might reasonably assume that the Debye length must be shorter than the Larmor radius for the average particle. In an ultra-relativistic plasma, the Debye length λ_D is given by (see the appendix)

$$\lambda_D^2 = \frac{2c^2}{a\omega_p^2}. \quad (51)$$

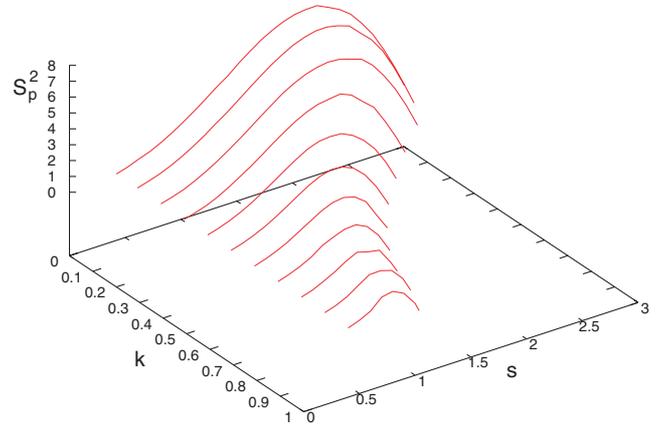


Figure 3. A plot of the dispersion parameter S_p^2 as a function of the normalized real frequency s and the wavenumber k .

Given that the mean speed of particles in the ultra-relativistic plasma is approximately c , the Larmor radius R_L is readily determined:

$$R_L^2 = \frac{\gamma^2 c^2}{\Omega_0^2} = \frac{25c^2}{a^2 \Omega_0^2} \quad (52)$$

where we have used the approximation $\gamma \approx 5/a$ when $a \ll 1$ in the Maxwell–Boltzmann–Jüttner distribution. For consistency, to ensure that there are no unbalanced electromagnetic forces at small scales, we need $R_L \gtrsim \lambda_D$, yielding

$$\frac{R_L^2}{\lambda_D^2} = \frac{25a}{2S_p^2}. \quad (53)$$

Hence $0 \leq S_p^2 \leq 25a/2$.

4. Presentation of results

In the previous section we have presented the behaviour of Bernstein waves in the limit of small wave numbers ($k \rightarrow 0$), showing the relationship between S_p^2 and the frequency parameter s . In this section we discuss the behaviour of plasma waves for a finite range of k . Our procedure is to start from equation (32), the imaginary part of the dispersion relation. For a sequence of values of k ($k = 0.1, 0.2, \dots, 1.0$) we obtain a relationship between s and v . Then, for a given pair (v, s) we use equation (28) to calculate the corresponding value of S_p^2 . Results, including also the case $k = 0$ considered in section 3, are summarized in figure 3. From these results we can construct the dispersion relations for across the full range of relevant values of S_p^2 ; this is shown in figure 4.

From figure 3, it is clear that the range of valid s values for various k diminishes as k grows over the range $k = 0$ to $k = 1$, accompanied by a monotonic decrease in the maximum associated value of S_p^2 . Long-wavelength modes have the maximum range of s and S_p values; short wavelength modes are significantly restricted in parameter space. In fact, to a good approximation, the s, S_p curves for arbitrary k all fit inside the case $k = 0$.

This is clearer in the contour plot in figure 4, where the data have been rearranged to produce a plot of s versus k —a dispersion relation in reciprocal frequency versus

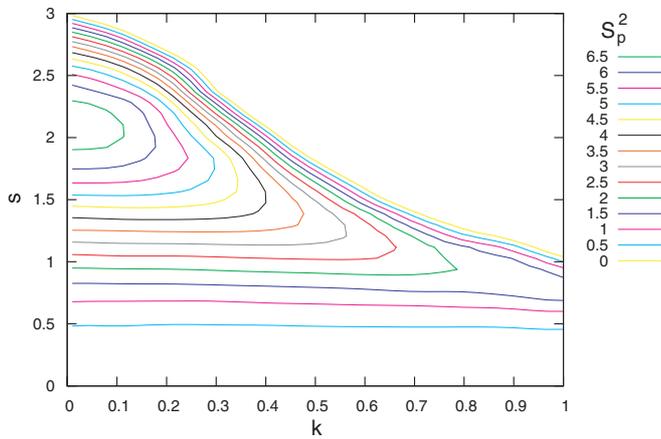


Figure 4. A plot of the dispersion relations showing k along the x -axis, s on the y -axis, for various values of the parameter S_p^2 , shown as contours.

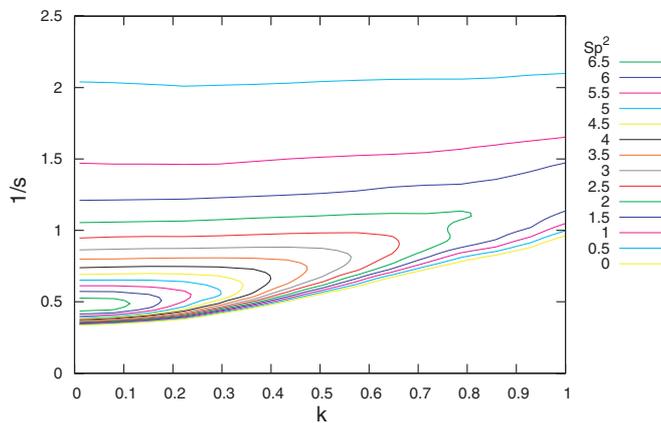


Figure 5. A plot of the dispersion relations showing k along the x -axis, $1/s$ on the y -axis, for various values of the parameter S_p^2 , shown as contours. Since $s^{-1} \sim \omega$, this is more directly a set of conventional dispersion curves.

wavenumber, in contours of S_p values. For completeness, the same data are plotted in figure 5 using $1/s$ rather than s , to portray more closely the form of the dispersion relation. (Note that we have not included the imaginary term in these plots, since if $v/s \ll 1$ then it is clear from equation (38) that to a good approximation s is essentially the normalized reciprocal frequency.) The power in this method is that S_p^2 contains within it the choice of ω_p and a , so that for a given a , different values of S_p^2 correspond to different choices of the plasma frequency, and vice versa.

These plots are for $a \rightarrow 0$ and so are the ultra-relativistic versions of earlier work for finite a [8, 9, 15]; note that the ‘fingering’ structure of the latter calculations, which are remnants of the fully classical cyclotron harmonics, is now absent from the ultra-relativistic calculation: the modes are compressed into a small region of parameter space, and show no cyclotron structure at all.

Note that these results generalize our earlier work [8, 9], in that not only is the range of relativistic parameter much larger, but that the compact nature of the formulation presented here obviates the need to specify the value of the normalized plasma

frequency $\hat{\omega}_p$, making the results more generally applicable. Note also that the work presented in this paper is able to model ultra-relativistic effects that go beyond the restricted discussion of [15]: this latter paper confines discussion to $a \geq 0.5$ only, and uses a mathematical formulation that makes the form of the dispersion relation for $k = 0$ problematic, in contrast to the work presented here: there is a singularity in the dispersion relation at $k = 0$, and the Landau contour may be incorrect (for example, see [9]). Moreover, in [15], the results need an explicit choice of $\hat{\omega}_p$.

5. Application to pulsar atmospheres

We consider the relevant value of S_p^2 here to approach zero if we wish to apply our theory of ultra-relativistic Bernstein waves to pulsar atmospheres. Bernstein modes, being travelling electrostatic disturbances, are attractive in the pulsar context since they offer the possibility of avoiding the intrinsic density instability that afflicts cold pair-plasma oscillations [20]; such static oscillations are known to generate a coherent electromagnetic response [21] that is implicated in pulsar emission mechanisms, and the electrodynamics of the pulsar atmosphere makes electrostatic disturbances inevitable [12]. However, electrostatic oscillations in the cold plasma limit are confined spatially, and do not travel beyond their initiation site; consequently finite amplitude disturbances deplete the ambient plasma density, leading to unphysical instabilities. Hence the characteristics of moving electrostatic disturbances are of considerable interest for developing a deeper insight into the underlying pulsar radiation physics.

We justify choosing $S_p^2 \ll 1$ by quoting some basic plasma parameters (using SI units):

$$\omega_p = \left(\frac{n_e e^2}{\epsilon_0 m_e} \right)^{1/2} = 56 n_e^{1/2} \quad (54)$$

$$\Omega_0 = \frac{e B_0}{m_e} = 17.6 \times 10^{10} B_0 \quad (55)$$

$$S_p^2 = \left(\frac{a \Omega_0}{\omega_p} \right)^2 = 10^{19} a^2 B_0^2 / n_e. \quad (56)$$

We assume, as typical pulsar examples, $B = 10^8$ [22, 23]. The value of the electron (or positron) number density n_e is not as certain as other pulsar parameters. Early literature [24] suggests that $n_e \sim 10^{20} \text{ m}^{-3}$; however, more recent research [25] deduces that the Goldreich–Julian electron density is a major underestimate of the true, n_e by a factor of 10^6 . Hence it is appropriate to assume n_e lies in the range $10^{25-27} \text{ m}^{-3}$. The corresponding range of S_p^2 can then be expressed (using equation (56)) as $S_p^2 = 10^{35} a^2 / n_e$, meaning that S_p can remain within its maximum limit if $a^2 / n_e \sim 10^{-34}$, suggesting that $a \sim 10^{-4}$ is an appropriate value, consistent with our original ultra-relativistic assumption. However, note that this yields an S_p^2 value around unity; if we apply the restriction that the Debye length must be less than the Larmor radius, this imposes a further limitation on appropriate parameter values. From equation (53), compatible values of a and S_p^2 are $a \sim 10^{-7}$

and $S_p^2 \sim 10^{-6}$, which is consistent with the very high values of $\gamma \sim 7 \times 10^7$ in the Crab Giant Pulses [25]. Of course if the plasma is not fully relaxed into its equilibrium, then the restriction on the Larmor radius being larger than the Debye length may not hold (though this could also make the analysis of Bernstein modes questionable in this context).

Acknowledgments

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Appendix. Properties of the relativistic Maxwellian

The generalization f_{MBJ} of the classical Maxwell–Boltzmann distribution has some very interesting properties, not least of which is the behaviour of the distribution at small a values, that is, as the plasma is more relativistic. As $a \rightarrow 0$, the distribution function when written in terms of speed u becomes a delta-function centred on c , the speed of light (see figure A1; for example, at $a = 1$, the maximum of f_{MBJ} lies at $u/c = 0.98$). Moreover, the value of relativistic $\gamma = (1 - u^2/c^2)^{-1/2}$ corresponding to the most probable speed satisfies

$$a^{-1} = \gamma \frac{\gamma^2 - 1}{5\gamma^2 - 3}, \quad (57)$$

which is true for all values of a ; for very large a , equation (57) yields $\gamma \approx 1 + 1/a$ implying $\frac{1}{2}mu^2 \approx k_B T$, as expected. For the ultra-relativistic case, $\gamma \approx 5/a$, corresponding to most particles moving close to c : $u^2/c^2 \approx 1 - a^2/25$.

5.1. Debye length, Larmor radius and S_p

The average kinetic energy \bar{E} of a plasma particle in the Jüttner distribution is given by [16]

$$\bar{E} = \frac{3mc^2}{a} + mc^2 \left[\frac{K_1(a)}{K_2(a)} - 1 \right]. \quad (58)$$

In the classical case, $a \gg 1$ and so we can use the asymptotic form for the Bessel functions:

$$K_\nu(z) \sim [\pi/(2z)]^{1/2} \left(1 + \frac{4\nu^2 - 1}{8z} \right) \quad (59)$$

and hence

$$\frac{K_1(a)}{K_2(a)} \sim 1 - \frac{3}{2a} \quad (60)$$

yielding

$$\bar{E} \approx \frac{3}{2} \frac{mc^2}{a} = \frac{3}{2} k_B T. \quad (61)$$

In the simple 1D electrostatic fluctuation, equating one third of \bar{E} to the potential well of a thermally produced charge imbalance yields the expression for the Debye length λ_D :

$$\frac{1}{2} \frac{n_0 e^2 \lambda_D^2}{\epsilon_0} = \frac{1}{2} k_B T \quad (62)$$

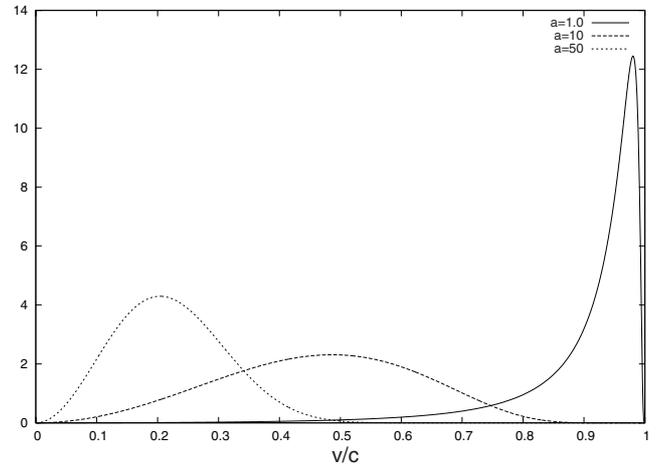


Figure A1. A plot of the Maxwell–Boltzmann–Jüttner distribution function for three values of a : $a = 50$ is classically Maxwellian-like, but $a = 1.0$ shows marked delta-function behaviour.

that is,

$$\lambda_D = \left(\frac{\epsilon_0 k_B T}{n_0 e^2} \right)^{1/2}. \quad (63)$$

Now consider the case $a \ll 1$, that is, the ultra-relativistic limit. Now the small-argument expansion is needed in the expression for the Bessel functions:

$$K_\nu(z) \sim \frac{1}{2} \frac{\Gamma(\nu)}{\left(\frac{1}{2}z\right)^\nu}. \quad (64)$$

Hence now we have

$$\frac{K_1(a)}{K_2(a)} \sim a/2 \quad (65)$$

and so the mean energy can be written in the form

$$\bar{E} \approx \frac{3mc^2}{a} (1 + a^2/6 - a/3) \approx 3k_B T, \quad (66)$$

retaining only the dominant term. Therefore, as $a \rightarrow 0$, the mean kinetic energy of the particles is twice that of the $a \rightarrow \infty$ case—a surprising result at first, except of course that the temperature used in the ultra-relativistic case is very much larger than that in the classical: the doubling is only an additional numerical factor on the variable dependences. The electrostatic potential calculation is just as before, hence we can state that the ultra-relativistic Debye length $\lambda_{D,\text{ur}}$ is

$$\lambda_{D,\text{ur}} = 2^{1/2} \lambda_D. \quad (67)$$

Another check on this result is to recall that in a classical plasma, the Debye length is the distance travelled by a sound wave in a plasma period: $\lambda_D = c_s/\omega_p$. In the ultra-relativistic case, the sound speed c_{ur} is given by $c_{\text{ur}} \approx c/\sqrt{3}$ [16], and so recalculating the distance travelled by a sound wave in a plasma period in the ultra-relativistic plasma, we have

$$\lambda_{D,\text{ur}}^2 \approx \frac{c^2}{3} \frac{\gamma m \epsilon_0}{n e^2} \approx \frac{5}{3} \frac{\epsilon_0 k_B T}{n e^2}, \quad (68)$$

where we have used $\gamma \approx 5/a$. Equations (68) and (67) are consistent (and indeed in agreement with [26]), and we shall take the ultra-relativistic Debye length to be given by equation (67). It is clear that $\lambda_{D,ur} \propto a^{-1/2}$, meaning that the relativistic Debye length grows as a becomes smaller.

Note also that the plasma parameter N (the number of particles in a Debye sphere) can be readily calculated for the ultra-relativistic plasma:

$$N = n\lambda_{D,ur}^3 = (na^3)^{-1/2} \left(\frac{2\epsilon_0 mc^2}{e^2} \right)^{3/2} \approx \frac{4 \times 10^{20}}{(na^3)^{1/2}}. \quad (69)$$

It is always prudent to check how the Larmor radius R_L of particles compares with the Debye length, since if the former approaches the latter, then the continuum equations need to be revisited because local charge neutrality is no longer an appropriate assumption. In the classical case,

$$R_L = \frac{v_T}{\Omega} = \frac{(mk_B T)^{1/2}}{eB} \quad (70)$$

and so the ratio of the Debye length and Larmor radius can be expressed as

$$\frac{\lambda_D^2}{R_L^2} = \frac{\epsilon_0 B^2}{n_0 m} = \frac{c_a^2}{c^2}. \quad (71)$$

Hence if the Alfvén speed is less than the speed of light, the Larmor radius is greater than the Debye length. In the ultra-relativistic plasma, the mean speed of the particles is c , resulting in the Larmor radius taking the form $R_{L,ur} = \gamma mc/(eB_0)$, that is, $R_{L,ur} \propto 1/a$. Thus the Larmor radius grows with decreasing a , at a faster rate than the Debye length in the ultra-relativistic limit. Hence we have

$$\left(\frac{\lambda_{D,ur}}{R_{L,ur}} \right)^2 = \frac{2\epsilon_0 k_B T}{n_0 e^2} \frac{e^2 B^2}{c^2 \gamma^2 m^2} \quad (72)$$

$$= \frac{4c_a^2}{\gamma^2 a c^2} \quad (73)$$

$$\approx \frac{4ac_a^2}{25c^2}. \quad (74)$$

Now

$$S_p^2 = (a/\hat{\omega}_p)^2 = 2a^2 \frac{c_a^2}{c^2}, \quad (75)$$

hence if the Larmor radius is to be greater than the Debye length, then

$$S_p^2 < \frac{25}{2} a. \quad (76)$$

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