METATHEORETIC RESULTS FOR A NON-TRANSITIVE LOGIC:

This is a companion to 'A Robustly Non-Transitive Logic', *Topoi* 2013, which focuses mainly on the details of the metatheory for the logic advocated there; it can be read independently of that article.

Outline

§I: Soundness for the Three-Valued System NC_{3^-};

§II: Completeness of the Three-Valued System NC_3;

§III: Robust Contraction-Freedom;

§IV The Infinite-Valued System NC_I;

§V The Infinitary Logic NC_∞;

§VI Naïve Truth Theory and Neo-Classical Logic.

§I Soundness for the Three-Valued System NC_{3^-}

The three-valued propositional logic NC_3 (for three-valued neo-classical logic) discussed in Weir (2013) uses the Łukasiewiczian semantics for the conditional $\rightarrow$ and the strong Kleene truth-functions for the other operators. That is, with 1 for the value true, 0 false, and ½ for the gap value, the conditional is interpreted by:

		Q		
	$P \rightarrow Q$	1	½	0
	1	1	½	0
P	½	1	1	½
	0	1	1	1

Conjunction $\wedge$, disjunction $\vee$ and negation $\neg$ are classical on classical 1/0 inputs; elsewhere negation maps gap to gap, a conjunction is false iff one conjunct is false, true iff both are true, disjunction the dual to conjunction. As usual, a propositional model is a function from the atoms into the value set, here {1,½,0} with the values for complex sentences determined by the truth-functions expressed by the connectives.

The definition of logical entailment is $\Delta \vDash_3 Q$ iff for every $P \in \Delta$, for every valuation v:

> if all of $\Delta - \{P\}$ are true in v then a) if P is true in v, Q is also true there and b) if Q is false in v, P is false.

The proof system utilised is sequent form natural deduction presented in Lemmon-style format. The introduction rules plus $\wedge$E are the usual classical rules. Thus for negation introduction rules, we have both the classical and intuitionistic rules:

X, $\neg$P	(1) $\perp$	Given
X	(2) P	1, $\neg$I

and

X, P	(1) $\perp$	Given
X	(2) $\neg$P	1, $\neg$I

Conditional introduction is the standard classical:

X, P	(1) Q	Given
X	(2) P $\rightarrow$ Q	1, $\rightarrow$I

and likewise the $\wedge$I, $\wedge$E and $\vee$I rules are classical, whilst the main structural rule of hypothesis, incorporating the reflexivity and monotonicity of derivability is

X	(1) A	H

where $A \in X$.

In addition we have the Minimax principle MM–

X	(1) A	Given
X	(2) B	1, MM

where B is a 'minimax' propositional consequence of A, that is, for all v^*_K $v^*_K(A) \leq v^*_K(B)$ where v^*_Kevaluates formulae under the Kleene interpretation of ∧, ∨ and ¬ but treats conditionals as atoms, expanding the evaluation to atoms given by v. Note that if X minimax entails A then X $\vDash_3$ A.

It is mechanically decidable whether a sequent follows from input sequents according to the minimax principle MM (where all sequents are finite). But we can also add purely syntactic rules to implement the minimax sub-relation of $\vDash_3$, for example by adding to the above introduction and elimination rules the 'Mingle' rule[1]: from X : A ∧ ¬A one can conclude X : B ∨ ¬B and the minimax sound distributivity principles for conjunction and disjunction (strictly only one direction each: from (A∨B) ∧ (A ∨ C) conclude A ∨ (B ∧ C) and from A ∧ (B ∨ C) conclude (A ∧ B) ∨ (A ∧ C)).

More fully, and with some redundancy in the three-valued case of NC_3, minimax soundness can be implemented by altering the MM rule to

X	(1) φ	Given
X	(2) φ[P/Q]	1 MM

where φ[P/Q] results from φ by uniform substitution of a sub-formula P by Q and where

P is	A	and Q is	¬¬A
or P is	¬¬A	and Q is	A
or P is	A ∧ B	and Q is	B ∧ A
or P is	A ∨ B	and Q is	B ∨ A

[1] See Anderson and Belnap, 1975, §29.5 especially the theorem ¬(A → A) → (B → B) together with theorem RM67 (p. 397) (A → A) ↔ (¬A ∨ A).

or P is	$A \wedge B$	and Q is	$\neg(\neg A \vee \neg B)$
or P is	$\neg(\neg A \vee \neg B)$	and Q is	$A \wedge B$
or P is	$A \vee B$	and Q is	$\neg(\neg A \wedge \neg B)$
or P is	$\neg(\neg A \wedge \neg B)$	and Q is	$A \vee B$
or P is	$(A \wedge (B \wedge C))$	and Q is	$((A \wedge B) \wedge C)$
or P is	$((A \wedge B) \wedge C)$	and Q is	$(A \wedge (B \wedge C))$
or P is	$(A \vee (B \vee C))$	and Q is	$((A \vee B) \vee C)$
or P is	$((A \vee B) \vee C)$	and Q is	$(A \vee (B \vee C))$
or P is	$(A \wedge (B \vee C))$	and Q is	$((A \wedge B) \vee (A \,\&\, C))$
or P is	$((A \wedge B) \vee (A \wedge C))$	and Q is	$(A \wedge (B \vee C))$
or P is	$(A \vee (B \wedge C))$	and Q is	$((A \vee B) \wedge (A \vee C))$
or P is	$((A \vee B) \wedge (A \vee C))$	and Q is	$(A \vee (B \wedge C))$

Minimax soundness is easily established by showing that the formulae on the left has the same value, in every model, as the formula on the right.

Thus far all is relatively standard. The deviant rules are the determinacy-restricted rules for $\vee$E, $\neg$E and $\rightarrow$E. The determinacy of a sentence P, abbreviated D(P), is defined by $D(P) \equiv \neg(P \leftrightarrow \neg P)$. The general idea in these rules is that whenever there is an overlap between the assumptions of the major premiss and of the minor premiss(es) we need additional premisses establishing that each such overlapping antecedent is determinate. Thus the negation elimination ex falso rule $\neg$E is modified to:

X	(1) P	Given
Y	(2) ¬P	Given
Z_i	(3.i) D(R_i)	$\forall R_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) C	1,2 [3.i] ¬E

whilst standard ∨E is modified to the following:

X	(1) P ∨ Q	Given
Y,P	(2) C	Given
Z,Q	(3) C	Given
W_i, $i \in I$	(4.i) D(R_i)	Given, $\forall R_i \in (X \cap (Y \cup Z))$
$X, Y, Z, \bigcup_{i \in I} W_i$	(5) C	1,2,3 [4.i, $i \in I$], ∨E.

Finally →E is:

X	(1) P → Q	Given
Y	(2) P	Given
Z_i	(3.i) D(R_i)	$\forall R_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.i] →E

Any linear chain of sequents, in Lemmon sequent natural deduction format and where the later follow from the earlier by one of the rules counts as a proof. If Δ is the set of antecedents of such a sequent in a genuine proof, A its succedent, then we write $\Delta \vdash_{3^-} A$ (the reason for the 3^- will emerge in the next section).

The soundness of the system NC_{3^-} can be proved by the usual induction on proof length, with the main business of the proof being in the inductive clauses for the various operational rules. Here are the steps for ∨E, →I and →E, the reasoning for the other cases, including the determinacy-restricted ¬E is similar .

The ∨E rule, to repeat, is:

X	(1) $P \vee Q$	Given
Y,P	(2) C	Given
Z,Q	(3) C	Given
W_i, $i \in I$	(4.i) $D(R_i)$	Given, $\forall R_i \in (X \cap (Y \cup Z))$
X,Y,Z, $\bigcup_{i \in I} W_i$	(5) C	1,2,3 [4.i, $i \in I$], $\vee$E

We have to prove both the truth-preservation downwards, and falsity preservation upwards clauses.

i) Truth preservation: this is much as in the classical case. Suppose all the given input sequents are entailments and that all of X,Y,Z, $\bigcup_{i \in I} W_i$, are true in valuation v. Indeed all we need is the truth of all of X,Y,Z. Then by line 1, $P \vee Q$ is true in v, so one or other disjunct is. Whichever is the case, line 2 or else line 3 establishes that C is true in v.

ii) Falsity preservation (upwards): suppose that C is false in v and all of X,Y,Z, $\bigcup_{i \in I} W_i$ are true in v but A. To prove: A is false. A cannot be true, else by truth-preservation C is true. Now each $D(R_i)$ is either true or false as can be checked by looking at the three-valued Łukasiewiczian truth table for $\neg(R_k \leftrightarrow \neg R_k)$. If any $D(R_k)$ is false then $A \in W_k$ and is false as required since 4.k is a correct entailment. So we may assume each $D(R_i)$ is true. By truth-preservation and the falsity of C, $A \in (X \cup Y \cup Z)$. If $A \in (X \cap (Y \cup Z))$ it is one of the R_i and since each $D(R_i)$ is true, A has a determinate truth value in v; since it is not true, it must be false as required. If $A \notin (X \cap (Y \cup Z))$ then either a) all of X are true in v or else b) all of $Y \cup Z$ are.

Case a):– by the correctness of line (1) $P \vee Q$ is true in v hence one of the disjuncts is, suppose without loss of generality that it is P. By the correctness of line (2) in the rule above, $A \in Y$ and is false.

Case b) By the correctness of lines (2) and (3) both P and Q are false in v hence so is $P \vee Q$. By the correctness of line (1), $A \in X$ and is false in v. ❑

As for the conditional rules, to repeat they are:

$\to$I:

X,P	(1) Q	Given
X	(2) P $\to$ Q	1 $\to$ I

and

$\to$E:

X	(1) P $\to$ Q	Given
Y	(2) P	Given
Z_i	(3.i) D(R_i)	$\forall R_i \in X \cap Y$
X, Y, $\bigcup_{i \in I} Z_i$	(4) Q	1,2 [3.i] $\to$E

The truth-preservation clause is essentially the same as the classical case for $\to$E. For $\to$I, suppose all of X are true in v. If Q is true then P $\to$ Q is true, as required, if Q is false, by line (1), P is false so P $\to$ Q is true, if Q is gappy then by line (1), P cannot be true, hence once again P $\to$ Q is true in v.

Falsity-preservation $\to$I: suppose P $\to$ Q is false in valuation v and all of X but A are true there. Then P is true at v and Q is false at v. By IH applied to line (1), A is false at v as required.

c) Falsity preservation $\to$E: suppose Q is false and all of X, Y, $\bigcup_{i \in I} Z_i$ but A are true. Each D(R_i) is either true or false. If any D(R_k) is false then A $\in$ Z_k and is false as required since 3.k is a correct entailment. So we may assume each D(R_i) is true. A $\in$ X $\cup$ Y else by truth-preservation Q is true at v. If A $\in$ X $\cap$ Y, then since D(A) is true, A is not gappy and since it is not true, is false as required. The remaining cases are:

a) A belongs to X but not Y. Then P is true, Q false so P $\to$ Q is false, whence by the correctness of line (1), A is false.

b) Similarly if A $\in$ Y but A $\notin$ X, P is false since P $\to$ Q is true and Q is false; hence A is false as required.

II Completeness for the Three-Valued System NC_3

The proof system NC_3- is thus sound with respect to the given notion of entailment $\vDash_3$. To secure completeness for the three-valued semantics we expand the proof theory to the system NC_3 by adding as axioms all instances of LEM with a determinacy disjunct; that is all instances of D(P) ∨ ¬D(P) are to be added as axioms– as noted, D(P) sentence are not gappy in any valuation hence each such instance of LEM is true in every valuation. Finally we need to add the following sound rule with I(P) – 'P is indeterminate' abbreviates ¬D(P):

X, P	(1) I(P)	Given
Y, Q	(2) I(Q)	Given
X,Y,P,Q	(3) ⊥	1,2, ⊥ rule

where P ≠ Q. For truth-preservation we note that if all the premisses are true then P and I(P) (or Q and I(Q)) are both true together, which is impossible. In the falsity-preservation direction, if all premisses but A are true then A must be in X or Y (or both) otherwise either P and I(P) are both true together or Q and I(Q) are true together but neither are possible. But then either P or Q are true, suppose without loss of generality P. Then I(P) is false hence by line (1), A ∈ X and is false.

The completeness theorem for NC_3:

If $\Delta \vDash_3 A$ then $\Delta \vdash_3 A$

we prove in the usual fashion. We start from a consistent set X, that is $X \nvdash_3 \bot$. We expand X by a Henkin-style construction:

if $\Delta_n, \varphi_n \nvdash_3 \bot$, $\Delta_{n+1} = \Delta_n, \varphi_n$;

otherwise $\Delta_{n+1} = \Delta_n$.

Our final Henkin set Δ is the union of the Δ_i.[2] The twist here is that classically, where we start from a consistent set, there is no third possibility of both $\Delta_n \vdash_3 \neg P_n$ and $\Delta_n \vdash_3 \neg\neg P_n$ given that $\Delta_n \nvdash_3 \bot$. Neo-classically, we need some finer discriminations. Let's say a set X is

weakly consistent just in case $X \nvdash_3 \bot$;

strongly consistent iff there is no P such that $X \vdash_3 P$ and $X \vdash_3 \neg P$;

negation-complete just in case for all P, $P \in X$ or $\neg P \in X$;

negation-complete$_1$ just in case for all P, $\Delta \vdash_3 P$ or $\Delta \vdash_3 \neg P$.

Now Δ is a negation-complete$_1$ theory. If $\varphi_n \in \Delta$ then clearly $\Delta \vdash_3 \varphi_n$, if not then $\Delta_n, \varphi_n \vdash_3 \bot$ hence $\Delta_n \vdash_3 \neg\varphi_n$ by ¬I. (Note that a weakly consistent negation-complete$_1$ theory X can have, for some P, both $X \vdash_3 P$ and $X \vdash_3 \neg P$.) Δ is not only negation-complete$_1$ but, as a straightforward inductive proof shows, at least weakly consistent (assuming our initial set X is weakly consistent). It is deductively closed iff it is strongly consistent.

The three-valued semantics and definition of determinacy in terms of → block, as we have seen, higher-order indeterminacy. $D(P) \equiv_{df.} \neg(P \leftrightarrow \neg P)$ can only take value true or false, regardless of the truth-value of P, likewise $I(P) \equiv_{df.} (P \leftrightarrow \neg P) \equiv \neg D(P)$ is bivalent.

<u>Preliminaries</u>.

<u>Proposition 1</u>: $\vdash_3 D^n(P)$ for all $n \geq 2$; $\vdash_3 D^m(I(P))$ for all $m \geq 1$. Proof: We have as axioms $\vdash_3 D(P) \vee \neg D(P)$. We have quite generally if $\vdash_3 A \vee \neg A$ then $\vdash_3 D(A)$ (Lemma 1– see below for proofs of the lemmata). Substituting $D^k(P)$ for A, $k \geq 1$, we get $\vdash D(D^k(P))$; substituting I(P) for P gives $D^m(I(P))$ for all $m \geq 2$, the case for $m = 1$ follows from Lemma 1 since we have $\vdash_3 I(P) \vee \neg I(P)$, that is $\vdash_3 \neg D(P) \vee \neg\neg D(P)$.

<u>Proposition 2</u>. Where we have proofs of the determinacy of the succedents of a premisses of a ¬E and →E rules, and both disjuncts of the major premiss

[2] If the language is uncountable, take unions at limits as usual.

succedent of the $\vee$E rule, then the full classical rule is neo-classically derivable, that is we do not need to establish the determinacy of the overlapping assumptions.

More fully, in the case of $\neg$E– if we have

Γ	(1) P	Given
Γ	(2) $\neg$P	Given.
Γ	(3) D(P)	Given

then we may conclude $\Gamma \vdash_3 \bot$ despite the overlap of the assumptions on which all the premiss sequents depend (where we have two different assumption sets X and Y for major and minor premiss we can bring them under this case by expanding both to $X \cup Y$). The same is true if we have $\Gamma \vdash D(\neg P)$ in line three. Similarly in $\rightarrow$E if we have

Γ	(1) P	Given
Γ	(2) $P \rightarrow Q$	Given.
Γ	(3) D(Φ)	Given

where Φ is replaced by P or by $P \rightarrow Q$, we can prove Q from Γ.

In the case of $\vee$E, the following is derivable:

Γ	(1) $P \vee Q$	Given
Γ, P	(2) C	Given.
Γ, Q	(3) C	Given
Γ	(4) D(P)	Given
Γ	(5) D(Q)	Given
Γ	(6) C	1:5 Derived $\vee$E

(If Γ is not needed in one of the sub-premisses, say at (3), then the corresponding determinacy clause, here (5), $\Gamma \vdash_3 D(Q)$, is not needed.)

Proof. Here is the case for $\rightarrow$E, where we have the determinacy of the minor premiss succedent. We use Lemma 2– if $X \vdash_3 D(P)$ then $X \vdash_3 P \vee \neg P$.

—	(1) $D(P) \vee I(P)$	Axiom
2	(2) $D(P)$	H
2	(3) $P \vee \neg P$	2 Lemma 2
4	(4) P	H
Δ	(5) $P \rightarrow Q$	Given
Δ, 4	(6) Q	4,5 $\rightarrow$E
7	(7) $\neg P$	H
Δ	(8) P	Given
Δ, 7	(9) Q	7,8, $\neg$E
2, Δ	(10) Q	3,6,9 $\vee$E
11	(11) $I(P)$	H
Δ	(12) $D(P)$	Given
Δ, 11	(13) Q	11, 12, $\neg$E
Δ	(14) Q	1,10, 13 $\vee$E

Here 'H' for 'Hypothesis' is the special case of the reflexivity axiom allowing sequent X: A at any line, $A \in X$, where $X = \{A\}$. There are a number of potential violations of neo-classical determinacy restrictions in this proof: i) line 6, if $P \in \Delta$; ii) line 9, if $\neg P \in \Delta$; iii) line 10 if $D(P) \in \Delta$ and iv) line 13, if $I(P) \in \Delta$. The third and fourth are straightforward to deal with since we have $\vdash_3 D^2(P)$ and $\vdash_3 D(I(P))$ which can be appealed to satisfy the determinacy constraints on these applications of $\vee$E and $\neg$E. For the occurrence of P as assumption at line 6 we bring in to play earlier the sub-proof captured in the sequent at line 12. Since this is readily extended to a proof of $D(\neg P)$ from Δ by simple minimax transformations a similar appeal attends to any determinacy requirement at line 9.

The case for $\neg$E is simpler, that of $\vee$E more convoluted, there are four nested applications of (neo-classical) $\vee$E but essentially the same strategy. I'll refer to the application of classical $\vee$E, $\neg$E or $\rightarrow$E justified by Proposition 2 (i.e.

where the union of all the, perhaps overlapping, assumptions for major and minor premiss also entail the determinacy of one of the succedent of one of the premisses for $\neg$E and $\rightarrow$E, and both disjuncts of the succedent of the major premiss for $\vee$E, as *derived classical* rules in the case and cite Proposition 2 when applying the derived rule.

Proposition 3: If $\Delta \vdash_3$ P and $\Delta \nvdash_3 \neg$P then $\Delta \vdash_3$ D(P). (And if $\Delta \vdash_3 \neg$P and $\Delta \nvdash_3$ P then $\Delta \vdash_3$ D(P)). Proof: Suppose for reductio, $\Delta \nvdash_3$ D(P). By negation-completeness$_1$ $\Delta \vdash_3 \neg$D(P). Hence we have the following proof:

—	(1) D(P) $\vee$ $\neg$D(P)	Axiom
2	(2) D(P)	H
Δ	(3) $\neg$D(P)	Given
Δ, 2	(4) $\neg$P	2,3 $\neg$E[3]
5	(5) $\neg$D(P)	H
5	(6) P $\leftrightarrow$ $\neg$P	5, MM
Δ	(7) P	Given
—	(8) D($\neg$D(P))	Prop. 1
Δ, 5	(9) $\neg$P	6,7, [8] $\leftrightarrow$ E
Δ	(10) $\neg$P	1,4,9 $\vee$E

contradicting $\Delta \nvdash_3 \neg$P.

Proposition 4: (primality). If $\Delta \vdash_3$ P $\vee$ Q then either $\Delta \vdash_3$ P or $\Delta \vdash_3$ Q.

Suppose $\Delta \vdash_3$ P $\vee$ Q, and yet one disjunct, say P, is not provable: $\Delta \nvdash_3$ P. Then P $\notin$ Δ and, by negation-completeness$_1$, $\Delta \vdash_3 \neg$P whilst by Proposition 3 $\Delta \vdash_3$ D(P). Neo-classically licit $\neg$E gives us P, $\Delta \vdash_3$ Q. But then $\Delta \vdash_3$ Q by derived classical $\vee$E from $\Delta \vdash_3$ P $\vee$ Q (with the sub-proof from Q being trivial, so no proof of D(Q) needed); the sub-proof from disjunct P is the derived classical leg which uses $\Delta \vdash_3$ D(P).

[3] If D(P) $\in$ Δ use $\vdash_3$ D^2(P) from Proposition 1.

We then prove a quasi-model existence theorem for (weakly) consistent sets of sentences by expanding them to our negation-complete$_1$ Δ and assigning truth values to atoms as follows:

> Atom A is true if $\Delta \vdash_3 A$, $\Delta \nvdash_3 \neg A$, false if $\Delta \vdash_3 \neg A$ $\Delta \nvdash A$, gappy otherwise.

A quasi-model or q-model for a set X is a model which assigns no member of X the value false. What we want to prove is (where T, F and G are true, false and neither in q-model M)

Proposition 5: For any q-model M generated from the consistent Henkin set Δ:

> i) ψ is T iff $\Delta \vdash_3 \psi$ and $\Delta \nvdash_3 \neg\psi$
>
> ii) ψ is F iff $\Delta \vdash_3 \neg\psi$ and $\Delta \nvdash_3 \psi$
>
> iii) ψ is G iff $\Delta \vdash_3 \psi$ and $\Delta \vdash_3 \neg\psi$

(negation-completeness$_1$ rules out the possibility of neither ψ nor its negation being provable).

Proof by induction:

Disjunction : ∨ i) L to R: If $P \vee Q$ is true in a valuation *υ*[4], one, say P is; by IH, $\Delta \vdash_3 P$ and $\Delta \nvdash_3 \neg P$. So $\Delta \vdash_3 P \vee Q$; if $\Delta \vdash_3 \neg(P \vee Q)$ then $\Delta \vdash_3 \neg P$; contradiction.

R to L: If $\Delta \vdash_3 (P \vee Q)$, $\Delta \nvdash_3 \neg(P \vee Q)$ then by Proposition 4, at least one of P, Q is provable from Δ, w.l.g. suppose P. If $\Delta \vdash_3 \neg P$ then $\Delta \nvdash_3 \neg Q$ else $\Delta \vdash_3 \neg(P \vee Q)$ Hence by negation-completeness$_1$ we have $\Delta \vdash_3 Q$ and so by IH Q is true, if P is not; hence $P \vee Q$ is true. $\nvdash_3$

∨ ii) L to R: If $P \vee Q$ is false, both P and Q are false so by IH $\Delta \vdash_3 \neg P$, $\Delta \vdash_3 \neg Q$, $\Delta \nvdash_3 P$, $\Delta \nvdash_3 Q$. Hence $\Delta \vdash_3 \neg(P \vee Q)$. If $\Delta \vdash_3 (P \vee Q)$ by primality one of $\Delta \vdash_3 P$, $\Delta \vdash_3$; contradiction.

R to L: Suppose $\Delta \vdash_3 \neg(P \vee Q)$, $\Delta \nvdash_3 (P \vee Q)$. Then $\Delta \vdash_3 \neg P$, $\vdash_3 \neg Q$, $\Delta \nvdash_3 P$, $\Delta \nvdash_3 Q$ hence by IH, both P and Q are false hence so is $(P \vee Q)$.

---

[4] For ease of writing, relativization of truth value to a valuation will be suppressed in what follows.

$\vee$ iii). If $P \vee Q$ is gappy then by since we have established the R to L truth clauses i) above we cannot have both $\Delta \vdash_3 (P \vee Q)$ and $\Delta \nvdash_3 \neg(P \vee Q)$, so either $\Delta \vdash_3 \neg(P \vee Q)$ or $\Delta \nvdash_3 (P \vee Q)$ but in the second case negation-completeness$_1$ also gives us $\Delta \vdash_3 \neg(P \vee Q)$. Similarly from R to L falsity clause ii) above $\Delta \vdash_3 \neg P \vee Q)$ hence $\Delta \vdash_3 (P \vee Q)$ and $\Delta \vdash_3 \neg(P \vee Q)$.

If $\Delta \vdash_3 (P \vee Q)$ and $\Delta \vdash_3 \neg(P \vee Q)$ then by L to R truth and falsity clauses proven in i) and ii) above, P is not true and not false.

The argument for conjunction is similar.

<u>Negation</u> $\neg$ i) Suppose $\neg P$ is true. Then P is false, by IH, $\Delta \vdash_3 \neg P$ and $\Delta \nvdash_3 P$ so $\Delta \nvdash_3 \neg\neg P$. In the other direction, if $\Delta \vdash_3 \neg P$ and $\Delta \nvdash_3 \neg\neg P$, then $\Delta \nvdash_3 P$ so by IH, P is false, $\neg P$ true.

$\neg$ ii) Very similar to i).

The clause for gappy $\neg P$ follows from i and ii as with $\vee$.

<u>The Conditional</u> $\rightarrow$ i) L to R: If $P \rightarrow Q$ is true, then either a) P is false, b) Q is true or c) P and Q are both gappy. In case a) $\Delta \vdash_3 \neg P$, $\Delta \nvdash_3 P$ by IH so $\Delta \vdash_3 P \rightarrow Q$ and $\Delta \vdash_3 D(P)$ (Proposition 3). If $\Delta \vdash_3 \neg(P \rightarrow Q)$, then $\Delta \vdash_3 P$ using $\Delta \vdash_3 D(\neg P)$ (from $\Delta \vdash_3 D(P)$) if $\neg P \in \Delta$. This contradicts, by IH, the falsity of P. Similarly with case b). In case c), by IH, $\Delta \vdash_3 P$ and $\Delta \vdash_3 \neg P$, hence $\Delta \vdash_3 I(P)$, likewise $\Delta \vdash_3 I(Q)$. From Lemma 3, we get, by two applications of $\rightarrow$I, $\vdash_3 I(P) \rightarrow (I(Q) \rightarrow (P \rightarrow Q))$. Since we have $\vdash_3 D(I(P))$ and $\vdash_3 D(I(Q))$ Proposition 2 and two applications of the derived classical $\rightarrow$E it legitimates, give us $\Delta \vdash_3 P \rightarrow Q$.

Suppose now for reductio $\Delta \vdash_3 \neg(P \rightarrow Q)$. We use Lemma 4: If $\Delta \vdash_3 \neg(P \rightarrow Q)$ then $\Delta \vdash_3 D(P) \vee D(Q)$. Proposition 4 then yields that one or other, say D(Q) is provable from Δ as well as $\Delta \vdash_3 I(Q)$. Classical $\neg$E is licit here, from Proposition 2, as we have $\vdash_3 D(I(Q))$ so $\Delta \vdash_3 \perp$ contradicting the consistency of Δ.

R to L: Suppose $\Delta \vdash_3 P \rightarrow Q$ and $\Delta \nvdash_3 \neg(P \rightarrow Q)$. Suppose for reductio that $P \rightarrow Q$ is not true. Then either a) P is true and Q is not or b) Q is false and P is not. In case a) by IH $\Delta \vdash_3 P$, $\Delta \nvdash_3 \neg P$ so $\Delta \vdash_3 D(P)$ hence by Proposition 2 we have $\Delta \vdash_3 Q$ using the derived classical $\rightarrow$E. But $\Delta \nvdash_3 \neg Q$ else $\Delta \vdash_3 \neg(P \rightarrow Q)$ so by IH, Q is true; $\bot$. In case b) similarly the determinacy of $\neg Q$ by Proposition 2 enables to use a derived classical modus tollens to get $\Delta \vdash_3 \neg P$; but $\Delta \nvdash_3 P$ contradicting the assumption that P is not false.

$\rightarrow$ ii) L to R: If $P \rightarrow Q$ is false, then P is true and Q is false. By IH $\Delta \vdash_3 P$, $\Delta \nvdash_3 \neg P$, $\Delta \vdash_3 \neg Q$, $\Delta \nvdash_3 Q$. Hence $\Delta \vdash_3 D(P)$, $\Delta \vdash_3 D(\neg Q)$ so applying Proposition 2 to the latter we get $\Delta \vdash_3 \neg(P \rightarrow Q)$ from $\Delta \vdash_3 P$ and $\Delta \vdash_3 \neg Q$ by derived classical $\rightarrow$E and $\neg$E, followed by $\neg$I. We need next to prove $\Delta \nvdash_3 (P \rightarrow Q)$. Suppose for reductio, $\Delta \vdash_3 P \rightarrow Q$. Since $\Delta \vdash_3 D(P)$, derived classical $\rightarrow$E yields $\Delta \vdash_3 Q$ contradicting $\Delta \nvdash_3 Q$.

R to L: Suppose $\Delta \vdash_3 \neg(P \rightarrow Q)$ and $\Delta \nvdash_3 P \rightarrow Q$. $\Delta \nvdash_3 \neg P$ else from $\vdash_3 \neg P \rightarrow (P \rightarrow Q)$, we get $\Delta \vdash_3 P \rightarrow Q$; by similar reasoning $\Delta \nvdash_3 Q$. We also have $\vdash_3 \neg(P \rightarrow Q) \rightarrow P$ and $\vdash_3 \neg(P \rightarrow Q) \rightarrow \neg Q$, hence $\Delta \vdash_3 P$ and $\Delta \vdash_3 \neg Q$ so by IH, P is true and Q is false, hence $P \rightarrow Q$ is false.

$\rightarrow$ iii) The clause for gappy $P \rightarrow Q$ follows from the truth and falsity clauses i and ii as with $\vee$ and $\neg$. ☐

The clauses of Proposition 5 give us our

<u>Q-Model Existence Theorem:</u> If $X \nvdash_3 \bot$ then there is a model of X in which no member is false. This follows immediately from the construction of the Henkin set and Proposition 5, the falsity clause in particular.

<u>Completeness Theorem</u>: If $X \vDash_3 \varphi$ then $X \vdash_3 \varphi$.

Proof: Suppose $X \nvdash_3 \varphi$; to prove: $X \nvDash \varphi$. So suppose for reductio $X \vDash \varphi$. Since $X \nvdash_3 \varphi$, the model existence theorem gives us a q-model M in which no member of Δ is false in M, where Δ is the Henkin set constructed from the consistent $X,\neg\varphi$. So φ is not true from which in turn it follows that not all of X can be true in M, since $X \vDash \varphi$. So at least one $P \in X$ is untrue.

Suppose there is another $Q \in X$ which is gappy in M. Then from Proposition 5 have $\Delta \vdash_3 P$, $\Delta \vdash_3 \neg P$, $\Delta \vdash_3 Q$, $\Delta \vdash_3 \neg Q$, with $P, Q \in \Delta$. But if so, $\Delta \vdash_3 \bot$ by this proof:

1	(1) P	H
Δ	(2) ¬P	Given
3	(3) D(P)	H
Δ, 1, 3	(4) ⊥	1,2 [3] ¬E
Δ,1	(5) I(P)	4, ¬I, ¬¬E
6	(6) Q	H
Δ,6	(7) I(Q)	As 1 to 5
Δ,1,6	(8) ⊥	5,7 ⊥ rule

Thus there is exactly one gappy sentence in X from which it follows that φ cannot be false, else $X \nvDash \varphi$; thus $\neg\varphi$ is gappy in M. Since it is a member of Δ we have $P \in \Delta$, $\Delta \vdash_3 \neg P$, $\neg\varphi \in \Delta$ and $\Delta \vdash_3 \varphi$. Applications of →I with vacuous antecedent give $\Delta \vdash_3 P \rightarrow \neg P$ and $\Delta \vdash_3 \neg P \rightarrow P$ hence $\Delta \vdash_3 I(P)$; similarly we have $\Delta \vdash_3 I(\varphi)$. We must have $\varphi \neq P$ since $P \in X$ but $X \nvdash_3 \varphi$; but then by a second application of the ⊥ rule, $\Delta \vdash_3 \bot$ thus reducing our supposition that $X \vDash \varphi$ to absurdity. □

There were four lemmata used above, proven as follows:

<u>Lemma 1</u>: If $\vdash_3 A \vee \neg A$ then $\vdash_3 D(A)$. Proof:

—	(1) $A \vee \neg A$	Given
2	(2) A	H
3	(3) $\neg A$	H
4	(4) $A \leftrightarrow \neg A$	H
3,4	(5) A	3,4 $\leftrightarrow$E
4	(6) A	1,2,5 $\vee$E
4	(7) $\neg A$	As 1 to 6.
4	(8) $A \wedge \neg A$	6,7 $\wedge$E
4	(9) $\neg(A \vee \neg A)$	8 MM
4	(10) $\bot$	1,9 $\neg$E
—	(11) D(A)	10 $\neg$I

Lemma 2: If $X \vdash_3 D(P)$ then $X \vdash_3 P \vee \neg P$:

1	(1) $P \wedge \neg P$	H
1	(2) P	1 $\wedge$E
1	(3) $\neg P$	1 $\wedge$E
1	(4)$\neg\neg(P \leftrightarrow \neg P)$	2,3 $\rightarrow$I $\times$2, $\wedge$I, DNI
X	(5) D(P)	Given
1,X	(6) $\bot$	4,5 $\neg$E
X	(7) $\neg(P \wedge \neg P)$	6, $\neg$I
X	(8) $P \vee \neg P$	7 MM

If $(P \wedge \neg P) \in X$ so that the application of $\neg$E at line (6) is illicit then the proof of LEM from X is even more direct, by $\wedge$E and $\vee$I.

Lemma 3: $I(P), I(Q) \vdash_3 P \rightarrow Q$

1	(1) $P \leftrightarrow \neg P$	H
2	(2) P	H
1,2	(3) $\neg P$	1,2 $\leftrightarrow$E
1,2	(4) $P \wedge \neg P$	2,3 $\wedge$I
1,2	(5) $Q \vee \neg Q$	4 MM
6	(6) Q	H
7	(7) $\neg Q$	H
8	(8) $Q \leftrightarrow \neg Q$	H
7,8	(9) Q	7, 8 $\leftrightarrow$E
1,2,8	(10) Q	5,6,9 $\vee$E
1,8	(11) $P \rightarrow Q$	10 $\rightarrow$I

Lemma 4: If $\Delta \vdash_3 \neg(P \rightarrow Q)$ then $\Delta \vdash_3 D(P) \vee D(Q)$.

Proof. We have $\vdash_3 D(P) \vee I(P)$, $\vdash_3 D(Q) \vee I(Q)$. A nested $\vee$E establishes $\Delta \vdash_3 D(P) \vee D(Q)$. The first sub-proof from assumption D(P) is immediate, the second sub-proof from I(P) contains a nested $\vee$E major premiss $D(Q) \vee I(Q)$, again the left disjunct sub-proof is immediate. So the key case is where we assume I(P) and I(Q). We use Lemma 3 and the final sub-proof consists in application of $\neg$E of the following form:

1	(1) $I(P)$	H
2	(2) $I(Q)$	H
1,2	(3) $P \rightarrow Q$	1,2 Lemma 3
Δ	(4) $\neg(P \rightarrow Q)$	Given
Δ, 1,2	(5) $D(P) \vee D(Q)$	3,4, $\neg$E

If I(P) or I(Q) are members of Δ then we use $\vdash_3 D(I(P))$ or $\vdash_3 D(I(Q))$ to satisfy the determinacy requirements.

§III Robust Contraction-Freedom for NC_3

The system NC_3 blocks not only the usual proofs of the Liar paradox and the Russell paradox but also the standard proofs of the Curry paradox (see Weir 2013). But Greg Restall (1993) points out that naïve theorists must not only worry about Curry-style paradoxes afflicting $\rightarrow$. We are in trouble if, in our language, we can define an operator $>$ satisfying

1. $>$I:

X	(1)	$A \rightarrow B$	Given
X	(2)	$A > B$	1 $>$I

2: $>$ Contraction:

X	(1)	$A > (A > B)$	Given
X	(2)	$A > B$	1 Contr.

3: $>$E:

X	(1)	$A > B$	Given
Y	(2)	A	Given
X,Y	(4)	B	1, 2 $>$E

However there is no 'contracting implication' $>$ in the three-valued system NC_3, no operator satisfying Rules 1, 2 and 3. For let A and B be atoms with A gappy, B false in a valuation v. Hence $A \rightarrow B$ is gappy in v, so by upwards falsity-preservation on $>$I (with $X = \{A \rightarrow B\}$) $A > B$ cannot be false. But by the soundness of $>$E, ($X = \{ A > B\}$, $Y = \{A\}$) $A > B$ cannot be true either, since B is false and A gappy. Now by $>$I, from $A \rightarrow (A > B)$ ($X = \{ A \rightarrow (A > B)\}$) we can conclude $A > (A > B)$. Since A and $A > B$ are both gappy, $A \rightarrow (A > B)$ is true in v according to the three-valued semantics; hence so is $A > (A > B)$. But then Rule 2 $>$Contr. is unsound, taking us from a true premiss to a gappy conclusion (more fully from a $\vDash_3$ correct sequent $A > (A > B) : A > (A > B)$ to an incorrect one $A > (A > B) : A > B$.) The package of Rules 1, 2 and 3 cannot be held together neo-classically.

We are not out of the woods yet, though. As Restall notes, if we have an operator satisfying

3:2 >Contraction:

X	(1)	$A > (A > (A > B))$	Given
X	(2)	$A > (A > B)$	3:2 >Contr.

(he calls this 3-2 contraction, with our original Rule 2, 2:1 contraction) then we are right back in trouble, if we also still have >I and >E. More generally define $A >_1 B$ by $A > B$, and $A >_{n+1} B$ by $A > (A >_n B)$. Then $n+1{:}n$ contraction also trivialises naïve theories. For example, we are faced with the following proof of absurdity and hence triviality from the instance D of naïve comprehension (a similar problem arises for naïve truth theory from iterated Curry sentences):

D: $\forall x(x \in d \leftrightarrow (x \in x >_n \bot))$

—	(1)	$d \in d \rightarrow (d \in d >_n \bot)$	Comp $\forall$E $\wedge$E
—	(2)	$d \in d > (d \in d >_n \bot)$	1 >I
—	(3)	$(d \in d >_n \bot)$	2 n+1:n Contr.
—	(4)	$(d \in d >_n \bot) \rightarrow d \in d$	Comp $\forall$E $\wedge$E
—	(5)	$d \in d$	3, 4 $\rightarrow$E
—	(6)	$\bot$	3,6 $>_n$ E

$>_n$E is a derived rule of the {>I, >E, $n+1{:}n$ Contr} system, by iterated applications of >E.

Fortunately, there can be no such $n+1{:}n$ contracting operator in our system as an argument of Restall's shows. He proves by induction that if we have $n+1{:}n$ contraction we have $m{:}n$ contraction for any $m > n$.[5] In particular, if we have $n+1{:}n$ contraction then we also must have $2n{:}n$ contraction. As noted we also have $>_n$E by iterated applications of >E. Finally, we have $>_n$I. For we have $\vdash_3 (A \rightarrow B) \rightarrow (A >_n B)$ by:

[5] The inductive step uses: $A >_{m+1} B = A >_{n+1} (A >_{m-n} B)$ from which formulae $n+1{:}n$ contraction yields $A >_n (A >_{m-n} B) = (A >_m B)$; IH tells us this contracts to $(A >_n B)$.

1	(1)	$(A \to B)$	H
1	(2)	$(A > B)$	1 >I
1, A	(3)	$(A > B)$	2 Exp.[6]
1	(4)	$A \to (A > B)$	3 →I
1	(5)	$A > (A > B)$	4 >I
1, A	(6)	$A > (A > B)$	6 Exp.
…	…	…	…
1	($3n$–1)	$(A >_n B)$	…
—	($3n$)	$(A \to B) \to (A >_n B)$	$3n$–1, →I

Given the theorem ($3n$), if we have $X \vdash_3 A \to B$ then a neo-classically correct →E yields $X \vdash (A >_n B)$. Thus $>_n$ is a 2:1 contraction operator, but we know that is impossible in the system. □

[6] For 'Expansion', a derived rule of the system allowing us to expand any assumption set. Any occurrences can be eliminated by suitably expanded applications of the Reflexivity axiom schema earlier in the proof.

<u>§IV The Infinite-Valued System NC_I</u>

However although NC_3 blocks the proofs of paradox from naïve truth axioms which appeal to contraction principles, this does not show that naïve truth theory is consistent in the neo-classical system; perhaps there are proofs of a different structure of absurdity from axioms of the unrestricted naïve truth schema. After all, the three-valued system semantics is incapable of handling iterated Curry Liars of the form

$$C: T(\langle C \rangle) \to (T(\langle C \rangle) \to \bot))$$

with 'T' a truth predicate and angled brackets representing some way of coding sentences. In a naïve theory C, that is $T(\langle C \rangle) \to (T(\langle C \rangle) \to \bot))$, must take the same value as $T(\langle C \rangle)$, but this is not possible in the three-valued system since if $T(\langle C \rangle)$ takes value 1, C takes value ½, if $T(\langle C \rangle)$ takes value ½ C takes value 1 and if $T(\langle C \rangle)$ takes value 0, C takes value 1. The multi-valued semantics, if it is to cope with Curry sentences, must also be able to handle higher-order indeterminacy, and the three-valued system cannot do so since all determinacy claims D(P) are themselves determinate.

The obvious move is to let the valuation space be the interval [0,1], either the rational, the real, or the surreal interval. We will focus on the most widely used case, the real interval [0,1], generalising our three-valued account of the logical operators to the Łukasiewiczian:

$$v(A \wedge B) = \min[v(A), v(B)]$$

$$v(A \vee B) = \max[v(A), v(B)]$$

$$v(\neg A) = 1 - v(A)$$

$$v(A \to B) = 1 \text{ when } v(A) \leq v(B),\ 1 - [v(A) - v(B)] \text{ otherwise.}$$

But how to define entailment in the infinite-valued system? To do this (see Weir, 2013 for motivation) let us introduce the notion of the *Value Sum*, relative to a valuation v, of a set of sentences X; we write this as $\Sigma_v(X)$. We leave $\Sigma_v(X)$ undefined if X has infinitely many members which have non-zero value in valuation v; otherwise $\Sigma_v(X)$ is just the arithmetic sum of the finitely many non-zero values of its members in υ. We then define for the infinitary system NC_I the entailment relation $X \vDash A$:

$$X \vDash A \text{ iff either } \Sigma_\upsilon\overline{X} \text{ is undefined or } \Sigma_\upsilon\overline{X} \geq \upsilon(\neg A).$$

where $\overline{X} = \{\neg B: B \in X\}$.

In defining the derivability for NC_I we take over the same set of rules of proof we used to characterise the basic system NC_{3^-} and its derivability relation $\vdash_{3^-}$ except that some additional qualifications are needed in the determinacy clauses, as specified in the soundness proofs below. The proof system $\vdash$ is a slight restriction of that for $\vdash_{3^-}$ determined by these rules.

The single premiss rules are unchanged and the proof of soundness is straightforward. Thus for classical ¬I:

X, ¬P	(1) ⊥	Given
X	(2) P	1, ¬I

the proof is as follows. We are given, in the non-trivial case: $\Sigma_\upsilon\overline{X} + \upsilon P \geq 1$; subtracting υP from both sides yields the required:

$$\Sigma_\upsilon\overline{X} \geq (1 - \upsilon P).$$

Similarly the Minimax rules are retained unchanged; the succedent formula of such a rule always has the same value as the succedent formula of the input sequent, it follows that neo-classically correctness (the set of antecedents bears ⊨ to the succedent formula) is transmitted from premiss sequent to conclusion sequent, as required.

The trickier cases are the multiple premiss rules, especially the determinacy-restricted E rules. Starting with the simplest case of ∧I

X	(1) P	Given
X	(2) Q	Given
X,Y	(3) P ∧ Q	1,2 ∧I

Given: a) $\Sigma_\upsilon\overline{X} \geq (1 - \upsilon P)$; b) $\Sigma_\upsilon\overline{Y} \geq (1 - \upsilon Q)$.

To prove: $\Sigma_v\overline{X \cup Y} \geq 1 - \min[vP, vQ]$

Suppose without loss of generality that $vP = \min[vP, vQ]$. The result follows by substituting this for vP in a) and from the fact that $\Sigma_v\overline{X \cup Y} \geq \Sigma_v\overline{X}$.

The conditional introduction rule $\rightarrow$I, which has no determinacy restrictions, is unchanged:

X, P	(1) Q	Given
X	(2) $P \rightarrow Q$	1, $\rightarrow$I

To prove the soundness of this rule assume line 1 is a correct entailment that is, for any v, either $\Sigma_v\overline{X \cup P}$ is undefined or

$$\text{a) } \Sigma_v\overline{X} + v(\neg P) \geq v(\neg Q),$$

In every case where the set of negated antecedents (here $\Sigma_v\overline{X}$) for one of the rule premisses is undefined, the correctness of the conclusion sequent trivially follows. So we will consider in the remainder, only the non-trivial cases where every antecedent value sum is defined and thus a finite sum. Thus in this case to prove (for each such v):

$$\Sigma_v\overline{X} \geq 1 - v(P \rightarrow Q)$$

If $vP \leq vQ$ then $v(P \rightarrow Q) = 1$ so suppose $vP > vQ$ and hence $v(P \rightarrow Q) = 1 - (vP - vQ)$. Hence we need to prove:

$$\Sigma_v\overline{X} \geq vP - vQ.$$

which we get by subtracting $(1 - vP)$ from each side of a).

Now for the determinacy-restricted rules starting with ¬E:

X	(1) P	Given
Y	(2) ¬P	Given
Z_i, $i \in I$	(3.i) D(R_i)	Given, $\forall R_i \in (X \cap Y)$
$X, Y, \bigcup_{i \in I} Z_i$	(5) C	1,2,3 [4.i, $i \in I$], ∨E

In the proof theory for the infinite-valued Łukasiewiczian system NC_I we add the additional clause that the Z_i are pairwise disjoint and each is disjoint from X and from Y. We have:

a) $\Sigma_v\overline{X} \geq (1 - v\text{P})$; b) $\Sigma_v\overline{Y} \geq v\text{P}$; c.$i$) $\Sigma_v\overline{Z_i} \geq (1 - v[\text{D}(R_i)])$ (for all $i \in$ I).

We take the case where C is ⊥ since if soundness holds for a case with consequent value 0 it holds for all cases. To prove, (given that each of the Z_i are disjoint both from X and Y and from each other and that all the antecedent sums are defined) for every v:

α) $\Sigma_v\overline{X \cup Y} + \Sigma_v\overline{\bigcup_{i \in I} Z_i} \geq 1.$

First re-arrange a) as:

d) $\Sigma_v\overline{X - Y} + \Sigma_v\overline{X \cap Y} \geq (1 - v\text{P})$

likewise b) as

e) $\Sigma_v\overline{Y - X} + \Sigma_v\overline{Y \cap X} \geq v\text{P}$

Adding d) and e) we get:

f) $\Sigma_v\overline{X - Y} + \Sigma_v\overline{Y - X} + 2\times\Sigma_v\overline{X \cap Y} \geq 1$

where what we want to prove is, after re-arranging:

g) $\Sigma_v\overline{X - Y} + \Sigma_v\overline{Y - X} + \Sigma_v\overline{X \cap Y} + \Sigma_v\overline{\bigcup_{i \in I} Z_i} \geq 1$

If there are no Z_i, hence no $X \cap Y$, then this is immediate from f). Suppose then $X \cap Y, \neq \varnothing$. By our assumption about antecedent sums, there are only finitely many Z_i with $\overline{Z_i} \neq 0$. Without loss of generality, let these be Z_1, ... Z_{n+m} and consider R_1, ... R_n, R_{n+1}, ... R_{n+m} with $v\text{D}(R_i) \leq vR_i$, $i \leq n$ $v\text{D}(R_j) > vR_j$, $n < j \leq n+m$.

Since $\Sigma_\upsilon\overline{Z_\iota} \geq (1 - \upsilon D(R_i))$, $\Sigma_\upsilon\overline{Z_k} \geq (1 - \upsilon R_k))$ for $k \leq n$. Hence from f), we get (setting aside the null $\overline{Z}_i$):

$\Sigma_\upsilon\overline{X - Y} + \Sigma_\upsilon\overline{Y - X} + \upsilon\neg R_1, \ldots \upsilon\neg R_n + \Sigma_\upsilon\overline{\bigcup_{\iota\leq n} Z_\iota} + 2\times\upsilon\neg R_{n+1}, + \ldots 2\times\upsilon\neg R_{n+m} \geq 1.$

If there are no R_{n+k} with $\upsilon D(R_{n+k}) > \upsilon R_{n+k}$, g) is satisfied and we are done. If there are, note that $\upsilon D(R_j) > \upsilon R_j$ only when $\upsilon R_j < 0.5$, where the slope of the D curve is $1 - 2x$, and hence in fact only when $x < 1/3$. So if there are any $\upsilon D(R_j) > \upsilon R_j$, $n < j \leq n+m$ then $2\times\upsilon\neg R_j > 4/3$ and all other terms in h) are redundant in terms of securing the weak inequality required.

Next $\rightarrow$E:

X	(1) P	Given
Y	(2) $P \rightarrow Q$	Given
Z_i	(3.i) $D(R_i)$	$\forall R_i \in X \cap Y$
$X, Y, \bigcup_{i\in I} Z_i$	(4) Q	1,2 [3.i] $\rightarrow$E

where the determinacy restrictions are that the Z_i are disjoint from $X \cup Y$ and each other. We have in the non-trivial case where none of the $\Sigma_\upsilon\overline{X}$, $\Sigma_\upsilon\overline{Y}$, $\Sigma_\upsilon\overline{Z}_i$, $i \in I$ are undefined:

a) $\Sigma_\upsilon\overline{X} \geq (1 - \upsilon P)$; b) $\Sigma_\upsilon\overline{Y} \geq 1 - \upsilon(P \rightarrow Q)$; c.i) $\Sigma_\upsilon\overline{Z}_i \geq (1 - \upsilon[D(R_i)])$; (for all $i \in I$).

To prove: β): $\Sigma_\upsilon\overline{X \cup Y} + \Sigma_\upsilon\overline{\bigcup_{\iota\in I} Z_\iota} \geq 1 - \upsilon Q$.

Suppose $\upsilon P \leq \upsilon Q$. Then from a) $\Sigma_\upsilon\overline{X} \geq (1 - \upsilon Q)$ and we are done.. So we need only consider the case where $\upsilon P > \upsilon Q$, that is where $\upsilon(P \rightarrow Q) = 1 - \upsilon P + \upsilon Q$. Adding a) and b) together, given that identity, we get:

c) $\Sigma_\upsilon\overline{X - Y} + \Sigma_\upsilon\overline{Y - X} + 2\times\Sigma_\upsilon\overline{X \cap Y} \geq 1 - \upsilon Q$.

Re-arranging β gives:

$$\Sigma_\upsilon\overline{X - Y} + \Sigma_\upsilon\overline{Y - X} + \Sigma_\upsilon\overline{X \cap Y} + \Sigma_\upsilon\overline{\bigcup_{\iota\in I} Z_\iota} \geq 1 - \upsilon Q$$

which we get from c) by the same argument as in the ¬E case.

Finally, using disjunctive syllogism as the disjunction elimination rule to make the proof here simpler, that is:

X	(1) $P \vee Q$	Given
Y	(2) $\neg Q$	Given
Z_i	(3.i) $D(R_i)$	$\forall R_i \in X \cap Y$
$X, Y, \bigcup_{i \in I} Z_i$	(4) P	1,2 [3.i] DS

the Z_i being disjoint from $X \cup Y$ and each other. We have in the non-trivial case :

a) $\Sigma_v\overline{X} \geq 1 - \max(vP, vQ)$; b) $\Sigma_v\overline{Y} \geq vQ$; c.i) $\Sigma_v\overline{Z_i} \geq (1 - v[D(R_i)])$; (for all $i \in I$).

To prove: γ) $\Sigma_v\overline{X \cup Y \cup \bigcup_{i \in I} Z_i} \geq 1 - vP$

If $vQ = \max(vP, vQ)$ then a) becomes

c) $\Sigma_v\overline{X} \geq 1 - vQ$. Adding b) and c) gives us

d) $\Sigma_v\overline{X - Y} + \Sigma_v\overline{Y - X} + 2\times\Sigma_v\overline{X \cap Y} \geq 1$. Applying the same argument as in the ¬E case we then have $\Sigma_v\overline{X \cup Y \cup \bigcup_{i \in I} Z_i} \geq 1 \geq 1 - vP$.

If, on the other hand, $vP = \max(vP, vQ)$ then a) becomes

e) $\Sigma_v\overline{X} \geq 1 - vP$. Adding b) and e) gives us

f) $\Sigma_v\overline{X - Y} + \Sigma_v\overline{Y - X} + 2\times\Sigma_v\overline{X \cap Y} \geq 1 + (vQ - vP)$. Applying the same argument as in the ¬E case we then have $\Sigma_v\overline{X \cup Y \cup \bigcup_{i \in I} Z_i} \geq 1 + (vQ - vP) \geq 1 - vP$.

The additional complications in the determinacy restrictions, that the Z_i assumptions used to prove the determinacy sentences $D(R_i)$ are pairwise disjoint and each is disjoint from the other premisses, mean that classical recapture (Weir, 2013) cannot be achieved quite so simply.[7] We cannot assume, as we did in NC_3, that we can simply add the determination D(P) for any assumption P which occurs both as a major and a minor premiss of →E, ¬E or ∨E (or DS), and carry on the proof as before with the extra assumption; for P may also occur in

[7] That is, where $\vdash_C$ is classical derivability and we have $\Delta \vdash_C A$ then there is a set Δ^* such that $\Delta^* \vdash_3 A$, where Δ^* is a superset of Δ expanding by adding sentences of the form D(φ).

other premisses in the application of the elimination rule. However we can always get round this problem by adding, instead of D(P), D(P) ∧ D(P) ∧ D(P) .... with enough iterations of D(P) to give us a self-conjunction which occurs nowhere thus far in the proof.

The semantics given above for ⊨ can not only be applied, scaling downwards, to the rational interval [0,1] but also, powering upwards, to any surreal interval [0,1]. (If we wish our valuation space and language to be members of other things, e.g. functions from the one to the other, to take as other theory of classes ZFC plus the axiom of inaccessibles:- choose an inaccessible cardinal θ_α and reserve 'set' and 'ordinal' for classes < θ_α in size, with a special membership relation defined by $[x \in y \wedge \mathrm{Set}(x)]$.)

There is one complication in the latter case. The clauses for conjunction and disjunction interpret them by min (greatest lower bound) and max (least upper bound) respectively. But the completeness principle for the reals does not hold for the surreals: a bounded set of surreal numbers need not have a least upper, or greatest lower, bound. So define more complex minimisation and maximisation functions MIN, MAX in terms of a choice function C over all subsets of [0,1]

MIN(X) = the g.l.b. of X, if it exists; otherwise

= C({x: x is a lower bound of X}).

MAX(1 – MIN({1 – x: $x \in$ X}))

We now need to show that ⊢ is sound in the surreal-valued semantics. For the rules other than the minimax rules, the argument is straightforward. For each such rule, in any valuation v in which the antecedent X of the conclusion sequent contains infinitely many sentences which are untrue, that is take values less than 1 in v, v does not constitute a counterexample to the correctness of the conclusion sequent. That is X ⊭ A, where A is the succedent since $\Sigma_v\overline{X}$ is undefined. If, on the other hand, there are only finitely many sentences in the conclusion antecedent which are untrue in the valuation then, by inspection of the rules, we can see that it is also the case that in each premiss sequent there are only finitely many untruths. But this means that there will always be greatest lower bounds (and least upper bounds) for the antecedent sets and so the same arguments as in the continuum-valued case establish soundness.

The other cases are the minimax rules. As before, soundness follows from the fact that the succedent of the conclusion of any such rule always has the same value in every valuation as the succedent of the premiss. This can be checked by inspection, noting the definition of MAX in terms of MIN which ensures that De Morgan duality holds. The demonstration of robust contraction-freedom in §III is specifically tailored to the three-valued system NC3. However a more general argument (see Weir, 2013, §IV) shows that NC_I is also contraction-free by consideration of any language for NC systems which includes a truth-constant which takes value ½ in every model.

V The Infinitary Logic NC_∞

Although one can implement the neo-classical notion of entailment in predicate logic fairly easily, by taking over standard ∀I, ∀E, and ∃I and amending ∃E with a determinacy constraint parallel to that of ∨E (roughly any assumptions common to both the major and minor premiss sequents must be determinate) it was argued in Weir (2013) that the neo-classical notion of entailment is best expressed by an infinitary conditional which permits infinitely many antecedents, a generalisation of right-associative binary conditionals such as $(A_1 \rightarrow (A_2 \ldots \rightarrow A_n)\ldots) \rightarrow B)$. So we must now consider an extension of the infinite-valued semantical systems $\vdash$ and $\vDash$ to infinitary languages, where derivability and consequence will be expressed by expanded notions $\vdash_\infty$ and $\vDash_\infty$.

A standard infinitary language (see Karp, 1964, Dickmann 1975) is designated in some such way as $L_{\kappa,\lambda}$ with κ a regular infinite cardinal which is a strict upper bound on the cardinality of conjunctions and disjunctions (considered as set-theoretic objects) and $\lambda \leq \kappa$ is the strict upper bound on the length of variable strings allowed in quantifier blocks of the form $\forall x_1, \ldots x_\gamma, \ldots$ or $\exists x_1, \ldots x_\gamma, \ldots$. However we wish to simplify by dispensing with quantifiers in favour of infinitary regular conjunctions and disjunctions. In this way the neo-classical logic generalises pretty immediately (with a qualification re distributivity noted in the footnote below) to the infinitary case. Generalisation is now to be effected by certain regular infinitary conjunctions and disjunctions, so we work only with languages of the form L_κ taking the 'domain' of the language just to be the set of referents of the singular terms. (I will require also that κ is inaccessible in order that the cardinality of the languages is suitable for the purposes at hand.[8]) Let us say that the subscript κ gives the *index* of the language. For κ regular, it also gives the depth of the language, the least upper bound on the depth of wffs in the language. Here depth is the recursively defined degree of wff complexity, atoms being of depth zero, a conjunction of the wffs in X having depth the least upper bound of the depths of the conjuncts in X and so on.

We can suppose that the atomic wffs are finite expression strings defined just as for conventional first-order languages. The 'substitution class' of singular terms determines the size of models; if we wish to use regular infinitary conjunction and disjunction to generalise over the entire domain, the cardinality $\|ST\|$ of the set of singular terms has to be fixed as of (infinite) cardinality $< \kappa$. The index of the language (and given only finitely many predicates, $\|ST\| < \kappa$ will fix the size of the language as κ also, for inaccessible κ). Since the size of the language is greater than the size of the set of singular terms, and it is through conjunctions and disjunctions of these that we express generality, this means that these small languages cannot even express their own syntax. However self-referentiality can be effected in a 'brute' fashion, as we will see by considering restrictions of these languages to the subset of formulae where each infinitely long wff is of size $< \|ST\|$.

The wffs of the language can be defined in the usual way as the inductive closure of the atoms under the operations of forming negations, and infinitary conjunction and disjunction which will be represented in the metalanguage in

[8] Standard infinitary propositional logic with 'ordinary' distributive laws is complete for L_κ, κ inaccessible. See Karp (1964) p. 52. These distributivity laws, then are added to the neo-classical 'minimax' principles when the logic goes transfinite.

such ways as $\wedge(A_{\alpha\ \alpha<\beta})$ and $\vee(A_{\alpha\ \alpha<\beta})$, where the ordinal β indexes the immediate constituents of the sentences. We can think of these sentences as pairs, $\langle\wedge, X\rangle$ and $\langle\vee, X\rangle$ with X a set of wffs < κ in size (the subscripting "α α < β" and so in is metalinguistic bookkeeping). Likewise negations are pairs of the form $\langle\neg, A\rangle$. However I will generally lapse from this notation into bracket notation.

To define the surrogates for quantification in our 'variable-free' language L_κ, it is useful to have to hand an auxiliary language L^*_κ, formed in the same way as L_κ but adding $\|ST\|$ new singular terms. An infinitary conjunction $\wedge(A_{\alpha\ \alpha<\beta})$ is a universal quantification iff

> there is a sentence φ of L^*_κ containing a 'new' singular term t such that each conjunct A_α results from φ by uniformly replacing t with a term in ST and every term u ∈ ST features in a conjunct in this fashion.

We can think of a formula φ of L^*_κ with a designated new term t as playing the role of an open sentence φx, t playing the role of free variable x. So although variables are no part of the language, we can abbreviate infinitary conjunctions and disjunctions in a slang form using variables. Thus $\forall x\varphi x$ is a metatheoretic abbreviation of a regular conjunction $\wedge(A_{\alpha\ \alpha<\beta})$ as above. Infinitary generalisations of the sentential rules ∧I, ∧E, ∨I and ∨E then perform the function of the (unfree) quantifier rules and the Łukasiewicz/Kleene account of the meaning of these operators generalises straightforwardly to the infinitary case. Models for these languages are essentially domains such that there is a function from the singular terms *onto* the domain, with predicates interpreted in the usual fashion. Proofs are strings < κ in length, κ the index of the language, built up from axioms by application of the sentential inference rules.

Thus far the language contains only the logical operators of negation and infinitary conjunction and disjunction but we also wish to introduce an infinitary conditional $[A_i]_{i\in I} \rightarrow B$, a triple whose first term is →, second term the (unordered set) of the A_i (of size less than the index of the language) and whose last term is B. It is an infinitary generalisation of a right-bracketing conditional of the form $A \to (B \to (C \to D))$ except that, as with conjunction and disjunction, we do not bother with the ordering of the antecedents.

We then keep the same definition of entailment as for $\vDash_I$ namely

$$X \vDash_\infty A \text{ iff } \Sigma_v \overline{X} \geq 1 - vB.$$

The truth conditions for infinitary conjunction and disjunction are the obvious generalisations of the binary case: the value of a conjunction is the minimum of the values of its conjuncts, that of a disjunction the maximum of the values of its disjuncts. For the infinitary conditional, we attempt to match the notion of logical consequence[9] by giving the conditional the truth-conditions:

$$v([A_i]_{i\in I} \to B) = 1 \text{ if } \Sigma_v \overline{A_{\iota\in I}} \text{ is undefined or } \Sigma_v \overline{A_{\iota\in I}} \geq 1 - vB\text{; otherwise}$$

$$v([A_i]_{i\in I} \to B) = (\Sigma_v \overline{A_{\iota\in I}} + vB).$$

We need then to augment the proof theory of $\vdash_I$ to the infinitary system $\vdash_\infty$ by adding infinitary versions of the $\wedge$ and $\vee$ rules. The infinitary generalisations of $\wedge$E and $\vee$I are obvious. For the other two rules they are $\wedge$I:

X_α	(1.α) A_α	Given for all $\alpha < \beta$
$\bigcup_{\alpha<\beta} X_\alpha$	(2) $\wedge A_{\alpha\ \alpha < \beta}$	1.α $\alpha<\beta$ $\wedge$I

whilst $\vee$E is:

X	(1) $\vee A_{\alpha\ \alpha < \beta}$	Given
Y_α, A_α	(2.α) C	Given, $\alpha < \beta$
X, $\bigcup_{\alpha<\beta} Y_\alpha$	(3) C	1, 2.α, $\alpha < \beta$ $\vee$E

[9] Actually a closer match would use a multi-valued notion of consequence in which we define intermediate degrees of consequence, in the cases where consequence does not determinately obtain, by the drop from the minimum premiss value to the conclusion value.

where $X \cap \bigcup_{\alpha<\beta} Y_\alpha = \varnothing$.

The rules for the infinitary → are:

→I:

$X, A_{i\ i \in I}$	(1) B	Given
X	(2) $[A_i]_{i\in I} \rightarrow B$	1 →I

→E:

X_i	(1.i) A_i	Given, $i \in I$
Y	(2) $[A_i]_{i\in I} \rightarrow B$	Given
Z_j	(3.j) D(R)	$\forall R_j$, $R_j \in \bigcup_{i\in I} X_i \cap Y$, or $R_j \in X_k \cap X_l$, $k \neq l$.
$\bigcup_{i\in I} X_i\ Y, \bigcup_{j\in J} Z_j$	(4) B	1.i $i \in I$, 2, [3.j, $j \in J$], →E

Where $(\bigcup_{i\in I} X_i \cup Y) \cap \bigcup_{j\in J} Z_j = \varnothing$ and $X_i \cap X_j =$ for $i \neq j$.

The soundness steps in the soundness proof are straightforward generalisations of the finitary case and we again skip over the trivial cases where an antecedent Value Sum is undefined. Thus the cases for the conditional are as follows.

Soundness for →I: We are given

a) $\Sigma_v\overline{X} + \Sigma_v \overline{A_{\iota\in I}} \geq 1 - vB$

and have to prove:

δ): $\Sigma_v\overline{X} \geq 1 - v([A_i]_{i\in I} \rightarrow B)$

Now if $\Sigma_v \overline{A_{\iota\in I}} \geq 1 - vB$ then $v([A_i]_{i\in I} \rightarrow B) = 1$ and we are done. So we suppose that $\Sigma_v \overline{A_{\iota\in I}} < 1 - vB$ in which case

i) $1 - v([A_i]_{i\in I} \rightarrow B) = 1 - (\Sigma_v \overline{A_{\iota\in I}} + vB)$.

so we need to prove

b) $\Sigma_v\overline{X} \geq 1 - (\Sigma_v \overline{A_{\iota\epsilon I}} + vB)$.

which we get by rearranging a). □

→E:

We are given, from the correctness of the premiss sequents:

a.i) $\Sigma_v\overline{X_\iota} \geq 1 - vA_i$; for all i;

b) $\Sigma_v\overline{Y} \geq 1 - v([A_i]_{i\in I} \to B)$;

c.j) $\Sigma_v\overline{Z_J} \geq 1 - vD(R_j)$, for all i.

To prove:

ε) $\Sigma_v\overline{\cup X_{i\in I} \cup Y} + \Sigma_v\overline{\bigcup_{J\in J} Z_J} \geq 1 - vB$.

The separation of the X and Y terms from the Z terms is justified because the latter sets are required to be disjoint from the major and minor assumption sets, the X_i and Y.

Proof: If $v([A_i]_{i\in I} \to B) = 1$ then $\Sigma_v \overline{A_{i\in I}} \geq 1 - vB$ so, since the X_i are pairwise disjoint, we have from the a.i that $\Sigma_v\overline{\bigcup X_{i\in I}}$, is greater than $\Sigma_v \overline{A_{i\in I}} \geq 1 - vB$, as required. (Note again that we are passing over the trivial case where $\overline{\bigcup X_{i\in I}}$ contains infinitely many non-zero terms and is undefined from which the result is immediate hence we can appeal to the ordinary monotonicity of addition.)

If $v([A_i]_{i\in I} \to B) \neq 1$ then $v([A_i]_{i\in I} \to B) = (\Sigma_v \overline{A_{\iota\epsilon I}} + vB)$. In the non-trivial case there are only finitely many (pairwise disjoint) X$i \in I^* \subseteq I$ each with finitely many non-zero terms and adding the equations a.i $i \in I^*$ together we get:

d) $\Sigma_v\overline{\cup X_{i\in I^*}} \geq (\Sigma_v \overline{A_{\iota\epsilon I}}$

Substituting $v([A_i]_{i\in I} \to B) = (\Sigma_v \overline{A_{\iota\epsilon I}} + vB)$ into b) and adding this to d) yields

$\Sigma_v\overline{X_1 - Y} + \Sigma_v\overline{Y - X_1} + 2\times\Sigma_v\overline{X_1 \cap Y}$

$+ \Sigma_v\overline{X_2 - Y} + \Sigma_v\overline{Y - X_2} + 2\times\Sigma_v\overline{X_2 \cap Y}$

$+ \Sigma_v\overline{X_{i\in I^*} - Y} + \Sigma_v\overline{Y - X_{i\in I^*}} + 2\times\Sigma_v\overline{X_i \cap Y} \geq 1 - vB$

going through all the finitely many non-zero-valued X_i. The same argument used with regard to the determinacy premisses in the finitary cases yields

$$\Sigma_{\upsilon}\overline{\cup X_{i \in I^*} \cup Y} + \Sigma_{\upsilon}\overline{\bigcup_{J \in J} Z_J} \geq 1 - \upsilon B.$$

and hence ε.

Most of the minimax rules generalise straightforwardly to the infinitary case. (The distributivity rules are an exception: if we wish to extend completeness results for standard languages (and standard classical semantics) any significant way into the transfinite one needs extra rules such as the 'ordinary' or 'Chang' distributivity laws (Karp, 1964 p. 41, Dickmann, 1975, p. 421).

VI Naïve Truth Theory and Neo-Classical Logic

We can prove, even from within a conventional meta-theoretic framework, such as a suitably strong set theory (ZFCI, I being the axiom schema of inaccessibles is the obvious candidate) in classical logic, that NC_∞ is a framework in which naïve truth theory is consistent, one indeed which allows for the expression of iterated liars of the form D^{alpha}: $\neg Def^{\alpha}(T\langle D^{alpha}\rangle)$ which say very roughly, 'I am not definitely true to degree α', where α can extend into the transfinite. However the self-referentiality is not effected by arithmetizing (or rather mathematicizing, via set theory for example) the syntax for as we have seen none of our infinitary languages can generalise over domains as large as the language itself.

There is, nevertheless, another more brute way to enable infinitary languages to refer to all their formulae.[10] Let the size of the set of singular terms (and hence of atoms) of language L* be some inaccessible κ. And consider now the sub-language L of L* in which κ is strict upper bound on the size of conjunctions, disjunctions and conditionals. L is itself of cardinality κ, hence

[10] Hartry Field shows, using the Brouwer fixed point theorem, that one can give an interpretation of the truth predicate in quantifier-free continuum-valued Łukasiewiczian logic in which the naïve interderivabilities come out as true (2008, Chapter 4, Section 2 and Appendix). The result does not apply to languages with infinitely long sentences, as envisaged here.

there will be models in which each sentence in L is named by a singular term of L*. The idea, though, is to let truth be directly applicable not to sentences but to some other class of truth-bearers, *statements* I will call them, with a sentence S true in an indirect sense if the statement it *expresses* is true.

We suppose, then, that for our language L there are κ-many names, perhaps structural-descriptive names, for each sentence in the language, each name designating the same sentence in each model. In addition to these and any other 'empirical' or 'non-semantic' singular terms there is another κ-sized special class of names which, in each model, name our truth-bearers, the κ-many statement. We can suppose that each statement name rigidly designates the same statement in each model, though this supposition is not essential to the truth theory. Each admissible model thus includes a domain with a subset containing all the sentences of the language and also a subset consisting of a set of statements of the same size as the sentences (which I will take to be disjoint from the subset of sentences). The non-semantic terms may include function terms which enable us to build complex singular terms out of simpler, for example expressions for successor, addition and multiplication (along with a term for zero). We could stipulate, therefore, that all our models contain a standard model of arithmetic.

The language L is to contain a truth predicate T, and a relation E which relates sentences to the statements they express. As truth is only directly predicable of statements, T(**c**) is false in a model for any **c** which does not refer to a statement, and E(**m**,**u**) false where **m** does not refer to a sentence and **u** a statement. The sentence:statement relation is many:many. A given statement can be expressed by many distinct sentences and also a sentence may express more than one statement.[11] We aren't interested in how fine-grained statements are, all that is necessary is that no two sentences with distinct truth values in a model can express the same statement.

A base model M_0 is given by the following:

[11] We might think of the sentence having a single meaning but making distinct statements in distinct contexts.

i) A one:one function mapping the statement names onto the statements and a similar bijection mapping the sentence names onto the sentences, the functions remaining constant through all the inductive successors of M_0.

ii) an assignment of one of the truth values to each atom not containing T and E; since the atoms are structured, this might be done by associating with each predicate a function from the appropriate product D^n of the domain into the valuation space. If $\langle \alpha_1, \ldots \alpha_n \rangle$ are the referents of $t_1, \ldots t_n$, respectively $\vDash_{M_0} F(t_1, \ldots t_n,) = f(\langle \alpha_1, \ldots \alpha_n \rangle)$ where f is the extension of F. (If we have function terms in the non-semantic part, we need a recursive account determining the referents of complex terms built up from them from simpler.)

iii) A partition of the *proper* T sentences, the ascriptions of truth to statements, into κ-many κ-sized value classes one for each value [0,1]. Write $\pi(T(\mathbf{m})) = q$ if the partition in M_0 assigns direct truth-ascription $T(\mathbf{m})$– thus **m** is a statement name– to $q \in [0,1]$. (The partition and assignment to truth values varies from model to model in general. A given statement can take different truth values in a different possible situation. However this partition and assignment will remain fixed through our successors to base model M_0.) For statement names **c**, $v[T(\mathbf{c})] = q$ where $\pi[T(\mathbf{c})] = q$. For any other T sentence $T(\mathbf{u})$, if **u** refers to a statement α with statement name **a** then $T(\mathbf{u})$ takes the same value as $T(\mathbf{a})$; otherwise $T(\mathbf{u})$ is false. We can, abusing the 'π' notion somewhat, derivatively assign a statement α a truth value $\pi(\alpha) = q$ in M_0 if $\pi[T(\mathbf{c})] = q$ in M_0, **a** being a name of α.

iv) E sentences are bivalent. $E(\mathbf{t},\mathbf{u})$ is false if **t** does not refer to a sentence or **u** to a statement, otherwise either true or false. Any and every assignment of 1 or 0 to the other cases is legitimate in any admissible model, the particular assignment for M_0 determining an extension for E mapping pairs of sentence/statement to 1 or 0 and mapping all other ordered pairs to F.

From these elements of the base model, we generate a valuation v using the Łukasiewiczian truth functions for the connectives. Once we have our valuation v in place we can link sentences to statements by an *external coding* C_0 relating each sentence P to one or more statements subject to the condition that each such statement, and hence each truth ascription T(**c**), **c** a name of one of the associated statements, has the same value in v as P.

Of course this post facto equalisation of truth values of ascriptions to sentences and ascriptions to statements is not very interesting in itself. For the governing idea is that though truth applies directly only to statements we can define truth for sentences **s** by ⌜**s** expresses **r** and **r** is true], i.e. $(E(\mathbf{s},\mathbf{r}) \wedge T(\mathbf{r}))$. But since the E relation is arbitrary in M_0 this will generally result in some sentences being assigned statements by E (as interpreted in M_0) which have a different truth value and thus do not line up with C_0. There will be base models in which $v[(E\ulcorner 0=1 \urcorner, \mathbf{r}) \wedge T(\mathbf{r})] = 1$.

So we now make the familiar move of generating an inductive sequence of models, jumping first to model M_1. The only change here is that we reinterpret E (over its non-trivial part) so that it agrees with C_0, that is $E(\mathbf{m},\mathbf{u})$ takes value 1, where **m** names a sentence and **u** a statement, just when C_0 relates that sentence to that statement. (Note the assignment of a truth value to all the T sentences stays the same, as does the assignment of statement names to statements and sentence names to sentences, indeed the assignment of referents to all terms of the language.) Model M_1 generates an interpretation v_1 in the same way as M_0. External coding C_0 gets things right in M_1 for all sentences except those in sub-language Σ, where Σ is the set of all sentences φ containing as sub-formula an E atom $E(\mathbf{t},\mathbf{u})$. For no atom in L–Σ changes truth value between the models, hence no complex sentence in the set either. C_0 however can assign a sentence φ of Σ to statements with truth values which differ from φ's

M_1 value.[12] So construct external coding C_1 so that it agrees with C_0 over L–Σ but assigns each Σ sentence to a statement which the same truth value as φ has in M_1.

It follows that E in M_1, agreeing with C_0, gets the truth-conditions right for L–Σ in M_1. But if $\varphi \in \Sigma$ has value q in M_0 so is assigned by C_0 a statement α with $\pi(\alpha) = q$ but φ has value $q_1 \neq q$ in M_1 then where $\upsilon_1[E(\varphi,\mathbf{p})] = 1$, $\upsilon[(E(\varphi,\mathbf{p}) \wedge T(\mathbf{p}))$ takes value q, not the 'correct' value q_1.

Onward, ever onward. At stage 2 revise the interpretation of E once more ensuring this time that E agrees with C_1, thus generating model M_2 and valuation υ_2. C_1 disagrees with C_0 only on sentences in Σ. Define Σ_1^A as the set of atoms $E(\mathbf{t},\mathbf{u})$, $\mathbf{t}$ naming a sentence in Σ and Σ_1 as the set of all sentences containing a Σ_1^A atom as a subformula. Then Σ_1 sentences are the only sentences which change truth value between M_2 and M_1 as no other atom changes truth value. Moreover, and most importantly, Σ_1 is a proper subset of Σ, since there are atoms in Σ not in Σ_1. We now construct a new coding C_2 as before (thus agreeing with C_1 over L–Σ_1) and at stage 3 revise the interpretation of E so it agrees with C_2, the revision only affecting sentences in Σ_2 built up from Σ_2^A atoms $E(\mathbf{t},\mathbf{u})$ where $\mathbf{u}$ names a sentence in Σ_1, $\Sigma_2 \subset \Sigma_1$.

This process generates a syntactically-determined sequence of Σ classes which we can extend into the transfinite by defining, for limit ordinal λ, $\Sigma_\lambda = \bigcap_{\alpha\in\lambda}\Sigma_\alpha$. We can prove inductively that if $\alpha < \beta$ and $\Sigma_\alpha \neq \varnothing$ then $\Sigma_\beta \subset \Sigma_\alpha$. For the successor steps γ+1, suppose the proper subset property holds for all $\alpha \leq \gamma$. If $\gamma = 0$, then the proof is as above: for any atom $E(\mathbf{t},\mathbf{p}) \in \Sigma_{\gamma+1}^A$ its first term t refers (in all models) to an atom in Σ but the members of $\Sigma_\gamma^A = \Sigma_0^A$ include atoms $E(\mathbf{u},\mathbf{q})$ where $\mathbf{u}$ does not refer to sentences or to a sentence containing E; hence $\Sigma_{\gamma+1} \subset \Sigma_\gamma$. If $\gamma \neq 0$ and Σ_γ is non-empty it contains atoms $E(\mathbf{m},\mathbf{r})$ where $\mathbf{m}$ refers to sentences in $\Sigma_\alpha - \Sigma_\gamma$ for some $\alpha < \gamma$ (there are such, by inductive hypothesis) hence $E(\mathbf{m},\mathbf{r}) \notin \Sigma_{\gamma+1}^A$ and $\Sigma_{\gamma+1} \subset \Sigma_\gamma$. If Σ_γ is empty so is $\Sigma_{\gamma+1}$ and so $\Sigma_{\gamma+1} = \Sigma_\gamma$ is, by

[12] Special cases of Σ sentences cannot change value, for example if the only occurrences of E take the form $E(\mathbf{a},\mathbf{b})$ where $\mathbf{a}$ is a statement name or $\mathbf{b}$ a sentence name, for all such occurrences are false in each model.

inductive hypothesis, is, a proper subset of every non-empty Σ_δ $\delta < \gamma+1$. At limit stages λ, since Σ_λ is non-empty, the inductive hypothesis tells is it is preceded by a chain of earlier sets $\Sigma_\delta \ldots \supset \ldots \Sigma_\varepsilon \ldots$, $\delta < \varepsilon$, and so as the intersection of such a chain is a proper subset of them all.

On the semantic side, we have set out the mechanism for generating model $M_{\alpha+1}$ from M_α by revising the interpretation of E so that in $M_{\alpha+1}$ it agrees with C_α, yielding a decreasing chain of $\Sigma_\alpha \supset \Sigma_{\alpha+1} \supset \Sigma_{\alpha+2} \ldots$ where the sentences in each Σ sets are the only ones which can change truth value as we move up through the successor models. At limit stages λ, consider the set $\bigcap_{\alpha\in\lambda}\Sigma_\alpha$, a class strictly smaller than each Σ_α. Every sentence φ not in $\bigcap_{\alpha\in\lambda}\Sigma_\alpha$ *stabilises* at some $\alpha \in \lambda$; that is, it takes the same value at each M_β, $\beta \geq \alpha$, from $\alpha+1$ on and thus φ is paired by each C_β with exactly the same statements from $\alpha+1$ upwards, with $\upsilon_{\alpha+1}(\varphi) = \pi_{\alpha+1}(\sigma)$, where σ is one of the paired statements. Hence the extension of E, restricted to φ[13] is the same in each such M_β. We characterise M_λ by fixing the extension of E at stage λ by stipulating that for each such sentence $\varphi \notin \bigcap_{\alpha\in\lambda}\Sigma_\alpha$, its E-relata at M_λ are the statements it is stably paired with in this way. As for the extension of E restricted to members of $\bigcap_{\alpha\in\lambda}\Sigma_\alpha$, it is arbitrary at stage λ.[14] It follows that the value of every sentence $\varphi \notin \bigcap_{\alpha\in\lambda}\Sigma_\alpha$ at M_λ is the value at which it stabilises before λ (trivially for sentences not containing E). Let C_λ agree on φ with $C_{\alpha+1}$, where φ's value stabilises at α, and let it assign to $\psi \in \bigcap_{\alpha\in\lambda}\Sigma_\alpha$ every statement with the same truth value q as φ has in M_λ, We then proceed as before, with $C_{\lambda+1}$ and $M_{\lambda+1}$ constructed by the rule for successor stages.

The usual cardinality considerations tell us that by some stage κ, $\Sigma_\kappa = \emptyset$. This has to be a limit stage. If $E_{\beta+1}$ is non-empty, then there is a non-empty set of atoms from which we generate $E_{\beta+2}$ at the next stage. Since $\bigcap_{\beta<\kappa}\Sigma_\beta = \emptyset$, M_κ thus assigns each E atom the value it will have stabilised at for all $\gamma > \beta$, for some $\beta < \kappa$. Every sentence has the value at M_κ it stabilised at earlier and hence E, at M_κ, agrees with C_κ. If $E(\ulcorner P \urcorner, \mathbf{a})$ is true at M_κ then, where α is the referent of **a**, we

---

13 I.e. the set of all ordered pairs $\langle\varphi, x\rangle$ in the extension of E.

14 Take $\{\langle X \times D^*\rangle: X \subset L, D^* \subseteq D)\}$ for some chosen subset D* of the domain D and let the extension of E at limit stage λ by $\bigcap_{\alpha\in\lambda}\Sigma_\alpha \times D^*$.

have $\upsilon_\kappa[\mathrm{T}(\mathbf{a})] = \pi_\kappa(\alpha) = \upsilon_\kappa[\mathrm{P}] = \upsilon_\kappa[\mathrm{E}(\ulcorner \mathrm{P} \urcorner, \mathbf{a}) \wedge \mathrm{T}(\mathbf{a})]$ (Here '$\ulcorner \mathrm{P} \urcorner$' is a name of P.) Hence every instance of

$$\mathrm{T}(\mathbf{a}) \leftrightarrow \mathrm{P},$$

every instance of

$$(\mathrm{E}(\ulcorner \mathrm{P} \urcorner, \mathbf{a}) \wedge \mathrm{T}(\mathbf{a})) \leftrightarrow \mathrm{P}$$

is true at M_κ.

The predicate $\exists y(\mathrm{E}(x,y) \wedge \mathrm{T}(y))$ is our naïve truth predicate. $\mathrm{E}(\mathbf{t},\mathbf{u})$ and hence $\mathrm{E}(\mathbf{t},\mathbf{u}) \wedge \mathrm{T}(\mathbf{u})$ takes value zero except (at most) when the referent α of $\mathbf{u}$ is a statement which is the image under C_κ of the referent, sentence P, of $\mathbf{t}$. If β is any statement which is the C_κ image of P and $\mathbf{v}$ refers to β then $\upsilon_\kappa[\mathrm{E}(\mathbf{t},\mathbf{v}) \wedge \mathrm{T}(\mathbf{v})] = \upsilon_\kappa[\mathrm{E}(\mathbf{t},\mathbf{u}) \wedge \mathrm{T}(\mathbf{u})] = \upsilon_\kappa[\mathrm{P}]$. Hence $\exists y(\mathrm{E}(x,y) \wedge \mathrm{T}(y))$ and P take the same value at M_κ and, given the Łukasiewiczian semantics for $\leftrightarrow$ every instance of the schema:

$$\exists y(\mathrm{E}(\mathbf{t},y) \wedge \mathrm{T}(y)) \leftrightarrow \mathrm{P}$$

holds, with P the referent of $\mathbf{t}$.

We thus get a naïve truth-theory for the continuum-valued (and surreal-valued) Łukasiewiczian semantics for non-monotonic $\rightarrow$. (Moreover in a suitable language we will be able to express generalisation over sub-models isomorphic to the natural number structure by regular conjunctions and disjunctions $\wedge[\varphi x/\mathrm{t}_i]$, $\vee[\varphi x/\mathrm{t}_i]$, $i \in \mathrm{N}$, N indexing all the numerical terms.[15])

§V.a Paradoxicality.

Having a language with its own truth predicate does not mean having a language in which we can express paradoxical liar-type sentences and since we do not have the expressive power of standard quantificational languages, we do not have the usual means of expressing self-reference, for example by diagonalization. However we can build such self-reference in 'by hand', in the coding at C_0 for sentences in L–Σ, a coding which remains fixed throughout the

[15] But of course since the language is uncountable, the syntax of the language cannot be arithmetized and so limitative results such as Restall (1992) Hájek, Paris, Sheperdson (2000) do not apply.

construction – no C_α, $\alpha > 0$ disagrees with C_0 over sentences in L–Σ. Thus let T(**a**) have value 0.5 throughout the models and let C_0 map ¬T(**a**) to the statement which is the referent of **a**. Then both of

$$T(\mathbf{a}) \leftrightarrow \neg T(\mathbf{a});\ \ (E(\ulcorner\neg T(\mathbf{a})\urcorner, \mathbf{a}) \wedge T(\mathbf{a})) \leftrightarrow \neg T(\mathbf{a})$$

are true at our fixed point model. Similarly if **b** is another name of a 0.5 statement and we let C_0 map (T(**b**) → ⊥) to the referent of **b** then

$$T(\mathbf{b}) \leftrightarrow (T(\mathbf{b}) \rightarrow \bot);\ \ (E(\ulcorner(T(\mathbf{b}) \rightarrow \bot\urcorner), \mathbf{b}) \wedge T(\mathbf{b})) \leftrightarrow (T(\mathbf{b}) \rightarrow \bot)$$

are true and we can continue in this vein for iterated Curry sentences T(**r**) such as $T(\mathbf{r}) \leftrightarrow (T(\mathbf{r}) \rightarrow (T(\mathbf{r}) \rightarrow \bot))$, where **r** names a statement valued 2/3, and so on. Similarly for 'revenge' liars expressed in terms of the notion of definite truth i.e. D^n: $\neg Def^n T(\mathbf{c})$. Where T(**c**) is part of the $2^n/(2^n+1)$ band of T sentences, we have the truths:

$$T(\mathbf{c}) \leftrightarrow \neg Def^n T(\mathbf{c});\ \ (E(\ulcorner\neg Def^n T(\mathbf{c})\urcorner, \mathbf{c}) \wedge T(\mathbf{c})) \leftrightarrow \neg Def^n T(\mathbf{c})$$

In general, any formula $\varphi(x)$ where φ expresses a truth-function f with a fixed point q, $q = f(q)$ will yield a truth $T(\mathbf{f}) \leftrightarrow \varphi(T(\mathbf{f}))$, T(**f**) being a q-valued T-sentence; hence also, given a suitable choice of C_0, a truth

$$(E(\ulcorner\varphi(T(\mathbf{f}))\urcorner, \mathbf{f}) \wedge T(\mathbf{f})) \leftrightarrow \varphi(T(\mathbf{f})).$$

To be sure, as Field points out (2008: 92-94), in a countable or continuum-valued valuation space there will be no self-referential formula $\neg Def^\omega T(\mathbf{e})$, this being the countable conjunction of all the $\neg Def^n T(\mathbf{e})$.[16] But if $\neg Def^0 T(\mathbf{e})$, ... $\neg Def^n T(\mathbf{e})$ ... take the values $1-\varepsilon$, $1-2\cdot\varepsilon$, ... $(1-2^n\cdot\varepsilon)$... for infinitesimal surreal number ε and the conjunction operation over the surreal interval [0,1] maps this decreasing set to ε then (notwithstanding Field's remark fn. 7, 2008: 93)

$$T(\mathbf{e}) \leftrightarrow \neg Def^\omega T(\mathbf{e});\ \ (E(\ulcorner\neg Def^\omega T(\mathbf{e})\urcorner, \mathbf{e}) \wedge T(\mathbf{e})) \leftrightarrow \neg Def^\omega T(\mathbf{e})$$

will both be true, where $Def^\omega(P)$ "really does mean", P is true to the ω^{th} degree of definiteness.

---

16 Field actually considers not genuine infinitary conjunctions but finitary quantificational formulae using the truth predicate and codings but this amounts to the same thing.

It has to be emphasised that the above systems are not in the least 'revenge-immune'. In particular, though the metatheoretic framework is set theoretic, the languages themselves do not have the means for expressing the set theory used in the metatheory. However I submit these are results are encouraging with respect to the programme of using neo-classical logic as a framework both for naïve truth and naïve set theories. I conjecture that the way forward to a revenge-immune solution to the paradoxes is to investigate using naïve set theory, in the above neo-classical framework but with respect to infinitary languages whose index is a naïve ordinal, such as order type of all the small ordinals, under the usual ordinal, the Burali-Forti order-type of the series of all ordinals, and its successors.

BIBLIOGRAPHY

Field, Hartry, 2008. *Saving Truth from Paradox.* Oxford: Oxford University Press.

Gonshor, Harry 1986. *An Introduction to the Theory of Surreal Numbers.* Cambridge, Cambridge University Press.

Hájek, P., Paris, J., Sheperdson, J, 2000. *Journal of Symbolic Logic* 65: 339-46.

Restall, Greg, 1992. Arithmetic and Truth in Łukasiewicz's Infinite-value Logic , *Logique et Analyse.* 140: 262-71.

Weir, Alan, 2013. 'A Robust Non-Transitive Logic', *Topoi.*