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AMBISKEW HOPF ALGEBRAS

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Abstract. Necessary and sufficient conditions are obtained for an ambiskew polynomial algebra A over a Hopf k-algebra R to possess the structure of a Hopf algebra extending that of R, in which the added variables X_+ and X_- are skew primitive. The coradical filtration is calculated, many examples are described, and properties determined.

1. Introduction

1.1. **Ambiskew algebras.** The ingredients to define an *ambiskew polynomial algebra* A are a field k, arbitrary except where otherwise stated; a k-algebra R, not necessarily commutative; a k-algebra automorphism σ of R; a central element h of R; and a nonzero element ξ of k. Define $A = A(R, X_+, X_-, \sigma, h, \xi)$ to be the k-algebra generated by R and indeterminates X_+ and X_-, subject to the relations

$$X_+ r = \sigma(r) X_+ \; ; \tag{1}$$

$$X_- r = \sigma^{-1}(r) X_- \; ; \tag{2}$$

$$X_+ X_- = h + \xi X_- X_+ \; , \tag{3}$$

for all $r \in R$. These algebras were given this name in [Jor00] (for the case of commutative R), although they had appeared previously in the literature - for example, in [JoW96]. As explained in [Jor00], they are closely related to down-up algebras [BeR98] and to generalised Weyl algebras [Bav92].

Let R and A be as above, and suppose that R is a Hopf k-algebra. Our main theorem gives necessary and sufficient conditions for A to be a Hopf algebra with R as Hopf subalgebra and such that X_+ and X_- are *skew primitive* - that is,

$$\Delta(X_\pm) = X_\pm \otimes r_\pm + l_\pm \otimes X_\pm, \tag{4}$$

for some $l_\pm, r_\pm \in R$. Before stating the result we need to fix some more notation.

1.2. **Notation.** Our standard reference for Hopf algebras will be [Mo]. Let H be a Hopf k-algebra. The comultiplication, counit and antipode maps are denoted Δ, ε and S, respectively. Unadorned tensor products are understood to be over k. For $h \in H$, we write the coproduct $\Delta(h)$ as $\sum h_1 \otimes h_2$. We shall assume always that S is bijective. The group of grouplike elements of H is denoted $G(H)$, and, for $g, w \in G(H)$ we write $P_{g,w}(H)$ for the g, w-primitive elements of H, that is the vector space of elements h of H for which $\Delta(h) = h \otimes g + w \otimes h$. Given a character χ of H, (that is, a k-algebra homomorphism from H to k), the *right winding automorphism* τ^r_χ of H is defined by $\tau^r_\chi(h) = \sum h_1 \chi(h_2)$ for $h \in H$; and

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τ^l_χ denotes the corresponding *left* winding automorphism, [BG2, I.9.25]. The *left adjoint action* of $y \in H$ on H is given by $\mathrm{ad}_l(y)(h) = \sum y_1 h S(y_2)$ for $h \in H$. The centre of a ring or group X is denoted by $Z(X)$.

1.3. **Main theorem.** First, one notes (Lemma 2.3) that, if A is an ambiskew Hopf k-algebra satisfying (4), then one can change the ambiskew variables in A to new ones, which we continue to denote $X_\pm$, for which

$$\Delta(X_\pm) = X_\pm \otimes 1 + y_\pm \otimes X_\pm, \tag{5}$$

for $y_+, y_- \in G(R)$. Using this choice of variables makes the statement of the theorem somewhat easier:

Theorem. *The ingredients which are necessary and sufficient to construct an ambiskew Hopf algebra $A = A(R, X_+, X_-, \sigma, h, \xi)$ over the Hopf algebra R, with ambiskew variables $X_\pm$ satisfying* (5)*, are as follows:*

- *a character χ of R;*
- *an element $z \in Z(R) \cap G(R)$;*
- *an element $h \in P_{1,z}(R) \cap Z(R)$;*
- *elements $y_+, y_- \in Z(G(R))$ with $z = y_+ y_-$ and $\chi(y_+) = \chi(y_-) =: \xi$;*
- *defining $\sigma := \tau^l_\chi$, it is required that*

$$\sigma \quad = \quad \mathrm{ad}_l(y_+) \circ \tau^r_\chi.$$

This is Theorem 2.4, with the rest of §2 containing the proof. The case where R is affine commutative over the algebraically closed field k was obtained previously as the main result of [Har08]. Even in the commutative case, however, we believe that the formulation given here makes it easier to determine the possible ambiskew extensions of a given Hopf algebra R.

1.4. **Consequences and comments.** (i) **Special cases.** The specialisations of the main theorem to the cases where R is commutative or cocommutative are stated respectively as Propositions 3.2 and 3.3. As mentioned above, Proposition 3.2 is a result of Hartwig.

(ii) **Examples.** As shown in Proposition 3.1, it is an easy consequence of the main theorem that, when A is an ambiskew Hopf algebra as above, then *either* $\xi = \pm 1$ *or h is a scalar multiple of* $z - 1$. This considerably helps the cataloguing of ambiskew Hopf extensions.

In (4.1) all the possibilities for A are listed when $R = k[t]$, the polynomial algebra in a single variable. One gets the enveloping algebra of $\mathfrak{sl}(2,k)$, the enveloping algebra of the 3-dimensional Heisenberg Lie algebra, and a 1-parameter family of enveloping algebras of 3-dimensional solvable Lie algebras.

In (4.2), the case $R = k[t^{\pm 1}]$ is described: the basic example here is the quantised enveloping algebra $A = U_q(\mathfrak{sl}(2,k))$. However there are a number of variants and degenerate cases to be considered.

In (4.3) we consider the case $R = U_q(\mathfrak{sl}(2,k))$. Examples occur where $y_\pm$ are not central in R, and interesting, apparently new, Hopf algebras are obtained as ambiskew extensions of $U_q(\mathfrak{sl}(2,k))$.

(iii) **Coradical filtration.** For R and A as in the main theorem, but assuming now that k has characteristic 0, the coradical filtration of A is described in Theorem 5.3. As special cases we recover the results for $A = U_q(\mathfrak{sl}(2))$ of [Mo2] and [B].

(iv) **Properties of ambiskew Hopf algebras.** In §6 a number of properties of ambiskew algebras are examined - ring-theoretic properties in Theorem 6.1, homological properties in Theorem 6.3 and Hopf-theoretic properties in Theorem 6.5. The properties covered include primeness and semiprimeness, and when the algebra is PI, AS-Gorenstein and AS-regular. Values for the PI-degree, the GK-dimension and the global and injective dimensions are given.

2. Hopf algebra structure

2.1. **Iterated skew structure.** Recall that, given a k-algebra T and algebra automorphisms α and β of T, an (α, β)-derivation δ of T is a k-linear endomorphism of T such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)\beta(b)$ for all $a, b \in T$.

Suppose here and throughout §2 that R and A are as in (1.1). Recall the definition of a skew polynomial algebra from [MR, 1.2.1]. Then A has the structure of an iterated skew polynomial ring over R, as follows. The subalgebra of A generated by R and X_+ is just the skew polynomial ring $R[X_+;\sigma]$. We can extend σ to an automorphism of $R[X_+;\sigma]$, which we denote by the same symbol, by defining $\sigma(X_+) = \xi X_+$. Define the $(\sigma^{-1}, \mathrm{id})$-derivation $\delta\colon R[X_+;\sigma] \to R[X_+;\sigma]$ by $\delta(R) = 0$, $\delta(X_+) = -\xi^{-1}h$. One checks routinely that this is well-defined. Then

$$A \cong R[X_+;\sigma][X_-;\sigma^{-1},\delta].$$

Consequently, A is a free left and right R-module, with basis

$$\{{X_+}^m {X_-}^n \colon m, n \geq 0\}. \tag{6}$$

In addition, $A \otimes A$ is a free left and right $R \otimes R$-module, with basis

$$\{{X_-}^m {X_+}^n \otimes {X_-}^p {X_+}^q \colon m, n, p, q \geq 0\}. \tag{7}$$

2.2. **The key lemma.**

Lemma. *Let R and A be as in (1.1).*

(i) *Suppose R is equipped with a bialgebra structure, which extends to a bialgebra structure on $A := (A, m, u, \Delta, \varepsilon)$ such that* (4) *holds. Then*
  (a) $\Delta(r_\pm) = r_\pm \otimes r_\pm$ *and* $\Delta(l_\pm) = l_\pm \otimes l_\pm$*;*
  (b) $r_+r_- = r_-r_+$ *and* $l_+l_- = l_-l_+$*;*
  (c) $\Delta(h) = h \otimes r_+r_- + l_+l_- \otimes h$*.*

(ii) *Suppose R is equipped with a Hopf algebra structure, which extends to a Hopf algebra structure on $A := (A, m, u, \Delta, \varepsilon, S)$ such that* (4) *holds. Define the character $\chi := \varepsilon \circ \sigma \colon R \to k$. Then*
  (a) $r_\pm$ *and* $l_\pm$ *are grouplike, and* r_+r_- *and* l_+l_- *are in* $Z(R)$*.*
  (b) $\sigma = \mathrm{ad}_l(l_+) \circ \tau^r_\chi = \mathrm{ad}_l(r_+) \circ \tau^l_\chi$*.*
  (c) $\sigma = \mathrm{ad}_r(l_-) \circ \tau^r_\chi = \mathrm{ad}_r(r_-) \circ \tau^l_\chi$*.*
  (d) $B := \langle r_+, r_-, l_+, l_- \rangle$ *is an abelian subgroup of* $G(R)$*.*
  (e) *For all* $g \in B$*,* $\sigma(g) = \chi(g)g$*.*
  (f) $\xi = \chi(l_-r_+) = \chi(l_+r_-)$*.*

Proof. (i) Suppose that A has a bialgebra structure extending that on R. (a) The counit condition for A implies that

$$m(\mathrm{id} \otimes \varepsilon)\Delta(X_\pm) = X_\pm\varepsilon(r_\pm) + l_\pm\varepsilon(X_\pm) = X_\pm,$$
$$m(\varepsilon \otimes \mathrm{id})\Delta(X_\pm) = \varepsilon(X_\pm)r_\pm + \varepsilon(l_\pm)X_\pm = X_\pm.$$

Rearranging these gives $(1-\varepsilon(r_\pm))X_\pm = l_\pm\varepsilon(X_\pm)$ and $(1-\varepsilon(l_\pm))X_\pm = \varepsilon(X_\pm)r_\pm$. Since A is a free R-module with basis (6), both sides must be zero, so ε satisfies the counit condition if and only if

$$\varepsilon(X_\pm) = 0 \text{ and } \varepsilon(l_\pm) = \varepsilon(r_\pm) = 1. \tag{8}$$

The coassociativity condition for Δ yields

$$(\mathrm{id}\otimes\Delta)\Delta(X_\pm) = (\Delta\otimes\mathrm{id})\Delta(X_\pm);$$

that is,

$$X_\pm \otimes (\Delta(r_\pm) - r_\pm\otimes r_\pm) = (\Delta(l_\pm) - l_\pm\otimes l_\pm)\otimes X_\pm.$$

Since $A\otimes A$ is a free $R\otimes R$-module on basis (7), and $\Delta(R)\subseteq R\otimes R$, both sides must be zero, so that the coassociativity of Δ implies (i)(a).

Using the fact that Δ preserves (3) combined with the freeness of $A\otimes A$ as an $R\otimes R$-module, (7), one deduces in a similar way to (a) that (i)(b) and (i)(c) hold, and also that

$$\sigma(l_-)\otimes r_+ = \xi l_-\otimes\sigma^{-1}(r_+), \tag{9}$$

$$l_+\otimes\sigma(r_-) = \xi\sigma^{-1}(l_+)\otimes r_-. \tag{10}$$

(ii) (a) Note first that if R is a Hopf algebra, the antipode condition combined with (i)(a) implies that $r_\pm$ and $l_\pm$ are invertible. So the first part of (a) follows from (i)(a). We postpone the proof of the last part of (a).

(b) For all $a\in R$, using (4) and (1),

$$\begin{aligned}\Delta(X_+)\Delta(a) &= (l_+\otimes X_+ + X_+\otimes r_+)\left(\sum a_1\otimes a_2\right)\\ &= \sum l_+a_1\otimes X_+a_2 + X_+a_1\otimes r_+a_2\\ &= \sum l_+a_1\otimes\sigma(a_2)X_+ + \sigma(a_1)X_+\otimes r_+a_2. \end{aligned}\tag{11}$$

Now Δ preserves (1) if and only if, for all $a\in R$,

$$\Delta(X_+)\Delta(a) = \Delta(\sigma(a))\Delta(X_+).$$

By (4) and (11), this is equivalent to

$$\sum l_+a_1\otimes\sigma(a_2)X_+ + \sigma(a_1)X_+\otimes r_+a_2 = \Delta(\sigma(a))(X_+\otimes r_+ + l_+\otimes X_+);$$

that is,

$$\begin{aligned}&\left(\sum l_+a_1\otimes\sigma(a_2) - \Delta(\sigma(a))(l_+\otimes 1)\right)(1\otimes X_+)\\ &\qquad = \left(\sum \Delta(\sigma(a))(1\otimes r_+) - \sigma(a_1)\otimes r_+a_2\right)(X_+\otimes 1).\end{aligned}$$

Therefore, Δ preserves (1) if and only if, for all $a\in R$,

$$\begin{aligned}&(\Delta(\sigma(a))(l_+\otimes 1) - (l_+\otimes 1)(\mathrm{id}\otimes\sigma)\Delta(a))(1\otimes X_+)\\ &\qquad + (\Delta(\sigma(a))(1\otimes r_+) - (1\otimes r_+)(\sigma\otimes\mathrm{id})\Delta(a))(X_+\otimes 1) = 0.\end{aligned}$$

Using the linear independence of the $R\otimes R$-basis (7) of $A\otimes A$, this is equivalent to the coefficient of each of the above two terms being 0. Since r_+ and l_+ are invertible, it follows that Δ preserves (1) if and only if

$$\Delta(\sigma(a)) = (\mathrm{ad}_l(l_+)\otimes\mathrm{id})(\mathrm{id}\otimes\sigma)\Delta(a), \tag{12}$$

and

$$\Delta(\sigma(a)) = (\mathrm{id} \otimes \mathrm{ad}_l(r_+))(\sigma \otimes \mathrm{id})\Delta(a). \tag{13}$$

Applying $m(\mathrm{id} \otimes \varepsilon)$ [resp. $m(\varepsilon \otimes \mathrm{id})$] to both sides of (12) [resp. (13)] yields (ii)(b).

(c) This is proved in the same way as (b), starting from (2) rather than (1).

(a) The last part of (a) now follows by comparing the two expressions for σ in terms of τ^l_χ [resp. τ^r_χ] obtained in (b) and (c).

(d),(e) Let $g \in S$ and let $\mu := \chi(g) \in k^*$. Now $g \in G(R)$ by (ii)(a), so that (ii)(b) and (c) yield

$$\sigma(g) = \mu\, \mathrm{ad}_l(l_+)(g) = \mu\, \mathrm{ad}_l(r_+)(g) = \mu\, \mathrm{ad}_r(l_-)(g) = \mu\, \mathrm{ad}_r(r_-)(g).$$

Taking $g \in \{l_\pm, r_\pm\}$ now implies (d), and then (e) follows at once.

(f) This follows by applying (e) to (9) and (10). □

2.3. **Relabelling the variables.** Let R be a Hopf algebra whose structure extends to a Hopf algebra structure on A such that (4) holds. Then r_+ and r_- are invertible, by Lemma 2.2 (ii)(a). We can equally well think of A as the ambiskew polynomial algebra with the variables $X_\pm$ replaced by $X_\pm {r_\pm}^{-1}$, with slight adjustments to σ, ξ and h, as in the following lemma. This has the advantage that the new variables are $(1, y_\pm)-$primitive, for suitable elements $y_\pm$ of $Z(G(R))$, thus significantly simplifying some later calculations.

Lemma. *Suppose that R and A are as in (1.1). Suppose that R is equipped with a Hopf algebra structure, which extends to a Hopf algebra structure on $A := (A, m, u, \Delta, \varepsilon, S)$ such that (4) holds. Set $\chi := \varepsilon \circ \sigma \colon R \to k$.*

(i) *Then there is an alternative ambiskew presentation of A,*

$$A = A(R, X_+{r_+}^{-1}, X_-{r_-}^{-1}, \widehat{\sigma}, \widehat{\xi}, \widehat{h}),$$

where
- $\widehat{\sigma} := \mathrm{ad}_r(r_+) \circ \sigma = \mathrm{ad}_l(r_-) \circ \sigma$;
- $\widehat{\xi} := \xi\chi(r_+r_-)^{-1}$;
- $\widehat{h} := \chi(r_+)^{-1}h(r_+r_-)^{-1}$.

(ii) *Setting $\widehat{X}_+ := X_+{r_+}^{-1}$, $\widehat{X}_- := X_-{r_-}^{-1}$, and $y_\pm := l_\pm r_\pm^{-1}$,*

$$\Delta(\widehat{X}_\pm) = \widehat{X}_\pm \otimes 1 + y_\pm \otimes \widehat{X}_\pm. \tag{14}$$

Proof. (i) It is straightforward to check using Lemma 2.2 that the stated data affords an alternative presentation of A as an ambiskew algebra.

(ii) This is clear from (4). □

2.4. **Main theorem.**

Theorem. *Let R be a Hopf k-algebra.*

(i) *Fix the following data:*
- *a character χ of R;*
- *elements $z \in Z(R) \cap G(R)$ and $h \in Z(R) \cap P_{1,z}(R)$;*
- *a factorisation $z = y_+y_- = y_-y_+$ with $y_\pm \in Z(G(R))$ and $\chi(y_+) = \chi(y_-) =: \xi$.*

Define $\sigma := \tau^l_\chi$ *and assume that the above data is chosen such that*

$$\sigma = \mathrm{ad}_l(y_+)\tau^r_\chi. \tag{15}$$

Then the ambiskew algebra $A = A(R, \widehat{X}_+, \widehat{X}_-, \sigma, h, \xi)$ *has a Hopf structure extending the given structure on* R*, and such that (14) holds. The antipode* S *of* R *extends to an antipode of* A*, with*

$$S(\widehat{X}_\pm) = -y_\pm \widehat{X}_\pm. \tag{16}$$

(ii) *Conversely, suppose that* A *is an ambiskew algebra constructed from* R *as in (1.1), and that* A *admits a structure of Hopf algebra extending the Hopf structure of* R*, and such that (4) holds. Then, after a change of variables as in Lemma 2.3,* A *can be presented with ambiskew variables satisfying (14), and such that the conditions of (i) are satisfied.*

Proof. (ii) Suppose that the ambiskew algebra (1.1) admits a structure of Hopf algebras extending that of R, with $\Delta(X_\pm)$ given by (4). Lemma 2.3 shows that we can change the variables to $\widehat{X}_\pm := X_\pm r_\pm^{\pm 1}$, and hence assume that the ambiskew variables satisfy (14), for some elements $y_\pm \in G(R)$. For convenience, however, we retain the notation σ, h, ξ, after the change of variables.

Now let χ be the character $\varepsilon \circ \sigma$ of R as in Lemma 2.2(ii). By Lemma 2.2(i)(b),(c) and (ii)(a),(b), $\langle y_+, y_- \rangle$ is a central subgroup of $G(R)$, with

$$z := y_+ y_- \in Z(R) \cap G(R),$$

and

$$\Delta(h) = h \otimes 1 + z \otimes h.$$

By Lemma 2.2(f),

$$\chi(y_+) = \chi(y_-) = \xi.$$

Finally, (15) follows from Lemma 2.2(ii)(b), and the formulae for $S(\widehat{X}_\pm)$ are determined by (14) and the fact that $y_\pm$ are group-like.

(i) Conversely, it is a routine (but fairly long) series of calculations to check that, given the data as listed in (i), the Hopf structure on R can be extended to a Hopf structure on A with $\Delta(\widehat{X}_\pm)$ given by (14), and with (16) holding. □

3. Consequences of the Main Theorem

3.1. We show first that once χ, z, y_+ and y_- have been chosen so as to satisfy the constraints of the Main Theorem, there is little room for manoeuvre left in the choice of h.

Proposition. *Let* R *be a Hopf algebra and let* $A = A(R, \widehat{X}_+, \widehat{X}_-, \sigma, \xi, h)$ *be an ambiskew Hopf algebra with* R *as a Hopf subalgebra and with* $\widehat{X}_\pm$ *satisfying (14) for suitable* $y_\pm$ *in* $G(R)$*. Recall that* $h \in P_{1,z}(R)$ *for* $z := y_- y_+$*, and* $\xi := \chi(y_+) = \chi(y_-)$ *by Theorem 2.4.*

Then (at least) one of the following holds:

(i) $\xi = \pm 1,\ \chi(h) = 0.$
(ii) $\xi = \pm 1,\ z = 1.$
(iii) $\xi \neq \pm 1,\ h = \frac{\chi(h)}{\xi^2 - 1}(z - 1).$

Proof. From Theorem 2.4, $\Delta(h) = h \otimes 1 + z \otimes h$. Thus, calculating $\sigma(h)$ in two ways using (15),

$$\begin{aligned}\sigma(h) = \tau^l_\chi(h) &= \chi(h) + \xi^2 h\\ &= \mathrm{ad}_l(y_+)(\tau^r_\chi(h))\\ &= \mathrm{ad}_l(y_+)\{h + z\chi(h)\}\\ &= h + z\chi(h).\end{aligned}$$

Hence,

$$(\xi^2 - 1)h \;=\; \chi(h)(z-1),$$

and the proposition follows. □

We'll give examples in §4 to show that all three possibilities occur.

3.2. Special case: R commutative. When Theorem 2.4 is specialised to the case where R is commutative we obtain the main result of [Har08], albeit stated somewhat differently from there:

Proposition. (Hartwig) *Let R be a commutative Hopf k-algebra and let χ be a character of R such that*

$$\sigma \;:=\; \tau^l_\chi \;=\; \tau^r_\chi. \tag{17}$$

Choose the following:

- *$y_\pm \in G(R)$ with $\xi := \chi(y_-) = \chi(y_+)$;*
- *$h \in P_{1,z}(R)$, where $z := y_- y_+$.*

Then the ambiskew algebra $A := A(R, \widehat{X}_+, \widehat{X}_-, \sigma, \xi, h)$ has a Hopf structure extending that of R, with the coproduct of $\widehat{X}_\pm$ given by (14). Conversely, every ambiskew Hopf algebra extending the Hopf structure of R and satisfying (4) arises in this way, after a change of ambiskew variables.

Many interesting examples of ambiskew Hopf algebras with R commutative are given in [Har08, §4]. See also 4.1 and 4.2 below.

When k has characteristic 0, so that R is semiprime [W], it is not hard to see that (17) holds if and only if $\chi \in Z(G(R^\circ))$. (Here, R° denotes the Hopf dual of R.) We don't know if the latter condition is sufficient in positive characteristic.

3.3. Special case: R cocommutative. When we are in the setting of the Main Theorem 2.4 and in addition R is cocommutative, then $\tau^l_\chi = \tau^r_\chi$, so that (15) forces $y_+ \in Z(R)$. Theorem 2.4 thus yields:

Proposition. *Let R be a cocommutative Hopf k-algebra. Choose the following:*

- *a character χ of R;*
- *$z, y_+, y_- \in G(R) \cap Z(R)$ with $z = y_+ y_-$ and $\xi := \chi(y_+) = \chi(y_-)$;*
- *either (i) $h \in k(z-1)$, or (ii) $h \in P(R)$. When (ii) holds, we also require $z = 1$, so that $y_- = y_+^{-1}$ and $\xi \in \{\pm 1\}$.*

Set $\sigma := \tau^l_\chi$. Then the ambiskew algebra $A := A(R, \widehat{X}_+, \widehat{X}_-, \sigma, \xi, h)$ has a Hopf structure extending that of R, with the coproduct of $\widehat{X}_\pm$ given by (14). Conversely, every ambiskew Hopf algebra extending the Hopf structure of R and satisfying (4) arises in this way, after a change of ambiskew variables.

Notice that the argument in the above proposition to deduce that y_+ (and therefore also y_-) are in $Z(R)$ applies in a similar way to the general case, allowing us to conclude that, in the setting of Theorem 2.4,

y_+ and y_- are in the centre of the subalgebra of cocommutative elements of R.

4. Examples

4.1. ***R* the polynomial algebra in a single variable.** Let $R = k[t]$, so $G(R) = \{1\}$ and $P(R) = kt$. Propositions (3.2) and (3.3) both apply. There are three cases, consisting of two singletons and a one-parameter infinite family.

(i) Take

$$z = y_+ = y_- = \xi = 1, \text{ and } h = t,$$

and $\chi \in \mathrm{char}(R)$ with $\chi(t) := \lambda \neq 0$. Thus

$$\sigma(t) = t + \lambda,$$

and

$$A \quad = \quad A(k[t], X_+, X_-, \sigma, t, 1) \quad = \quad k\langle t, X_+, X_- \rangle,$$

with relations

$$\begin{aligned} X_+t &= (t+\lambda)X_+, \\ X_-t &= (t-\lambda)X_-, \\ X_+X_- - X_-X_+ &= t. \end{aligned}$$

After an obvious change of variables, $A \cong U(\mathfrak{sl}(2,k))$.

(ii) Fix the parameters as in (i), except that $\lambda = \chi(t) = 0$. Thus σ is the identity map and $A = k\langle t, X_+, X_- \rangle$, with relations

$$[X_+, t] = [X_-, t] = 0, \quad [X_+, X_-] = t,$$

so A is the enveloping algebra of the 3-dimensional Heisenberg Lie algebra.

(iii) Fix the parameters as in (i) or (ii), except that $h = 0$. Then $A = k\langle t, X_+, X_- \rangle$, with relations

$$[t, X_+] = -\lambda X_+, \quad [t, X_-] = \lambda X_-, \quad [X_+, X_-] = 0.$$

So A is the enveloping algebra of a 3-dimensional solvable Lie algebra with adjoint eigenvalues $\{\pm\lambda\}$, abelian when $\lambda = 0$.

4.2. ***R* the Laurent polynomial algebra in a single variable.** In this case the generic example is $U_q(\mathfrak{sl}(2,k))$, but there are more variants than in (4.1). Let $R = k[t^{\pm 1}]$, so $G(R) = \langle t \rangle$. Propositions (3.2) and (3.3) apply. Start by fixing a character χ of R, with $\chi(t) := \eta \in k^*$, so that $\sigma(t) = \eta t$. Choose also a group-like element

$$z = t^m, \qquad (m \in \mathbb{Z}),$$

and a factorisation of z,

$$z \quad := \quad t^\ell t^n,$$

so that

$$y_+ := t^\ell, \quad y_- := t^n; \quad (\ell, n \in \mathbb{Z},\, m = \ell + n).$$

Since we must have $\chi(y_+) = \chi(y_-)$, we require

$$\eta^{\ell - n} \quad = \quad 1. \tag{18}$$

Case (i): Suppose that

$$\ell = n. \tag{19}$$

That is, $y_+ = y_- = t^\ell$ and

$$m = 2\ell,$$

with

$$\chi(y_+) = \eta^\ell =: \xi. \tag{20}$$

In the proposition below we initially assume that

$$m \neq 0; \quad \textit{that is,} \quad z \neq 1. \tag{21}$$

Then Proposition 3.3 tells us that, for some $\lambda \in k$,

$$h = \lambda(z-1).$$

Again, we avoid degenerate cases initially by assuming that

$$\lambda \neq 0. \tag{22}$$

Let A be the ambiskew algebra with these parameters,

$$A = A(k[t^{\pm 1}], X_+, X_-, \sigma, \lambda(t^m - 1), \eta^\ell).$$

Proposition. *Fix the notation as at the start of the paragraph, and assume (19).*

(i) *Assume (21) and (22), and that η is not an ℓth root of 1. Then A is the Hopf algebra $A = k\langle t^{\pm 1}, E, F\rangle$, with*

$$Et = \eta t E, \qquad Ft = \eta^{-1} t F,$$
$$EF - FE = \frac{t^\ell - t^{-\ell}}{\eta^{\ell/2} - \eta^{-\ell/2}};$$

moreover,

$$\Delta(E) = E \otimes 1 + t^\ell \otimes E,$$
$$\Delta(F) = F \otimes t^{-\ell} + 1 \otimes F.$$

That is, A is the variant of the standard quantised $\mathfrak{sl}(2)$ which is free of rank ℓ as a module over its subHopf algebra $U_q(\mathfrak{sl}(2,k))$, where $q = \eta^{\frac{\ell}{2}}$.

(ii) *Assume (21) and (22), and that η is an ℓth root of 1. Then A is as in (i), except that*

$$EF - FE = t^\ell - t^{-\ell}.$$

(iii) *Assume that $\lambda = 0$ so that $h = 0$. Allow arbitrary $m \in \mathbb{Z}$. Then A is as in (i), except that*

$$EF = FE.$$

Thus A is a quantum affine 3-space with a line removed.

Proof. (i) Make the change of variables $E := X_+,\ F := \lambda^{-1}(\eta^{\ell/2} - \eta^{-\ell/2})^{-1} X_- t^{-\ell}$.
(ii) Put $E := X_+,\ F := \lambda^{-1} X_- t^{-\ell}$.
(iii) Put $E := X_+,\ F := X_- t^{-\ell}$. □

Case (ii): Suppose that

$$\ell \neq n, \tag{23}$$

so that, by (18),

$$\eta \text{ is a primitive } \omega\text{th root of 1, where } \omega | (\ell - n). \tag{24}$$

Changing the ambiskew variables to $E := X_+$, $F := X_- t^{-n}$, we obtain:

Proposition. *Fix the notation as at the start of the paragraph, and assume (23), so that (24) also holds. We can normalise so that either $\lambda = 0$ or $\lambda = 1$. Then $A = k\langle t^{\pm 1}, E, F\rangle$ has relations*

$$Et = \eta t E; \qquad Ft = \eta^{-1} t F,$$
$$EF - FE = \lambda(t^{\ell} - t^{-n}),$$

and coproduct given by

$$\Delta(E) = E \otimes 1 + t^{\ell} \otimes E,$$
$$\Delta(F) = F \otimes t^{-n} + 1 \otimes F.$$

4.3. ***R* the quantised enveloping algebra of $\mathfrak{sl}_2$.** We work with the standard presentation of R, namely

$$k\langle E, F, K^{\pm 1} : KE = q^2 EK$$
$$KF = q^{-2} FK$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\rangle$$

where $q \in k$, $q \neq 0, \pm 1$. The coproduct is defined by requiring $K \in G(R)$, $E \in P_{1,K}(R)$, $F \in P_{K^{-1},1}(R)$. There are two characters: (i) χ defined by $\chi(E) = \chi(F) = 0, \chi(K) = -1$, and (ii) the counit ε.

Case (i): The non-trivial character χ. In this case

$$\sigma = \tau^{\ell}_{\chi} : K \mapsto -K, \quad E \mapsto -E, \quad F \mapsto F;$$

whereas

$$\tau^{r}_{\chi} : K \mapsto -K, \quad E \mapsto E, \quad F \mapsto -F.$$

To satisfy (15), we thus need

$$\mathrm{ad}_{\ell}(y_+) : K \mapsto K, \quad E \mapsto -E, \quad F \mapsto -F.$$

For a group-like element $y_+ = K^r$ of R, this holds if and only if

$$q^{2r} = q^{-2r} = -1.$$

Thus, for some positive integer $\widehat{\ell}$,

$$q \text{ is a primitive } 4\widehat{\ell}^{th} \text{ root of } 1,$$

with

$$r = \widehat{\ell} s$$

for some odd positive integer s. In particular,

$$Z(R) \cap G(R) = \langle K^{2\widehat{\ell}}\rangle, \tag{25}$$

so we can fix

$$z := K^{2\widehat{\ell} m},$$

for some $m \in \mathbb{Z}$. Then

$$y_- = zy_+^{-1} = K^{2\widehat{\ell}m - \widehat{\ell}s}.$$

Therefore, noting that s and $p := 2m - s$ are odd,

$$\xi := \chi(y_+) = \chi(y_-) = (-1)^{\widehat{\ell}}.$$

Finally, choose $h \in Z(R) \cap P_{1,z}(R)$; that is,

$$h = \lambda(1 - z)$$

for some $\lambda \in k$, which, after normalising, we can take to be either 0 or 1.

Case (ii): $\chi = \varepsilon$, **the counit.** Thus σ is the identity map on R, and (15) forces $y_+ \in Z(R) \cap G(R)$. Since z must always be central, we get $z, y_+, y_- \in Z(R) \cap G(R)$ with $z = y_+y_-$, and $\xi := \chi(y_+) = \chi(y_-) = 1$.

Define r by

$$r := \begin{cases} \ell & \textit{if } q \textit{ is a primitive } \ell^{th} \textit{ root of 1, } \ell \textit{ odd} \\ \ell/2 & \textit{if } q \textit{ is a primitive } \ell^{th} \textit{ root of 1, } \ell \textit{ even} \\ 0 & \textit{otherwise.} \end{cases}$$

Thus

$$Z(R) \cap G(R) = \langle K^r \rangle,$$

so we can choose

$$z := K^{rm},$$

for some $m \in \mathbb{Z}$. Then we define

$$y_+ = K^{rs} \quad \textit{and} \quad y_- = K^{rp},$$

for $s, p \in \mathbb{Z}$ with $s + p = m$. Finally, we set

$$h = \lambda(1 - z)$$

for some $\lambda \in k$ which after normalisation we can take to be either 0 or 1.

Summarising the above calculations, we have proved:

Proposition. *Let R be the quantised enveloping algebra $U_q(\mathfrak{sl}_2)$, where $q \in k$, $q \neq 0, \pm 1$, as presented at the start of (4.3). The following is a complete list of the ambiskew Hopf algebras $A = \langle R, X_+, X_- \rangle$ satisfying (4), after a change of variables so that (14) holds.*

(i) *Let $\chi = \varepsilon$, the counit, with q arbitrary. Set*

$$A = R[X_+, X_-] = R \otimes k[X_+, X_-],$$

with $X_+, X_- \in P(A)$.

(ii) *Let $\chi = \varepsilon$, the counit, with q a primitive ℓ^{th} root of 1. Set r to be ℓ if ℓ is odd and $\ell/2$ if ℓ is even. Fix integers m, s and p with $m = s + p$, and $\lambda \in \{0, 1\}$. Define $A = \langle R, X_+, X_- \rangle$,, where X_+ and X_- commute with R,*

$$X_+X_- - X_-X_+ = \lambda(1 - K^{rm}),$$

and

$$X_+ \in P_{1,K^{rs}}(A), \qquad X_- \in P_{1,K^{rp}}(A).$$

(iii) *Let χ be the non-trivial character of R, with q a primitive $4\widehat{\ell}^{th}$ root of 1. Choose integers m, s and p, with s and p odd and $s + p = 2m$, and $\lambda \in \{0, 1\}$. Set $\xi := (-1)^{\widehat{\ell}}$. Define $A = \langle R, X_+, X_- \rangle$,, where*

$$X_{\pm}E = -EX_{\pm}, \quad X_{\pm}F = FX_{\pm}, \quad X_{\pm}K = -KX_{\pm},$$

with

$$X_+X_- - (-1)^{\widehat{\ell}}X_-X_+ = \lambda(1 - K^{2\widehat{\ell}m}),$$

where

$$X_+ \in P_{1,K^{\widehat{\ell}s}}(A), \qquad X_- \in P_{1,K^{\widehat{\ell}p}}(A).$$

Note that in part (iii) of the proposition, $y_{\pm}$ are not in the centre of R, in view of (25). Thus the statement in the Main Theorem that $y_{\pm} \in Z(G(R))$ cannot in general be improved.

5. Coradical filtration

5.1. Hypotheses, notation and definitions. The *coradical filtration* $\{C_n : n \geq 0\}$ of a coalgebra C is defined by taking C_0 to be the *coradical* of C, that is the sum of the simple subcoalgebras of C, and setting, for all $n \geq 0$,

$$C_{n+1} := \{c \in C \colon \Delta(c) \in C_n \otimes C + C \otimes C_0\}.$$

The apparent left-right asymmetry in this definition is not genuine, as can be easily deduced from "associativity of the wedge", [Mo, (5.2.5)]. Recall that a *coalgebra filtration* of a coalgebra C is a family $\{F_n : n \geq 0\}$ of subspaces of C such that (i) $F_n \subseteq F_{n+1}$, (ii) $C = \cup_{n \geq 0} F_n$ and (iii) $\Delta(F_n) \subseteq \sum_{i=0}^{n} F_i \otimes F_{n-i}$. We record below some key facts we shall need; other basic properties can be found in [Mo, Section 5.2], for example.

Lemma. *Let C be a coalgebra with coradical filtration $\{C_n : n \geq 0\}$.*

(i) *$\{C_n\}$ is a coalgebra filtration of C.*
(ii) *If $\{F_n : n \geq 0\}$ is any coalgebra filtration of C with $F_0 \subseteq C_0$, then $F_n \subseteq C_n$ for all $n \geq 0$.*
(iii) *If D is any subcoalgebra of C, then the nth term of the coradical filtration of D is $D \cap C_n$.*
(iv) *Let $\{A_i : i \in \mathcal{I}\}$ be subcoalgebras of C with $\sum_{i \in \mathcal{I}} A_i = C$. For each $i \in \mathcal{I}$ and each $n \geq 0$ let $A_{i,n}$ be the nth term of the coradical filtration of A_i. Then, for all $n \geq 0$ and for all $i \in \mathcal{I}$,*

$$C_n = \sum_{i \in \mathcal{I}} A_{i,n}.$$

Proof. (i) This is [Mo, Theorem 5.2.2].

(ii) This is a straightforward induction on n.

(iii) This is [Mo, Lemma 5.2.12].

(iv) This follows from (iii) and by an induction on n, using the fact [Mo, Proposition 5.2.9(2)] that C_n is the nth term in the socle series of C as a left C^*-module. □

We shall assume throughout §5 that R is a Hopf k-algebra, and $A = \langle R, X_{\pm} \rangle$ is an ambiskew algebra such that the Hopf structure on R extends to A, so that (4) holds. In fact, as shown in Lemma 2.3, we may and will assume throughout that the variables have been changed if necessary so that (14) holds. The notation for the constituents of A will be as in the Main Theorem, except that we write $X_\pm$ for the ambiskew variables. Let $\{R_a\colon a \geq 0\}$ denote the coradical filtration of R, $\{A_t\colon t \geq 0\}$ the coradical filtration of A.

We shall further assume throughout the rest of §5 that

$$k \text{ has characteristic } 0. \tag{26}$$

There is no doubt that this additional hypothesis could be removed by refinements of the arguments below, along the lines of [B] and [Mu].

In the above set-up we have in particular that ξ is a non-zero element of k. When ξ is a primitive dth root of unity for some $d > 1$, define, for each non-negative integer m, non-negative integers q_m and r_m by

$$m := dq_m + r_m,$$

with $0 \leq r_m < d$. If $\xi = 1$ or ξ is not a root of unity, set $q_m := m$ and $r_m := 0$.

Define a partial order $\prec$ on $\mathbb{Z}_{\geq 0}$; this is dependent on d, but we suppress that in the notation. Set $p \prec m$ if $q_p \leq q_m$ and $r_p \leq r_m$, for any $p, m \in \mathbb{Z}_{\geq 0}$. Thus if $\xi = 1$ or ξ is not a non-trivial root of unity, then $p \prec m$ if and only if $p \leq m$, for all non-negative integers m and p. Set $\widehat{m} := q_m + r_m$. It is easy to check that, for all non-negative integers m, p with $p \prec m$,

$$\widehat{m - p} = \widehat{m} - \widehat{p}. \tag{27}$$

Let $0, 1 \neq q \in k$. For any integer $n > 0$, let

$$(n)_q := 1 + \ldots + q^{n-1} = \frac{q^n - 1}{q - 1}.$$

The *q-factorial* is defined by setting $(0)!_q := 1$ and

$$(n)!_q := (1)_q(2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}.$$

Thus $(n)!_q$ is a polynomial in q with integer coefficients. For $0 \leq i \leq n$, we define the *Gaussian binomial coefficient* or *q-binomial coefficient* by

$$\binom{n}{i}_q = \frac{(n)!_q}{(i)!_q(n-i)!_q}.$$

This is also a polynomial in q with integer coefficients [Kas95, Proposition IV.2.1(a)]. There are analogues of the identities for binomial coefficients [Kas95, Proposition IV.2.1(c)]:

$$\binom{n}{i}_q = \binom{n-i}{i}_q + q^{n-i}\binom{n-1}{i-1}_q = \binom{n-1}{i-1}_q + q^i\binom{n-1}{i}_q. \tag{28}$$

If q is a primitive nth root of unity, then

$$\binom{n}{i}_q = 0, \quad \text{for all } 1 \leq i \leq n-1. \tag{29}$$

5.2. Comultiplication formulas. The q-binomial theorem [BG2, I.6.1] at once yields (i) of the following; (ii) follows easily.

Lemma. *Keep the hypotheses and notation of (5.1), and let m and n be non-negative integers.*

(i)
$$\Delta({X_\pm}^m) = \sum_{j=0}^{m} \binom{m}{j}_{\xi^{\pm 1}} {y_\pm}^{m-j} {X_\pm}^j \otimes {X_\pm}^{m-j}.$$

(ii)
$$\Delta({X_+}^m {X_-}^n) = \sum_{j=0}^{m}\sum_{k=0}^{n} \binom{m}{j}_{\xi} \binom{n}{k}_{\xi^{-1}} \xi^{j(n-k)} {y_+}^{m-j} {y_-}^{n-k} {X_+}^j {X_-}^k \otimes {X_+}^{m-j} {X_-}^{n-k}.$$

(iii) *There are non-zero scalars $\alpha_p \in k$, for all $p \prec m$, such that*
$$\Delta({X_\pm}^m) = \sum_{0 \le p \prec m} \alpha_p {y_\pm}^{m-p} {X_\pm}^p \otimes {X_\pm}^{m-p}.$$

(iv) *There are non-zero scalars $\beta_{v,p} \in k$, for all $v \prec m$ and $p \prec n$, such that*
$$\Delta({X_+}^m {X_-}^n) = \sum_{v \prec m}\sum_{p \prec n} \beta_{v,p} {y_+}^{m-v} {y_-}^{n-p} {X_+}^v {X_-}^p \otimes {X_+}^{m-v} {X_-}^{n-p}.$$

Proof. (iii) When $\xi = 1$ or ξ is not a non-trivial root of unity, this is just (i). Suppose now that ξ is a primitive dth root of unity for $d > 1$. Using (29), (i) implies
$$\Delta({X_\pm}^d) = {y_\pm}^d \otimes {X_\pm}^d + {X_\pm}^d \otimes 1.$$
Furthermore, by Theorem 2.4(i), $\sigma^{\pm 1}({y_\pm}^d) = \xi^{\pm d} {y_\pm}^d = {y_\pm}^d$, and so ${y_\pm}^d$ commutes with $X_\pm$. Therefore, for all integers $a \ge 0$,
$$\Delta({X_\pm}^d)^a = \sum_{i=0}^{a} \binom{a}{i} {y_\pm}^{d(a-i)} {X_\pm}^{di} \otimes {X_\pm}^{d(a-i)},$$
where $\binom{a}{i}$ denotes the ordinary binomial coefficient. Similarly, $X_\pm^d$ commutes with $y_\pm$. Setting $m := dq_m + r_m$ and using the above equation together with (i) gives
$$\begin{aligned}
\Delta({X_\pm}^m) &= \Delta({X_\pm}^d)^{q_m} \Delta({X_\pm}^{r_m}) \\
&= \left(\sum_{i=0}^{q_m} \binom{q_m}{i} {y_\pm}^{d(q_m-i)} {X_\pm}^{di} \otimes {X_\pm}^{d(q_m-i)} \right) \\
&\qquad \cdot \left(\sum_{j=0}^{r_m} \binom{r_m}{j}_{\xi^{\pm 1}} {y_\pm}^{r_m-j} {X_\pm}^j \otimes {X_\pm}^{r_m-j} \right) \\
&= \sum_{i=0}^{q_m}\sum_{j=0}^{r_m} \binom{q_m}{i} \binom{r_m}{j}_{\xi^{\pm 1}} {y_\pm}^{d(q_m-i)+r_m-j} {X_\pm}^{di+j} \otimes {X_\pm}^{d(q_m-i)+r_m-j} \\
&= \sum_{i=0}^{q_m}\sum_{j=0}^{r_m} \binom{q_m}{i} \binom{r_m}{j}_{\xi^{\pm 1}} {y_\pm}^{m-(di+j)} {X_\pm}^{di+j} \otimes {X_\pm}^{m-(di+j)} \\
&= \sum_{0 \le p \prec m} \alpha_p {y_\pm}^{m-p} {X_\pm}^p \otimes {X_\pm}^{m-p},
\end{aligned}$$

where $\alpha_p := \binom{q_m}{i}\binom{r_m}{j}_{\xi^{\pm 1}}$ for $p := di + j$, with $0 \le i \le q_m$ and $0 \le j \le r_m \le d-1$. Our hypotheses now ensure that $\alpha_p \neq 0$, for all $p \prec m$.

(iv) This follows easily from (iii). □

5.3. The filtration. Let $\{R_q : q \ge 0\}$ denote the coradical filtration of R, and define subspaces $\{F_t : t \ge 0\}$ of A by

$$F_t := \sum_{q+\widehat{m}+\widehat{n}\le t} R_q {X_+}^m {X_-}^n.$$

Lemma. *Keep the hypotheses and notation of (5.1), and let t, m, n, ℓ, v be non-negative integers.*

(i) *$\{F_t : t \ge 0\}$ as defined above is a coalgebra filtration of A.*

(ii) *If $v \prec \ell \prec m$, then $\ell - v \prec m$.*

(iii) *Suppose that $\widehat{m} + \widehat{n} \le t$, and define $a := t - \widehat{m} - \widehat{n}$. Then*

$$F_{t,(m,n)} := \sum_{\ell \prec m, w \prec n} R_a X_+^{\ell} X_-^{w}$$

is a subcoalgebra of F_t.

Proof. (i) It is clear that $F_t \subseteq F_{t+1}$ for all $t \ge 0$ and that $A = \bigcup_{t\ge 0} F_t$, so it remains to prove that $\Delta(F_t) \subseteq \sum_{i=0}^{t} F_i \otimes F_{t-i}$. Let $r{X_+}^m{X_-}^n \in F_t$, so $r \in R_a$, with $a + \widehat{m} + \widehat{n} \le t$. Lemma 5.2(iv) shows that $\Delta(r{X_+}^n{X_-}^m)$ is a sum of terms of the form

$$\beta_{v,p} r_1 {y_+}^{m-v} {y_-}^{n-p} {X_+}^v {X_-}^p \otimes r_2 {X_+}^{m-v} {X_-}^{n-p}, \tag{30}$$

where $v \prec n$, $p \prec m$, $\beta_{v,p} \in k^*$. By definition of the coradical filtration of R, we have $\Delta(r) \in \sum_{j=0}^{a} R_j \otimes R_{a-j}$, so without loss of generality we can suppose that $r_1 \in R_j$ and $r_2 \in R_{a-j}$ for some j, $0 \le j \le a$. Since $y_\pm \in G(R)$, $r_1{y_+}^{m-v}{y_-}^{n-p} \in R_j$. Therefore, (30) is contained in $F_u \otimes F_w$, where $u = j + \widehat{v} + \widehat{p}$ and $w = a - j + \widehat{m-v} + \widehat{n-p}$. It is enough to show that $u + w \le t$. We have, using (27),

$$u + w = a + \widehat{v} + \widehat{p} + \widehat{m-v} + \widehat{n-p} = a + \widehat{m} + \widehat{n} \le t,$$

as required.

(ii) This follows routinely from the definition of $\prec$.

(iii) As in the proof of (ii), if $r \in R_a$, $\ell \prec m$ and $w \prec n$, so that $rX_+^{\ell}X_-^{w} \in F_{t,(m,n)}$, then $\Delta(rX_+^{\ell}X_-^{w})$ is a sum of terms of the form

$$\beta_{v,p} r_1 y_+^{\ell-v} y_-^{w-p} X_+^v X_-^p \otimes r_2 X_+^{\ell-v} X_-^{w-p}, \tag{31}$$

where $v \prec \ell, p \prec w, \beta_{v,p} \in k^*, r_1 \in R_j, r_2 \in R_{a-j}$, for some j, $0 \le j \le a$. Since, in view of (ii), both tensorands in (31) are in $F_{t,(m,n)}$, the result follows. □

Theorem. *Keep the hypotheses and notation of (5.1). Denote the coradical filtration of A by $\{A_t : t \ge 0\}$. Then*

$$A_t = \sum_{q+\widehat{m}+\widehat{n}\le t} R_q {X_+}^m {X_-}^n.$$

Proof. As before, set $F_t := \sum_{a+\widehat{m}+\widehat{n}} R_a X_+^m X_-^n$. We must show that $F_t = A_t$ for all $t \ge 0$.

Since $F_0 = R_0 = A_0$, Lemmas 5.3(i) and 5.1(ii) show that $F_t \subseteq A_t$ for all $t \ge 0$.

Now we show by induction on t that $F_t = A_t$ for all $t \geq 0$. Suppose that $t \geq 0$, that we have proved that $F_i = A_i$ for all $i \leq t$, but that

$$F_{t+1} \subsetneq A_{t+1}. \tag{32}$$

Since $\cup_{i\geq 0} F_i = A$, there exists $t_0 \geq t+2$, t_0 minimal such that

$$F_{t+1} \subsetneq F_{t_0} \cap A_{t+1}. \tag{33}$$

For each ordered pair (m,n) of non-negative integers with $\widehat{m}+\widehat{n} \leq t_0$, write $E_{t_0,(m,n)}$ for $F_{t_0,(m,n)} + F_{t_0-1}$. Observe that, by choice of t_0,

$$F_{t_0-1} \cap A_{t+1} = F_{t+1}. \tag{34}$$

Each $E_{t_0,(m,n)}$ is a subcoalgebra of F_{t_0}, by Lemma 5.3(iii), and

$$F_{t_0} = \sum_{m,n} E_{t_0,(m,n)}.$$

By Lemma 5.1(iii), the $(t+1)$st term of the coradical filtration of $E_{t_0,(m,n)}$ is $E_{t_0,(m,n)} \cap A_{t+1}$, which contains F_{t+1}. By Lemma 5.1(iv) and (33), there exists at least one ordered pair (m,n) with

$$\begin{aligned} F_{t+1} &\subsetneq A_{t+1} \cap E_{t_0,(m,n)} \\ &= A_{t+1} \cap (F_{t_0,(m,n)} + F_{t_0-1}) \\ &= (A_{t+1} \cap F_{t_0,(m,n)}) + (A_{t+1} \cap F_{t_0-1}) \end{aligned}$$

The second equality above follows from Lemmas 5.3(iv) and 5.1(iv). Using (34), we deduce that

$$F_{t+1} \subsetneq (A_{t+1} \cap F_{t_0,(m,n)}) + F_{t+1} \subseteq E_{t_0,(m,n)}. \tag{35}$$

Define the non-negative integer a by

$$a + \widehat{m} + \widehat{n} = t_0. \tag{36}$$

Now (35) shows that there exist $w, w' \in A$ and $r \in R_a \setminus R_{a-1}$ such that

$$w \quad = \quad rX_+^m X_-^n + w',$$

where w' is a sum of terms $r'X_+^i X_-^j$ where r' has coradical degree a' with

$$a' + \widehat{i} + \widehat{j} \leq t_0 - 1.$$

By definition of the coradical filtration and by the induction hypothesis,

$$\Delta(w) \in A_t \otimes A + A \otimes A_0 = F_t \otimes A + A \otimes F_0.$$

Now $\Delta(w)$ has the form

$$\Delta(r) \sum_{v \prec m} \sum_{p \prec n} \beta_{v,p} {y_+}^{m-v} {y_-}^{n-p} {X_+}^v {X_-}^p \otimes {X_+}^{m-v} {X_-}^{n-p}, \tag{37}$$

and we can find a term $x \otimes z$ in the expansion of $\Delta(r)$ with $x \in R_a \setminus R_{a-1}$, $z \neq 0$. Consider now the corresponding term (38) in the expansion of $\Delta(w)$, noting first that it cannot cancel with any terms from the expansion of $\Delta(w')$:

$$\beta_{v,p} x {y_+}^{m-v} {y_-}^{n-p} {X_+}^v {X_-}^p \otimes z {X_+}^{m-v} {X_-}^{n-p} \tag{38}$$

A summand of (37) is in $A \otimes F_0$ if and only if $v = m$ and $p = n$; all other terms must be in $F_t \otimes A$. So, by definition of F_t, we have

$$a + \widehat{v} + \widehat{p} \leq t \tag{39}$$

for all $v \prec m$ and $p \prec n$ except $(v,p) = (m,n)$. As in (5.1), write $m = q_m d + r_m$ and $n = q_n d + r_n$ with $q_m, q_n \in \mathbb{Z}_{\geq 0}$, $0 \leq r_m, r_n \leq d-1$. It is easy to check that we cannot have $m = n = 0$; so we assume without loss of generality that $m \geq 1$. There are now five cases to consider.

(i) Suppose that $\xi = 1$ or ξ is not a non-trivial root of unity. Thus $\prec$ is $\leq$. Consider the term of form (38) given by $v = m-1$, $p = n$, which is in $F_t \otimes A$; then (39) yields

$$a + m - 1 + n \leq t,$$

which by (36) implies

$$t_0 - 1 \leq t,$$

contradicting $t_0 \geq t+2$.

In the remaining four cases, ξ is a primitive dth root of unity for some $d > 1$.

(ii) Suppose that $r_m = r_n = 0$. Therefore, $m = q_m d$ and $n = q_n d$ for some $q_m \geq 1$ and $q_n \geq 0$. Consider the term of form (38) obtained by taking $v := (q_m - 1)d$ and $p := n$, which is contained in $F_t \otimes A$. Then $\widehat{v} = \widehat{m} - 1$, so (39) yields

$$a + \widehat{m} - 1 + \widehat{n} \leq t,$$

yielding thanks to (36)

$$t_0 - 1 \leq t,$$

again a contradiction.

(iii) Suppose that $r_m \geq 1$ and $r_n = 0$. Consider the summand of (37) given by taking $v := m-1$ and $p := n$, which is contained in $F_t \otimes A$. Then $\widehat{v} = \widehat{m} - 1$ and (39) gives $a + \widehat{m} - 1 + \widehat{n} \leq t$, a contradiction.

Cases (iv) $r_m = 0$, $r_n \geq 1$, and (v) $r_m, r_n \geq 1$ are similar to (iii).

Thus in all cases the hypothesis (32) produces a contradiction. So $F_{t+1} = A_{t+1}$, completing the proof of the induction step, and with it, the theorem. □

For convenience we state the following immediate consequences of the theorem:

Corollary. *Assume the hypotheses of Theorem 5.3.*

(i) $A_0 = R_0$. *In particular, A is pointed if and only if R is pointed.*

(ii)

$$A_1 = \begin{cases} R_1 \oplus R_0 X_{\pm}, & \xi = 1 \text{ or } \xi \text{ not a root of 1;} \\ R_1 \oplus R_0 X_{\pm} \oplus R_0 {X_{\pm}}^d, & \xi \text{ a primitive } d\text{th root of 1.} \end{cases}$$

6. Properties of ambiskew Hopf algebras

6.1. Ring-theoretic properties. The additional terminology used throughout §6 is standard and can be checked as required in the references provided. However, to state in Theorem 6.1(vi) the conditions required for A to satisfy a polynomial identity we need some further notation, as follows. Suppose that k is algebraically closed of characteristic 0, and that the automorphism σ of R has finite order n. Let η be a primitive nth root of unity in k. Then

$$R = \bigoplus_{i=0}^{n-1} R_i,$$

where $R_i = \{r \in R : \sigma(r) = \eta^i r\}$. In particular,

$$h = \sum_{i=0}^{n-1} h_i, \tag{40}$$

with $h_i \in R_i$ for all i.

Theorem. *Let R be a k-algebra and let $A = A(R, X_+, X_-, \sigma, h, \xi)$ be an ambiskew algebra over R as in 1.1.*

(i) *A is noetherian if and only if R is noetherian.*
(ii) *A is a domain if and only if R is a domain.*
(iii) *A is semiprime left Goldie if and only if R is semiprime left Goldie.*
(iv) *If R is prime then A is prime.*
(v) *Suppose that every finite subset of R is contained in a finite dimensional $\sigma-$invariant subspace of R. Then A has finite Gel'fand-Kirillov dimension if and only if R does also. In this case*

$$\mathrm{GK}-\dim(A) = \mathrm{GK}-\dim(R) + 2. \tag{41}$$

(vi) *Suppose that k is algebraically closed of characteristic 0 and that R is a commutative affine domain. Then the following conditions are equivalent:*
  (a) *A satisfies a polynomial identity.*
  (b) *A is a finite module over its centre.*
  (c) *The following conditions hold:*
    - *$\sigma_{|R}$ has finite order n;*
    - *ξ has finite order t;*
    - *if $t|n$, so that $\xi = \eta^j$ for some j, $0 \leq j \leq n-1$, then $h_j = 0$ in (40).*

  When the above equivalent conditions hold, the P.I. degree of A is $2mn$, where $m = l.c.m.(n,t)$.

Proof. (i),(ii) [MR, Theorem 1.2.9(iv),(i)].

(iii) [Ma, Theorem 2.6].

(iv) [MR, Theorem 1.2.9(iii)].

(v) Using the iterated skew structure of A as explained in 2.1, set $B := R[X_+; \sigma]$. Then the result follows by two applications of [Z, Lemma 4.1]. For the second application, that is to pass from B to A, one has to check first that σ is locally finite dimensional when extended to B, and then that every finite set of elements of B is contained in a finite dimensional subspace V for which the subalgebra $k\langle V\rangle$ of B is $\delta-$stable. The first statement is true because σ is extended to B by setting $\sigma(X_+) = \xi X_+$, as discussed in 2.1, and the second is clear since $\delta(R) = 0$ and δ is locally nilpotent on B.

(vi) This is proved in [M, § 5.4.2]. □

6.2. Remarks: (i) The converse of (iv) is false, as shown by taking $R = k \oplus k$ and $\sigma((a,b)) = (b,a)$, with (say) $h = 0$ and $\xi = 1$.

(ii) We have omitted the proof of (vi) because it is rather long and technical, and its inclusion would take us too far away from the main theme of this paper. The hypothesis there that k has characteristic 0 is necessary, as is illustrated by the first Weyl algebra $A_1(k)$, whose centre is $k[X^p, Y^p]$ when k has characteristic $p > 0$. In this case $h = h_0 = 1 \neq 0$. It is somewhat surprising to us that the question of when an ambiskew algebra satisfies a polynomial identity does not seem to have been previously addressed in the literature.

6.3. Homological properties. For the definitions of the Auslander-Gorenstein and Auslander-regular properties, see for example [E] or [BG2].

Theorem. *Let R be a k-algebra and let $A = A(R, X_+, X_-, \sigma, h, \xi)$ be an ambiskew algebra over R as in 1.1.*

(i) *If R has finite global dimension (*gl.dim.*) then so also does A, and in this case*

$$\text{gl.dim.}(R) + 1 \leq \text{gl.dim}(A) \leq \text{gl.dim.}(R) + 2. \tag{42}$$

(ii) *If R has finite injective dimension (*inj.dim.*) then so does A, and in this case*

$$\text{inj.dim.}(R) + 1 \leq \text{inj.dim}(A) \leq \text{inj.dim.}(R) + 2. \tag{43}$$

(iii) *If R is Auslander-Gorenstein then so is A.*
(iv) *If R is Auslander-regular then so is A.*

Proof. (i) With the notation introduced in the proof of Theorem 6.1(v), gl.dim.$(B) =$ gl.dim.$(R) + 1$ by [MR, Theorem 7.5.3(iii)]. Now $A = B[X_-; \sigma^{-1}, \delta]$, so that gl.dim.$(B) \leq$ gl.dim.$(A) \leq$ gl.dim.$(B) + 1$ by [MR, Theorem 7.5.3(i)]. Combining these yields the desired inequalities.

(ii), (iii) These are similar to (i), using [E, Theorem 4.2].

(iv) Immediate from (i) and (iii). □

6.4. Remark: Both inequalities in (42) (and hence in (43)) can be attained. Take $R = k$ and $A = k[X, Y]$ to get equality at the higher end, and $R = k$ (of characteristic 0) and $A = A_1(k)$, the first Weyl algebra, to get equality at the lower end.

6.5. Hopf algebraic properties. We recall from [BZ] that a noetherian k-algebra T with a fixed augmentation $\varepsilon : T \longrightarrow k$ is *AS-Gorenstein* if the left T-module T has finite injective dimension d, with $\text{Ext}^i_T(k, T) = \delta_{id}k$, and the same condition holds on the right. Then T is *AS-regular* if, in addition, gl.dim.$(T) < \infty$. Part (ii) of the result below should be read in the context of the (currently open) question [BG1, 1.15], [Br, Question E] whether every noetherian Hopf algebra is AS-Gorenstein.

Theorem. *Let R be a k-algebra and let $A = A(R, X_+, X_-, \sigma, h, \xi)$ be an ambiskew algebra over R as in 1.1. Suppose that A admits a Hopf algebra structure extending a Hopf structure on R, and such that* (4) *holds.*

(i) *The second inequality in* (42) *is an equality.*
(ii) *A is AS-Gorenstein if and only if R is AS-Gorenstein. In this case, the second inequality in* (43) *is an equality.*
(iii) *A is AS-regular if and only if R is AS-regular.*
(iv) *A has finite Gel'fand-Kirillov dimension if and only if R does also. In this case* (41) *holds.*

Proof. (i) By [LL], the global dimension of a Hopf k-algebra H is the projective dimension of its trivial module k. It then follows by an easy application of the long exact sequence of cohomology that, when finite,

$$\text{gl.dim.}(H) = \max\{i : \text{Ext}^i_H(k, H) \neq 0\}. \tag{44}$$

Since k is manifestly a finitely generated module over B and R, two applications of [S, Theorem 8] yield

$$\mathrm{Ext}_A^{i+2}(k, A) \cong \mathrm{Ext}_R^i(k, R) \tag{45}$$

for all $i \geq 0$. (Note that there are typos in the statement of this theorem in [S].) The result follows from (44) and (45).

(ii) Immediate from (43) and (45).

(iii) Immediate from (42) and (45).

(iv) We have to show that the hypothesis on σ needed for Theorem 6.1(v) is satisfied. This follows from the fact, proved in the Main Theorem 2.4, that σ is a winding automorphism, combined with the basic property of coalgebras that every finite set of elements of a coalgebra is contained in a finite dimensional subcoalgebra, [Mo, Theorem 5.1.1]. □

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