FACTORING IN THE HYPERELLIPTIC TORELLI GROUP

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Abstract. The hyperelliptic Torelli group is the subgroup of the mapping class group consisting of elements that act trivially on the homology of the surface and that also commute with some fixed hyperelliptic involution. Putman and the authors proved that this group is generated by Dehn twists about separating curves fixed by the hyperelliptic involution. In this paper, we introduce an algorithmic approach to factoring a wide class of elements of the hyperelliptic Torelli group into such Dehn twists, and apply our methods to several basic types of elements. As one consequence, we answer an old question of Dennis Johnson.

1. Introduction

Let s be a hyperelliptic involution of a compact, orientable surface S_g^1 of genus g with one boundary component; see Figure 1. The hyperelliptic Torelli group $\mathcal{SI}(S_g^1)$ is the group of connected components of the group of homeomorphisms of S_g^1 that commute with s, restrict to the identity on ∂S_g^1, and act trivially on $H_1(S_g^1)$. This group arises as the fundamental group of the branch locus of the period mapping from Torelli space to the Siegel upper half-plane and also as the kernel of the Burau representation of the braid group B_{2g+1} evaluated at $t = -1$; see [8].

Figure 1. Rotation by π about the indicated axis is a hyperelliptic involution.

The simplest nontrivial element of $\mathcal{SI}(S_g^1)$ is a Dehn twist T_c about a symmetric separating curve c, that is, a separating curve fixed by s. Hain conjectured $\mathcal{SI}(S_g^1)$ is generated by such elements, and the authors proved this conjecture with Putman [4]. There are two other basic types of elements of $\mathcal{SI}(S_g^1)$ (refer to Figure 2):

(1) A *symmetric simply intersecting pair map* is a commutator $[T_x, T_y]$ where x and y are symmetric nonseparating curves with geometric intersection number $i(x, y) = 2$ and algebraic intersection number $\hat{\imath}(x, y) = 0$.
(2) A *symmetrized simply intersecting pair map* is a commutator $[T_{u_1}T_{u_2}, T_{v_1}T_{v_2}]$ where u_1, v_1, u_2, and v_2 are nonseparating curves with $i(u_1, v_1) = 2$, $\hat{\imath}(u_1, v_1) = 0$, $s(u_1) = u_2$, $s(v_1) = v_2$, $i(u_1, u_2) = 0$, and $i(u_1, v_2) = 0$.

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The topological type of a symmetric simply intersecting pair is determined by its genus, that is, the genus of the complementary component not containing ∂S_g^1; the pair in Figure 2 has genus 0. Similarly, the topological type of a symmetrized simply intersecting pair is determined by the genera of the three complementary regions not containing ∂S_g^1. The example in Figure 2 has all three genera equal to 0.

When the authors first learned of Hain's conjecture, it seemed intractable because we did not know how to factor the above types of elements into Hain's proposed generators. In this paper, we not only give simple factorizations for both types, but we also give an algorithm for factoring a much wider class of elements. We expect that our relations will play a role for $\mathcal{SI}(S_g^1)$ analogous to the critical role that the classical lantern relation has played in our understanding of the full Torelli group.

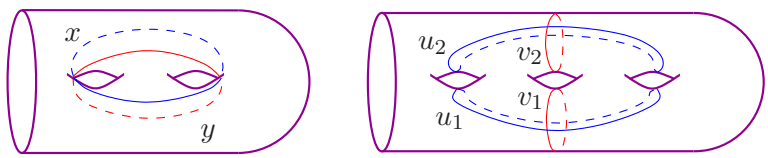

FIGURE 2. Left: The curves x and y form a symmetric simply intersecting pair. Right: the curves u_1, v_1, u_2, and v_2 form a symmetrized simply intersecting pair.

Three applications. We now describe three applications of our factorizations.

Application 1: a question of Dennis Johnson. Let S_2 denote the closed surface of genus two, let $\mathrm{Mod}(S_2)$ denote its mapping class group, the group of homotopy classes of orientation preserving homeomorphisms of S_2, and let $\mathcal{I}(S_2)$ denote the Torelli subgroup, the subgroup consisting of elements that act trivially on $H_1(S_2)$.

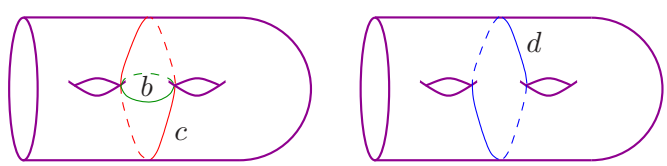

FIGURE 3. The curves used to define Johnson's mapping class.

Consider the simple closed curves c and $d = T_b(c)$ in S_2 as shown in Figure 3. The curves c and d are homologous (after choosing appropriate orientations), and so it follows that $T_c T_d^{-1}$ is a nontrivial element of $\mathcal{I}(S_2)$ (see, e.g., [6, Section 6.5.2]). Let J denote the subgroup of $\mathcal{I}(S_2)$ generated by all conjugates of $T_c T_d^{-1}$ in $\mathrm{Mod}(S_2)$.

Joan Birman communicated to us the following question of Dennis Johnson, which he asked during a seminar at Columbia University in the early 1980s:

Do the conjugates of $T_c T_d^{-1}$ in $\mathrm{Mod}(S_2)$ generate $\mathcal{I}(S_2)$?

As $T_c T_d^{-1}$ can be regarded as the symmetric simply intersecting pair map $[T_c, T_b]$, we will be able to apply our factorization of the latter to answer Johnson's question in the negative. Let J denote the subgroup of $\mathcal{I}(S_2)$ generated by the conjugates of $T_c T_d^{-1}$ in $\mathrm{Mod}(S_2)$.

Theorem 1.1. *The quotient $\mathcal{I}(S_2)/J$ is isomorphic to $\mathbb{Z}$.*

It follows from the theorem that J contains the commutator subgroup of $\mathcal{I}(S_2)$ and that $\mathcal{I}(S_2)$ is generated by J plus a single Dehn twist about a separating curve.

Application 2: higher genus twists. The genus of a separating curve in S_g^1 is the genus of the complementary component not containing ∂S_g^1. In our earlier paper [5], we showed that a Dehn twist about a symmetric separating curve of arbitrary genus is equal to a product of Dehn twists about symmetric separating curves of genus 1 and 2. In particular, by our theorem with Putman, $\mathcal{SI}(S_g^1)$ is generated by such Dehn twists. As another application of the methods of this paper, we give an explicit factorization of the Dehn twist about any genus $k \geq 3$ symmetric separating curve into Dehn twists about symmetric separating curves of smaller genus.

Application 3: the Burau representation. The group $\mathcal{SI}(S_g^1)$ is isomorphic to the kernel of the Burau representation of the braid group B_{2g+1} evaluated at $t = -1$. In another paper [3], we use the idea from our factorization algorithm to derive a "squared lantern relation" and we use it to give a topological description of the kernel of the Burau representation at $t = -1$, modulo 4: it is equal to the group generated by squares of all Dehn twists.

If $\sigma_1, \ldots, \sigma_{2g}$ are the standard generators for B_{2g+1}, then Artin's generators for the pure braid group are the $a_{i,i+k} = (\sigma_{i+1} \cdots \sigma_{i+(k-1)})^{-1} \sigma_i^2 (\sigma_{i+1} \cdots \sigma_{i+(k-1)})$. With this notation, the squared lantern relation is:

$$[a_{12}, a_{23}] = a_{13}^{-2}(a_{13}a_{12}a_{23})^2(a_{13}a_{12}^{-2}a_{13}^{-1})a_{23}^{-2}.$$

This relation equates the commutator of two Artin generators with a product of squares of Dehn twists. We use this relation to show that the group generated by squares of Dehn twists contains the commutator subgroup of the pure braid group.

Algorithmic factorizations. We now explain the general framework that will provide the desired factorizations. Let a be a symmetric nonseparating curve in S_g^1, and denote by $\mathcal{SI}(S_g^1, a)$ the stabilizer of the isotopy class of a in $\mathcal{SI}(S_g^1)$. There is an s-equivariant inclusion $S_g^1 - a \to S_{g-1}^1$ and this induces a surjective homomorphism $\mathcal{SI}(S_g^1, a) \to \mathcal{SI}(S_{g-1}^1)$ [5, Proposition 6.6]. We denote the kernel by $\mathcal{SIBK}(S_g^1, a)$:

$$1 \to \mathcal{SIBK}(S_g^1, a) \to \mathcal{SI}(S_g^1, a) \to \mathcal{SI}(S_{g-1}^1) \to 1.$$

As we will explain in Section 2, the group $\mathcal{SI}(S_g^1)$ is naturally isomorphic to a subgroup of the braid group B_{2g+1}, and so any element of $\mathcal{SIBK}(S_g^1, a)$ can be written as a word in the standard generators for B_{2g+1}.

Theorem 1.2. *There is an explicit algorithm for factoring arbitrary elements of $\mathcal{SIBK}(S_g^1, a)$ into Dehn twists about symmetric separating curves of genus 1 and 2. If the input is a word of length N in the standard generating set for B_{2g+1}, the output is a product of at most*

$$\tfrac{5}{8}3^{2(N+1)} - \tfrac{1}{4}3^{N+2}$$

Dehn twists about symmetric separating simple closed curves.

The idea is to identify $\mathcal{SIBK}(S_g^1, a)$ with a subgroup of the fundamental group of a disk with $2g-1$ points removed. Then the problem of factoring elements of $\mathcal{SIBK}(S_g^1, a)$ into Dehn twists about symmetric separating curves is translated into a problem of finding special factorizations of certain elements of this free group.

We emphasize that the existence of the algorithm from Theorem 1.2 does not guarantee that one can find a simple factorization for a given element of $\mathcal{SIBK}(S_g^1, a)$. The factorizations we give in this paper were only found after much trial and error (cf. the earlier version of this paper [2]). Their relative tameness suggests that $\mathcal{SI}(S_g^1)$ is more tractable than was originally believed.

We can obtain factorizations in the hyperelliptic Torelli group of a closed surface S_h by including S_g^1 into S_h where $h \geq g$; the induced map $\mathcal{SI}(S_g^1) \to \mathcal{SI}(S_h)$ is injective if $h > g$ and has cyclic kernel $\langle T_{\partial S_g^1} \rangle$ otherwise; see [5, Theorem 4.2].

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2. The factoring algorithm

In this section we explain how to algorithmically factor an arbitrary element of $\mathcal{SIBK}(S_g^1, a)$ as per Theorem 1.2.

The setup. We will rephrase our problem about factoring elements of $\mathcal{SIBK}(S_g^1, a)$ into a problem about certain factorizations in a free group. To this end, let D_{2g-1} denote a disk with $2g-1$ marked points and D_{2g-1}° the punctured disk obtained by removing the marked points. The fundamental group of D_{2g-1}° is a free group F_{2g-1}; we take the generators $x_1, \ldots, x_{2g-1}$ for F_{2g-1} to be simple loops in D_{2g-1} each surrounding one marked point; see Figure 4.

Let F_{2g-1}^{even} denote the kernel of the map $F_{2g-1} \to \mathbb{Z}/2\mathbb{Z}$ given by $x_i \mapsto 1$ for all i; this group is generated by the $x_i^{\delta_i} x_j^{\delta_j}$ with $i \leq j$ and $\delta_i \in \{-1, 1\}$. Denote the generators for $\mathbb{Z}^{2g-1}$ by $e_1, \ldots, e_{2g-1}$. Let

$$\epsilon : F_{2g-1}^{even} \to \mathbb{Z}^{2g-1}$$

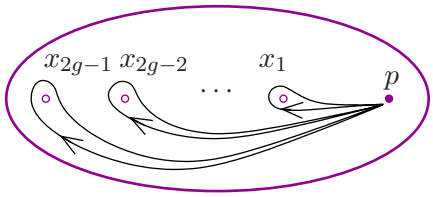

FIGURE 4. The generators x_i for $\pi_1(D^\circ_{2g-1})$.

be the homomorphism given by $x_i^{\delta_i} x_j^{\delta_j} \mapsto e_i - e_j$. We will require the following two facts, explained below.

(1) There is an isomorphism $\Psi : \mathcal{SIBK}(S^1_g, a) \to \ker \epsilon$.
(2) $\ker \epsilon$ is generated by squares of simple loops in D°_{2g-1} about 1 or 3 punctures.

Once we define Ψ, it will be easy to see that squares of simple loops in D°_{2g-1} surrounding 1 or 3 punctures correspond to products of Dehn twists about symmetric separating curves in S^1_g of genus 1 and 2. After discussing the above two facts, we proceed to explain the factorization algorithm of Theorem 1.2.

The isomorphism Ψ. The isomorphism Ψ was given in our earlier paper [5, Theorem 1.2]; we recall the construction. In what follows, the mapping class group of a surface S with marked points is the group $\mathrm{Mod}(S)$ of connected components of the group of homeomorphisms of S that restrict to the identity on ∂S and preserve the set of marked points.

The quotient $S^1_g/\langle s\rangle$ is a disk D_{2g+1} with $2g+1$ marked points, and $\mathrm{Mod}(D_{2g+1})$ is isomorphic to the braid group B_{2g+1}. Let $\mathrm{SMod}(S^1_g)$ be the subgroup of $\mathrm{Mod}(S^1_g)$ with elements represented by s-equivariant homeomorphisms. Birman–Hilden proved the natural map $\Theta : \mathrm{SMod}(S^1_g) \to \mathrm{B}_{2g+1}$ is an isomorphism [6, Theorem 9.1].

The group $\mathcal{SI}(S^1_g, a)$ maps to $\mathrm{Mod}(D_{2g+1}, \bar{a})$, the stabilizer of the isotopy class of the arc $\bar{a}$, the image of a in D_{2g+1}. By collapsing $\bar{a}$ to a marked point p and removing the other $2g-1$ marked points to obtain $2g-1$ punctures, we obtain a homomorphism $\Xi : \mathrm{Mod}(D_{2g+1}, \bar{a}) \to \mathrm{Mod}(D_{2g-1}, p)$. Since the kernel of $\Xi \circ \Theta$ is generated by T_a, the restriction $\Psi : \mathcal{SI}(S^1_g, a) \to \mathrm{Mod}(D^\circ_{2g-1}, p)$ is injective.

We then arrive at the following special case of the Birman exact sequence:

$$1 \to \pi_1(D^\circ_{2g-1}, p) \to \mathrm{Mod}(D^\circ_{2g-1}, p) \to \mathrm{Mod}(D^\circ_{2g-1}) \to 1.$$

The first nontrivial map here is actually an anti-homomorphism, as the usual orders of operation in the two groups do not agree. Therefore, relations in $\pi_1(D^\circ_{2g-1}, p)$ will translate to the reverse relations in $\mathrm{Mod}(D^\circ_{2g-1}, p)$.

The image of $\mathcal{SIBK}(S^1_g, a)$ under Ψ lies in the kernel $\pi_1(D^\circ_{2g-1}, p)$ of the Birman exact sequence. In our earlier paper [5, Lemma 4.5] we showed that for $\alpha \in \pi_1(D^\circ_{2g-1}, p)$, the action of the lift $(\Xi \circ \Theta)^{-1}(\alpha)$ on $H_1(S^1_g; \mathbb{Z})$ is exactly given by $\epsilon(\alpha)$, and so the image of $\mathcal{SIBK}(S^1_g, a)$ under Ψ is precisely $\ker \epsilon$.

Squares of simple loops. If $\alpha \in \pi_1(D^\circ_{2g-1}, p)$ is a simple loop surrounding k punctures, where k is odd, then α^2 lies in $\ker \epsilon = \mathrm{Im}\Psi$ and $\Psi^{-1}(\alpha^2)$ is equal to

$T_c T_d^{-1}$, where c and d are the preimages in S_g^1 of the curves obtained by pushing α off of p to the left and right, respectively, and positive Dehn twists are to the left. The curves c and d are separating curves of genus $(k+1)/2$ and $(k-1)/2$ (not necessarily in that order) and a lies in the genus 1 subsurface between them. When $k = 1$, note that one of the two separating curves is inessential.

We will now show that these α^2 generate $\ker \epsilon$. To begin, the image of ϵ is $\mathbb{Z}_{bal}^{2g-1}$, the kernel of the map $\mathbb{Z}^{2g-1} \to \mathbb{Z}$ recording the coordinate sum [5, Lemma 5.1]. Also, the group F_{2g-1}^{even} is generated by elements of the form x_i^2 and the $x_j x_1$, since

$$x_i x_j = (x_i x_1)(x_j x_1)^{-1}(x_j^2) \quad \text{and} \quad x_i x_j^{-1} = (x_i x_1)(x_j x_1)^{-1},$$

and $\mathbb{Z}_{bal}^{2g-1}$ has a presentation whose generators are the images of these generators:

$$\langle \epsilon(x_1^2), \ldots, \epsilon(x_{2g-1}^2), \epsilon(x_2 x_1), \ldots, \epsilon(x_{2g-1} x_1) \mid \epsilon(x_i^2), [\epsilon(x_i x_1), \epsilon(x_j x_1)] \rangle.$$

It follows that $\ker \epsilon$ is normally generated by the set

$$\{x_i^2 \mid 1 \leq i \leq 2g-1\} \cup \{[x_i x_1, x_j x_1] \mid 1 < i < j \leq 2g-1\}.$$

We notice the following relation in $\ker \epsilon$:

$$[x_i x_1, x_j x_1] = [x_j^{-2}(x_j x_i x_1)^2 (x_i^{-2})^{x_1^{-1}} x_1^{-2}]^{x_j},$$

where x^y denotes yxy^{-1}. It now follows that $\ker \epsilon$ is normally generated by

$$\{x_i^2 \mid 1 \leq i \leq 2g-1\} \cup \{(x_j x_i x_1)^2 \mid 1 < i < j \leq 2g-1\}.$$

Referring to Figure 4, we see that each $x_j x_i x_1$ is a simple closed curve in D_{2g-1}°. In particular, $\ker \epsilon$ is generated by

$$\{\alpha^2 \mid \alpha \text{ is a simple loop surrounding 1 or 3 punctures}\}.$$

It follows that $\mathcal{SIBK}(S_g^1, a)$ is generated by maps of the form T_c where c is a symmetric separating curve of genus 1 with a lying on the genus 1 side of c and of the form $T_c T_d^{-1}$ where c and d are symmetric separating curves of genus 1 and 2, respectively, with a lying in the genus 1 subsurface between.

The algorithm. We now give the factoring algorithm described in Theorem 1.2. Suppose we are given some $f \in \mathcal{SIBK}(S_g^1, a)$ as a word of length N in the generators $\sigma_1, \ldots, \sigma_{2g}$ for B_{2g+1} (we can use Artin's combing algorithm to check whether the product fixes $\bar{a}$, hence a, and whether the corresponding element of $\mathrm{Mod}(D_{2g-1}, p)$ lies in the image of $\pi_1(D_{2g-1}^{\circ}, p)$, and it is easy to check if this element further lies in $\ker \epsilon$; in other words, we can easily check if f indeed lies in $\mathcal{SIBK}(S_g^1, a)$).

As a word in the standard generators for B_{2g}, the image $\bar{f}$ in $\mathrm{Mod}(D_{2g-1}, p)$ has length at most N (this word is obtained by deleting all occurrences of σ_{2g}. As above, since $f \in \mathcal{SIBK}(S_g^1, a)$, we know that $\bar{f}$ lies in the kernel of the above Birman exact sequence. We can then use Artin's combing algorithm for pure braids [1] to write $\bar{f}$ as a word w_0 in the $x_i^{\pm 1}$.

We claim that the length of w_0 is bounded above by 3^N. This upper bound follows easily from the standard combing argument. Indeed we can first rewrite w_0 as a word of length N in $\sigma_1, \ldots, \sigma_{2g-2}, x_1, \ldots, x_{2g-1}$ (see [7, Proof of Proposition

3.1]). Then we push the $\sigma_i^{\pm 1}$ in this word to the left; each time we push a $\sigma_i^{\pm 1}$ past an x_j, we replace x_j with $\sigma_i^{\mp 1} x_j \sigma_i^{\pm 1}$. Such a conjugate is equal to a word of length at most three in the x_i (specifically, x_j, x_{j+1}, or $x_j^{-1} x_{j+1} x_j$). After pushing the σ_i to the left, the resulting word in $\sigma_1, \ldots, \sigma_{2g-2}$ represents the trivial element of $\mathrm{B}_{2g-1} \subseteq \mathrm{B}_{2g}$ by assumption on $\bar{f}$. The claimed bound follows.

The word w_0 lies in F_{2g-1}^{even}, and so it equals some word w in the x_i^2 and the $x_i x_1$. Using the rewriting system above, the length of w is at most $(3/2)3^N = 3^{N+1}/2$.

Since $\Psi(f) \in \ker \epsilon$, the word w maps to a relator in $\mathbb{Z}_{bal}^{2g-1}$ with respect to the presentation given above, that is, w maps to a word in the generators $\epsilon(x_i^2)$ and $\epsilon(x_i x_1)$ that equals the identity. Therefore, there is a sequence of commutations, free cancellations, and cancellations of $\epsilon(x_i^2)$-terms transforming $\epsilon(w)$ into the empty word. By the correspondence between relators for $\mathbb{Z}_{bal}^{2g-1}$ and normal generators for $\ker \epsilon$, we obtain a factorization of w into a product of conjugates of the x_i^2, the $[x_i x_1, x_j x_1]$, and their inverses. We already explained above how to factor $[x_i x_1, x_j x_1]$ into a product of squares of simple loops surrounding 1 or 3 punctures. This gives the desired factorization of f into squares of Dehn twists.

To complete the proof of the theorem we need an upper bound for the length of the corresponding product of Dehn twists about symmetric separating curves. If $\epsilon(w)$ is a word of length n in the $\epsilon(x_i^2)$ and $\epsilon(x_i x_1)$, then we need to apply at most n relators of the form $\epsilon(x_i^2)$—corresponding to at most n Dehn twists—and at most $n(n-1)/2$ commutations—corresponding to at most $5n(n-1)/2$ Dehn twists. As the length of $\epsilon(w)$ is bounded by $3^{N+1}/2$, we obtain the upper bound in the statement of Theorem 1.2. This completes the proof of the theorem.

Relations in genus two. Above, we explained how the commutator $[x_i x_1, x_j x_1]$ corresponds to an element of $\mathcal{SI}(S_2^1)$ and we factored this into a product of five Dehn twists about symmetric separating curves in S_2^1 (see the previous version of this paper [2, Theorem 3.1] for a picture of the curves). If we cap the boundary of S_2^1 with a disk, $[x_i x_1, x_j x_1]$ maps to $[T_c^2 T_e^{-2}, T_b^2 T_d^{-2}] T_a^8$ in $\mathcal{SI}(S_2)$; see Figure 5. The related (but simpler) element $[T_c T_e^{-1}, T_b T_d^{-1}] T_a^2$ also lies in $\mathcal{SI}(S_2)$ and so is a product of Dehn twists about symmetric separating curves. Surprisingly, it equals a single (left) Dehn twist.

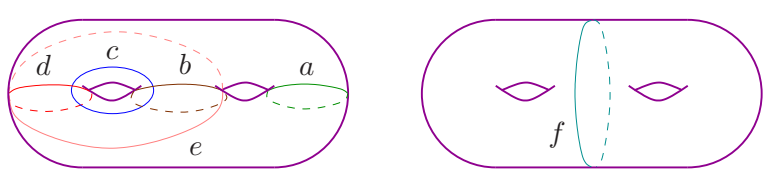

FIGURE 5. The curves a, b, c, d, e, and f from Theorem 2.1.

Theorem 2.1. *Let a, b, c, d, e, and f be as in Figure 5. We have:*

$$[T_c T_e^{-1}, T_b T_d^{-1}] T_a^2 = T_f.$$

One can check the relation in Theorem 2.1 using the Alexander Method [6, Section 2.3]; see [2, Section 4.1] for a conceptual proof.

3. Factorizations and applications

In this section, we give explicit factorizations of symmetric simply intersecting pair maps and symmetrized simply intersecting pair maps into Dehn twists about symmetric separating curves using the ideas of Section 2. We also give an explicit factorization of the Dehn twist about any genus $k \geq 3$ symmetric separating curve into Dehn twists about symmetric separating curves of smaller genus. Along the way, we use our factorization of symmetric simply intersecting pair maps to answer Johnson's question from the introduction.

3.1. **Factoring symmetric simply intersecting pair maps.** We start by writing the symmetric simply intersecting pair map from Figure 6 as a product of Dehn twists about symmetric separating curves.

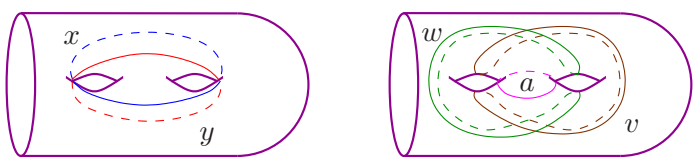

Figure 6. The curves used in Theorem 3.1.

Theorem 3.1. *Every symmetric simply intersecting pair map is the difference of two Dehn twists about symmetric simple closed curves. In particular, if x, y, v, and w are the simple closed curves shown in Figure 6, we have:*

$$[T_x, T_y] = T_v^{-1} T_w.$$

Note that the first statement follows immediately from the second statement and the change of coordinates principle [6, Section 1.3].

We give two proofs of Theorem 3.1. The first is an easy application of the lantern relation, a relation between (left) Dehn twists about 7 curves lying in a subsurface homeomorphic to a sphere with four boundary components; see [6, Section 5.1.1]. In both cases we restrict to the case where the symmetric simply intersecting pair map has genus zero; for a symmetric simply intersecting pair map of genus k the curves v and w would be replaced with separating curves of genus $k+1$.

First proof of Theorem 3.1. By the lantern relation, we have $T_v T_x T_y = M$ and $T_w T_y T_x = M'$, where M and M' are the products of twists about the boundary curves of the corresponding four-holed spheres. Since x and y appear in both lantern relations and since a regular neighborhood of $x \cup y$ is a sphere with four holes, the four-holed spheres in the two lantern relations are equal, and so $M = M'$. Thus,

$$[T_x, T_y] = (T_x T_y)(T_x^{-1} T_y^{-1}) = (T_v^{-1} M)(M^{-1} T_w) = T_v^{-1} T_w,$$

as desired. □

We now give a proof of Theorem 3.1 that is intrinsic to the braid group.

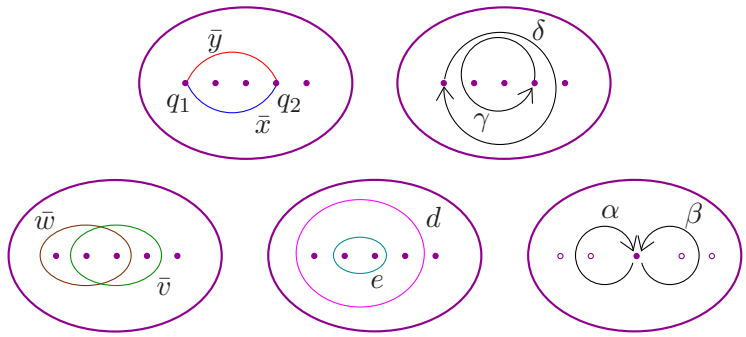

FIGURE 7. Curves, loops, and arcs used in the second proof of Theorem 3.1

Second proof of Theorem 3.1. The images of x and y in D_{2g+1} are arcs $\bar{x}$ and $\bar{y}$; denote their endpoints by q_1 and q_2 (throughout, refer to Figure 7). As above, T_x and T_y correspond to the half-twists $H_{\bar{x}}$ and $H_{\bar{y}}$ in $\mathrm{Mod}(D_{2g+1})$.

As a loop in the space of configurations of $2g+1$ points in the disk (see [6, Theorem 9.1]), the product $H_{\bar{x}}H_{\bar{y}}$ is given by the motion of points where q_1 and q_2 move around δ and γ, respectively (we multiply half-twists right to left). These motions correspond to the mapping classes $T_{\bar{v}}^{-1}T_d$ and $T_{\bar{v}}^{-1}T_e$, respectively. Similarly, $H_{\bar{x}}^{-1}H_{\bar{y}}^{-1}$ corresponds to $(T_{\bar{w}}T_d^{-1})(T_{\bar{w}}T_e^{-1})$. Since T_d and T_e commute with all the other twists, the original commutator $[T_x, T_y]$ in $\mathcal{SI}(S_g^1)$ corresponds to $T_{\bar{v}}^{-2}T_{\bar{w}}^2$ in $\mathrm{Mod}(D_{2g+1})$. The preimage under Ψ is $T_v^{-1}T_w$ in $\mathrm{SMod}(S_g^1)$, as desired. $\square$

The second proof of Theorem 3.1 has a connection with the algorithm from Section 2. Let a be the symmetric simple closed curve in S_g^1 shown in Figure 6. Assuming Theorem 3.1, and referring to Figure 7, we can see that $T_{\bar{v}}^{-2}T_{\bar{w}}^2$, the image of $T_v^{-1}T_w$ under Ψ, is $\alpha^2\beta^2$, where α and β are as shown in Figure 7. This is a product of squares of simple loops, each surrounding one puncture, as per Section 2.

3.2. **Johnson's question.** We will now use Theorem 3.1 to prove Theorem 1.1 from the introduction which states the the quotient of $\mathcal{I}(S_2)$ by the normal closure in $\mathrm{Mod}(S_2)$ of $T_cT_d^{-1}$ is isomorphic to $\mathbb{Z}$.

Proof of Theorem 1.1. The first observation is that $T_cT_d^{-1}$ is equal to the symmetric simply intersecting pair map $[T_c, T_b]$ where the curve b is as shown in the right-hand side of Figure 3:

$$[T_c, T_b] = T_c(T_bT_c^{-1}T_b^{-1}) = T_cT_{T_b(c)}^{-1} = T_cT_d^{-1}.$$

By Theorem 3.1 Johnson's mapping class $T_cT_d^{-1}$ is equal to the difference of two Dehn twists about separating curves, say $T_u^{-1}T_z$.

Mess proved that $\mathcal{I}(S_2)$ is a free group and that it is generated by Dehn twists about separating curves; there is one generator for each conjugacy class of such Dehn twists in $\mathcal{I}(S_2)$. There is thus a surjective homomorphism

$$\nu : \mathcal{I}(S_2) \to \mathbb{Z}$$

where each (positive) Dehn twist about a separating curve maps to 1.

Since $T_c T_d^{-1}$ is equal to $T_u^{-1} T_z$, it lies in the kernel of ν and so $J \subseteq \ker(\nu)$. In other words, ν factors through a surjective map $\mathcal{I}(S_2)/J \to \mathbb{Z}$. It remains to show that $\mathcal{I}(S_2)/J$ is cyclic. For this it suffices to show that for any two separating curves v and w in S_2 the Dehn twists T_v and T_w have the same image in $\mathcal{I}(S_2)/J$.

Let X be the simplicial complex whose vertices correspond to isotopy classes of separating simple closed curves in S_2 and whose edges connect vertices with geometric intersection equal to four. The complex X is known to be connected; see Putman's elegant proof [9, Theorem 1.4].

By an application of the change of coordinates principle [6, Section 1.3], $\mathrm{Mod}(S_2)$ acts transitively on the edges of X. As the curves u and z used in the factorization of $T_c T_d^{-1}$ correspond to an edge in X, this means that every edge in X corresponds to a symmetric simply intersecting pair map, and hence to an element of J.

More specifically if v and w are adjacent vertices in X, then $T_v T_w^{-1}$ lies in J and so T_v and T_w have the same image in $\mathcal{I}(S_2)/J$. Since X is connected, all Dehn twists in $\mathcal{I}(S_2)$ have the same image in $\mathcal{I}(S_2)/J$, and we are done. □

3.3. **Factoring symmetrized simply intersecting pair maps.** We now address the symmetrized simply intersecting pair maps.

Theorem 3.2. *Every symmetrized simply intersecting pair map is equal to a product of six Dehn twists about symmetric separating curves.*

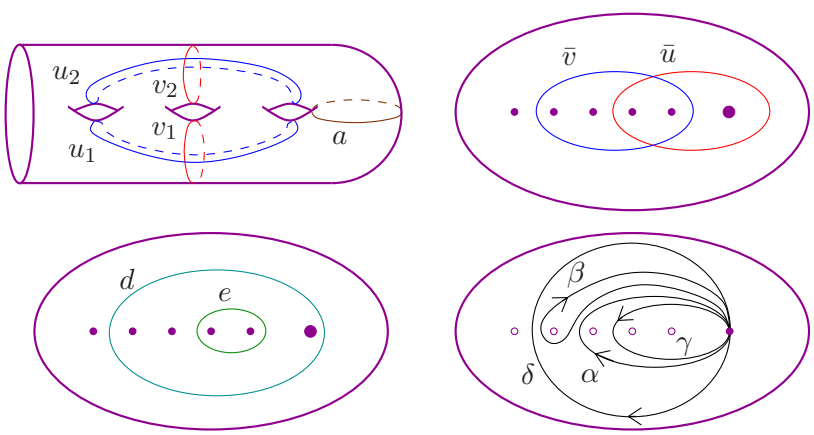

FIGURE 8. The curves and loops used in the proof of Theorem 3.2.

Proof. Consider the symmetrized simply intersecting pair map shown in Figure 8 (throughout we refer to this figure). First we notice that $[T_{u_1} T_{u_2}, T_{v_1} T_{v_2}]$ lies in

$\mathcal{SIBK}(S_g^1, a)$. We claim that the image of this commutator under the map Ψ from Section 2 is $[\delta, \gamma]$. Indeed, we have $\Psi(T_{u_1}T_{u_2}) = T_{\bar{u}}$ and $\Psi(T_{v_1}T_{v_2}) = T_{\bar{v}}$, and so the claim follows from the fact that the images of δ and γ in $\mathrm{Mod}(D^\circ_{2g-1}, p)$ are $T_{\bar{v}}^{-1}T_{\bar{v}'}$ and $T_{\bar{u}}^{-1}T_{\bar{u}'}$ and the fact that $T_{\bar{u}'}$ and $T_{\bar{v}'}$ commute with all other twists in the commutator (remember that the order of multiplication gets reversed!).

Now that we have written $\Psi([T_{u_1}T_{u_2}, T_{v_1}T_{v_2}])$ as an element of the free group $\pi_1(D^\circ_{2g-1}, p)$, we observe the following factorization in this free group:

$$[\delta, \gamma] = [\beta^2(\beta^{-1}\gamma)^2(\alpha\gamma)^{-2}\alpha^2]^\alpha$$

As in Section 2, this is a product of squares of simple loops in $\pi_1(D^\circ_{2g-1}, p)$ surronding 1 or 3 punctures, and hence the preimage under Ψ is a product of Dehn twists about symmetric separating curves of genus 1 and 2 in S_g^1. The first and fourth loops each correspond to a single Dehn twist, while the second and third loops each correspond to a pair of Dehn twists. □

From the proof of Theorem 3.2, it is straightforward, though not necessarily enlightening, to draw the six symmetric separating simple closed curves whose Dehn twists factorize the symmetrized simply intersecting pair map shown in Figure 2. For an explicit picture, see the first version of this paper [2].

In terms of the x_i from Figure 4, we can also write the Ψ-image of a symmetrized simply intersecting pair map as $[x_4x_3, x_2x_1]$. In the free group, this factors as:

$$[x_4x_3, x_2x_1] = [(x_3x_2x_1)^2(x_3^{-2})^{(x_2x_1)^{-1}}(x_4x_2x_1)^{-2}(x_4^2)]^{x_4}.$$

Again, the right-hand side in this equality is a product of squares of simple loops and so we obtain an alternate factorization.

The relation given in Theorem 3.2 involves 14 Dehn twists, twice the number of Dehn twists involved in the lantern relation. However, there is no way to rearrange our relation into a product of two lantern relations.

An arbitrary symmetrized simply intersecting pair map (where the genera of the complementary regions are greater than zero) can be factored into a product of five Dehn twists about symmetric separating curves in exactly the same way as in the proof of Theorem 3.2. To do this we simply replace the third and fifth marked points (from the left) in Figure 8 with odd numbers of marked points, and we add any even number of marked points to the region containing p.

3.4. **Factoring higher genus twists.** Finally, we obtain the factorization of a Dehn twist about an arbitrary symmetric separating curve into a product of Dehn twists about symmetric separating curves, each having genus 1 or 2, by applying the following theorem inductively.

Theorem 3.3. *Let d denote the boundary of S_g^1, and let c denote a symmetric separating curve of genus $g-1$. The product $T_dT_c^{-1}$ is equal to a product of 11 Dehn twists about symmetric separating curves in S_g^1, each of genus at most $g-1$.*

Proof. Let a be a symmetric nonseparating curve in S_g^1 lying between c and d. The image of d in D_{2g+1} is the boundary of the disk, and if we choose the identification of $S_g^1/\langle s\rangle$ with D_{2g+1} appropriately, the image of c in D_{2g+1} is a round circle surrounding the $2g-1$ leftmost marked points and the image of a is a straight arc connecting the other two marked points.

Let y_5 denote $x_{2g-1}x_{2g}\cdots x_5$. It follows from the previous paragraph that the image of $T_dT_c^{-1}$ under Ψ is equal to the image of $(y_5x_4x_3x_2x_1)^2$ under the point pushing map $\pi_1(D^\circ_{2g-1},p)\to \mathrm{Mod}(D_{2g-1},p)$.

As in Section 2, we factor $(y_5x_4x_3x_2x_1)^2$ into a product of simple loops each surrounding an odd number of punctures:

$$[(x_2x_1y_5)^2(x_2^{-2})^{y_5^{-1}x_1^{-1}}(x_3x_1y_5)^{-2}x_3^2]^{y_5x_4x_3}[(x_1y_5x_4)^2(x_4^{-2})(x_4x_3x_2)^2]^{x_1^{-1}}.$$

This is a product of 11 Dehn twists about symmetric separating curves, three of genus $g-1$, three of genus $g-2$, four of genus 1, and one of genus 2. □

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