FACTORIZATION IN GENERALIZED CALOGERO-MOSER SPACES

GWYN BELLAMY

Abstract. Using a recent construction of Bezrukavnikov and Etingof, [BE], we prove that there is a factorization of the Etingof-Ginzburg sheaf on the generalized Calogero-Moser space associated to a complex reflection group. In the case $W = S_n$, this confirms a conjecture of Etingof and Ginzburg, [EG].

1. Introduction

In this paper we apply a recent construction of Bezrukavnikov and Etingof, [BE], to the study of the centres of the rational Cherednik algebras at $t = 0$. The affine varieties $X_{\mathbf{c}}(W)$, corresponding to these centres are called the *generalized Calogero-Moser spaces* and are known to influence the representation theory of the algebras. We show that there exists an isomorphism of schemes

$$\Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0), \tag{1}$$

where $\pi_W : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W$ and W_b is a parabolic subgroup of W associated to the orbit $b \in \mathfrak{h}/W$ (for precise definitions see Section 2). In order to relate the representation theory of the algebras $H_{0,\mathbf{c}}$ to the varieties $X_{\mathbf{c}}$, Etingof and Ginzburg introduced the coherent sheaf $\mathcal{R}[W]$, defined by $\Gamma(X_{\mathbf{c}}, \mathcal{R}[W]) = H_{0,\mathbf{c}}\mathbf{e}$. Our main result describes the pushforward of $\mathcal{R}[W]_{|_{\pi_W^{-1}(b)}}$ by Φ.

Theorem. On $\pi_{W_b}^{-1}(0)$ there is an isomorphism of W-equivariant sheaves

$$\Phi_* \left(\mathcal{R}[W]_{|_{\pi_W^{-1}(b)}} \right) \simeq \mathtt{Ind}_{W_b}^{W} \mathcal{R}[W_b]_{|_{\pi_{W_b}^{-1}(0)}}$$

In particular, this theorem proves that the sheaf $\mathcal{R}[W]$ on the Calogero-Moser space associated to the symmetric group factorizes as conjectured by Etingof and Ginzburg, [EG, page 319].

2. The rational Cherednik algebra

2.1. **Definitions and notation.** Let W be a complex reflection group, $\mathfrak{h}$ its reflection representation over $\mathbb{C}$ with rank $\mathfrak{h} = n$, and $\mathcal{S}$ the set of all complex reflections in W. The idempotent in $\mathbb{C}W$ corresponding to the trivial representation will be denoted $\mathbf{e}_W$. Let $(\cdot,\cdot) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ be the natural pairing defined by $(y, x) = x(y)$. For $s \in S$, fix $\alpha_s \in \mathfrak{h}^*$ to be a basis of the one dimensional space $\mathrm{Im}(s-1)|_{\mathfrak{h}^*}$ and $\alpha_s^\vee \in \mathfrak{h}$ a basis of the one dimensional space $\mathrm{Im}(s-1)|_{\mathfrak{h}}$ such that $\alpha_s(\alpha_s^\vee) = 2$. Choose $\mathbf{c} : \mathcal{S} \to \mathbb{C}$ to be a W-invariant function and t a complex number. The *rational Cherednik algebra*, $H_{t,\mathbf{c}}(W)$, as introduced by Etingof and Ginzburg, [EG, page 250], is the quotient of the skew group algebra of the tensor algebra, $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes W$, by the ideal generated by the relations

$$(2) \qquad [x_1, x_2] = 0 \qquad [y_1, y_2] = 0 \qquad [x_1, y_1] = t(y_1, x_1) - \sum_{s \in S} \mathbf{c}(s)(y_1, \alpha_s)(\alpha_s^\vee, x_1)s$$

$\forall x_1, x_2 \in \mathfrak{h}^*$ and $y_1, y_2 \in \mathfrak{h}$.

Since there is an isomorphism $H_{\lambda t, \lambda \mathbf{c}}(W) \cong H_{t,\mathbf{c}}(W)$ for any $\lambda \in \mathbb{C}^*$, we can restrict ourselves to considering the cases $t = 0$ or 1.

2.1.1. *Parabolic subgroups.* For a point $b \in \mathfrak{h}$, the stabilizer subgroup of W with respect to b will be denoted W_b. By a theorem of Steinberg, [S2, Theorem 1.5], W_b is itself a complex reflection group. If $(\mathfrak{h}^{*W_b})^\perp$ denotes the vector subspace of $\mathfrak{h}$ consisting of all vectors y in $\mathfrak{h}$ such that $x(y) = 0 \quad \forall y \in \mathfrak{h}^{*W_b}$ then $\mathfrak{h} = \mathfrak{h}^{W_b} \oplus (\mathfrak{h}^{*W_b})^\perp$ is a decomposition of $\mathfrak{h}$ as a W_b-module. Note that $(\mathfrak{h}^{*W_b})^\perp$ is a faithful reflection representation of W_b of minimal rank.

2.1.2. *Centralizer algebras.* We recall the centralizer construction described in [BE, 3.2]. Let A be an $\mathbb{C}$-algebra equipped with a homomorphism $H \longrightarrow A^\times$, where H is a finite group. Let G be another finite group such that H is a subgroup of G. The algebra $C(G, H, A)$ is defined to be the centralizer of A in the right A-module $P := \mathrm{Fun}_H(G, A)$ of H-invariant, A-valued functions on G. By making a choice of left coset representatives of H in G, $C(G, H, A)$ is realized as the algebra of $|G/H|$ by $|G/H|$ matrices over A. For $w, g \in G$ and $f \in \mathrm{Fun}_H(G, A)$, $w \cdot f(g) := f(gw)$ defines, by linearity, an embedding $\iota : \mathbb{C}G \hookrightarrow C(G, H, A)$. Let $\mathbf{e}_G \in \mathbb{C}G$ and $\mathbf{e}_H \in \mathbb{C}H$ denote the idempotents corresponding to the trivial representation of G and H respectively, where $\mathbb{C}H$ is considered as a subalgebra of A.

Lemma 2.1. *There are isomorphisms of $\mathbb{C}G$-$Z(A)$-bimodules*

$$C(G, H, A) \cdot \iota(\mathbf{e}_G) \simeq \mathit{Fun}_H(G, A\mathbf{e}_H) \simeq \mathtt{Ind}_H^G A\mathbf{e}_H,$$

where $Z(A)$ denotes the centre of A. Here $\mathbb{C}G$ acts on $C(G, H, A)$ by multiplication on the left via ι and on the left of $\mathit{Fun}_H(G, A\mathbf{e}_H)$ also via ι.

Proof. The second isomorphism is clear from the definition of $\mathrm{Fun}_H(G, A)$. Let $\delta \in \mathrm{Fun}_H(G, A)$ be the function defined by $\delta(g) = \mathbf{e}_H$, for all $g \in G$. We define a linear map ζ from $C(G, H, A) \cdot \iota(\mathbf{e}_G)$ to $\mathrm{Fun}_H(G, A\mathbf{e}_H)$ and a map η in the opposite direction by

$$\zeta : \quad M \cdot \iota(\mathbf{e}_G) \mapsto M(\delta)$$

$$\eta : \quad f \mapsto \left(h(-) \mapsto f(-) \sum_{g \in G} h(g) \right),$$

where $M \in C(G, H, A)$, $f \in \mathrm{Fun}_H(G, A\mathbf{e}_H)$ and $h \in \mathrm{Fun}_H(G, A)$. After fixing left coset representatives of H in G, a direct calculation shows that η is both a left and right inverse to ζ. The G-equivariance of ζ is clear since

$$g \cdot \zeta(M\iota(\mathbf{e}_G)) = g \cdot M(\delta) = \iota(g)(M(\delta)) = (\iota(g)M)(\delta) = \zeta(g \cdot M\iota(\mathbf{e}_G))$$

The $Z(A)$-equivariance of ζ is similarly clear. $\square$

2.2. **Completing the rational Cherednik algebra.** For each point $b \in \mathfrak{h}$, the completion, $\widehat{H}_{1,\mathbf{c}}(W)_b$, at the orbit $W \cdot b \in \mathfrak{h}/W$ of $H_{1,\mathbf{c}}(W)$ is defined in [BE, 2.4]. However, the notion of completion at $W \cdot b$ works equally well when $t = 0$ because $H_{0,\mathbf{c}}(W)$ can be thought of as a sheaf of algebras on the affine variety $\mathfrak{h}/W$. Therefore, if $\mathbb{C}[[\mathfrak{h}/W]]_b$ denotes the completion of $\mathbb{C}[\mathfrak{h}/W]$ at $W \cdot b$, we define the completion of $H_{0,\mathbf{c}}(W)$ at b to be

$$\widehat{H}_{0,\mathbf{c}}(W)_b := \mathbb{C}[[\mathfrak{h}/W]]_b \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_{0,\mathbf{c}}(W) \tag{3}$$

Crucially, we note that [BE, Theorem 3.2] is independent of the parameter t and hence can be applied to the case $t = 0$. We state it here for completeness.

Theorem 2.2 ([BE], Theorem 3.2)**.** *Let $b \in \mathfrak{h}$, and define $\mathbf{c}'$ to be the restriction of $\mathbf{c}$ to the set S_b of reflections in W_b. Then one has an isomorphism*

$$\theta : \widehat{H}_{t,\mathbf{c}}(W, \mathfrak{h})_b \to C(W, W_b, \widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0), \tag{4}$$

defined by the following formulas. Suppose that $f \in P = Fun_{W_b}(W, \widehat{H}_{t,\mathbf{c}'}(W_b, \mathfrak{h})_0)$. Then

$$(\theta(u)f)(w) = f(wu), u \in W;$$

for any $\alpha \in \mathfrak{h}^*$,

$$(\theta(x_\alpha)f)(w) = (x^{(b)}_{w\alpha} + (w\alpha, b))f(w),$$

where $x_\alpha \in \mathfrak{h}^* \subset H_{t,\mathbf{c}}(W, \mathfrak{h}), x^{(b)}_{w\alpha} \in H_{t,\mathbf{c}'}(W_b, \mathfrak{h})$; and for any $a \in \mathfrak{h}$,

$$(\theta(y_a)f)(w) = y^{(b)}_{wa} f(w) + \sum_{s \in S : s \notin W_b} \frac{2c_s}{1 - \lambda_s} \frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s} + \alpha_s(b)} (f(sw) - f(w)).$$

where $y_a \in \mathfrak{h} \subset H_{t,\mathbf{c}}(W, \mathfrak{h})$ and $y^{(b)}_a$ the same vector considered now as an element of $H_{t,\mathbf{c}'}(W_b, \mathfrak{h})$.

3. The Etingof-Ginzburg Sheaf

Let $Z_{\mathbf{c}}(W)$ denote the centre of $H_{0,\mathbf{c}}(W)$ and $X_{\mathbf{c}}(W) = \mathrm{maxspec}(Z_{\mathbf{c}}(W))$, the corresponding affine variety. The space $X_{\mathbf{c}}(W)$ is called the *generalized Calogero-Moser space* associated to the complex reflection group W at the parameter $\mathbf{c}$. For $b \in \mathfrak{h}$, the maximal ideal of $\mathbb{C}[\mathfrak{h}/W]$ corresponding to $W \cdot b$ will be written $\mathfrak{m}(b)$ and the two-sided ideal of $H_{0,\mathbf{c}}(W)$ generated by the elements of $\mathfrak{m}(b)$ will be denoted $\langle \mathfrak{m}(b) \rangle$. Similarly, the maximal ideal of $\mathbb{C}[\mathfrak{h}/W_b]$ corresponding to $W_b \cdot p$, $p \in \mathfrak{h}$, will be written $\mathfrak{n}(p)$.

Let $\mathcal{A}$ denote the set of reflecting hyperplanes of W in $\mathfrak{h}$ and, for each $H \in \mathcal{A}$, let $L_H \in \mathfrak{h}^*$ be a linear functional whose kernel is H (e.g. $\alpha_s \in \mathfrak{h}^*$ if s is a reflection about H). Choose homogeneous algebraically independent generators $F_1, \ldots F_n$ of $\mathbb{C}[\mathfrak{h}]^W$ and $P_1, \ldots, P_n$ of $\mathbb{C}[\mathfrak{h}]^{W_b}$. The following description of the Jacobian is due to Steinberg, [S1, Lemma].

$$\Pi_W := \det \left(\frac{\partial F_i}{\partial x_j} \right) = k \prod_{H \in \mathcal{A}} L_H^{e_H - 1} \tag{5}$$

where e_H is the order of the cyclic group W_H of elements of W that fix H pointwise and k a non-zero scalar.

Lemma 3.1. *For each $b \in \mathfrak{h}$ the map $\Psi : \mathbb{C}[[\mathfrak{h}/W_b]]_0 \longrightarrow \mathbb{C}[[\mathfrak{h}/W_b]]_0$ defined by*

$$P_i(\mathbf{x}) \mapsto F_i(\mathbf{x}+b) - F_i(b)$$

is an automorphism.

Proof. Since $F_i(\mathbf{x}+b)-F_i(b) \in \mathfrak{n}(0)$ for all i there exist polynomials $Q_1, \ldots, Q_n$ such that $F_i(\mathbf{x}+b)-F_i(b) = Q_i(P_1, \ldots, P_n)$. The chain rule gives

$$D := \det\left(\frac{\partial(F_i(\mathbf{x}+b) - F_i(b))}{\partial x_j}\right) = \det\left(\frac{\partial Q_i}{\partial P_k}\right)\det\left(\frac{\partial P_k}{\partial x_j}\right)$$

However, $D = \Pi_W(\mathbf{x}+b)$ and this gives

$$\prod_{H\in\mathcal{A}} L_H^{e_H-1}(\mathbf{x}+b) = \det\left(\frac{\partial Q_i}{\partial P_k}\right)\prod_{H\in\mathcal{A} \text{ with } b\in H} L_H^{e_H-1}(\mathbf{x})$$

Since $L_H(\mathbf{x}+b) = L_H(\mathbf{x})$ if and only if $b \in H$, we get

$$\det\left(\frac{\partial Q_i}{\partial P_k}\right) = \prod_{H\in\mathcal{A} \text{ with } b\notin H} L_H^{e_H-1}(\mathbf{x}+b)$$

and

$$\det\left(\frac{\partial Q_i}{\partial P_k}\right)(0) = \prod_{H\in\mathcal{A} \text{ with } b\notin H} L_H^{e_H-1}(b) \neq 0.$$

Hence, by [E, Exercise 7.25], Ψ is an isomorphism. □

As a consequence of Theorem 2.2, we have an isomorphism of quotient algebras.

Corollary 3.2. *Let $\theta : \widehat{H}_{0,\mathbf{c}}(W,\mathfrak{h})_b \to C(W, W_b, \widehat{H}_{0,\mathbf{c}'}(W_b,\mathfrak{h})_0)$ be the isomorphism (4). Then θ descends to an isomorphism*

$$\theta : \frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle\mathfrak{m}(b)\rangle} \xrightarrow{\sim} C\left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle\mathfrak{n}(0)\rangle}\right). \tag{6}$$

Proof. For $a \in \mathfrak{h}$, $\alpha \in \mathfrak{h}^*$ and $w \in W$, $(x_{w\cdot\alpha} + (w\alpha, b))(a) = (w\alpha, a) + (w\alpha, b) = (w\cdot x_\alpha)(a+b)$. Therefore $\theta(g)(f(w)) = (w\cdot g)(\mathbf{x}+b)f(w) = g(\mathbf{x}+b)f(w)$ for all $g \in \mathbb{C}[\mathfrak{h}]^W \subset \mathbb{C}[[\mathfrak{h}]]_b$ and $f \in \mathrm{Fun}_{W_b}(W, \widehat{H}_c(W_b,\mathfrak{h})_0)$. Now choose $u \in W_b$, then

$$u\cdot g(\mathbf{x}+b) = g(u^{-1}\cdot\mathbf{x}+b) = g(u^{-1}\cdot(\mathbf{x}+b)) = g(\mathbf{x}+b)$$

shows that $g(\mathbf{x}+b) \in \mathbb{C}[\mathfrak{h}]^{W_b}$. Hence, if $g \in \mathfrak{m}(b) \lhd \mathbb{C}[\mathfrak{h}]^W$, then $g(\mathbf{x}+b) \in \mathfrak{n}(0) \lhd \mathbb{C}[\mathfrak{h}]^{W_b}$. This shows that $\theta(g)f(w) \in \mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b)_0$ and

$$\theta(\mathfrak{m}(b)\widehat{H}_{t,\mathbf{c}}(W,\mathfrak{h})_b) \subseteq C(W, W_b, \mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b,\mathfrak{h})_0). \tag{7}$$

The ideal $\mathfrak{m}(b)$ in $\mathbb{C}[\mathfrak{h}]^W$ is generated by $F_1(\mathbf{x})-F_1(b),\dots,F_n(\mathbf{x})-F_n(b)$ and we have $\theta(F_i(\mathbf{x})-F_i(b))f(w) = (F_i(\mathbf{x}+b)-F_i(b))f(w)$. The statement of Lemma 3.1 is equivalent to the fact that

$$\{F_1(\mathbf{x}+b)-F_1(b),\dots,F_n(\mathbf{x}+b)-F_n(b))\}\cdot\mathbb{C}[[\mathfrak{h}/W_b]]_0 = \mathfrak{n}(0)\mathbb{C}[[\mathfrak{h}/W_b]]_0.$$

This, together with (7), implies that

$$\theta(\mathfrak{m}(b)\widehat{H}_{t,\mathbf{c}}(W,\mathfrak{h})_b) = C(W,W_b,\mathfrak{n}(0)\widehat{H}_{t,\mathbf{c}'}(W_b,\mathfrak{h})_0).$$

and the isomorphism follows. □

By [EG, Proposition 4.15], $\mathbb{C}[\mathfrak{h}]^W$ is contained in the centre of $H_{0,\mathbf{c}}(W,\mathfrak{h})$ and the embedding defines a surjective morphism $\pi_W : X_{\mathbf{c}}(W) \twoheadrightarrow \mathfrak{h}/W$. The algebra $Z_{0,\mathbf{c}}(W)/\langle\mathfrak{m}(b)\rangle$ is the coordinate ring of the scheme-theoretic pull-back $\pi_W^{-1}(b)$. Comparing the centres of the algebras in Corollary 3.2 gives an isomorphism of (non-reduced) schemes.

Corollary 3.3. *For $b\in\mathfrak{h}$, there is a scheme-theoretic isomorphism*

$$\Phi : \pi_W^{-1}(b) \xrightarrow{\sim} \pi_{W_b}^{-1}(0) \tag{8}$$

Proof. The Satake isomorphism, [EG, Theorem 3.1], is the map $Z_{0,\mathbf{c}}(W)\to \mathbf{e}_W\, H_{0,\mathbf{c}}(W,\mathfrak{h})\,\mathbf{e}_W$ defined by $z\mapsto z\,\mathbf{e}_W$. Since $\mathfrak{m}(b)H_{0,\mathbf{c}}(W,\mathfrak{h})$ is a centrally generated ideal in $H_{0,\mathbf{c}}(W,\mathfrak{h})$,

$$\mathfrak{m}(b)H_{0,\mathbf{c}}(W,\mathfrak{h})\cap \mathbf{e}_W\, H_{0,\mathbf{c}}(W,\mathfrak{h})\,\mathbf{e}_W = \langle \mathbf{e}_W\,\mathfrak{m}(b)\rangle,$$

where the right-hand side is considered as an ideal in $\mathbf{e}_W\, H_{0,\mathbf{c}}(W,\mathfrak{h})\,\mathbf{e}_W$. Therefore the Satake isomorphism descends to an isomorphism

$$\mathsf{S}_{W,b} : \frac{Z_{0,\mathbf{c}}(W)}{\mathfrak{m}(b)Z_{0,\mathbf{c}}(W)} \xrightarrow{\sim} \mathbf{e}_W\left(\frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle\mathfrak{m}(b)\rangle}\right)\mathbf{e}_W. \tag{9}$$

As noted in [BE, Lemma 3.1 (ii)], the isomorphism (6) restricts to an isomorphism of subalgebras

$$\theta : \mathbf{e}_W\left(\frac{H_{0,\mathbf{c}}(W,\mathfrak{h})}{\langle\mathfrak{m}(b)\rangle}\right)\mathbf{e}_W \xrightarrow{\sim} \mathbf{e}_{W_b}\left(\frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle\mathfrak{n}(0)\rangle}\right)\mathbf{e}_{W_b}, \tag{10}$$

where $\theta(\mathbf{e}_W)=\mathbf{e}_{W_b}$. Here we have implicitly identified the spherical algebra on the right-hand side with a subalgebra of $C\left(W,W_b,\frac{H_{0,\mathbf{c}'}(W_b,\mathfrak{h})}{\langle\mathfrak{n}(0)\rangle}\right)$. It is possible, though uninformative, to give an explicit description of this identification. Combining the isomorphisms of (9) and (10) produces the comorphism

$$(\Phi^*)^{-1} = \mathsf{S}_{W_b,0}^{-1}\circ\theta\circ S_{W,b} : \frac{Z_{0,\mathbf{c}}(W)}{\mathfrak{m}(b)Z_{0,\mathbf{c}}(W)} \xrightarrow{\sim} \frac{Z_{0,\mathbf{c}'}(W_b)}{\mathfrak{n}(0)Z_{0,\mathbf{c}'}(W_b)} \tag{11}$$

corresponding to Φ. □

Etingof and Ginzburg, [EG, page 247], introduced an important coherent sheaf on $X_{\mathbf{c}}(W)$, which we now recall.

Definition 3.4. The *Etingof-Ginzburg sheaf* is the coherent sheaf $\mathcal{R}[W]$ on $X_{\mathbf{c}}(W)$ corresponding to the finitely generated $Z_{0,\mathbf{c}}(W)$-module $H_{0,\mathbf{c}}(W)\,\mathbf{e}_W$.

The coordinate ring of a Zariski-open subset $U\subseteq X_{\mathbf{c}}(W)$ will be written $Z_{0,\mathbf{c}}(W)_U$. We now conclude;

Theorem 3.5. *Let $\mathcal{R}[W]$ be the Etingof-Ginzburg sheaf on $X_{\mathbf{c}}(W)$ and $\mathcal{R}[W_b]$ the Etingof-Ginzburg sheaf on $X_{\mathbf{c}'}(W_b)$. For $b \in \mathfrak{h}/W$ we have an isomorphism of W-equivariant sheaves on $\pi_{W_b}^{-1}(0)$*

$$\Phi_* \left(\mathcal{R}[W]_{|_{\pi_W^{-1}(b)}} \right) \simeq \mathit{Ind}_{W_b}^{W} \mathcal{R}[W_b]_{|_{\pi_{W_b}^{-1}(0)}}. \tag{12}$$

Proof. Since $\pi_W^{-1}(b)$ is an affine scheme, to show that we have an isomorphism of W-equivariant sheaves as stated in (12) it suffices to show that the global sections are isomorphic as $(W, Z_{0,\mathbf{c}'}(W_b) \,/\, \langle \mathfrak{n}(0) \rangle =: \mathsf{Z})$-bimodules. Taking global sections gives

$$\Phi_* \left(\mathcal{R}[W]_{|_{\pi_W^{-1}(b)}} \right) (\pi_{W_b}^{-1}(0)) = \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W$$

and

$$\mathtt{Ind}_{W_b}^{W} \mathcal{R}[W_b]_{|_{\pi_{W_b}^{-1}(0)}} (\pi_{W_b}^{-1}(0)) = \mathtt{Ind}_{W_b}^{W} \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \mathbf{e}_{W_b}.$$

Thus we must show that

$$\mathsf{He} := \left(\frac{H_{0,\mathbf{c}}(W)}{\langle \mathfrak{m}(b) \rangle} \right) \mathbf{e}_W \simeq \mathtt{Ind}_{W_b}^{W} \left(\frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \mathbf{e}_{W_b}$$

as (W, Z)-bimodules. Applying the isomorphism θ (of (6)) to He, and noting that the restriction of θ to $\mathbb{C}W$ is the map ι, gives

$$\theta \,:\, \mathsf{He} \simeq C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \iota(\mathbf{e}_W).$$

However, we now have two different actions of Z on He. It acts on He, viewed as global sections, via the map Φ^*, but acts on the right of $C \left(W, W_b, \frac{H_{0,\mathbf{c}'}(W_b, \mathfrak{h})}{\langle \mathfrak{n}(0) \rangle} \right) \iota(\mathbf{e}_W)$ via θ^{-1}. These two actions are the same: as stated in (11),

$$\Phi^* = \mathsf{S}_{W,b}^{-1} \circ \theta^{-1} \circ S_{W_b,0},$$

therefore

$$h\, \mathbf{e}_W \cdot \Phi^*(z) = h\, \mathbf{e}_W \cdot \mathsf{S}_{W,b}^{-1} \circ \theta \circ S_{W_b,0}(z) = h\, \mathbf{e}_W \cdot \mathbf{e}_W\, \theta^{-1}(\mathbf{e}_{W_b} \cdot z) = h\, \mathbf{e}_W \cdot \theta^{-1}(z),$$

where $z \in \mathsf{Z}$ and $h\, \mathbf{e}_W \in \mathsf{He}$ (recall that $\theta(\mathbf{e}_W) = \mathbf{e}_{W_b}$, c.f. (11)). Noting that Z is a subalgebra of the centre of $H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) \,/\, \langle \mathfrak{n}(0) \rangle$, the required bimodule isomorphism is given by Lemma 2.1 where $G = W$, $H = W_b$ and $A = H_{0,\mathbf{c}'}(W_b, \mathfrak{h}) \,/\, \langle \mathfrak{n}(0) \rangle$. □

Example 3.6. In the case $W = S_n, \mathfrak{h} = \mathbb{C}^n$, the Calogero-Moser space $X_{\mathbf{c}}(S_n)$ has been shown by Etingof and Ginzburg, [EG, Theorem 1.23], to be isomorphic to the classical Calogero-Moser space as introduced by Kazhdan, Kostant and Sternberg and studied by Wilson, [W]. It is known to be smooth for $\mathbf{c} \neq 0$ ([EG, Corollary 16.2] or [W, Proposition 1.7]), therefore [EG, Theorem 1.7 (i)] implies that $\mathcal{R}[S_n]$ is a vector bundle of rank $n!$ on $X_{\mathbf{c}}(S_n)$. Identifying $\mathbb{C}^n/S_n$ with $Sym^n(\mathbb{C})$, a point of $\mathbb{C}^n/S_n$ has the form $n_1x_1 + \cdots + n_kx_k$, where $n_1 + \cdots + n_k = n$ and $x_1, \ldots, x_k \in \mathbb{C}$ are pairwise distinct. Given $b \in \mathbb{C}^n$ such that $S_n \cdot b = n_1x_1 + \cdots + n_kx_k$, the stabilizer $(S_n)_b$ is conjugate to $S_{n_1} \times \cdots \times S_{n_k}$. For $W = S_n$, the isomorphism of Corollary 3.3 induces, after factoring out nilpotent elements, an isomorphism of varieties

$$\pi_{S_n}^{-1}(b) \simeq \pi_{S_{n_1}}^{-1}(0) \times \cdots \times \pi_{S_{n_k}}^{-1}(0). \tag{13}$$

In [W, Lemma 7.1], Wilson explicitly constructs an isomorphism between the subvarieties of the classical Calogero-Moser space coinciding with the varieties of (13). Let $\boxtimes$ denote the external tensor product of vector bundles, then Theorem 3.5 implies that

$$\Phi_* \left(\mathcal{R}[S_n]_{|_{\pi_{S_n}^{-1}(b)}} \right) \simeq \mathtt{Ind}_{S_{n_1} \times \cdots \times S_{n_k}}^{S_n} \left(\mathcal{R}[S_{n_1}]_{|_{\pi_{S_{n_1}}^{-1}(0)}} \boxtimes \ldots \boxtimes \mathcal{R}[S_{n_k}]_{|_{\pi_{S_{n_k}}^{-1}(0)}} \right)$$

as S_n-equivariant vector bundles. This confirms the conjectured factorization given in [EG, 11.27].

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School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, James Clerk Maxwell Building, Kings Buildings, Mayfield Road, Edinburgh EH9 3JZ, Scotland

E-mail address: `G.E.Bellamy@sms.ed.ac.uk`