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LINE-TENSION MODEL FOR PLASTICITY AS THE Γ-LIMIT OF A NONLINEAR DISLOCATION ENERGY

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Abstract. In this paper we rigorously derive a line-tension model for plasticity as the Γ-limit of a *nonlinear* mesoscopic dislocation energy, without resorting to the introduction of an *ad hoc* cut-off radius. The Γ-limit we obtain as the length of the Burgers vector tends to zero has the same form as the Γ-limit obtained by starting from a *linear, semi-discrete* dislocation energy. The nonlinearity, however, creates several mathematical difficulties, which we tackled by proving suitable versions of the Rigidity Estimate in non-simply-connected domains and by performing a rigorous two-scale linearisation of the energy around an equilibrium configuration.

Keywords: Nonlinear elasticity, dislocations, plasticity, strain gradient theories, rigidity estimate, Γ-convergence.

1. Introduction

In this paper we rigorously derive a line-tension model for plasticity from a nonlinear mesoscopic dislocation model. Since the motion of dislocations is regarded as the main cause of plastic deformation, a large literature is focused on the problem of deriving plasticity theories from more fundamental dislocation models. There are a number of strain-gradient crystal plasticity models available in the engineering literature (see, for example, [4, 14, 19, 20] and [26]), all derived phenomenologically. A mathematically rigorous derivation of a plasticity model à la Fleck and Hutchinson [14] has been obtained in [16], starting from a *semi-discrete* dislocation model. A line-tension plasticity model has been obtained by Garroni and Müller (see [17, 18]) starting from a phase-field model for dislocations introduced in [21] (see also more recent papers by Cacace and Garroni [6] and by Conti, Garroni and Müller [9]).

Although a dislocation is a lattice defect, in most dislocation models it has been described in the framework of a continuum theory, in which the positions of the atoms are averaged out. Indeed this reduces enormously the total number of degrees of freedom: From all atom positions to a few geometric quantities (displacement/deformation, dislocation line, slip planes, *etc.*).

The starting point of our derivation is also a continuum dislocation model. The main novelty of our approach is that we consider a *nonlinear* dislocation energy, whereas most mathematical and engineering papers treat only a quadratic dislocation energy (see, e.g.,[7, 16, 24]), so that the constitutive relation between stress and strain is *linear*. These models are referred to as *semi-discrete* dislocation models. Clearly, the linear constitutive relation is not satisfactory close to the dislocations' cores, where the strains are too large for the linear approximation to hold. Moreover, since the internal stress field caused by a dislocation decays as $1/r$, with the distance r from the dislocation, the associated strain energy blows up at a dislocation. The conventional way of fixing this problem is to exclude in the computation of the energy the contribution in a small tube (or disc in the two-dimensional case) around the dislocation. Therefore an internal scale, proportional to the interatomic distance, needs to be introduced. This is the so called *core radius*. The resulting strain-energy then diverges logarithmically in the core radius and has to be rescaled in order to obtain a finite energy in the limit of vanishing core radius. We remark that semi-discrete dislocation models are continuum models, but contrary to classical continuum models they are inherently size dependent, since they contain a small parameter, the core radius, which is reminiscent of the discrete structure of the crystal lattice. The core radius, though, is introduced only for mathematical reasons and it provides a poor representation of the structure of the lattice close to the dislocation.

A possible approach to overcome this problem is to start from a purely discrete (linear) model for dislocations (see [2, 24]) and to compute the continuum limit as the lattice size tends to zero. Another possibility is to consider a more general, nonlinear constitutive relation between stress and strain. Indeed, the blow-up of the strain energy in the linear case is due to the fact that the energy density exhibits a non-integrable singularity at zero, being essentially $1/r^2$ (in the $2D$-case). Therefore, intuitively, it is possible to choose nonlinear stress-strain constitutive relations for which the strain energy in the core regions around the dislocations is finite. Here we follow this approach. Namely, we consider a nonlinear energy density W satisfying the following mixed growth conditions:

$$C_1\Big(\mathrm{dist}^2(F,SO(2))\wedge(|F|^p+1)\Big)\le W(F)\le C_2\Big(\mathrm{dist}^2(F,SO(2))\wedge(|F|^p+1)\Big), \tag{1.1}$$

with $C_1, C_2 > 0$ and $1 < p < 2$ (see [22], where similar energy densities are considered to study a dimensional reduction problem for bi-phase materials with dislocations at the interface).

To our knowledge, the present paper is the first one in which a Γ-convergence analysis of a nonlinear dislocation energy is carried out.

We treat idealised dislocations, *pure edge dislocations*, and we assume that the dislocation lines are straight and parallel. Therefore the problem is two-dimensional, living in the plane orthogonal to the direction of the dislocation lines, and involves only two components of the deformation. More precisely, given a displacement u of a domain $\Omega\subset\mathbb{R}^2$ and denoting with $\beta\colon\Omega\to\mathbb{R}^{2\times 2}$ the elastic part of the strain ∇u, the nonlinear elastic energy is given by

$$\int_\Omega W(\beta)\,dx. \tag{1.2}$$

In this work we analyse the case of a finite number $N\ge 1$ of fixed dislocations, that we identify with N points in Ω representing the singularities of the strain field β.

Notice that the mixed growth conditions (1.1) satisfied by W allow us to define the strain energy (1.2) in the whole domain Ω, hence also close to the dislocations. In fact, our choice of W entails that the nonlinear strain energy has a quadratic growth only for small strains; *i.e.,* far from the dislocations, while it has a p-growth close to the dislocations. Therefore, since $W(\beta)\sim\frac{1}{r^p}$ is integrable at zero for $1 < p < 2$, the energy contribution in the *core regions* is finite.

At a first look, the nonlinear energy (1.2) does not contain explicitly the small parameter, say $\varepsilon > 0$, describing the underlying lattice in the original discrete dislocation problem. This lattice parameter is indeed recovered via the incompatibility condition that the strain β has to satisfy. The above condition asserts that the circulation of β around each dislocation is proportional to the Burgers vector (which is of the order of the lattice spacing ε); *i.e.,*

$$\operatorname{Curl}\beta = \varepsilon \sum_{i=1}^{N} \hat{b}_i \delta_{x_i}, \tag{1.3}$$

where $\hat{b}_1, \dots, \hat{b}_N \in S^1$ are the directions of the Burgers vectors corresponding to a system of fixed dislocations located at $x_1, \dots, x_N \in \Omega$. With this choice the dislocation density $\frac{\operatorname{Curl}\beta}{\varepsilon}$ is fixed and therefore the energy (1.2) depends only on the strain β.

It turns out that the dislocation energy (1.2) associated to a strain β satisfying the admissibility condition (1.3) is of order $\varepsilon^2|\log\varepsilon|$ in the lattice parameter ε. We notice that this scaling is the same as that of the linear dislocation energy (see, e.g. [24]). In the linear case, however, one can equivalently assume that the Burgers vectors are fixed, of length one, and scale the dislocation energy by $|\log\varepsilon|$.

The scaled elastic energy corresponding to a strain β satisfying (1.3) is then defined by

$$\frac{1}{\varepsilon^2|\log\varepsilon|} \int_{\Omega} W(\beta)\, dx. \tag{1.4}$$

The logarithmic scaling of the energy and the topological singularities of the strain resemble some features of the Ginzburg-Landau model for vortices [5, 25], but in the considered nonlinear setting the connection with this model is quite formal.

Some of the technical difficulties arising in the study of the asymptotic behaviour of (1.4) originate from the nonlinear nature of the model, and some from the specific growth assumptions (1.1) on the energy density W. In order not to deal with all of them at once, we start our analysis focusing on a model case in which the energy density W is nonlinear, but exhibits a *quadratic* nonlinearity. Hence, this model case needs to be "regularised" removing from the domain N core regions of radius $\varepsilon > 0$ around the N point defects $x_1, \dots, x_N$. If the strains β satisfy a suitable variant of (1.3) (see (2.4)), the corresponding scaled-energy is then defined by

$$\frac{1}{\varepsilon^2|\log\varepsilon|} \int_{\Omega_\varepsilon} W(\beta)\, dx, \tag{1.5}$$

where W now behaves essentially as $\operatorname{dist}^2(F, SO(2))$ (see Section 2 for more details) and $\Omega_\varepsilon := \Omega \setminus \cup_{i=1}^N B_\varepsilon(x_i)$. Due to this nonlinear *quadratic* constitutive stress-strain relation, the model case still contains the core radius ε and, therefore, it is only partially satisfactory. Nevertheless, it sets the stage for the subsequent analysis of more general energy densities satisfying the mixed growth conditions (1.1).

The strategy to analyse both the model case and the general case is to *rigorously* reduce to a linear problem, in the spirit of [12], and then to apply the convergence results obtained in the linear setting by Cermelli and Leoni [7], and by Garroni, Leoni and Ponsiglione [16]. This linearisation step is highly non-trivial requiring in particular, for a given strain, a fine estimate of the global deviation from being a rotation in terms of the local deviation (see the Rigidity Estimate [15, Theorem 3.1]). Moreover, since the mixed growth assumptions (1.1) on W introduce further technical difficulties due to the weak regularity of the admissible strains, to focus on the linearisation step and to illustrate it in a clear way we start our analysis with the model case (1.5).

In Section 3 we study the asymptotic behaviour of the energy functionals defined in (1.5) via Γ-convergence (see [11] for a comprehensive introduction to this variational convergence).

The linearisation procedure for this functional is possible thanks to a uniform rigidity estimate in non-simply connected domains with "small" holes, like Ω_ε (see Lemma 3.1).

Indeed, by virtue of this estimate we can prove a convergence result for suitably renormalised sequences of strains with equi-bounded energy (Proposition 3.5) as well as a rigorous second-order Taylor expansion of the energy around an equilibrium configuration, on two different scales: the meso-scale $\varepsilon\sqrt{|\log\varepsilon|}$ of the strain, and the micro-scale ε of the dislocation measure in (1.3). This two-scale linearisation is achieved by means of a careful partition of the domain Ω_ε into disjoint annuli (with fixed outer radii) around each dislocation and the rest of the domain, which is connected. Then, each annulus surrounding a dislocation is in turn split up into annuli via a suitable dyadic decomposition and a delicate energy estimate is performed (see Proposition 3.11).

The limiting macroscopic functional (see Theorem 3.9 and Corollary 3.10) has the same form as the Γ-limit obtained in [16, Theorem 15] by Garroni, Leoni and Ponsiglione in the linear setting (compare also with [7]), in the subcritical regime. Then, since (1.5) can be seen as a nonlinear counterpart of the model introduced in [7] by Cermelli and Leoni, the Γ-converge result established in Theorem 3.9 justifies the usual linear approximation of the energy far from the defects.

Using the techniques developed to study the case model, in Section 4 we analyse the asymptotic behaviour of the general nonlinear functionals defined in (1.4) with W satisfying (1.1).

This is the most physically relevant part of the paper, since the model does not suffer from the same deficiency of the linear theory and of the nonlinear quadratic case treated in Section 3, where an *ad hoc* cut-off radius around each dislocation needs to be introduced.

From the mathematical point of view, a substantial difference with the nonlinear quadratic case is the proof of the compactness result Proposition 4.4. This relies on a version of the Rigidity Estimate valid in the case of mixed growth conditions which has been proved in [22] (see also [8]). Moreover, in this case the linearisation procedure shows some additional difficulties, as (1.1) guarantees, *a priori*, a weaker regularity of the strain field if compared with the quadratic model. Eventually, the Γ-limit of (1.4) is the same line-tension model obtained in the nonlinear quadratic case (Theorem 4.6). Therefore the Γ-convergence results Theorem 3.9 and Theorem 4.6 can be summarized as follows: The two functionals defined in (1.5) and (1.4) (and extended to $L^2(\Omega;\mathbb{R}^{2\times 2})$ and $L^p(\Omega;\mathbb{R}^{2\times 2})$, respectively) Γ-converge to a limiting energy $\mathcal{E}\colon L^2(\Omega;\mathbb{R}^{2\times 2})\times SO(2)\to[0,+\infty]$ of the form

$$\mathcal{E}(\beta,R) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbb{C}(R)\beta\colon \beta\,dx + \sum_{i=1}^{N}\psi(R,\hat{b}_i) & \text{if } \operatorname{Curl}\beta = 0,\\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^{2\times 2})\times SO(2), \end{cases}$$

where $\mathbb{C}(R)=\frac{\partial^2 W}{\partial F^2}(R)$ and ψ is defined through a suitable cell formula.

We remark that the Γ-limit $\mathcal{E}$ depends explicitly on a rotation R. The presence of a rotation in the limit energy is genuinely nonlinear: The Γ-limit of quadratic energies derived in [16] does not contain any rotation. However, in the case of a nonlinear *isotropic* strain energy (namely $W(F)=W(FR)$ for every rotation R), the dependency on the rotation in the limit energy $\mathcal{E}$ is dropped.

In conclusion, our result provides a rigorous justification of a line-tension model by showing that such a model can be derived without necessarily resorting to the introduction of semi-discrete models containing an *ad hoc* cut-off radius.

2. Notation and Setting of the problem

In this section we introduce the two models we are going to study. Namely, we define two nonlinear dislocation energies associated to the (elastic part of the) deformation strain in presence of a *finite system* of fixed *edge dislocations*.

We restrict our analysis to the case of *plane isotropic elasticity*, so that straight edge dislocations orthogonal to the plane of strain can be modeled by point defects in $\mathbb{R}^2$.

2.1. **The reference configuration.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded, Lipschitz domain. Let $N \geq 1$ denote the number of dislocations, and $x_1, \dots, x_N$ their positions in Ω. Every dislocation is characterized by its position and by the Burgers vector, which represents the circulation of the deformation strain close to the dislocation. The size of the Burgers vector (or, equivalently, the interatomic distance in the discrete lattice) is denoted by $\varepsilon > 0$. Let $S \subset S^1$ denote the set of admissible directions for the Burgers vectors; therefore, the Burgers vectors associated with the system of dislocations located at $x_1, \dots, x_N \in \Omega$ can be written as $\varepsilon\hat{b}_1, \dots, \varepsilon\hat{b}_N$, where $\hat{b}_i \in S$, for every $i = 1, \dots, N$.

2.2. **The dislocation energy density.** Let $W : \mathbb{R}^{2\times 2} \to [0, +\infty]$ satisfy the usual assumption of nonlinear elasticity; *i.e.,* W has a single well at $SO(2)$, where $SO(2) := \{R \in \mathbb{R}^{2\times 2} : R^T R = I, \det R = 1\}$ denotes the set of rotations in $\mathbb{R}^{2\times 2}$.

Since, as previously stated, we are going to analyse the asymptotic behaviour of two different nonlinear dislocation energies, which are in particular defined through energy densities satisfying two different growth conditions, here we first list the common assumptions to the two models, then we specify the two different growth conditions (g-2) and (g-p).

The common assumptions on W are the following:

(i) $W \in C^0(\mathbb{R}^{2\times 2})$, $W \in C^2$ in a neighbourhood of $SO(2)$;
(ii) $W(I) = 0$ (stress-free reference configuration);
(iii) $W(RF) = W(F)$ for every $F \in \mathbb{R}^{2\times 2}$ and every $R \in SO(2)$ (frame indifference).

The two models differ in their growth conditions as follows:

(g-2) there exists two constants $C_1, C_2 > 0$ such that for every $F \in \mathbb{R}^{2\times 2}$
$$C_1 \mathrm{dist}^2(F, SO(2)) \leq W(F) \leq C_2 \,\mathrm{dist}^2(F, SO(2));$$
(g-p) there exist $1 < p < 2$ and two constants $C_1, C_2 > 0$ such that for every $F \in \mathbb{R}^{2\times 2}$
$$C_1\Big(\mathrm{dist}^2(F, SO(2)) \wedge (|F|^p + 1)\Big) \leq W(F) \leq C_2\Big(\mathrm{dist}^2(F, SO(2)) \wedge (|F|^p + 1)\Big).$$

We observe that (g-p) requires that the energy density W satisfies a more restrictive bound from above than the one in (g-2). This additional requirement ensures that the dislocation cores have finite energy and is used in the proof of the lim sup-inequality in Theorem 4.6.

The upper bound for the energy density W, though, is unsatisfactory, since it rules out the physically relevant conditions that the deformations are orientation-preserving and that the energy blows up if the body is compressed to zero volume.

2.3. **The model case.** The energy density W satisfies assumptions (i)-(iii) and the quadratic growth condition (g-2). Due to (g-2) the strain energy associated to a deformation is singular at the dislocations, and therefore it is well defined only in the domain Ω_ε obtained removing from Ω a disc of radius ε around each dislocation $x_1, \dots, x_N$. More precisely, we set

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^{N} B_\varepsilon(x_i). \tag{2.1}$$

The effect of the presence of dislocations can be measured in terms of a dislocation density, which represents the failure of the condition of being a gradient for the strain. In the case of a finite number of point defects this dislocation density reads as

$$\mu_\varepsilon := \varepsilon \sum_{i=1}^{N} \hat{b}_i \delta_{x_i}, \tag{2.2}$$

where δ_{x_i} is the Dirac mass centred in x_i.

Then, the class of the admissible strains associated with μ_ε is defined by all functions $\beta \in L^2(\Omega_\varepsilon; \mathbb{R}^{2\times 2})$ satisfying

$$\operatorname{Curl} \beta = 0 \ \text{ in } \Omega_\varepsilon \quad \text{and} \quad \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \varepsilon \hat{b}_i, \text{ for } i = 1, \dots, N,$$

where the equality $\operatorname{Curl}\beta = 0$ is intended in the sense of distributions.[1] The vector t above denotes the oriented tangent vector[2] to $\partial B_\varepsilon(x_i)$ and the integrand $\beta \cdot t$ is intended in the sense of traces (see [13, Theorem 2, pag. 204]).

The scaled elastic energy corresponding to an admissible strain β is defined by

$$\frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{\Omega_\varepsilon} W(\beta) \, dx. \tag{2.3}$$

In what follows it is useful to extend the admissible strains β to the whole domain Ω. There are different possible extensions compatible with our model. Here we decide to consider $\beta = I$ in the discs $B_\varepsilon(x_i)$, for $i = 1, \dots, N$. Therefore, from now on the class of admissible strains associated with the measure μ_ε in (2.2) is given by

$$\mathcal{AS}_\varepsilon^{(2)} := \Big\{ \beta \in L^2(\Omega; \mathbb{R}^{2\times 2}) \colon \beta \equiv I \text{ in } \cup_{i=1}^{N} B_\varepsilon(x_i), \ \operatorname{Curl} \beta = 0 \text{ in } \Omega_\varepsilon, \\ \int_{\partial B_\varepsilon(x_i)} \beta \cdot t \, ds = \varepsilon \hat{b}_i, \text{ for } i = 1, \dots, N \Big\}. \tag{2.4}$$

Since from now on β is extended by the identity outside Ω_ε, by (ii) we can rewrite (2.3) as

$$E_\varepsilon(\beta) := \frac{1}{\varepsilon^2 |\log \varepsilon|} \int_{\Omega} W(\beta) \, dx, \tag{2.5}$$

and we also define the elastic energy induced by the measure μ_ε as

$$F_\varepsilon(\mu_\varepsilon) := \inf_{\beta \in \mathcal{AS}_\varepsilon^{(2)}} E_\varepsilon(\beta). \tag{2.6}$$

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[1] For a matrix $\beta \in \mathbb{R}^{2\times 2}$, $\operatorname{Curl} \beta$ is the vector field of $\mathbb{R}^2$ defined as $\operatorname{Curl} \beta = (\partial_1 \beta_{12} - \partial_2 \beta_{11}, \partial_1 \beta_{22} - \partial_2 \beta_{21})$.

[2] We choose $t = \nu^\perp$ to be a counterclockwise $\pi/2$-rotation of the outward normal ν to ∂B_ε.

Moreover, in view of (2.4) we may also extend E_ε to the space $L^2(\Omega;\mathbb{R}^{2\times 2})$. We set

$$\mathcal{E}_\varepsilon^{(2)}(\beta) := \begin{cases} E_\varepsilon(\beta) & \text{if } \beta \in \mathcal{AS}_\varepsilon^{(2)}, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^{2\times 2}). \end{cases} \tag{2.7}$$

2.4. **The general case.** In this case W satisfies assumptions (i)-(iii) and the mixed growth conditions (g-p).

Let μ_ε be as in (2.2); we define the class of admissible strains associated to the dislocation density μ_ε as follows

$$\mathcal{AS}_\varepsilon^{(p)} := \left\{\beta \in L^p(\Omega;\mathbb{R}^{2\times 2}) : \operatorname{Curl}\beta = \mu_\varepsilon \text{ in } \Omega\right\}. \tag{2.8}$$

Then, the strain energy corresponding to $\beta \in L^p(\Omega;\mathbb{R}^{2\times 2})$ is given by

$$\mathcal{E}_\varepsilon^{(p)}(\beta) := \begin{cases} \dfrac{1}{\varepsilon^2|\log\varepsilon|}\displaystyle\int_\Omega W(\beta)dx & \text{if } \beta \in \mathcal{AS}_\varepsilon^{(p)}, \\ +\infty & \text{otherwise in } L^p(\Omega;\mathbb{R}^{2\times 2}). \end{cases} \tag{2.9}$$

3. The model case: Quadratic growth conditions

In this section we study the asymptotic behaviour, via Γ-convergence, of the sequence of functionals defined in (2.7) (hence under the assumptions (i)-(iii) and (g-2) on W).

We stress once more that the dislocation energy (2.7) is a model case for the more general nonlinear energy we will consider in Section 4. In fact, it shares with the general case some difficulties that are due do the common nonlinear nature of the energies. On the other hand, due to the quadratic growth conditions (g-2), it serves as an *intermediate step* between linear models for dislocations and the more general nonlinear model defined by (2.9).

3.1. **Compactness.** This subsection is devoted to prove a compactness result for sequences $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(2)}$ with equi-bounded energy $\mathcal{E}_\varepsilon^{(2)}$. To this purpose, we start proving a suitable version of the Friesecke, James and Müller Rigidity Estimate [15, Theorem 3.1] in a domain with"small holes" (see [23, Section 4] for analogous results in the linear setting).

Since the Rigidity Estimate holds true in any space dimension $n \geq 2$ and for any exponent $q \in (1,+\infty)$ (see [15, Theorem 3.1] and [10, Section 2.4]), we think worth proving the following lemma in this more general setting.

With a little abuse of notation, we denote by Ω_ε the n-dimensional analogue of (2.1).

Lemma 3.1 (Rigidity with holes)**.** *Let $1 < q < +\infty$, let $n \geq 2$ and let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. There exists a constant $C = C(\Omega, n, q) > 0$ with the following property: Let $\varepsilon > 0$ be sufficiently small, then for every $u \in W^{1,q}(\Omega_\varepsilon;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla u - R\|_{L^2(\Omega_\varepsilon;\mathbb{R}^{n\times n})} \leq C\|\operatorname{dist}(\nabla u, SO(n))\|_{L^2(\Omega_\varepsilon)}. \tag{3.1}$$

Proof. Throughout the proof C is a positive constant independent of ε.

We divide the proof into two steps.

Step 1: Extension to Ω. In this step we extend $u \in W^{1,q}(\Omega_\varepsilon;\mathbb{R}^n)$ to a function $\tilde{u} \in W^{1,q}(\Omega;\mathbb{R}^n)$ satisfying

$$\int_\Omega \mathrm{dist}^q(\nabla\tilde{u}, SO(n))\,dx \le C \int_{\Omega_\varepsilon} \mathrm{dist}^q(\nabla u, SO(n))\,dx, \tag{3.2}$$

for some $C > 0$.

To this end, for every $i = 1, \dots, N$, we apply the Rigidity Estimate [15, Theorem 3.1] in $B_{2\varepsilon}(x_i) \setminus B_\varepsilon(x_i)$ to find a constant $C > 0$ (which is independent of ε thanks to the invariance of the rigidity estimate under uniform scaling of the domain) and N rotations $R^i_\varepsilon \in SO(n)$, $i = 1, \dots, N$, such that

$$\int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} |\nabla u - R^i_\varepsilon|^q\,dx \le C \int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} \mathrm{dist}^q(\nabla u, SO(n))\,dx. \tag{3.3}$$

For $i = 1, \dots, N$, let $\hat{u}^i$ be the restriction of u to $B_{2\varepsilon}(x_i) \setminus B_\varepsilon(x_i)$ and set

$$\hat{u}_{R^i_\varepsilon} := \hat{u}^i - R^i_\varepsilon x, \quad i = 1, \dots, N.$$

Then, consider the functions $v_{R^i_\varepsilon} \in W^{1,q}(B_2(x_i) \setminus B_1(x_i);\mathbb{R}^n)$ defined as

$$v_{R^i_\varepsilon}(y) := \varepsilon^{\frac{n-q}{q}}\, \hat{u}_{R^i_\varepsilon}(\varepsilon y), \quad i = 1, \dots, N.$$

Notice that for every $i = 1, \dots, N$

$$\int_{B_2(x_i)\setminus B_1(x_i)} |\nabla v_{R^i_\varepsilon}|^q\,dy = \int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} |\nabla \hat{u}_{R^i_\varepsilon}|^q\,dx. \tag{3.4}$$

Appealing to the extension result for Sobolev functions [1, Lemma 2.6], for every $i = 1, \dots, N$ we find $T^i(v_{R^i_\varepsilon}) \in W^{1,q}(B_2(x_i);\mathbb{R}^n)$ such that $T^i(v_{R^i_\varepsilon}) \equiv v_{R^i_\varepsilon}$ in $B_2(x_i) \setminus B_1(x_i)$ and

$$\int_{B_2(x_i)} |\nabla T^i(v_{R^i_\varepsilon})|^q\,dy \le C \int_{B_2(x_i)\setminus B_1(x_i)} |\nabla v_{R^i_\varepsilon}|^q\,dy, \tag{3.5}$$

with C depending on q and n.

Eventually, we define the functions $v^i \in W^{1,q}(B_{2\varepsilon}(x_i);\mathbb{R}^n)$ as

$$v^i(x) := \varepsilon^{\frac{q-n}{q}} T^i(v_{R^i_\varepsilon})\left(\frac{x}{\varepsilon}\right), \quad \text{for } i = 1, \dots, N.$$

Notice that, by definition

$$\int_{B_{2\varepsilon}(x_i)} |\nabla v^i|^q\,dx = \int_{B_2(x_i)} |\nabla T^i(v_{R^i_\varepsilon})|^q\,dy, \quad i = 1, \dots, N. \tag{3.6}$$

Hence, if we set

$$\tilde{v}^i := v^i + R^i_\varepsilon x, \quad \text{for } i = 1, \dots, N,$$

it is immediate to check that $\tilde{v}^i \in W^{1,q}(B_{2\varepsilon}(x_i);\mathbb{R}^n)$ and $\tilde{v}^i \equiv \hat{u}^i \equiv u$ in $B_{2\varepsilon}(x_i) \setminus B_\varepsilon(x_i)$. Moreover, combining (3.3), (3.4), and (3.6) we find

$$\int_{B_{2\varepsilon}(x_i)} |\nabla\tilde{v}^i - R^i_\varepsilon|^q\,dx = \int_{B_{2\varepsilon}(x_i)} |\nabla v^i|^q\,dx$$
$$\le C \int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} |\nabla \hat{u}_{R^i_\varepsilon}|^q\,dx \le C \int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} \mathrm{dist}^q(\nabla u, SO(n))\,dx,$$

therefore

$$\int_{B_{2\varepsilon}(x_i)} \operatorname{dist}^q(\nabla\tilde v^i, SO(n))\,dx \le C\int_{B_{2\varepsilon}(x_i)\setminus B_\varepsilon(x_i)} \operatorname{dist}^q(\nabla u, SO(n))\,dx, \tag{3.7}$$

for every $i = 1,\dots,N$. Finally, we define

$$\tilde u := \begin{cases} \tilde v^i & \text{in} \quad B_\varepsilon(x_i), \quad i=1,\dots,N \\ u & \text{in} \quad \Omega_\varepsilon. \end{cases}$$

Clearly, $\tilde u$ extends u to Ω and $\tilde u \in W^{1,q}(\Omega;\mathbb{R}^n)$. Moreover, since (3.7) entails (3.2), the first step is achieved.

Step 2: Rigidity estimate. Now we apply the rigidity estimate to the function $\tilde u \in W^{1,q}(\Omega;\mathbb{R}^n)$ constructed in the previous step. This provides us with a constant $C > 0$ and a rotation $R \in SO(n)$ with the property that

$$\int_\Omega |\nabla\tilde u - R|^q\,dx \le C\int_\Omega \operatorname{dist}^q(\nabla\tilde u, SO(n))\,dx. \tag{3.8}$$

Therefore, in view of (3.2) we have

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla u - R|^q\,dx &\le \int_\Omega |\nabla\tilde u - R|^q\,dx\\ \le C\int_\Omega \operatorname{dist}^q(\nabla\tilde u, SO(n))\,dx &\le C\int_{\Omega_\varepsilon} \operatorname{dist}^q(\nabla u, SO(n))\,dx,\end{aligned}$$

and the claim is proved. □

Lemma 3.1 is a key tool to establish a compactness result for sequences of strains with equi-bounded energy $\mathcal{E}_\varepsilon^{(2)}$.

As $\mathcal{E}_\varepsilon^{(2)}$ is defined in Ω_ε, which contains a finite number of holes with small radius ε, the relevance of Lemma 3.1 is clear. Nevertheless, this lemma cannot be directly applied to a sequence of strains $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(2)}$, as it is not a sequence of gradients. Then, we achieve the compactness result Proposition 3.5 by exploiting the specific singularity of the strains belonging to $\mathcal{AS}_\varepsilon^{(2)}$, and by applying Lemma 3.1 to a new curl-free field $\beta_\varepsilon - \tilde\beta_\varepsilon$, with $\tilde\beta_\varepsilon$ suitably chosen.

Another possible strategy to prove compactness is suggested by observing that β_ε are in fact gradients in a suitable simply connected subset of Ω_ε obtained removing from Ω_ε a finite number of segments. Therefore, a compactness result can be as well a consequence of a variant of Lemma 3.1 for domains with “holes” and “cuts”. Since this alternative approach is suitable for more general types of singularities, we find it interesting to discuss here at least a special case, which is, moreover, an easy consequence of Lemma 3.1.

We consider the case $\Omega = B_1 \subset \mathbb{R}^2$, where B_1 is the unit disc centred at the origin. We assume that there is a single dislocation located at the origin, so that the dislocation density μ_ε (2.2) reads as $\mu_\varepsilon = \varepsilon\hat b\,\delta_0$, with $\hat b \in S$. Let $\beta \in \mathcal{AS}_\varepsilon$, where $\mathcal{AS}_\varepsilon$ is defined as

$$\mathcal{AS}_\varepsilon := \Big\{\beta \in L^2(B_1;\mathbb{R}^{2\times2})\colon \beta \equiv I \text{ in } B_\varepsilon,\ \operatorname{Curl}\beta = 0 \text{ in } B_1\setminus B_\varepsilon,\ \int_{\partial B_\varepsilon} \beta\cdot t\,ds = \varepsilon\hat b\Big\}, \tag{3.9}$$

in analogy with (2.4). We cut the annulus $\Omega_\varepsilon = B_1 \setminus B_\varepsilon$ with the segment $L_\varepsilon := \{(z,0)\colon \varepsilon < z < 1\}$; in this way we obtain the simply connected set $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$. Since $\operatorname{Curl}\beta = 0$ in $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$ by definition, there exists a function $u \in H^1((B_1 \setminus B_\varepsilon) \setminus L_\varepsilon; \mathbb{R}^2)$ such that $\nabla u = \beta$ in $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$.

At this point we prove a modified version of Lemma 3.1, namely a rigidity estimate with a uniform constant in a set with a "hole" and a "cut". Corollary 3.2 below will allow us to prove this result.

We first set some notation. Let $\vartheta \in (0, 2\pi)$; we denote by $S_\varepsilon(\vartheta)$ the open sector of $B_1 \setminus B_\varepsilon$ of angle ϑ; *i.e.*,

$$S_\varepsilon(\vartheta) := \{(r,\theta)\colon \varepsilon < r < 1,\ 0 < \theta < \vartheta\}. \tag{3.10}$$

Corollary 3.2. *Let $1 < q < +\infty$ and let $S_\varepsilon(\vartheta)$ be as in* (3.10)*. There exists a constant $C = C(q,\vartheta) > 0$ with the following property: Let $\varepsilon > 0$ be sufficiently small, then for every $u \in W^{1,q}(S_\varepsilon(\vartheta); \mathbb{R}^2)$ there is an associated rotation $R \in SO(2)$ such that*

$$\|\nabla u - R\|_{L^q(S_\varepsilon(\vartheta);\mathbb{R}^{2\times 2})} \leq C \|\operatorname{dist}(\nabla u, SO(2))\|_{L^q(S_\varepsilon(\vartheta))}. \tag{3.11}$$

Proof. The proof follows exactly the line of that of Lemma 3.1. □

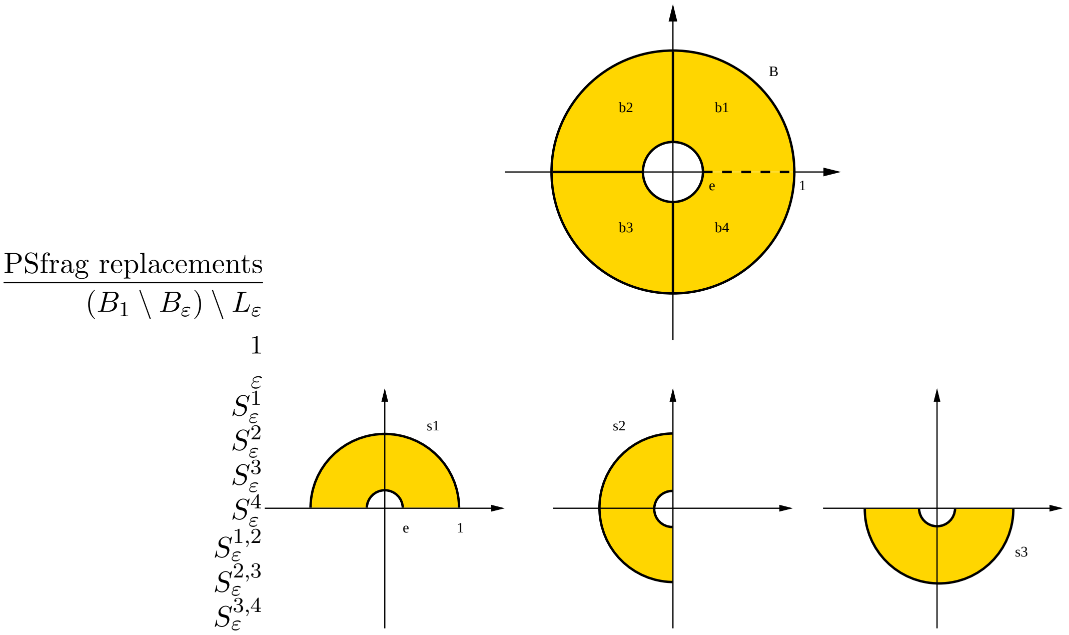

FIGURE 1. Different coverings of $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$.

We have the following uniform rigidity estimate in $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$.

Proposition 3.3 (Rigidity with a "hole" and a "cut"). *Let $1 < q < +\infty$. There exists a constant $C = C(q) > 0$ with the following property: Let $\varepsilon > 0$ be sufficiently small, then for every $u \in W^{1,q}((B_1 \setminus B_\varepsilon) \setminus L_\varepsilon; \mathbb{R}^2)$ there is an associated rotation $R \in SO(2)$ such that*

$$\|\nabla u - R\|_{L^q((B_1\setminus B_\varepsilon)\setminus L_\varepsilon;\mathbb{R}^{2\times 2})} \leq C \|\operatorname{dist}(\nabla u, SO(2))\|_{L^q((B_1\setminus B_\varepsilon)\setminus L_\varepsilon)}. \tag{3.12}$$

Proof. Let $u \in W^{1,q}((B_1 \setminus B_\varepsilon) \setminus L_\varepsilon; \mathbb{R}^2)$. Define $S^1_\varepsilon := S_\varepsilon(\pi/2)$ and let $S^2_\varepsilon, S^3_\varepsilon, S^4_\varepsilon$ be the sets obtained through a rotation of S^1_ε as in Figure 1. Let $R_1, R_2, R_3, R_4 \in SO(2)$ be the constant rotations provided by Corollary 3.2; *i.e.,*

$$\int_{S^i_\varepsilon} |\nabla u - R_i|^q \, dx \le C \int_{S^i_\varepsilon} \operatorname{dist}^q(\nabla u, SO(2)) \, dx, \tag{3.13}$$

for some $C > 0$ and for every $i = 1, \dots, 4$.

We show that (3.12) holds true with $R = R_2$. To do this it is enough to prove that

$$|S^i_\varepsilon||R_2 - R_i|^q \le C \int_{(B_1 \setminus B_\varepsilon)\setminus L_\varepsilon} \operatorname{dist}^q(\nabla u, SO(2)) \, dx, \quad \text{for } i = 1, 3, 4.$$

We start considering $|S^1_\varepsilon||R_1 - R_2|^q$. To this end, we introduce the set $S^{1,2}_\varepsilon := S_\varepsilon(\pi)$ (see Figure 1) and, appealing to Corollary 3.2, the corresponding rotation $R_{1,2} \in SO(2)$; *i.e.,* the constant rotation matrix such that

$$\int_{S^{1,2}_\varepsilon} |\nabla u - R_{1,2}|^q \, dx \le C \int_{S^{1,2}_\varepsilon} \operatorname{dist}^q(\nabla u, SO(2)) \, dx, \tag{3.14}$$

for some $C > 0$. Notice that $S^1_\varepsilon \cup S^2_\varepsilon \subset S^{1,2}_\varepsilon \subset (B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$. As $|S^1_\varepsilon| = |S^2_\varepsilon|$, we immediately deduce

$$\int_{S^1_\varepsilon} |R_1 - R_2|^q \, dx \le C\Big(\int_{S^1_\varepsilon} |R_1 - R_{1,2}|^q \, dx + \int_{S^2_\varepsilon} |R_2 - R_{1,2}|^q \, dx \Big). \tag{3.15}$$

Now we estimate only the first term in the right end side of (3.15), the other being analogous.

Combining (3.13) and (3.14) we get

$$\begin{aligned} \int_{S^1_\varepsilon} |R_1 - R_{1,2}|^q \, dx &\le C\Big(\int_{S^1_\varepsilon} |\nabla u - R_1|^q \, dx + \int_{S^1_\varepsilon} |\nabla u - R_{1,2}|^q \, dx \Big) \\ &\le C \int_{S^1_\varepsilon} \operatorname{dist}^q(\nabla u, SO(2)) \, dx + C \int_{S^{1,2}_\varepsilon} \operatorname{dist}^q(\nabla u, SO(2)) \, dx. \end{aligned}$$

Then, considering the sets $S^{2,3}_\varepsilon$ and $S^{3,4}_\varepsilon$ as in Figure 1 and noticing that $S^{2,3}_\varepsilon, S^{3,4}_\varepsilon \subset (B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$ we can easily proceed as above to estimate $|S^3_\varepsilon||R_2 - R_3|^q$ and $|S^3_\varepsilon||R_3 - R_4|^q$ (and therefore also $|S^4_\varepsilon||R_2 - R_4|^q$). □

Remark 3.4 (Heuristics for the scaling)**.** We note that the definition of the class of admissible deformations (2.4) ensures that the strain energy $\int_\Omega W(\beta)\, dx$ associated to an admissible deformation β is bounded from below by $\varepsilon^2 |\log \varepsilon|$ up to a multiplicative constant, which justifies the scaling in (2.5). Here we prove this bound in the special case $\Omega = B_1$, assuming that there is a single dislocation located at the centre of the disc; *i.e.,* $\mu_\varepsilon = \varepsilon \hat{b}\, \delta_0$. Let $\beta \in \mathcal{AS}_\varepsilon$, where $\mathcal{AS}_\varepsilon$ is defined in (3.9). We cut the annulus $B_1 \setminus B_\varepsilon$ with L_ε and we consider a function $u \in H^1((B_1 \setminus B_\varepsilon) \setminus L_\varepsilon; \mathbb{R}^2)$ such that $\nabla u = \beta$ in $(B_1 \setminus B_\varepsilon) \setminus L_\varepsilon$. Then, Proposition 3.3 (with $q = 2$) applied to u provides us with a constant $C > 0$ and a rotation $R \in SO(2)$ such that

$$\int_{B_1 \setminus B_\varepsilon} |\beta - R|^2 \, dx \le C \int_{B_1 \setminus B_\varepsilon} \operatorname{dist}^2(\beta, SO(2)) \, dx \le C \int_{B_1} W(\beta) \, dx,$$

the last inequality being a consequence of the assumption (g-2) on W. Moreover, since $\beta \in \mathcal{AS}_\varepsilon$ we find

$$\int_{B_1\setminus B_\varepsilon} |\beta - R|^2\,dx \geq \int_\varepsilon^1 \frac{1}{2\pi r}\left|\int_{\partial B_r}(\beta - R)\cdot t\,ds\right|^2 dr$$

$$= \int_\varepsilon^1 \frac{1}{2\pi r}\left|\int_{\partial B_r}\beta\cdot t\,ds\right|^2 dr = \int_\varepsilon^1 \frac{\varepsilon^2}{2\pi r}|\hat{b}|^2\,dr = \frac{\varepsilon^2|\log\varepsilon|}{2\pi}.$$

We now go back to the case of N dislocations in a generic domain Ω.

Appealing to Lemma 3.1, we are in a position to prove a compactness result for suitably renormalised sequences of admissible strains β_ε.

The renormalisation factor for strains β_ε with equi-bounded energy is dictated by the scaling of the energy and by the quadratic growth condition (g-2) on the energy density, and is $\varepsilon\sqrt{|\log\varepsilon|}$. Then, since the natural scaling for the dislocation density μ_ε is ε, the effect of the renormalisation of the strains is that the admissibility condition valid for β_ε disappears in the limit. More precisely, we find that the limit strains β are always gradients; *i.e.,* $\operatorname{Curl}\beta = 0$ (cf. [16, Theorem 15 (i)]).

Proposition 3.5 (Compactness). *Let $\varepsilon_j \to 0$ and let $(\beta_j) \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ be a sequence such that $\sup_j \mathcal{E}^{(2)}_{\varepsilon_j}(\beta_j) < +\infty$. Then there exist a sequence of constant rotations $(R_j) \subset SO(2)$ and a function $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta = 0$ such that (up to subsequences)*

$$\frac{\beta_j - R_j}{\varepsilon_j\sqrt{|\log\varepsilon_j|}} \rightharpoonup \beta \quad in \quad L^2(\Omega;\mathbb{R}^{2\times 2}). \tag{3.16}$$

Proof. Let $(\beta_j) \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ be as in the statement; therefore, in view of assumption (g-2) on W, we have

$$\int_{\Omega_{\varepsilon_j}} \operatorname{dist}^2(\beta_j, SO(2))\,dx \leq \frac{M}{C}\varepsilon_j^2|\log\varepsilon_j|, \tag{3.17}$$

for every j. In $\mathbb{R}^2 \setminus \{x_1,\dots,x_N\}$ we define the function

$$\eta := \sum_{i=1}^N \frac{1}{2\pi}\hat{b}_i \otimes J\frac{x - x_i}{|x - x_i|^2},$$

where J is the clockwise rotation of $\pi/2$; then we set $\tilde{\beta}_j := \varepsilon_j\eta\,\chi_{\Omega_{\varepsilon_j}}$. Clearly $\tilde{\beta}_j$ is defined in the whole Ω; moreover it is immediate to check that

$$\int_\Omega |\tilde{\beta}_j|^2\,dx \leq C\varepsilon_j^2|\log\varepsilon_j|. \tag{3.18}$$

By construction we have $\operatorname{Curl}(\beta_j - \tilde{\beta}_j) = 0$ in Ω_{ε_j} and $\int_{\partial B_\varepsilon(x_i)}(\beta_j - \tilde{\beta}_j)\cdot t\,ds = 0$, for every $i = 1,\dots,N$. Hence, there exists $u_j \in H^1(\Omega_{\varepsilon_j};\mathbb{R}^2)$ such that $\beta_j - \tilde{\beta}_j = \nabla u_j$ in Ω_{ε_j}. Then, Lemma 3.1 (with $q = n = 2$) provides us with a constant $C > 0$ independent of j, and a sequence $(R_j) \subset SO(2)$ such that

$$\begin{aligned}\int_{\Omega_{\varepsilon_j}} |\beta_j - \tilde{\beta}_j - R_j|^2\,dx &= \int_{\Omega_{\varepsilon_j}} |\nabla u_j - R_j|^2\,dx \\ \leq C\int_{\Omega_{\varepsilon_j}} \operatorname{dist}^2(\nabla u_j, SO(2))\,dx &= C\int_{\Omega_{\varepsilon_j}} \operatorname{dist}^2(\beta_j - \tilde{\beta}_j, SO(2))\,dx \\ &\leq C\int_{\Omega_{\varepsilon_j}} \operatorname{dist}^2(\beta_j, SO(2))\,dx + C\int_{\Omega_{\varepsilon_j}} |\tilde{\beta}_j|^2\,dx.\end{aligned}$$

Thus, appealing to (3.17) and (3.18), the previous estimate yields

$$\int_{\Omega_{\varepsilon_j}} \frac{|\beta_j - R_j|^2}{\varepsilon_j^2 |\log\varepsilon_j|}\,dx \leq C,$$

for every j. Finally, recalling that $\beta_j \equiv I$ in $\bigcup_{i=1}^N B_{\varepsilon_j}(x_i)$ we deduce that, up to subsequences,

$$\frac{\beta_j - R_j}{\varepsilon_j\sqrt{|\log\varepsilon_j|}} \rightharpoonup \beta \quad \text{in } L^2(\Omega;\mathbb{R}^{2\times 2}).$$

Now we prove that $\operatorname{Curl}\beta = 0$ in Ω, in the sense of distributions. To this end, let $\phi \in C_0^1(\Omega)$ and let $(\phi_j) \subset H_0^1(\Omega)$ be a sequence converging to ϕ uniformly and strongly in $H_0^1(\Omega)$ and such that $\phi_j \equiv \phi(x_i)$ in $B_{\varepsilon_j}(x_i)$, for $i = 1,\dots,N$. Then we have

$$\begin{aligned}\langle \operatorname{Curl}\beta, \phi\rangle &= \lim_{j\to 0} \frac{1}{\sqrt{|\log\varepsilon_j|}}\langle \operatorname{Curl}\frac{\beta_j - R_j}{\varepsilon_j}, \phi_j\rangle \\ = \lim_{j\to 0} \frac{1}{\sqrt{|\log\varepsilon_j|}}\langle \operatorname{Curl}\frac{\beta_j}{\varepsilon_j}, \phi_j\rangle &= \lim_{j\to 0} \frac{\sum_{i=1}^N \phi(x_i)\,\hat{b}_i}{\sqrt{|\log\varepsilon_j|}} = 0.\end{aligned}$$

□

In view of Proposition 3.5 we give the following notion of (L^2-)convergence for sequences of admissible strains (β_ε).

Definition 3.6. *A sequence $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(2)}$ converges to a pair $(\beta, R) \in L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ if there exists a sequence of rotations $(R_\varepsilon) \subset SO(2)$ such that*

$$\frac{\beta_\varepsilon - R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} \rightharpoonup \beta \quad \text{in} \quad L^2(\Omega;\mathbb{R}^{2\times 2}) \quad \text{and} \quad R_\varepsilon \to R. \tag{3.19}$$

3.2. **Γ-convergence.** The compactness result proved in Proposition 3.5 and Definition 3.6 suggest that the Γ-limit of the energies $\mathcal{E}_\varepsilon^{(2)}$ is a function of a pair: A gradient β and a rotation R, representing, respectively, the macroscopic strain and the rotation acting on the Burgers directions $\hat{b}_1,\dots,\hat{b}_N$ to bring the system back in the reference configuration.

For later references, it is convenient to introduce a new class of admissible (scaled) strains. For $0 < r_1 < r_2 < 1$ and $\xi \in S^1$ we define

$$\mathcal{AS}_{r_1,r_2}(\xi) := \left\{\eta \in L^2(B_{r_2}\setminus B_{r_1})\colon \operatorname{Curl}\eta = 0 \text{ in } B_{r_2}\setminus B_{r_1},\ \int_{\partial B_{r_1}} \eta\cdot t\,ds = \xi\right\}.$$

In the special case $r_2 = 1$ we will simply write $\mathcal{AS}_{r_1}(\xi)$ instead of $\mathcal{AS}_{r_1,1}(\xi)$.

For $\delta \in (0,1)$, $Q \in SO(2)$ we set the following notation:

$$C_\delta := B_1 \setminus B_\delta, \quad L_\delta := \{(z,0)\colon \delta < z < 1\}, \quad \widetilde{C}_\delta := C_\delta \setminus L_\delta,$$

and

$$\psi(Q,\xi,\delta) := \min\left\{\frac{1}{2}\int_{C_\delta} \mathbb{C}(Q)\eta : \eta\, dx,\ \eta \in \mathcal{AS}_\delta(\xi)\right\}$$
$$= \min\left\{\frac{1}{2}\int_{\widetilde{C}_\delta} \mathbb{C}(Q)\nabla v : \nabla v\, dx,\ v \in H^1(\widetilde{C}_\delta;\mathbb{R}^2),\ [v] = \xi \ \text{ on } \ L_\delta\right\}, \tag{3.20}$$

where $\mathbb{C}(Q) = \frac{\partial^2 W}{\partial F^2}(Q)$ and $[v]$ is the jump of v.

We recall the following fundamental result (see [16, Corollary 6, Remark 7]).

Proposition 3.7. *For $\xi \in S^1$, $\delta \in (0,1)$, and $Q \in SO(2)$ let*

$$\psi_\delta(Q,\xi) := \frac{\psi(Q,\xi,\delta)}{|\log\delta|},$$

with $\psi(Q,\xi,\delta)$ as in (3.20). Then, the functions ψ_δ converge pointwise to the function $\psi\colon SO(2)\times S^1 \to \mathbb{R}^+$ defined by

$$\psi(Q,\xi) := \lim_{\delta\to 0}\frac{1}{|\log\delta|}\frac{1}{2}\int_{C_\delta} \mathbb{C}(Q)\eta_0 : \eta_0\, dx, \tag{3.21}$$

where $\eta_0 : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ is a distributional solution of

$$\begin{cases} \operatorname{Curl}\eta = \xi\,\delta_0 & \text{in} \quad \mathbb{R}^2, \\ \operatorname{Div}\mathbb{C}(Q)\eta = 0 & \text{in} \quad \mathbb{R}^2. \end{cases}$$

Remark 3.8. Let $0 < \delta < r < 1$ be fixed and let $\bar\psi_\delta$ be defined through the following minimisation problem

$$\bar\psi_\delta(Q,\xi) := \frac{1}{|\log\delta|}\min\left\{\frac{1}{2}\int_{B_r\setminus B_\delta} \mathbb{C}(Q)\eta : \eta\, dx,\ \eta \in \mathcal{AS}_{\delta,r}(\xi)\right\}.$$

Then $\bar\psi_\delta = \psi_\delta(1 + o(\delta))$, as $\delta \to 0$ (see [16, Proposition 8]).

The following theorem is the main result of this section.

Theorem 3.9 (Γ-convergence)**.** *The functionals $\mathcal{E}^{(2)}_\varepsilon$ defined in (2.7) Γ-converge with respect to the convergence of Definition 3.6 to the functional $\mathcal{E}$ defined on $L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ by*

$$\mathcal{E}(\beta,R) := \begin{cases} \displaystyle\frac{1}{2}\int_\Omega \mathbb{C}(R)\beta\colon \beta\, dx + \varphi_{\mathbf{b}}(R) & \text{if } \operatorname{Curl}\beta = 0, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2), \end{cases}$$

where $\mathbb{C}(R) = \frac{\partial^2 W}{\partial F^2}(R)$, $\varphi_{\mathbf{b}}(R) := \sum_{i=1}^N \psi(R,\hat b_i)$, with ψ as in (3.21) and $\mathbf{b} := (\hat b_1,\ldots,\hat b_N)$.

As an immediate consequence of Theorem 3.9, we can deduce the following convergence result for the elastic energy induced by the dislocation measure μ_ε.

Corollary 3.10 (Convergence of $F_\varepsilon(\mu_\varepsilon)$)**.** *Let* μ_ε *be as in* (2.2)*. The following convergence holds true for the sequence* $(F_\varepsilon(\mu_\varepsilon))$ *defined in* (2.6)

$$\lim_{\varepsilon\to 0} F_\varepsilon(\mu_\varepsilon) = \inf_{R\in SO(2)} \varphi_{\mathbf{b}}(R). \tag{3.22}$$

Proof. By virtue of Proposition 3.5 and Theorem 3.9, (3.22) is a straightforward consequence of the fundamental property of Γ-convergence. □

To shorten the notation, in what follows we always write B^i_s (for $s>0$) in place of $B_s(x_i)$.

Proposition 3.11 (Γ-lim inf inequality)**.** *For every* $\beta\in L^2(\Omega;\mathbb{R}^{2\times 2})$ *with* $\operatorname{Curl}\beta=0$*, for every* $R\in SO(2)$*, and for every sequence* $(\beta_\varepsilon)\subset L^2(\Omega;\mathbb{R}^{2\times 2})$ *converging to* (β,R) *in the sense of Definition 3.6, we have*

$$\liminf_{\varepsilon\to 0} \mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon) \geq \mathcal{E}(\beta,R).$$

Proof. Let $\beta\in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta=0$, let $R\in SO(2)$, and let $(\beta_\varepsilon)\subset\mathcal{AS}^{(2)}_\varepsilon$ be a sequence with equi-bounded energy $\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon)$ such that

$$\frac{\beta_\varepsilon-R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} \rightharpoonup \beta \quad \text{in} \quad L^2(\Omega;\mathbb{R}^{2\times 2}), \tag{3.23}$$

for some sequence of constant rotations $(R_\varepsilon)\subset SO(2)$ such that $\lim_{\varepsilon\to 0}R_\varepsilon=R$.

We study separately the asymptotic behavior of the energy concentrated in regions surrounding the dislocations and of the energy diffused in the remaining part of the domain, far from the dislocations. To this end, let $r_{\min}>0$ denote the minimum distance between two dislocations; *i.e.,* $r_{\min}:=\min\{|x_i-x_j|, i,j=1,\dots,N, i\neq j\}$, and let $r'_{\min}>0$ denote the distance between the set $\{x_1,\dots,x_N\}$ and $\partial\Omega$; *i.e.,* $r'_{\min}:=\min_{1\leq i\leq N}\operatorname{dist}(x_i,\partial\Omega)$. Let $0<r<\min\{r_{\min},r'_{\min}\}/2$ and define $\Omega_r:=\Omega\setminus\cup_{i=1}^N B^i_r$; we have

$$\begin{aligned}\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon) &= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_r} W(\beta_\varepsilon)\,dx + \frac{1}{\varepsilon^2|\log\varepsilon|}\sum_{i=1}^N\int_{B^i_r\setminus B^i_\varepsilon} W(\beta_\varepsilon)\,dx\\ &=: \mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon;\Omega_r)+\sum_{i=1}^N \mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon;B^i_r\setminus B^i_\varepsilon).\end{aligned}$$

We divide the proof into two main steps.

Step 1: Lower bound far from the core-regions.

The idea is to linearise the energy density W around the identity.

By a Taylor expansion of order two we get $W(I+F)=\frac{1}{2}\mathbb{C}(I)F:F+\sigma(F)$, where $\sigma(F)/|F|^2\to 0$ as $|F|\to 0$. Setting $\omega(t):=\sup_{|F|\leq t}|\sigma(F)|$, we have

$$W(I+\varepsilon\sqrt{|\log\varepsilon|}F) \geq \frac{1}{2}\varepsilon^2|\log\varepsilon|\mathbb{C}(I)F:F-\omega(\varepsilon\sqrt{|\log\varepsilon|}|F|), \tag{3.24}$$

with $\omega(t)/t^2\to 0$ as $t\to 0$.

Let

$$G_\varepsilon := \frac{\beta_\varepsilon-R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}},$$

and define the characteristic function

$$\chi_\varepsilon := \begin{cases} 1 & \text{if} \quad |G_\varepsilon| \le \varepsilon^{-1/2} \\ 0 & \text{otherwise in } \Omega. \end{cases} \tag{3.25}$$

By the boundedness of (G_ε) in $L^2(\Omega;\mathbb{R}^{2\times 2})$ it easily follows that $\chi_\varepsilon \to 1$ boundedly in measure. Therefore, in view of (3.23) we deduce that

$$\widetilde{G}_\varepsilon := \chi_\varepsilon G_\varepsilon \rightharpoonup \beta \quad \text{in} \quad L^2(\Omega;\mathbb{R}^{2\times 2}). \tag{3.26}$$

Using the frame indifference of W and (3.24) we get

$$\begin{aligned} \mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon;\Omega_r) &\ge \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_r}\chi_\varepsilon W(\beta_\varepsilon)\,dx \\ &= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_r}\chi_\varepsilon W(R_\varepsilon^T\beta_\varepsilon)\,dx \\ &= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_r}\chi_\varepsilon W(I+\varepsilon\sqrt{|\log\varepsilon|}R_\varepsilon^T G_\varepsilon)\,dx \\ &\ge \int_{\Omega_r}\Big(\frac{1}{2}\mathbb{C}(I)R_\varepsilon^T\widetilde{G}_\varepsilon : R_\varepsilon^T\widetilde{G}_\varepsilon - \chi_\varepsilon\frac{\omega(\varepsilon\sqrt{|\log\varepsilon|}|G_\varepsilon|)}{\varepsilon^2|\log\varepsilon|}\Big)\,dx. \end{aligned} \tag{3.27}$$

Then, the first term in (3.27) is lower semicontinuous with respect to the convergence (3.26). On the other hand, the second term converges to zero, which can be easily seen multiplying its numerator and denominator by $|G_\varepsilon|^2$. Indeed, $|G_\varepsilon|^2\cdot\chi_\varepsilon\omega(\varepsilon\sqrt{|\log\varepsilon|}|G_\varepsilon|)/(\varepsilon\sqrt{|\log\varepsilon|}|G_\varepsilon|)^2$ is the product of a bounded sequence in $L^1(\Omega)$ and a sequence tending to zero in $L^\infty(\Omega)$, since $\varepsilon\sqrt{|\log\varepsilon|}|G_\varepsilon| \le \varepsilon^{1/2}\sqrt{|\log\varepsilon|}$ whenever $\chi_\varepsilon \ne 0$. Combining these two facts, we eventually obtain

$$\liminf_{\varepsilon\to 0}\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon;\Omega_r) \ge \frac{1}{2}\int_{\Omega_r}\mathbb{C}(I)R^T\beta : R^T\beta\,dx = \frac{1}{2}\int_{\Omega_r}\mathbb{C}(R)\beta:\beta\,dx, \tag{3.28}$$

for every $0 < r < \min\{r_{\min}, {r'}_{\min}\}/2$, where the last equality follows by frame indifference.

Step 2: Lower bound close to the core-regions.

The idea is to divide the annulus $B^i_r \setminus B^i_\varepsilon$ (for $i = 1,\dots,N$) into dyadic annuli in order to rewrite $\mathcal{E}_\varepsilon(\beta_\varepsilon; B^i_r\setminus B^i_\varepsilon)$ as the sum of $|\log\varepsilon|$ contributions. Then, for each of these contributions we provide a linearisation argument analogous to that performed in Step 1. Finally, we conclude by means of the Γ-convergence results established in [16], in the linear framework.

By (3.23) we have that

$$\int_\Omega |\beta_\varepsilon - R_\varepsilon|^2\,dx \le C\varepsilon^2|\log\varepsilon|, \tag{3.29}$$

for some $C > 0$ and for every sufficiently small $\varepsilon > 0$.

Fix $\delta \in (0,1/2)$; for every $i = 1,\dots,N$, we divide $B^i_r\setminus B^i_\varepsilon$ into dyadic annuli $C^{k,i} := B^i_{r\delta^{k-1}}\setminus B^i_{r\delta^k}$, and we consider only those annuli $C^{k,i}$ corresponding to the indices $k = 1,\dots,\tilde{k}_\varepsilon$, where

$$\tilde{k}_\varepsilon := \lfloor k_\varepsilon \rfloor \quad \text{and} \quad k_\varepsilon := (1-\rho)\frac{\log\varepsilon}{\log\delta}, \tag{3.30}$$

for some fixed $\rho \in (0,1)$ ($\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$). Notice that the smallest inner radius of the dyadic annuli, namely $r\delta^{\tilde{k}_\varepsilon}$, is much bigger than ε; indeed,

$$r\,\delta^{\tilde{k}_\varepsilon} \geq r\,\delta^{k_\varepsilon} = r\,\delta^{(1-\rho)\frac{\log\varepsilon}{\log\delta}} = r\,\delta^{\log_\delta(\varepsilon^{1-\rho})} = r\,\varepsilon^{1-\rho} \gg \varepsilon. \tag{3.31}$$

Therefore, we have

$$\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon; B^i_r \setminus B^i_\varepsilon) \geq \frac{1}{|\log\varepsilon|}\sum_{k=1}^{\tilde{k}_\varepsilon}\int_{C^{k,i}} \frac{W(\beta_\varepsilon)}{\varepsilon^2}\,dx, \tag{3.32}$$

for every $i = 1, \dots, N$.

The main point of this step is proving a lower bound (uniform in k) for each term in the sum in (3.32).

Let $\psi(R, \hat{b}_i, \delta)$ be as in (3.20). We claim that there exists a positive sequence (σ_ε), infinitesimal for $\varepsilon \to 0$, such that

$$\int_{C^{k,i}} \frac{W(\beta_\varepsilon)}{\varepsilon^2}\,dx \geq \psi(R, \hat{b}_i, \delta) - \sigma_\varepsilon, \tag{3.33}$$

for every $i = 1, \dots, N$, for every $k = 1, \dots, \tilde{k}_\varepsilon$, and for every $\varepsilon > 0$.

We establish (3.33) arguing by contradiction. If (3.33) does not hold true, then there exists a sequence of positive numbers $\varepsilon_j \to 0$ as $j \to +\infty$ such that, for every positive infinitesimal sequence (ς_j) there exist an index $i \in \{1, \dots, N\}$ and an index $k \in \{1, \dots, \tilde{k}_{\varepsilon_j}\}$ such that

$$\int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2}\,dx < \psi(R, \hat{b}_i, \delta) - \varsigma_j, \tag{3.34}$$

for every $j \in \mathbb{N}$, where we set $\beta_j := \beta_{\varepsilon_j}$ for brevity.

By assumption (g-2) on W, (3.34) yields in particular

$$\int_{C^{k,i}} \operatorname{dist}^2(\beta_j, SO(2))\,dx < C\psi(R, \hat{b}_i, \delta)\,\varepsilon_j^2.$$

Therefore, Proposition 3.3 gives the existence of a constant $C > 0$ (independent of i, k and δ) and a sequence of constant rotations $(\overline{R}_j) \subset SO(2)$ such that

$$\int_{C^{k,i}} |\beta_j - \overline{R}_j|^2\,dx \leq C\psi(R, \hat{b}_i, \delta)\varepsilon_j^2. \tag{3.35}$$

Set $R_j := R_{\varepsilon_j}$ where (R_{ε_j}) is the sequence satisfying (3.29). We have

$$\lim_{j\to+\infty} \overline{R}_j = \lim_{j\to+\infty} R_j = R. \tag{3.36}$$

Indeed, the following estimate holds true

$$\begin{aligned}|\overline{R}_j - R_j|^2 &\leq \frac{2}{\pi r^2\delta^{2k}\left(\frac{1}{\delta^2} - 1\right)}\Big(\int_{C^{k,i}} |\overline{R}_j - \beta_j|^2\,dx + \int_{C^{k,i}} |R_j - \beta_j|^2\,dx\Big)\\ &\leq C\Big(\frac{\varepsilon_j}{r\delta^k}\Big)^2\psi(R, \hat{b}_i, \delta) + C\Big(\frac{\varepsilon_j}{r\delta^k}\Big)^2|\log\varepsilon_j|,\end{aligned}$$

where in the last inequality we used (3.35) and the fact that (β_j) satisfies (3.29). Then, since by (3.31) we have

$$\frac{\varepsilon_j}{r\delta^k} \leq \frac{\varepsilon_j}{r\delta^{\tilde{k}_\varepsilon}} \leq \frac{\varepsilon_j^\rho}{r} \tag{3.37}$$

for $\rho > 0$, we infer (3.36).

Set

$$\eta_j := \frac{\beta_j - \overline{R}_j}{\varepsilon_j};$$

we cut the annulus $C^{k,i}$ with the segment $L^{k,i} := \{x_i + (z,0)\colon r\delta^k < z < r\delta^{k-1}\}$, thus obtaining the simply connected set $C^{k,i} \setminus L^{k,i}$. Then, let $(v_j) \subset H^1(C^{k,i} \setminus L^{k,i}; \mathbb{R}^2)$ denote the sequence with zero average such that $\eta_j = \nabla v_j$ in $C^{k,i} \setminus L^{k,i}$. Notice that $[v_j] = \hat{b}_i$ on $L^{k,i}$. Moreover, in view of (3.35) we deduce that

$$\int_{C^{k,i} \setminus L^{k,i}} |\nabla v_j|^2 \, dx \le C\psi(R, \hat{b}_i, \delta).$$

Then, setting $\tilde{v}_j(x) := v_j(r\delta^{k-1}(x - x_i))$ we immediately get that $(\nabla \tilde{v}_j)$ is bounded in $L^2(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})$ uniformly in j. The latter combined with $\int_{\widetilde{C}_\delta} \tilde{v}_j \, dx = 0$, yields

$$\tilde{v}_j \rightharpoonup \tilde{v} \quad \text{in} \quad H^1(\widetilde{C}_\delta; \mathbb{R}^2). \tag{3.38}$$

Moreover, since on L_δ $[\tilde{v}_j] = \hat{b}_i$, it follows that $[\tilde{v}] = \hat{b}_i$ on L_δ.

We are going to linearise the energy density W around the identity, in analogy to what we did in Step 1. For the convenience of the reader we define $\lambda_j^k := \varepsilon_j/(r\delta^{k-1})$ and we follow closely the steps leading to formula (3.27), with $\varepsilon_j\sqrt{|\log \varepsilon_j|}$ replaced by λ_j^k. We notice that (3.37) provides a bound (independent of k) for the sequence (λ_j^k), which is infinitesimal for $j \to +\infty$.

First of all, we define the sequence (χ_j) of characteristic functions

$$\chi_j := \begin{cases} 1 & \text{if} \quad |\nabla \tilde{v}_j| \le \varepsilon_j^{-\rho/2} \\ 0 & \text{otherwise in } \widetilde{C}_\delta. \end{cases} \tag{3.39}$$

By the boundedness of $(\nabla \tilde{v}_j)$ in $L^2(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})$ it follows that $\chi_j \to 1$ boundedly in measure so that, by (3.38), $\nabla \tilde{v}_j \chi_j \rightharpoonup \nabla \tilde{v}$ in $L^2(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})$. Using the frame indifference of W we may perform a second order Taylor expansion of the energy density around the identity, obtaining

$$\begin{aligned} \int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2} \, dx &\ge \int_{C^{k,i} \setminus L^{k,i}} \chi_j \frac{W(I + \varepsilon_j \overline{R}_j^T \nabla v_j)}{\varepsilon_j^2} \, dx = \int_{\widetilde{C}_\delta} \chi_j \frac{W(I + \lambda_j^k \overline{R}_j^T \nabla \tilde{v}_j)}{(\lambda_j^k)^2} \, dx \\ &\ge \int_{\widetilde{C}_\delta} \left(\frac{1}{2} \mathbb{C}(I) \big(\overline{R}_j^T \nabla \tilde{v}_j \chi_j \big) : \big(\overline{R}_j^T \nabla \tilde{v}_j \chi_j \big) - \chi_j \frac{\omega(\lambda_j^k |\nabla \tilde{v}_j|)}{(\lambda_j^k)^2} \right) dx, \end{aligned} \tag{3.40}$$

where ω is defined as in Step 1.

The first term in (3.40) is lower semicontinuous with respect to the $L^2(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})$-convergence, therefore there exists a positive infinitesimal sequence (ς_j^1) such that

$$\begin{aligned} \int_{\widetilde{C}_\delta} \frac{1}{2} \mathbb{C}(I) \big(\overline{R}_j^T \nabla \tilde{v}_j \chi_j \big) : \big(\overline{R}_j^T \nabla \tilde{v}_j \chi_j \big) \, dx &\ge \int_{\widetilde{C}_\delta} \frac{1}{2} \mathbb{C}(I) R^T \nabla \tilde{v} : R^T \nabla \tilde{v} \, dx - \varsigma_j^1 \\ &= \int_{\widetilde{C}_\delta} \frac{1}{2} \mathbb{C}(R) \nabla \tilde{v} : \nabla \tilde{v} \, dx - \varsigma_j^1. \end{aligned} \tag{3.41}$$

Moreover, the second term in (3.40) converges to zero as $j \to +\infty$. In fact, we can rewrite its integrand as

$$\chi_j \frac{\omega(\lambda_j^k |\nabla \tilde{v}_j|)}{(\lambda_j^k)^2} = |\nabla \tilde{v}_j|^2 \cdot \chi_j \frac{\omega(\lambda_j^k |\nabla \tilde{v}_j|)}{(\lambda_j^k |\nabla \tilde{v}_j|)^2},$$

which is the product of a bounded sequence in $L^1\big(\widetilde{C}_\delta\big)$ and a sequence converging to zero in $L^\infty\big(\widetilde{C}_\delta\big)$, since $\lambda_j^k |\nabla \tilde{v}_j| \leq C\, \varepsilon_j^{\rho/2}$ for every k, when $\chi_j \neq 0$. Therefore, setting

$$\varsigma_j^2 := \sup_{k \in \{1,\dots,\tilde{k}_{\varepsilon_j}\}} \int_{\widetilde{C}_\delta} \chi_j \frac{\omega(\lambda_j^k |\nabla \tilde{v}_j|)}{(\lambda_j^k)^2}\, dx, \tag{3.42}$$

and combining (3.40), (3.41) and (3.42) we have

$$\int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2}\, dx \geq \frac{1}{2} \int_{\widetilde{C}_\delta} \mathbb{C}(R) \nabla \tilde{v} : \nabla \tilde{v}\, dx - \varsigma_j$$

for every j, where we set $\varsigma_j := \varsigma_j^1 + \varsigma_j^2$. Finally, taking the infimum over all the $\tilde{v} \in H^1(\widetilde{C}_\delta; \mathbb{R}^2)$ with $[\tilde{v}] = \hat{b}_i$ on L_δ, and recalling (3.20), we get

$$\int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2}\, dx \geq \psi(R, \hat{b}_i, \delta) - \varsigma_j$$

for every j, and thus the contradiction, since (ς_j) is infinitesimal as $j \to +\infty$.

Once (3.33) is proved, by (3.32) and (3.30) we have

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i) &\geq \liminf_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \sum_{k=1}^{\tilde{k}_\varepsilon} (\psi(R, \hat{b}_i, \delta) - \sigma_\varepsilon) \\
&\geq (1-\rho) \frac{1}{|\log \delta|} \liminf_{\varepsilon \to 0} (\psi(R, \hat{b}_i, \delta) - \sigma_\varepsilon) \\
&= (1-\rho) \frac{1}{|\log \delta|} \psi(R, \hat{b}_i, \delta) = (1-\rho)\, \psi_\delta(R, \hat{b}_i),
\end{aligned}$$

for every $i = 1, \dots, N$. Then, appealing to Proposition 3.7, we pass to the limit on δ, and we get

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i) \geq (1-\rho) \psi(R, \hat{b}_i).$$

Therefore, summing over $i = 1, \dots, N$ we have

$$\liminf_{\varepsilon \to 0} \sum_{i=1}^N \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i) \geq (1-\rho) \sum_{i=1}^N \psi(R, \hat{b}_i). \tag{3.43}$$

Finally, combining (3.28) and (3.43) entails

$$\begin{aligned}
\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon) &\geq \liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon; \Omega_r) + \liminf_{\varepsilon \to 0} \sum_{i=1}^N \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i) \\
&\geq \frac{1}{2} \int_{\Omega_r} \mathbb{C}(R) \beta : \beta\, dx + (1-\rho) \varphi_{\mathbf{b}}(R),
\end{aligned}$$

therefore the lim inf-inequality is achieved letting r and ρ tend to zero. □

The following proposition states the lim sup-inequality for the Γ-limit.

Proposition 3.12 (Γ-lim sup inequality). *Given $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta = 0$ and $R \in SO(2)$, there exists a sequence $(\beta_\varepsilon) \subset L^2(\Omega;\mathbb{R}^{2\times 2})$ converging to (β, R) in the sense of Definition 3.6 such that*

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon^{(2)}(\beta_\varepsilon) \le \mathcal{E}(\beta, R).$$

Proof. Let $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta = 0$ and let $R \in SO(2)$. By standard density arguments, it suffices to prove the claim for $\beta \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$.

For every $i = 1,\dots,N$ let $\eta_i \colon \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ be a distributional solution of

$$\begin{cases} \operatorname{Curl}\eta = \hat{b}_i\delta_0 & \text{in } \mathbb{R}^2, \\ \operatorname{Div}\mathbb{C}(R)\eta = 0 & \text{in } \mathbb{R}^2. \end{cases}$$

In polar coordinates the planar strain η_i has the form

$$\eta_i(r,\theta) = \frac{1}{r}\,\Gamma_{\hat{b}_i}(\theta), \tag{3.44}$$

where the function $\Gamma_{\hat{b}_i}$ depends on $\hat{b}_i$ and on the elasticity tensor $\mathbb{C}(R)$, and satisfies the bound $|\Gamma_{\hat{b}_i}(\theta)| \le C$ for every $\theta \in [0, 2\pi)$ (see e.g. [3]). Moreover, by Proposition 3.7 and Remark 3.8 we have that for every $r > 0$

$$\lim_{\varepsilon\to 0} \frac{1}{|\log\varepsilon|}\frac{1}{2}\int_{B_r\setminus B_\varepsilon} \mathbb{C}(R)\eta_i : \eta_i\,dx = \psi(R, \hat{b}_i). \tag{3.45}$$

Let $\hat{\eta}_i(x) := \eta_i(x - x_i)$. We assert that the following

$$\beta_\varepsilon := \left(R + \varepsilon\sqrt{|\log\varepsilon|}\,\beta + \varepsilon\sum_{i=1}^N \hat{\eta}_i\right)\chi_{\Omega_\varepsilon} + I\chi_{\bigcup_{i=1}^N B_\varepsilon^i}$$

is a recovery sequence. Clearly $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(2)}$. Moreover, it satisfies (3.19) with $R_\varepsilon = R$ for every ε. Indeed, we have

$$\frac{\beta_\varepsilon - R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} = \left(\beta + \sum_{i=1}^N \frac{\hat{\eta}_i}{\sqrt{|\log\varepsilon|}}\right)\chi_{\Omega_\varepsilon} + \frac{(I - R)}{\varepsilon\sqrt{|\log\varepsilon|}}\,\chi_{\bigcup_{i=1}^N B_\varepsilon^i};$$

as the last term converges to zero strongly in $L^2(\Omega;\mathbb{R}^{2\times 2})$, it remains only to prove that the sequences $\left(\frac{\hat{\eta}_i}{\sqrt{|\log\varepsilon|}}\,\chi_{\Omega_\varepsilon}\right)$ converge to zero weakly in $L^2(\Omega;\mathbb{R}^{2\times 2})$ for every $i = 1,\dots,N$. These sequences are bounded in $L^2(\Omega;\mathbb{R}^{2\times 2})$ and converge to zero strongly in $L^1(\Omega;\mathbb{R}^{2\times 2})$, hence the claim.

To prove the lim sup-inequality for $\mathcal{E}_\varepsilon^{(2)}$ we first notice that, as $\beta_\varepsilon = I$ in $\cup_{i=1}^N B_\varepsilon^i$, the energy contribution in $\cup_{i=1}^N B_\varepsilon^i$ is identically zero.

Now we fix $\rho \in (0,1)$ and we set $\Omega_{\varepsilon^{1-\rho}} := \Omega \setminus \cup_{i=1}^N B^i_{\varepsilon^{1-\rho}}$. Then, we have

$$\begin{aligned}\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon) &= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_\varepsilon} W\left(R + \varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat\eta_i\right) dx \\ &= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_{\varepsilon^{1-\rho}}} W\left(R + \varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat\eta_i\right) dx \\ &+\frac{1}{\varepsilon^2|\log\varepsilon|}\sum_{i=1}^N\int_{B^i_{\varepsilon^{1-\rho}}\setminus B^i_\varepsilon} W\left(R + \varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat\eta_i\right) dx =: I^1_\varepsilon + I^2_\varepsilon.\end{aligned}$$

We now estimate I^1_ε and I^2_ε. Regarding I^1_ε, by a Taylor expansion of order two of W around the rotation R we get

$$\begin{aligned}I^1_\varepsilon &= \frac{1}{2}\int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\beta : \beta\, dx + \frac{1}{|\log\varepsilon|}\sum_{i=1}^N \frac{1}{2}\int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\hat\eta_i : \hat\eta_i\, dx \\ &+\frac{1}{\sqrt{|\log\varepsilon|}}\sum_{i=1}^N\int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\beta : \hat\eta_i\, dx + \frac{1}{|\log\varepsilon|}\sum_{i,j=1, i\neq j}^N \int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\hat\eta_i : \hat\eta_j\, dx \\ &+\int_{\Omega_{\varepsilon^{1-\rho}}} \frac{\sigma\left(\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N\hat\eta_i\right)}{\varepsilon^2|\log\varepsilon|}\, dx,\end{aligned}$$

where $\sigma(F)/|F|^2 \to 0$ as $|F| \to 0$.

Recalling that $\beta \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$, by virtue of (3.44) we immediately get

$$\lim_{\varepsilon\to 0}\frac{1}{\sqrt{|\log\varepsilon|}}\int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\beta : \hat\eta_i\, dx = 0, \quad \text{for every } i = 1,\dots,N. \tag{3.46}$$

We also claim that for every $i,j = 1,\dots,N$, with $i \neq j$,

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_{\Omega_{\varepsilon^{1-\rho}}} \mathbb{C}(R)\hat\eta_i : \hat\eta_j\, dx = 0. \tag{3.47}$$

Indeed, let $r_{\min}$ and $r'_{\min}$ be as in the proof of Proposition 3.11 and let $0 < r < \min\{r_{\min}, r'_{\min}\}/2$. Then $\hat\eta_i$ is bounded in $\Omega \setminus B^i_r$ for every $i = 1,\dots,N$. Therefore the claim follows as in (3.46).

We now show that the reminder in the Taylor expansion tends to zero as $\varepsilon \to 0$.

Since by (3.44)

$$\left|\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat\eta_i\right| \le C(\varepsilon\sqrt{|\log\varepsilon|} + \varepsilon^\rho) \quad \text{in} \quad \Omega_{\varepsilon^{1-\rho}},$$

setting $\chi_\varepsilon := \chi_{\Omega_{\varepsilon^{1-\rho}}}$ and $\omega(t) := \sup_{|F|\leq t} |\sigma(F)|$, we have

$$\lim_{\varepsilon\to 0}\left|\int_{\Omega_{\varepsilon^{1-\rho}}} \frac{\sigma\left(\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat{\eta}_i\right)}{\varepsilon^2|\log\varepsilon|}\,dx\right|$$
$$\leq \lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon \frac{\omega\left(|\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat{\eta}_i|\right)}{|\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat{\eta}_i|^2}\cdot\frac{|\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{i=1}^N \hat{\eta}_i|^2}{\varepsilon^2|\log\varepsilon|}\,dx = 0. \tag{3.48}$$

In fact the above integrand is the product of a sequence converging to zero in $L^\infty(\Omega)$ and a bounded sequence in $L^1(\Omega)$. Thus, combining (3.45), (3.46), (3.47) and (3.48), we get

$$\limsup_{\varepsilon\to 0} I^1_\varepsilon \leq \mathcal{E}(\beta, R). \tag{3.49}$$

By the growth assumption on W, by the definition of $\hat{\eta}_i$, and by the $L^\infty(\Omega;\mathbb{R}^{2\times 2})$-bound on β we find

$$\begin{aligned} I^2_\varepsilon &\leq \frac{C_2}{\varepsilon^2|\log\varepsilon|}\sum_{i=1}^N \int_{B^i_{\varepsilon^{1-\rho}}\setminus B^i_\varepsilon} \left|\varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{k=1}^N \hat{\eta}_k\right|^2 dx \\ &\leq C\Big(||\beta||^2_{L^\infty(\Omega;\mathbb{R}^{2\times 2})}(\varepsilon^{2-2\rho} - \varepsilon^2) + \rho\Big), \end{aligned}$$

then, as $\rho < 1$, we get

$$\limsup_{\varepsilon\to 0} I^2_\varepsilon \leq \rho. \tag{3.50}$$

Since

$$\limsup_{\varepsilon\to 0}\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon) \leq \limsup_{\varepsilon\to 0} I^1_\varepsilon + \limsup_{\varepsilon\to 0} I^2_\varepsilon,$$

in view of (3.49) and (3.50) we have

$$\limsup_{\varepsilon\to 0}\mathcal{E}^{(2)}_\varepsilon(\beta_\varepsilon) \leq \mathcal{E}(\beta, R) + \rho,$$

hence the thesis follows by the arbitrariness of $\rho \in (0,1)$. □

Remark 3.13. The Γ-convergence result stated in Theorem 3.9 can be extended with minor changes to the more general case of a dislocation density $\tilde{\mu}_\varepsilon$ of the form (cf. (2.2))

$$\tilde{\mu}_\varepsilon := \varepsilon\sum_{i=1}^N \hat{b}_i\delta_{x_i^\varepsilon}, \tag{3.51}$$

under the assumption that $|x^\varepsilon_k - x^\varepsilon_j| \geq 2\varrho^\varepsilon$ for every $k \neq j$, where $\varrho^\varepsilon/\varepsilon^s \to +\infty$ as $\varepsilon \to 0$, for every fixed $s \in (0,1)$.

If the number N of dislocations becomes increasingly large as $\varepsilon \to 0$ a different approach needs to be considered, which will be the subject of a forthcoming paper.

4. Beyond the model case: Mixed growth conditions

In this section we study the asymptotic behaviour, via Γ-convergence, of the sequence of functionals defined in (2.9) (hence under the assumptions (i)-(iii) and (g-p) on W).

In this case the energy is quadratic for small strains and of order $p \in (1, 2)$ for big strains; *i.e.,* quadratic far from the dislocations (as in Section 3), and of order p in the *core regions* around each dislocation.

4.1. Compactness.

The compactness result proved in the case of the quadratic growth relies on a suitable version of the rigidity estimate in a domain with small holes (Lemma 3.1), or in a domain with small holes and cuts (Proposition 3.3).

In this case we need a rigidity estimate in a domain with a cut, where the cut is a (simple) path through the dislocation points. Moreover, due to the mixed growth conditions (g-p) we make use of a variant of the rigidity estimate proved in [22] (see also [8]). For the reader's convenience, here we recall the precise statement.

Proposition 4.1. [22, Proposition 2.3] *Let $1 \le p < 2$, let $n \ge 2$, and let $U \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there exists a positive constant $C(U)$ such that for each $u \in W^{1,p}(U; \mathbb{R}^n)$ there exists $R \in SO(n)$ such that*

$$\int_U |\nabla u - R|^2 \wedge (|\nabla u|^p + 1)\, dx \le C(U) \int_U \operatorname{dist}^2(\nabla u, SO(n)) \wedge (|\nabla u|^p + 1) dx. \tag{4.1}$$

We want to use Proposition 4.1 to prove a compactness result for sequences $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(p)}$ with equi-bounded energy $\mathcal{E}_\varepsilon^{(p)}$. Proposition 4.1, though, cannot be directly applied to the sequence (β_ε) as it is not a sequence of gradients. This problem can be overcome observing that β_ε is a gradient in any simply connected subset of $\Omega \setminus \{x_1, \dots, x_N\}$, and suitably choosing one of such subsets in which Proposition 4.1 still holds true.

The idea is very simple, in fact, in the case $\Omega = B_s(0) \subset \mathbb{R}^2$, for $s > 0$, with only one singularity located at 0. Indeed, we can just "cut" the disc with a radius L to obtain the simply connected domain $B_s(0) \setminus L$. Then, arguing as in Proposition 3.3 we easily derive the following result.

Proposition 4.2. *Let $1 \le p < 2$. There exists a constant $C = C(s) > 0$ such that for every $u \in W^{1,p}(B_s(0) \setminus L; \mathbb{R}^2)$ there is an associated rotation $R \in SO(2)$ such that*

$$\int_{B_s(0)\setminus L} |\nabla u - R|^2 \wedge (|\nabla u|^p + 1)\, dx \le C \int_{B_s(0)\setminus L} \operatorname{dist}^2(\nabla u, SO(2)) \wedge (|\nabla u|^p + 1) dx. \tag{4.2}$$

Proof. The proof can be derived easily from that of Proposition 3.3 (with $\varepsilon = 0$) and using Proposition 4.1. □

Now we consider the general case of a simply connected, bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ containing $N \ge 1$ singularity points for the strain β. In all that follows we denote by S a simple path through $x_1, \dots, x_N$ and such that $\Omega \setminus S$ is simply connected.

We prove the following rigidity estimate in $\Omega \setminus S$.

Proposition 4.3. *Let $1 \leq p < 2$. There exists $C > 0$ such that for every $u \in W^{1,p}(\Omega \setminus S; \mathbb{R}^2)$ there is an associated rotation $R \in SO(2)$ such that*

$$\int_{\Omega\setminus S} |\nabla u - R|^2 \wedge (|\nabla u|^p + 1)\, dx \leq C \int_{\Omega\setminus S} \mathrm{dist}^2(\nabla u, SO(2)) \wedge (|\nabla u|^p + 1) dx. \tag{4.3}$$

Proof. We first observe that there exists a bi-lipschitz transformation of Ω into a new domain $\hat{\Omega}$ which maps S into a segment L and such that $\hat{\Omega} \setminus L$ is simply connected.

Then, we notice that the result in Proposition 4.2 can be easily extended to the case of a general Lipschitz domain with a straight cut. Indeed, the constant in (4.1) can be chosen independent of the domain for a finite number of sets that are bi-lipschitz images of a half disc. Therefore, the constant provided by Proposition 4.2 turns out to be invariant under bi-lipschitz transformation and since estimate (4.2) holds true for the domain $\hat{\Omega} \setminus L$, it is in turn true for its bi-lipschitz image $\Omega \setminus S$. □

Notice that from estimate (4.3) we can deduce that there is a constant $C_p > 0$ such that

$$\int_{\Omega\setminus S} |\nabla u - R|^2 \wedge C_p(|\nabla u - R|^p + 1)\, dx \leq C \int_{\Omega\setminus S} \mathrm{dist}^2(\nabla u, SO(2)) \wedge (|\nabla u|^p + 1) dx \tag{4.4}$$

We are now ready to prove a compactness result for strains with equi-bounded energy.

Proposition 4.4 (Compactness). *Let $1 < p < 2$. Let $\varepsilon_j \to 0$ and let $(\beta_j) \subset L^p(\Omega; \mathbb{R}^{2\times 2})$ be a sequence such that $\sup_j \mathcal{E}^{(p)}_{\varepsilon_j}(\beta_j) < +\infty$. Then there exist a sequence of constant rotations $(R_j) \subset SO(2)$ and a function $\beta \in L^2(\Omega; \mathbb{R}^{2\times 2})$ with* $\mathrm{Curl}\,\beta = 0$ *such that (up to subsequences)*

$$\frac{\beta_j - R_j}{\varepsilon_j \sqrt{|\log \varepsilon_j|}} \rightharpoonup \beta \quad in \quad L^p(\Omega; \mathbb{R}^{2\times 2}). \tag{4.5}$$

Proof. Let $(\beta_j) \subset \mathcal{AS}^{(p)}_\varepsilon$ be a sequence with equi-bounded energy $\mathcal{E}^{(p)}_{\varepsilon_j}$. There exists $u_j \in W^{1,p}(\Omega \setminus S; \mathbb{R}^2)$ such that $\beta_j = \nabla u_j$ in $\Omega \setminus S$. Then, Proposition 4.3 together with (4.4) and assumption (g-p) on W guarantee the existence of a sequence of constant rotations $(R_j) \subset SO(2)$ such that

$$\int_{\Omega\setminus S} |\nabla u_j - R_j|^2 \wedge C_p(|\nabla u_j - R_j|^p + 1)\, dx \leq C\varepsilon_j^2 |\log \varepsilon_j|, \tag{4.6}$$

for some $C > 0$.

Hence, if we set

$$G_j := \frac{\beta_j - R_j}{\varepsilon_j \sqrt{|\log \varepsilon_j|}},$$

estimate (4.6) easily yields the following bound

$$\int_\Omega |G_j|^2 \wedge C_p \left(\frac{|G_j|^p}{(\varepsilon_j \sqrt{|\log \varepsilon_j|})^{2-p}} + \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \right) dx \leq C. \tag{4.7}$$

We provide a partition of Ω considering the two sets $A^2_{\varepsilon_j}$ and $A^p_{\varepsilon_j}$ defined as follows

$$A^2_{\varepsilon_j} := \left\{ x \in \Omega : |G_j(x)|^2 \le C_p \left(\frac{|G_j(x)|^p}{(\varepsilon_j \sqrt{|\log \varepsilon_j|})^{2-p}} + \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \right) \right\}, \tag{4.8}$$

$$A^p_{\varepsilon_j} := \left\{ x \in \Omega : |G_j(x)|^2 > C_p \left(\frac{|G_j(x)|^p}{(\varepsilon_j \sqrt{|\log \varepsilon_j|})^{2-p}} + \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \right) \right\}. \tag{4.9}$$

Therefore (4.7) can be rewritten as

$$\int_{A^2_{\varepsilon_j}} |G_j|^2 dx + C_p \int_{A^p_{\varepsilon_j}} \left(\frac{|G_j|^p}{(\varepsilon_j \sqrt{|\log \varepsilon_j|})^{2-p}} + \frac{1}{\varepsilon_j^2 |\log \varepsilon_j|} \right) dx \le C. \tag{4.10}$$

We claim that there exists a function $\beta \in L^p(\Omega; \mathbb{R}^{2\times 2})$ such that $G_j \rightharpoonup \beta$ in $L^p(\Omega; \mathbb{R}^{2\times 2})$. In order to prove it, we need to show that the sequence (G_j) has equi-bounded $L^p(\Omega; \mathbb{R}^{2\times 2})$-norm. Since by (4.10) we have that the $L^2(A^2_{\varepsilon_j}; \mathbb{R}^{2\times 2})$-norm of (G_j) is bounded, it remains to provide an L^p-bound for (G_j) in $A^p_{\varepsilon_j}$. This bound easily follows as we notice that (4.10) in particular implies

$$\int_{A^p_{\varepsilon_j}} |G_j|^p dx \le C \big(\varepsilon_j \sqrt{|\log \varepsilon_j|} \big)^{2-p}. \tag{4.11}$$

Now we show that the limit function β is actually in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Indeed, denoting by $\chi_{A^2_{\varepsilon_j}}$ the characteristic function of the set $A^2_{\varepsilon_j}$, (4.10) implies that the sequence $(G_j \chi_{A^2_{\varepsilon_j}})$ is equi-bounded in $L^2(\Omega; \mathbb{R}^{2\times 2})$; therefore it converges weakly in $L^2(\Omega; \mathbb{R}^{2\times 2})$ to a function $\hat{\beta}$. Hence, it remains to prove that $\hat{\beta} = \beta$. This follows since the set $A^2_{\varepsilon_j}$ has asymptotically full measure as $j \to +\infty$, as (4.10) implies that

$$|A^p_{\varepsilon_j}| \le C \varepsilon_j^2 |\log \varepsilon_j| \to 0,$$

as $j \to +\infty$. Therefore $\chi_{A^2_{\varepsilon_j}} \to 1$ boundedly in measure and this yields

$$G_j \chi_{A^2_{\varepsilon_j}} \rightharpoonup \beta \quad \text{in } L^p(\Omega; \mathbb{R}^{2\times 2}),$$

hence $\hat{\beta} = \beta$ a.e. in Ω.

Finally, we prove that $\operatorname{Curl} \beta = 0$ in Ω in the sense of distributions. Let $\phi \in C^1_0(\Omega)$; then we have

$$\begin{aligned} \langle \operatorname{Curl} \beta, \phi \rangle &= \lim_{j \to 0} \frac{1}{\sqrt{|\log \varepsilon_j|}} \langle \operatorname{Curl} \frac{\beta_j - R_j}{\varepsilon_j}, \phi \rangle \\ &= \lim_{j \to 0} \frac{1}{\sqrt{|\log \varepsilon_j|}} \langle \operatorname{Curl} \frac{\beta_j}{\varepsilon_j}, \phi \rangle = \lim_{j \to 0} \frac{\sum_{i=1}^N \phi(x_i) \, \hat{b}_i}{\sqrt{|\log \varepsilon_j|}} = 0, \end{aligned}$$

which completes the proof. □

Before stating the Γ-convergence result, it is convenient to give the following definition of (L^p-)convergence of admissible strains.

Definition 4.5. *A sequence* $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(p)}$ *converges to a pair* $(\beta, R) \in L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ *if there exists a sequence* $(R_\varepsilon) \subset SO(2)$ *such that*

$$\frac{\beta_\varepsilon - R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} \rightharpoonup \beta \quad \text{in} \quad L^p(\Omega;\mathbb{R}^{2\times 2}) \quad \text{and} \quad R_\varepsilon \to R. \tag{4.12}$$

4.2. **Γ-convergence.** This subsection contains the main result of this paper, namely Theorem 4.6, in which we prove that the sequence of functionals $\mathcal{E}_\varepsilon^{(p)}$ has the same Γ-limit as the sequence $\mathcal{E}_\varepsilon^{(2)}$ (cf. Theorem 3.9).

Theorem 4.6 (Γ-convergence)**.** *The functionals* $\mathcal{E}_\varepsilon^{(p)}$ *defined in* (2.9) *Γ-converge with respect to the convergence of Definition 4.5 to the functional* $\mathcal{E}$ *defined in* $L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2)$ *by*

$$\mathcal{E}(\beta, R) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbb{C}(R)\beta \colon \beta \, dx + \varphi_{\mathbf{b}}(R) & \text{if } \operatorname{Curl}\beta = 0, \\ +\infty & \text{otherwise in } L^2(\Omega;\mathbb{R}^{2\times 2}) \times SO(2), \end{cases}$$

where $\mathbb{C}(R) = \frac{\partial^2 W}{\partial F^2}(R)$, $\varphi_{\mathbf{b}}(R) := \sum_{i=1}^N \psi(R, \hat{b}_i)$, *with* ψ *as in* (3.21) *and* $\mathbf{b} = (\hat{b}_1, \ldots, \hat{b}_N)$.

Proof. We divide the proof into two main steps: In the first step we show that $\mathcal{E}$ is a lower bound for the functionals $\mathcal{E}_\varepsilon^{(p)}$, while in the second step, for every target function β, we exhibit a recovery sequence for $\mathcal{E}(\beta)$. Since the proofs of the two steps are similar to the proofs of Proposition 3.11 and Proposition 3.12, we illustrate in detail only the main differences from the previous case, and refer to the proofs of Proposition 3.11 and Proposition 3.12 for the rest. For the reader's sake we employ the same notation used in the proofs of Proposition 3.11 and Proposition 3.12.

Γ-lim inf *inequality.* Let $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(p)}$ be a sequence such that $\sup_{\varepsilon>0} \mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon) < +\infty$. Then, assumption (g-p) on W together with Proposition 4.3 yield the existence of a sequence of constant rotations $(R_\varepsilon) \subset SO(2)$ such that

$$\int_\Omega |\beta_\varepsilon - R_\varepsilon|^2 \wedge (|\beta_\varepsilon|^p + 1)\, dx \le C\varepsilon^2 |\log\varepsilon|, \tag{4.13}$$

for some $C > 0$. Moreover, by Proposition 4.4 we infer that (up to subsequences)

$$G_\varepsilon := \frac{\beta_\varepsilon - R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} \rightharpoonup \beta \quad \text{in} \quad L^p(\Omega;\mathbb{R}^{2\times 2}), \tag{4.14}$$

for some $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta = 0$. Let $R := \lim_{\varepsilon\to 0} R_\varepsilon$.

We study separately the asymptotic behavior of the energy concentrated in regions surrounding the dislocations and of the energy diffused in the remaining part of the domain, far from the dislocations. To this end, we choose $r > 0$ as in the proof of Proposition 3.11 and we define Ω_r accordingly. We have

$$\mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon) \ge \mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; \Omega_r) + \sum_{i=1}^N \mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i).$$

We start proving a lower bound for $\mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; \Omega_r)$.

Let χ_ε be as in (3.25) and set $E_\varepsilon := \{x \in \Omega \colon \chi_\varepsilon = 1\}$. Notice that $E_\varepsilon \subset A^2_\varepsilon$ with A^2_ε as in (4.8). Indeed, for every $x \in E_\varepsilon$ and for sufficiently small ε

$$|G_\varepsilon|^2 \leq \frac{1}{\varepsilon} \leq \frac{C_p}{\varepsilon^2 |\log \varepsilon|} \leq C_p \left(\frac{|G_\varepsilon|^p}{(\varepsilon \sqrt{|\log \varepsilon|})^{2-p}} + \frac{1}{\varepsilon^2 |\log \varepsilon|} \right).$$

Therefore, appealing to the proof of Proposition 4.4 we deduce that $\widetilde{G}_\varepsilon := G_\varepsilon \chi_\varepsilon$ is bounded in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Moreover, since by (4.14) the sequence (χ_ε) converges to 1 boundedly in measure, we immediately deduce that $\widetilde{G}_\varepsilon \rightharpoonup \beta$ in $L^2(\Omega; \mathbb{R}^{2\times 2})$. Then we can perform a linearisation of W around the identity exactly as we did in Proposition 3.11, Step 1, obtaining

$$\liminf_{\varepsilon \to 0} \mathcal{E}^{(p)}_\varepsilon(\beta_\varepsilon; \Omega_r) \geq \frac{1}{2} \int_{\Omega_r} \mathbb{C}(R)\beta : \beta \, dx. \tag{4.15}$$

Now we provide a lower bound on $\mathcal{E}^{(p)}_\varepsilon(\beta_\varepsilon; B^i_r \setminus B^i_\varepsilon)$, for every $i = 1, \dots, N$.

To this end fix $\delta \in (0,1)$ and, for every $i = 1, \dots, N$, divide $B^i_r \setminus B^i_\varepsilon$ into dyadic annuli $C^{k,i} := B^i_{r\delta^{k-1}} \setminus B^i_{r\delta^k}$. For $i = 1, \dots, N$, we have

$$\mathcal{E}^{(p)}_\varepsilon(\beta_\varepsilon; B^i_r \setminus B^i_\varepsilon) \geq \frac{1}{|\log \varepsilon|} \sum_{k=1}^{\tilde{k}_\varepsilon} \int_{C^{k,i}} \frac{W(\beta_\varepsilon)}{\varepsilon^2} \, dx, \tag{4.16}$$

with $\tilde{k}_\varepsilon$ as in the proof of Proposition 3.11 Step 2.

Let $\psi(R, \hat{b}_i, \delta)$ be as in (3.20). We claim that there exists a positive sequence (σ_ε), infinitesimal for $\varepsilon \to 0$, such that

$$\int_{C^{k,i}} \frac{W(\beta_\varepsilon)}{\varepsilon^2} \, dx \geq \psi(R, \hat{b}_i, \delta) - \sigma_\varepsilon, \tag{4.17}$$

for every $i = 1, \dots, N$, for every $k = 1, \dots, \tilde{k}_\varepsilon$, and for every $\varepsilon > 0$.

We establish (4.17) arguing by contradiction. If (4.17) does not hold true, then there exists a sequence of positive numbers $\varepsilon_j \to 0$ as $j \to +\infty$ such that, for every positive infinitesimal sequence (ς_j) there exist an index $i \in \{1, \dots, N\}$ and an index $k \in \{1, \dots, \tilde{k}_{\varepsilon_j}\}$ such that

$$\int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2} \, dx < \psi(R, \hat{b}_i, \delta) - \varsigma_j, \tag{4.18}$$

for every $j \in \mathbb{N}$, where we set $\beta_j := \beta_{\varepsilon_j}$ for brevity.

Hence, assumption (g-p) on W combined with a suitable variant of Proposition 4.2 applied in the domain $C^{k,i} \setminus L^{k,i}$ yields the existence of a sequence of rotations $(\overline{R}_j) \subset SO(2)$ for which

$$\int_{C^{i,k}} |\beta_j - \overline{R}_j|^2 \wedge (|\beta_j|^p + 1) \, dx \leq C\varepsilon_j^2, \tag{4.19}$$

for some $C > 0$ depending on $\delta > 0$ (but not on j). Since β_j satisfies also (4.13), putting together the latter and (4.19) it is easy to show that $\lim_{j \to +\infty} \overline{R}_j = R$.

Now define

$$\eta_j := \frac{\overline{\beta}_j - R_j}{\varepsilon_j};$$

we can rewrite (4.19) in terms of η_j as

$$\int_{C^{i,k}} |\eta_j|^2 \wedge C_p\left(\frac{|\eta_j|^p}{\varepsilon_j^{2-p}} + \frac{1}{\varepsilon_j^2}\right) dx \le C, \tag{4.20}$$

for some $C_p > 0$.

Let $v_j \in W^{1,p}(C^{k,i} \setminus L^{k,i}; \mathbb{R}^2)$ be a function with zero average such that $\nabla v_j = \eta_j$ in $C^{k,i} \setminus L^{k,i}$; then, $[v_j] = \hat{b}_i$ on $L^{k,i}$. In view of (4.19) we deduce that

$$\int_{C^{i,k} \setminus L^{k,i}} |\nabla v_j|^2 \wedge C_p\left(\frac{|\nabla v_j|^p}{\varepsilon_j^{2-p}} + \frac{1}{\varepsilon_j^2}\right) dx \le C,$$

hence, setting $\tilde{v}_j(x) := v_j(r\delta^{k-1}(x - x_i))$ we get

$$\int_{\widetilde{C}_\delta} |\nabla \tilde{v}_j|^2 \wedge C_p\left(|\nabla \tilde{v}_j|^p \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^{2-p} + \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^2\right) dx \le C. \tag{4.21}$$

We provide a partition of $\widetilde{C}_\delta$ considering the two sets C_j^2 and C_j^p defined as follows

$$C_j^2 := \left\{ x \in \widetilde{C}_\delta : |\nabla \tilde{v}_j(x)|^2 \le C_p\left(|\nabla \tilde{v}_j(x)|^p \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^{2-p} + \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^2\right)\right\},$$

$$C_j^p := \left\{ x \in \widetilde{C}_\delta : |\nabla \tilde{v}_j(x)|^2 > C_p\left(|\nabla \tilde{v}_j(x)|^p \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^{2-p} + \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^2\right)\right\}.$$

Therefore, (4.21) can be rewritten as

$$\int_{C_j^2} |\nabla \tilde{v}_j|^2 \, dx + C_p \int_{C_j^p} \left(|\nabla \tilde{v}_j(x)|^p \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^{2-p} + \Big(\frac{r\delta^{k-1}}{\varepsilon_j}\Big)^2\right) dx \le C. \tag{4.22}$$

By (4.22) we immediately get

$$\int_{C_j^2} |\nabla \tilde{v}_j|^2 \, dx \le C, \quad \int_{C_j^p} |\nabla \tilde{v}_j|^p \, dx \le C\Big(\frac{\varepsilon_j}{r\delta^{k-1}}\Big)^{2-p}, \quad \text{and} \quad |C_j^p| \le C\Big(\frac{\varepsilon_j}{r\delta^{k-1}}\Big)^2.$$

Recalling that, by our choice of $\tilde{k}_{\varepsilon_j}$, $\varepsilon_j / r\delta^{k-1} \to 0$ as $j \to +\infty$, we deduce $\|\nabla \tilde{v}_j\|_{L^p(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})} \le C$. This combined with $\int_{\widetilde{C}_\delta} \tilde{v}_j \, dx = 0$ yields $\tilde{v}_j \rightharpoonup \tilde{v}$ in $W^{1,p}(\widetilde{C}_\delta; \mathbb{R}^2)$, and as a consequence $[\tilde{v}] = \hat{b}_i$ on L_δ. Moreover, arguing as in the proof of Proposition 4.4 it can be easily proved that, in fact, $\nabla \tilde{v} \in L^2(\widetilde{C}_\delta; \mathbb{R}^{2\times 2})$. Therefore, we can proceed exactly as in Proposition 3.11, Step 2 linearising the following energy contribution

$$\int_{C^{k,i}} \frac{W(\beta_j)}{\varepsilon_j^2} \, dx = \int_{\widetilde{C}_\delta} \frac{W(I + \lambda_j^k \overline{R}_j^T \nabla \tilde{v}_j)}{\big(\lambda_j^k\big)^2} \, dx,$$

to obtain a contradiction.

Once (4.17) is established, we can conclude as in Proposition 3.11, Step 2 deducing

$$\liminf_{\varepsilon \to 0} \sum_{i=1}^N \mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; B_r^i \setminus B_\varepsilon^i) \ge (1 - \rho)\varphi_{\mathbf{b}}(R),$$

for every $\rho \in (0,1)$. Thus, we finally achieve the Γ-lim inf inequality by recalling (4.15).

Γ-lim sup *inequality.* Let $\beta \in L^2(\Omega;\mathbb{R}^{2\times 2})$ with $\operatorname{Curl}\beta = 0$ and let $R \in SO(2)$.
By standard density arguments, it suffices to prove the claim for $\beta \in L^\infty(\Omega;\mathbb{R}^{2\times 2})$.
For every $i = 1,\dots,N$ let $\hat\eta_i$ be as in the proof of Proposition 3.12; we assert that

$$\beta_\varepsilon := R + \varepsilon\sqrt{|\log\varepsilon|}\,\beta + \varepsilon\sum_{i=1}^N \hat\eta_i \tag{4.23}$$

is a recovery sequence. Clearly $(\beta_\varepsilon) \subset \mathcal{AS}_\varepsilon^{(p)}$; moreover, it satisfies (4.12) with $R_\varepsilon = R$ for every ε. Indeed, we have

$$\frac{\beta_\varepsilon - R_\varepsilon}{\varepsilon\sqrt{|\log\varepsilon|}} = \beta + \sum_{i=1}^N \frac{\hat\eta_i}{\sqrt{|\log\varepsilon|}},$$

and $\hat\eta_i/\sqrt{|\log\varepsilon|}$ converges to zero strongly in $L^p(\Omega;\mathbb{R}^{2\times 2})$, for every $i = 1,\dots,N$.

In view of (4.23) and appealing to the proof of Proposition 3.12, to achieve the lim sup-inequality for $\mathcal{E}_\varepsilon^{(p)}$ it is enough to show that the energy contribution in each B_ε^i vanishes as $\varepsilon \to 0$.

By virtue of assumption (g-p) on W and by (4.23), for every $i = 1,\dots,N$ we get

$$\begin{aligned}
\mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; B_\varepsilon^i) &:= \frac{1}{\varepsilon^2|\log\varepsilon|}\int_{B_\varepsilon^i} W\Big(R + \varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{k=1}^N \hat\eta_k\Big)\,dx \\
&\le \frac{C_2}{\varepsilon^2|\log\varepsilon|}\int_{B_\varepsilon^i}\Big(\Big|R + \varepsilon\sqrt{|\log\varepsilon|}\beta + \varepsilon\sum_{k=1}^N \hat\eta_k\Big|^p + 1\Big)\,dx \\
&\le C\Bigg(\frac{1}{|\log\varepsilon|} + \int_{B_\varepsilon^i}\frac{|\beta|^p}{(\varepsilon\sqrt{|\log\varepsilon|})^{2-p}}\,dx + \int_{B_\varepsilon^i}\frac{|\hat\eta_i|^p}{\varepsilon^{2-p}|\log\varepsilon|}\,dx + \sum_{k=1,k\neq i}^N\int_{B_\varepsilon^i}\frac{|\hat\eta_k|^p}{\varepsilon^{2-p}|\log\varepsilon|}\,dx\Bigg) \\
&\le C\Bigg(\frac{1}{|\log\varepsilon|} + \|\beta\|^p_{L^\infty(\Omega;\mathbb{R}^{2\times 2})}\frac{\varepsilon^p}{\sqrt{|\log\varepsilon|}^{2-p}} + \sum_{k=1,k\neq i}^N \|\hat\eta_k\|^p_{L^\infty(B_\varepsilon^i;\mathbb{R}^{2\times 2})}\frac{\varepsilon^p}{|\log\varepsilon|}\Bigg).
\end{aligned}$$

Therefore, for every $i = 1,\dots,N$ we may deduce that

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon^{(p)}(\beta_\varepsilon; B_\varepsilon^i) = 0,$$

and the lim sup-inequality is achieved. □

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