ON UNITARY SUBMODULES IN THE POLYNOMIAL REPRESENTATIONS OF RATIONAL CHEREDNIK ALGEBRAS

M. FEIGIN, C. SHRAMOV

Abstract. We consider representations of rational Cherednik algebras which are particular ideals in the ring of polynomials. We investigate convergence of the integrals which express the Gaussian inner product on these representations. We derive that the integrals converge for the minimal submodules in types B and D for the singular values suggested by Cherednik with at most one exception, hence the corresponding modules are unitary. The analogous result on unitarity of the minimal submodules in type A was obtained by Etingof and Stoica, we give a different proof of convergence of the Gaussian product in this case. We also obtain partial results on unitarity of the minimal submodule in the case of exceptional Coxeter groups and group B with unequal parameters.

Contents

1. Introduction

Let $\mathcal{R} \subset \mathbb{R}^N$ be an irreducible Coxeter root system, let W be the corresponding Coxeter group which is generated by orthogonal reflections s_α with respect to the hyperplanes $(\alpha, x) = 0$ where $\alpha \in \mathcal{R}$, $x = (x_1, \ldots, x_N)$ and $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{R}^N$ (see [12]). Let $c : \mathcal{R} \to \mathbb{R}$ be a W-invariant function. The corresponding rational Cherednik algebra $H_c(W)$ (see [8]) is generated by the group algebra $\mathbb{C}W$ and two commutative polynomial subalgebras

$\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$, $\mathbb{C}[y_1, \dots, y_N]$. The algebra can be defined by its faithful representation ϕ in the space of polynomials $\mathbb{C}[x]$. In this representation $\phi|_{\mathbb{C}W}$ is the reflection representation of the group algebra $\mathbb{C}W$, $\phi(p(x))$ is the operator of multiplication by $p(x)$, and $\phi(p(y_1, \dots, y_N))$ is the operator $p(\nabla_1, \dots, \nabla_N)$ where ∇_i are (commuting) Dunkl operators [4] corresponding to the basis vectors $\xi = e_i$:

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in \mathcal{R}_+} \frac{c(\alpha)(\alpha, \xi)}{(\alpha, x)}(1 - s_\alpha), \tag{1.1}$$

where $\mathcal{R}_+$ is the set of positive roots.

The study of unitary representations of the algebra $H_c(W)$ was initiated in the paper by Etingof, Stoica, Griffeth [9] (see also [2]). Recall that category $\mathcal{O}$ consists of finitely generated modules such that all Dunkl operators act locally nilpotently [6]. The simple objects L_τ in category $\mathcal{O}$ are parametrized by the irreducible modules τ for the corresponding Coxeter group W. The module L_τ carries a W-invariant nondegenerate Hermitian form $(\cdot, \cdot)_\tau$ satisfying

$$(x_i u, v)_\tau = (u, y_i v)_\tau$$

for any $u, v \in L_\tau$, for any $i = 1, \dots, N$. This form is unique up to proportionality. The unitary modules are such that this form can be scaled to be positive definite.

Of particular interest there is Cherednik's question on unitarity of the minimal submodule $\mathbb{S}_c$ in the polynomial representation $\mathbb{C}[x]$ (see [9, Section 4.6] and [2]). This submodule has the form $\mathbb{S}_c \cong L_{\tau_c}$ where τ_c is an irreducible W-module which might depend on c. Submodule $\mathbb{S}_c$ is unique and it is non-trivial only for the so-called singular multiplicities c when the polynomial representation is reducible. The singular multiplicities were completely determined in [5]. In the case of constant multiplicity function they are special rational numbers with the denominators d_i which are degrees of the corresponding Coxeter group. Cherednik's question is whether the minimal submodule $\mathbb{S}_c$ is unitary when $c = 1/d_i$.

It is shown in [9, Proposition 4.12] (see also [2]) that unitarity of the minimal submodule follows from the convergence of the integral

$$\gamma_c(f) = \int_{\mathbb{R}^N} |f(x)|^2 e^{-\frac{1}{2}|x|^2} \prod_{\alpha \in \mathcal{R}_+} |(\alpha, x)|^{-2c(\alpha)} dx \tag{1.2}$$

for all $f \in \mathbb{S}_c$. This is due to the observation that

$$(f, f)_{\tau_c} = \lambda \gamma_c(e^{-\frac{1}{2}\sum_{i=1}^N \nabla_i^2} f)$$

for some constant $\lambda \in \mathbb{R}$ independent of $f \in \mathbb{S}_c$, and to the obvious inequality $\gamma_c(f) \geqslant 0$. Thus the related question posed in [2, 9] is on convergence of the integral (1.2) which in such case is called the Gaussian inner product. It is shown in [9, Theorem 5.14] that this integral does converge in the case $\mathcal{R} = A_{N-1}$ hence the questions have positive answer in this case.

In this paper we show unitarity of the minimal submodules in the polynomial representations for the algebras $H_c(W)$ in certain cases by establishing the convergence of the above integral, in particular we give another proof of convergence for the A_{N-1} case (c.f. suggestions in [2]). More exactly we show that

$$\Phi_f = |f(x)| \prod_{\alpha \in \mathcal{R}_+} |(\alpha, x)|^{-c(\alpha)}$$

is locally L^2-integrable in $\mathbb{R}^N$ for any $f \in M$ where M is an appropriate ideal. This implies, in particular, that the Cherednik's question has positive answer in types B and D except for the singular value $1/N$ in the case of D_N with odd N (see Theorem 5.17 which is our main result). In the latter case the answer actually happens to be negative ([16]; see Proposition 7.1 below).

The structure of the paper is as follows. In Section 2 we consider special ideals in the ring $\mathbb{C}[x]$ and find their generators which are singular polynomials for the corresponding rational Cherednik algebras of type $H_c(G(m, p, N))$. In Section 3 we recall the algebro-geometric technique of checking local integrability and apply it to our situation by producing an explicit log resolution of the hyperplane arrangement corresponding to the poles of Φ_f. The explicit estimates for particular cases are gathered in Section 4. In Section 5 we complete the proof of convergence of integrals (1.2) for A, B, D cases and deduce unitarity of the corresponding minimal representations. In Section 6 we present a few results on the convergence of the Gaussian product (1.2) mainly for the case of exceptional Coxeter groups (see Propositions 6.7, 6.11, and also Proposition 6.12). In the last section we discuss a few examples when the minimal submodule is not unitary or when at least the integral (1.2) is not convergent on the minimal submodule.

2. H_c-invariant ideals

In this section we discuss special ideals in the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_N]$ which are invariant under certain appropriate rational Cherednik algebra. We specify singular polynomials generating these representations.

Let $\Delta(x_1,\ldots,x_p)$ be the Vandermonde determinant, that is

$$\Delta(x_1,\ldots,x_p)=\prod_{i<j}^{p}(x_i-x_j)$$

for $2\leqslant p\leqslant N$ and $\Delta(x_1)=1$.

Let $\nu=(\nu_1,\ldots,\nu_l)$ be a partition of N, that is $\nu_i\geqslant\nu_{i+1}$, $\nu_i\in\mathbb{Z}_+$ and $\sum\nu_i=N$. Let $l(\nu)=l$ be the length of the partition. Define the associated polynomial

$$(2.1)\quad p_\nu(x)= \\ =\Delta(x_1,\ldots,x_{\nu_1})\cdot\Delta(x_{\nu_1+1},\ldots,x_{\nu_1+\nu_2})\cdot\ldots\cdot\Delta(x_{\nu_1+\ldots+\nu_{l-1}+1},\ldots,x_N).$$

Let k be an integer, $1\leqslant k<N$. Consider the ideal I_k in the ring $\mathbb{C}[x_1,\ldots,x_N]$ consisting of polynomials $p(x)$ such that $p(x)=0$ whenever $x_{i_1}=x_{i_2}=\ldots=x_{i_{k+1}}$ for some indexes $1\leqslant i_1<i_2<\ldots<i_{k+1}\leqslant N$.

It is clear that the image of $p_\nu(x)$ under any $\sigma\in\mathrm{S}_N$ is contained in I_k if the length $l(\nu)\leqslant k$. Moreover, the following proposition is contained in [9].

Proposition 2.2 ([9, Section 5.3])**.** *Let $N=kq+s$ where $q,s\in\mathbb{Z}_{\geqslant0}$, $s<k$. Let ν_N^k be the partition $\nu_N^k=((q+1)^s,q^{k-s})$. Then the ideal I_k in the ring $\mathbb{C}[x]$ is generated by the S_N-images of the polynomial $p_{\nu_N^k}(x)$.*

Indeed, it is shown in [9, Theorem 5.10] that I_k is an irreducible module over the rational Cherednik algebra $H_c(\mathrm{S}_N)$ with the parameter $c=1/(k+1)$. Therefore it has to be generated as ideal in $\mathbb{C}[x]$ by its lowest homogeneous component. It is determined in [9] (see the proof of Proposition 5.16) that the lowest homogeneous component of the module I_k is linearly generated by the S_N-orbit of $p_{\nu_N^k}(x)$ (under the geometric action of S_N in $\mathbb{C}[x]$).

Consider now the ideal $I_k^{\pm}$ in $\mathbb{C}[x]$ which consists of the polynomials vanishing on the union of planes

$$(2.3)\qquad \varepsilon_{i_1}x_{i_1}=\varepsilon_{i_2}x_{i_2}=\ldots=\varepsilon_{i_{k+1}}x_{i_{k+1}}$$

where $\varepsilon_{i_s}=\pm1$ and the indexes $1\leqslant i_1<i_2<\ldots<i_{k+1}\leqslant N$. This ideal is a module over the rational Cherednik algebra $H_c(D_N)$ with the parameter $c=1/(k+1)$ [10, Section 4.3]. It is also a module over $H_c(B_N)$ with the parameters $c(e_i\pm e_j)=1/(k+1)$, $c(e_i)$ is arbitrary (see [10, Section 4.2]).

Proposition 2.4. *The ideal* $I_k^{\pm} \subset \mathbb{C}[x]$ *is generated by the* S_N*-images of the polynomial* $p_{\nu_N^k}(x_1^2, \ldots, x_N^2)$.

We actually prove the following slightly more general result.

Proposition 2.5. *Let* $m \geqslant 2$ *be an integer. Consider the ideal* $I_k^{(m)} \subset \mathbb{C}[x]$ *consisting of polynomials vanishing on the planes (2.3), where* $\varepsilon_{i_s}^m = 1$ *and the indexes* $1 \leqslant i_1 < i_2 < \ldots < i_{k+1} \leqslant N$. *This ideal is generated by the* S_N*-images of the polynomial* $p_{\nu_N^k}(x_1^m, \ldots, x_N^m)$.

Proof. Consider first the lowest homogeneous component $M \subset I_k^{(m)}$, and let $q \in M$. Let E_i for $1 \leqslant i \leqslant N$ be the idempotents

$$E_i = \frac{1}{m} \sum_{k=0}^{m-1} s_i^k,$$

where s_i multiplies the basis vector e_i by $\xi = e^{2\pi \mathrm{i}/m}$, while $s_i(e_l) = e_l$ for $l \neq i$. Consider the difference

$$q(x) - E_i q(x) = x_i r_i(x), \quad i = 1, \ldots, N, \tag{2.6}$$

where $r_i(x)$ are some polynomials. The collection of planes (2.3) is invariant with respect to reflections s_i and therefore $q(x) - E_i q(x) \in I_k^{(m)}$. Since x_i is not identically zero on the planes from (2.3) we conclude that $r_i(x) \in I_k^{(m)}$. By minimality of the degree of $q(x)$ it follows that $q(x) = E_i q(x)$, and therefore $s_i q(x) = q(x)$. Thus

$$q(x) = \tilde{q}(y_1, \ldots, y_N),$$

where $y_i = x_i^m$, $i = 1, \ldots, N$, and $\tilde{q}$ is a polynomial. Now $\tilde{q} \in I_k \subset \mathbb{C}[y]$, and therefore by Proposition 2.2 the polynomial $q(x)$ is a linear combination of the S_N-images of the polynomial $p_{\nu_N^k}(x_1^m, \ldots, x_N^m)$.

The rest of the Proposition follows by induction on the degree of a polynomial $q(x) \in I_k^{(m)}$. Indeed we again apply the relations (2.6). They imply by induction that $q(x) - \hat{q}(x)$ has the required form where $\hat{q}(x) = \prod_{i=1}^N E_i q(x)$. Since $s_i \hat{q}(x) = \hat{q}(x)$, we have $\hat{q}(x) \in I_k \subset \mathbb{C}[y]$. So $\hat{q}$ has the required form, and hence the statement for $q(x)$ also follows. □

Recall that the complex reflection group $G(m, p, N)$ is defined when $p|m$, it is generated by the elements s_{ij}^k for $1 \leqslant i < j \leqslant N$, $k = 0, \ldots, m-1$, and the elements τ_i for $i = 1, \ldots, N$. The element τ_i acts on the basis coordinate functions as $\tau_i(x_i) = \eta x_i$, where $\eta = e^{2\pi \mathrm{i} p/m}$ and $\tau_i(x_j) = x_j$ for $j \neq i$. The elements s_{ij}^k defined for $i \neq j$ act as $s_{ij}^k(x_j) = \xi^{-k} x_i$, $s_{ij}^k(x_i) = \xi^k x_j$, where $\xi = e^{2\pi \mathrm{i}/m}$, and $s_{ij}^k(x_l) = x_l$ for $l \neq i, j$.

It follows from [10, Section 7] that the ideal $I_k^{(m)}$ is a module over the rational Cherednik algebra $H_c(G(m,p,N))$ when $c_1 = 1/(k+1)$, therefore Proposition 2.5 has the following corollary.

Corollary 2.7. *The polynomials* $p_{\nu_N^k}(x_1^m, \ldots, x_N^m)$, $1 \leqslant k \leqslant N-1$, *are singular polynomials for the rational Cherednik algebra* $H_c(G(m,p,N))$. *More exactly,*

$$\nabla_i p_{\nu_N^k}(x_1^m, \ldots, x_N^m) = 0$$

for all $i = 1, \ldots, N$, *where* ∇_i *is the Dunkl–Opdam operator* (see [6])

$$\nabla_i = \partial_i - c_1 \sum_{\substack{j=1 \\ j \neq i}}^{N} \sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} - \sum_{t=1}^{\frac{m}{p}-1} c_{t+1} \sum_{s=0}^{\frac{m}{p}-1} \frac{\eta^{-st} \tau_i^s}{x_i} \tag{2.8}$$

with $c_1 = 1/(k+1)$.

Define now the ideal $J_k = J_k^N \subset \mathbb{C}[x_1, \ldots, x_N]$, $0 \leqslant k \leqslant N-1$, which consists of the polynomials vanishing on the union of planes

$$x_{i_1} = \ldots = x_{i_{k+1}} = 0$$

for arbitrary indexes $1 \leqslant i_1 < \ldots < i_{k+1} \leqslant N$.

Proposition 2.9. *The ideal* $J_k \subset \mathbb{C}[x]$ *is generated by the* S_N*-images of the polynomial* $x_1 \cdot \ldots \cdot x_{N-k}$.

Proof. Let f be an element from J_k. Consider the Taylor expansion with respect to the variable x_N:

$$f = \sum_{i=0}^{\deg f} x_N^i g_i(x_1, \ldots, x_{N-1}).$$

The polynomials g_i then have to satisfy $g_0 \in J_{k-1}^{N-1}$, $g_i \in J_k^{N-1}$ for $i > 0$. The statement follows by induction on the dimension. □

The ideal J_k, $0 \leqslant k \leqslant N-1$, is a representation of the rational Cherednik algebra $H_c(G(m,p,N))$ if (and only if) multiplicities satisfy the relation $kc_1 + p^{-1}c_2 = m^{-1}$ (where in $p = m$ case one assumes $c_i = 0$ for $i \geqslant 2$) by [10, Proposition 9]. In particular the ideal J_k for $1 \leqslant k \leqslant N-1$ is a module over the rational Cherednik algebra $H_c(D_N)$ with $c = \frac{1}{2k}$ [10, Section 4.3]. Also for any $0 \leqslant k \leqslant N-1$ the ideal J_k is a module over $H_c(B_N)$ if the parameters satisfy the relation $2kc_1 + 2c_2 = 1$ where $c_1 = c(e_i \pm e_j)$ and $c_2 = c(e_i)$ [10, Section 4.2].

Proposition 2.9 has the following corollary.

Corollary 2.10. *The polynomials $x_1 \cdot \ldots \cdot x_k$, $1 \leqslant k \leqslant N$, are singular with respect to $H_c(G(m,p,N))$. More exactly,*

$$\nabla_i(x_1 \cdot \ldots \cdot x_k) = 0$$

for all $i = 1, \ldots, N$, where ∇_i is the Dunkl–Opdam operator (2.8) and the multiplicities satisfy $m(N-k)c_1 + mp^{-1}c_2 = 1$.

Remark 2.11. Corollary 2.10 for $k = 1$ is contained in [3, Proposition 4.1] where it is generalized in a different direction.

We are going to construct some more singular polynomials for the rational Cherednik algebra $H_c(G(m,p,N))$. Firstly we need the following lemma.

Lemma 2.12. *Let L be the operator*

$$L = \sum_{j=2}^{n+1} \sum_{k=0}^{m-1} \frac{1 - s_{1j}^k}{x_1 - \xi^k x_j}.$$

Then

$$L\left(x_1^{mk} \Delta(x_2^m, \ldots, x_{n+1}^m)\right) = \partial_{x_1}\left(x_1^{mk} \Delta(x_2^m, \ldots, x_{n+1}^m)\right), \tag{2.13}$$

for $0 \leqslant k \leqslant n$,

$$L\left(x_1^{mk+1} \Delta(x_2^m, \ldots, x_{n+1}^m)\right) = (\partial_{x_1} + \frac{m-1}{x_1})\left(x_1^{mk+1} \Delta(x_2^m, \ldots, x_{n+1}^m)\right), \tag{2.14}$$

for $0 \leqslant k \leqslant n-1$,

$$L\left(x_1^{mk} \Delta(x_2^m, \ldots, x_{n+1}^m) \prod_{j=2}^{n+1} x_j\right) = \partial_{x_1}\left(x_1^{mk} \Delta(x_2^m, \ldots, x_{n+1}^m) \prod_{j=2}^{n+1} x_j\right), \tag{2.15}$$

for $0 \leqslant k \leqslant n$, and

$$L\left(x_1^{mk+1} \Delta(x_2^m, \ldots, x_{n+1}^m) \prod_{j=2}^{n+1} x_j\right) = \tag{2.16}$$

$$= (\partial_{x_1} - \frac{1}{x_1})\left(x_1^{mk+1} \Delta(x_2^m, \ldots, x_{n+1}^m) \prod_{j=2}^{n+1} x_j\right),$$

for $0 \leqslant k \leqslant n$.

Proof. We rewrite Vandermonde determinant using anti-symmetrization with respect to the group S_n acting by permutations of the variables

$x_2, \dots, x_{n+1}$:

$$\Delta(x_2^m, \dots, x_{n+1}^m) = Alt \prod_{j=1}^{n} x_{j+1}^{m(j-1)},$$

where $Alt = \sum_{g \in S_n} sign(g) g$. Note that the operator L is $G(m,1,n)$-invariant, where the group $G(m,1,n)$ is generated by s_{ij}^k, $2 \leqslant i < j \leqslant n+1$ and τ_i, $2 \leqslant i \leqslant n+1$. Therefore

$$L\left(x_1^{mk}\Delta(x_2^m, \dots, x_{n+1}^m)\right) = Alt\Big(L\big(x_1^{mk}\prod_{j=1}^{n} x_{j+1}^{m(j-1)}\big)\Big) =$$
$$= \sum_{i=2}^{n+1} Alt\Big(\sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} x_1^{mk} \prod_{j=1}^{n} x_{j+1}^{m(j-1)}\Big)$$

and the right-hand side is polynomial in $x_2^m, \dots, x_{n+1}^m$. Now

$$Alt\Big(\sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} x_1^{mk} \prod_{j=1}^{n} x_{j+1}^{m(j-1)}\Big) =$$
$$= \begin{cases} m x_1^{mk-1}\Delta(x_2^m, \dots, x_{n+1}^m), & 2 \leqslant i \leqslant k+1, \\ 0, & k+2 \leqslant i \leqslant n+1, \end{cases}$$

hence the statement (2.13) follows. Similarly

$$Alt\Big(\sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} x_1^{mk+1} \prod_{j=1}^{n} x_{j+1}^{m(j-1)}\Big) =$$
$$= \begin{cases} m x_1^{mk}\Delta(x_2^m, \dots, x_{n+1}^m), & 2 \leqslant i \leqslant k+2, \\ 0, & k+3 \leqslant i \leqslant n+1, \end{cases}$$

hence the statement (2.14) holds. The statements (2.15), (2.16) follow analogously. □

Proposition 2.17. *Let $N = \sum_{i=1}^{r-1} \nu_i$ where $\nu_i \in \mathbb{Z}_{>0}$ and $|\nu_i - \nu_j| \in \{0,1\}$ for $1 \leqslant i,j \leqslant r-1$. Denote*

$$\mathcal{I}_i = \{m \in \mathbb{Z}_+ | \sum_{j=1}^{i-1} \nu_j + 1 \leqslant m \leqslant \sum_{j=1}^{i} \nu_j\}.$$

Let $T \subset \{1, \dots, N\}$ be a subset of indexes of size $|T| = r-s-1$ for some $0 \leqslant s \leqslant r-1$. Let $N = (r-1)q + t$ with $0 \leqslant t < r-1$ so that $\nu_i = q$ or $\nu_i = q+1$. Assume that if there exists $i \in T$ such that $\nu_i = q+1$ then for all j such that $\nu_j = q$ one has $j \in T$. Then the *polynomial*

$$p^{(m)}_{\nu,T} = p_\nu(x_1^m, \dots, x_N^m) \prod_{i\in T}\prod_{j\in\mathcal{I}_i} x_j$$

where p_ν is defined by (2.1), is $G(m,p,N)$-singular. More exactly one has

$$\nabla_i p^{(m)}_{\nu,T} = 0 \tag{2.18}$$

for $1 \leqslant i \leqslant N$, where ∇_i is the $G(m,p,N)$ Dunkl operator (2.8) with $m > p$ and $c_1 = 1/r$, $c_2 = \frac{p}{m}(1-\frac{sm}{r})$. In the case $m = p$ the polynomial $p^{(m)}_{\nu,T}$ satisfies (2.18) if $c_1 = 1/r$ and $r = ms$.

Proof. By symmetry it is sufficient to establish that $\nabla_1 p^{(m)}_{\nu,T} = 0$. Consider firstly the case when $1 \notin T$. We have

$$\begin{aligned}
&\nabla_1 p^{(m)}_{\nu,T} = \\
&= \partial_1 p^{(m)}_{\nu,T} - c_1 \sum_{j\in\mathcal{I}_1;k=0}^{m-1} \frac{1-s^k_{1j}}{x_1-\xi^k x_j} p^{(m)}_{\nu,T} - c_1 \sum_{i=2}^{r-1}\sum_{j\in\mathcal{I}_i;k=0}^{m-1} \frac{1-s^k_{1j}}{x_1-\xi^k x_j} p^{(m)}_{\nu,T} = \\
&\qquad = \partial_1 p^{(m)}_{\nu,T} - 2c_1\partial_1 p^{(m)}_{\nu,T} - c_1(r-2)\partial_1 p^{(m)}_{\nu,T}
\end{aligned}$$

by Lemma 2.12. Since $c_1 = 1/r$ the value of the last expression is 0.

Consider now the case $1 \in T$. Assume $m > p$. We have

$$\begin{aligned}
\nabla_1 p^{(m)}_{\nu,T} &= \partial_1 p^{(m)}_{\nu,T} - \frac{mc_2}{px_1} p^{(m)}_{\nu,T} - c_1 \sum_{j\in\mathcal{I}_1;k=0}^{m-1} \frac{1-s^k_{1j}}{x_1-\xi^k x_j} p^{(m)}_{\nu,T} - \\
&\quad - c_1 \sum_{i=2}^{r-1}\sum_{j\in\mathcal{I}_i;k=0}^{m-1} \frac{1-s^k_{1j}}{x_1-\xi^k x_j} p^{(m)}_{\nu,T} = \\
&= \partial_1 p^{(m)}_{\nu,T} - \frac{mc_2}{px_1} p^{(m)}_{\nu,T} - 2c_1\partial_1 p^{(m)}_{\nu,T} + \frac{2c_1}{x_1} p^{(m)}_{\nu,T} - \\
&\quad - c_1 \sum_{i\in T, i\neq 1} (\partial_1 - \frac{1}{x_1}) p^{(m)}_{\nu,T} - c_1 \sum_{i\notin T} (\partial_1 + \frac{m-1}{x_1}) p^{(m)}_{\nu,T}
\end{aligned}$$

by Lemma 2.12. Therefore

$$\begin{aligned}
\nabla_1 p^{(m)}_{\nu,T} &= \partial_1 p^{(m)}_{\nu,T}(1 - 2c_1 - c_1(r-2)) + \\
&\quad + \frac{1}{x_1}\big(-\frac{m}{p}c_2 + 2c_1 + c_1(r-s-2) - c_1 s(m-1)\big) p^{(m)}_{\nu,T} = 0
\end{aligned}$$

as required. The case $m = p$ also follows. □

We will need later a version of the previous proposition for the cases D_N and B_N. We formulate this corollary now. Let ν be a partition of N of length $l(\nu) \leqslant k$. Let T be a subset of indexes $T \subset \{1, \dots, k\}$. Define the polynomial

$$p_{\nu,T}(x) = p_\nu(x_1^2, \dots, x_N^2) \prod_{j \in T} \prod_{i=1}^{\nu_j} x_{\nu_1 + \ldots + \nu_{j-1} + i}, \tag{2.19}$$

where p_ν is given by (2.1) and for $j > l(\nu)$ we put $\nu_j = 0$. Let $K_{\nu,T}$ be the ideal generated by S_N-images of the polynomial $p_{\nu,T}$.

Let now $\nu = \nu_N^k$ be the partition defined in Proposition 2.2. We define $K_{k,s} = K_{\nu_N^k, T_{k,s}}$, where $T_{k,s} = \{k, k-1, \dots, k-s+1\}$. For $k = 2r+1$ define $K_k = K_{k,r}$.

As a corollary from Proposition 2.17 we have the following.

Proposition 2.20. *For $1 \leqslant k \leqslant N-1$, $0 \leqslant s \leqslant k$, the ideal $K_{k,k-s}$ is $H_c(B_N)$-invariant if $c(e_i) = \frac{1}{2} - \frac{s}{k+1}$ and $c(e_i \pm e_j) = 1/(k+1)$. For odd k, $1 \leqslant k \leqslant N-1$, the ideal K_k is $H_{1/(k+1)}(D_N)$-invariant.*

Below we will also need some other ideals (which we don't claim to be representations of any interesting algebras). Namely, for $0 \leqslant s \leqslant k$ define the ideal $\mathcal{K}_{k,s}$ to be generated by S_N-images of all polynomials (2.19) with $l(\nu) \leqslant k$ and $|T| = s$. Note that for $l(\nu) \leqslant k$ and $|T| = s$ one has $K_{\nu,T} \subset \mathcal{K}_{k,s}$. In particular, $K_{k,s} \subset \mathcal{K}_{k,s}$ and $K_{2r+1} \subset \mathcal{K}_{2r+1,r}$. Note also that $\mathcal{K}_{k,s} \subset \mathcal{K}_{k+1,s}$ and $\mathcal{K}_{k,s} \subset \mathcal{K}_{k+1,s+1}$.

Remark 2.21. The inclusions $K_{k,s} \subset \mathcal{K}_{k,s} \subset I_k^{\pm} \cap J_{k-s}$ are obvious and it would be interesting to clarify if any or both of them are actually equalities.

3. Local integrability

Let X be a smooth variety of dimension $N \geqslant 2$ defined over a field $\Bbbk$ of $\mathrm{char}(\Bbbk) = 0$, and D a $\mathbb{Q}$-divisor on X. Write $D = \sum d_j D_j$, where D_j are pairwise different prime divisors. For a rational function $\Phi \in \Bbbk(X)$ we denote by (Φ) the divisor defined by Φ. By $X(\Bbbk)$ we denote the set of $\Bbbk$-points of X. Recall that in the case $\Bbbk = \mathbb{R}$ the set $X(\mathbb{R})$ has a structure of a C^∞-manifold provided that $X(\mathbb{R}) \neq \varnothing$.

Definition 3.1 (see e. g. [13, Definition 3.3]). Let $\pi : Y \to X$ be a birational morphism. Write

$$K_Y + \pi^{-1}(D) \sim_{\mathbb{Q}} \pi^*(K_X + D) + \sum a(E_i) E_i,$$

where K_X and K_Y are the canonical classes of X and Y, respectively, π^{-1} and π^* stand for a proper transform and a pull-back, and E_i are the exceptional divisors of π. The coefficients $a(E_i) = a(X, D, E_i) \in \mathbb{Q}$ are called *discrepancies.*

Convention 3.2 (see e. g. [13, Convention 3.3.2])**.** Define the discrepancy of a (non-exceptional!) divisor D_j to equal $a(D_j) = -d_j$.

Example 3.3. Let Φ be a rational function on X and $c \in \mathbb{Q}$. Let $\pi : Y \to X$ be a birational morphism from a smooth variety Y. Take an exceptional divisor E of π. Choose the local coordinates $y_1, \ldots, y_N$ in a neighborhood of a point $Q \in E$ so that $y_1 = 0$ is a local equation of E, and the local coordinates $x_1, \ldots, x_N$ in a neighborhood of the point $P = \pi(Q)$. Put

$$m = \mathrm{mult}_{(y_1=0)} \Phi \circ \pi \quad \text{and} \quad e = \mathrm{mult}_{(y_1=0)} \det \left(\frac{\partial y_i}{\partial x_j} \right).$$

Then $a(X, c(\Phi), E) = e - cm$. (Note that this formula agrees with Convention 3.2.)

Example 3.4. Let $X = \mathbb{A}^N_{\Bbbk}$, and let $\pi : Y \to X$ be a blow-up of a subvariety $Z \subset X$ of dimension d with an exceptional divisor E. Let Φ be a rational function on X and $c \in \mathbb{Q}$. Then

$$a(X, c(\Phi), E) = N - d - 1 - c \cdot \mathrm{mult}_Z(\Phi).$$

Definition 3.5 (see e. g. [13, 1.1.3])**.** Let $\pi : Y \to X$ be a birational morphism. We call it a *log-resolution* of the pair (X, D), if Y is smooth and the union of the (support of the) strict transform $\pi^{-1}(D)$ of D on Y and the exceptional locus $Exc(\pi)$ is a normal crossing divisor.

Definition 3.6 (see e. g. [13, Definition 3.5])**.** Assume that $\Bbbk = \bar{\Bbbk}$. The pair (X, D) is called *Kawamata log-terminal* (or *klt* for short) if for any log-resolution $\pi : Y \to X$ the inequalities $a(F) > -1$ hold, where F is any exceptional divisor E_i of π or any component D_j of the divisor D.

Definition 3.7. The pair (X, D) is called Kawamata log-terminal if such is the pair $(X_{\bar{\Bbbk}}, D_{\bar{\Bbbk}})$.

Remark 3.8. If the pair (X, D) is klt, one has $d_j < 1$ for all j.

It appears that to check the klt condition it is not necessary to consider all possible log-resolutions.

Theorem 3.9 (see e. g. Lemmas 3.10.2 and 3.12 in [13])**.** *Assume that $d_j < 1$ for all j, and that there exists a log resolution $\pi : Y \to X$ of the pair (X, D) such that the discrepancy of any exceptional divisor appearing on Y is greater than -1. Then the pair (X, D) is klt.*

Recall that in the case $\Bbbk = \mathbb{R}$ (or $\Bbbk = \mathbb{C}$) a function Ψ is said to be *locally* L^1*-integrable* (or just *locally integrable*) at a point $P \in X(\Bbbk)$ if for a sufficiently small (analytic) neighborhood $P \in U_P \subset X(\Bbbk)$ the integral

$$\int\limits_{U_P} |\Psi| dV < \infty.$$

A function Ψ is said to be *locally* L^2*-integrable* if the function Ψ^2 is locally integrable.

One of the important applications of klt singularities is provided by the following theorem (cf. [15, Corollary 2]).

Theorem 3.10. *Let $\Bbbk = \mathbb{R}$. Let Φ be a rational function on X and $c \in \mathbb{Q}$. Assume that the pair*

$$\Big(X, c(\Phi)\Big)$$

is klt. Then Φ^{-c} is locally integrable on X.

Proof. The idea of the proof is standard (see e. g. the proofs of [14, 2.11] or [13, Proposition 3.20]), but since it is usually given in the case $\Bbbk = \mathbb{C}$ (and locally L^2-integrable functions), we will reproduce it here for convenience of the reader.[1]

Let $\dim(X) = N$. We may assume that $X(\mathbb{R}) \neq \varnothing$. Choose the local coordinates $x_1, \ldots, x_N$ in a neighborhood of some point $P \in X(\mathbb{R})$, and put $dV = dx_1 \wedge \ldots \wedge dx_N$. The function Φ^{-c} is locally integrable near P if and only if for some open subset $P \in U \subset X(\mathbb{R})$ the integral

$$\int_U \big|\Phi\big|^{-c} dV < \infty.$$

Let $\pi : Y \to X$ be a log resolution of the pair $\big(X, (\Phi)\big)$. Then

$$\int_U \big|\Phi\big|^{-c} dV = \int_{\pi^{-1}(U)} \big|\Phi \circ \pi\big|^{-c} \pi^* dV.$$

Choose a point $Q \in \pi^{-1}(U)$ such that $\pi(Q) = P$, and the local coordinates $y_1, \ldots, y_N$ in the neighborhood of Q. Then

$$\Phi \circ \pi = \Xi \prod y_i^{m_i} \quad \text{and} \quad \pi^* dV = \Theta \prod y_i^{e_i} dy_1 \wedge \ldots \wedge dy_N$$

for some functions Ξ and Θ that are invertible in a neighborhood of Q, and

$$m_i = \mathrm{mult}_{(y_i=0)} \Phi \circ \pi, \quad e_i = \mathrm{mult}_{(y_i=0)} \det\left(\frac{\partial y_t}{\partial x_j}\right).$$

[1] Another minor difference is that we will need to work with rational rather than regular functions, but this does not influence the proof at all.

Thus the initial integral is finite if and only if for any choice of Q such is the integral

$$\int \dots \int_{U_1 \times \ldots \times U_N} \prod |y_i|^{e_i - cm_i} dy_1 \wedge \ldots \wedge dy_N,$$

where $U_i \subset \mathbb{R}$ is some open subset. The latter holds provided that each of the integrals

$$\int_{U_i} |y_i|^{e_i - cm_i} dy_i < \infty,$$

that is when $e_i - cm_i > -1$ for all i. Now the assertion follows by Example 3.3 and Remark 3.8. □

Remark 3.11 (cf. [15, Theorem 1]). Unlike the case $\Bbbk = \mathbb{C}$, the statement of Theorem 3.10 is not invertible. For example, take $X = \mathbb{R}^3$ with coordinates x_1, x_2, x_3 and define the divisor D by the equation $\Phi = x_1^2 + x_2^2 + x_3^2 = 0$. Then for $1 \leqslant c < 3/2$ the function Φ^{-c} is integrable, but the pair (X, cD) is not klt.

On the other hand, the converse to the statement of Theorem 3.10 does hold in some important particular cases, for example when $X = \mathbb{R}^N$ and the poles of Φ^{-c} are supported on the real hyperplanes (cf. Remark 5.19 and Section 7).

Consider now a collection of hyperplanes given by the equations $l_i = 0$, $i = 1, \ldots, M$, where l_i are some non-zero covectors in $\mathbb{R}^N$. Recall that this collection defines a semi-lattice $\mathcal{L}$ which is the minimal set of linear subspaces of $\mathbb{R}^N$ containing all the hyperplanes $l_i = 0$ and closed with respect to intersection.

Corollary 3.12. *Let $\mathcal{F} \subset \mathbb{R}[x_1, \ldots, x_N]$ be a finite set of polynomials, and $\bar{\mathcal{F}}$ be the ideal generated by $\mathcal{F}$. Choose the numbers $c_i \in \mathbb{Q}$, $i = 1, \ldots, M$. For a linear subspace $L \subset \mathbb{R}^N$ define $m(L) = m_{\mathcal{F}}(L)$ to be the minimal multiplicity of a function $f \in \mathcal{F}$ along L, and $\kappa(L) = \sum_{L \subset l_i} c_i$. Then for any $f \in \bar{\mathcal{F}}$ the function*

$$f \prod_{i=1}^{M} l_i^{-c_i}$$

is locally L^2-integrable at any point $P \in \mathbb{R}^N$ provided that

$$\kappa(L) < \frac{\operatorname{codim}(L)}{2} + m(L)$$

for any $L \in \mathcal{L}$.

Proof. Choose a nonzero function $f \in \bar{\mathcal{F}}$ and put

$$\Phi_f = \frac{\prod l_i^{c_i}}{f}.$$

By Theorems 3.9, 3.10 it is enough to check that the discrepancies

$$a(\mathbb{R}^N, (\Phi_f^2), E_j) > -1$$

for all exceptional divisors E_i of a partial log-resolution $\pi : Y \to \mathbb{R}^N$ such that $\pi^{-1}(\bigcup l_i) \cup Exc(\pi)$ is a normal crossing divisor. To construct such resolution put $\pi = \pi_{N-2} \circ \ldots \circ \pi_0$, where $\pi_0 : Y_0 \to \mathbb{R}^N$ is the blow-up of the point $\mathbf{0} \in \mathbb{R}^N$, and $\pi_d : Y_d \to Y_{d-1}$ for $d \geqslant 1$ is the blow-up of the strict transforms of all subspaces $L \in \mathcal{L}$ such that $\dim(L) = d$. Note that these strict transforms are disjoint on Y_{d-1}, so that $\pi : Y = Y_{N-2} \to \mathbb{R}^N$ indeed enjoys the desired property. Note that $\pi_{d-1} : Y_{d-1} \to \mathbb{R}^N$ is an isomorphism at a neighborhood of a general point $P \in \pi_{d-1}^{-1}(L)$ for $L \in \mathcal{L}$ with $\dim(L) = d$. Hence the discrepancy $a(\mathbb{R}^N, (\Phi_f^2), E_L)$ of the exceptional divisor E_L whose center on $\mathbb{R}^N$ is L equals the discrepancy of the exceptional divisor of the blow-up of $\mathbb{R}^N$ along L, which in turn equals

$$\begin{aligned} a_L = \mathrm{codim}(L) - 1 - 2\sum c_i \mathrm{mult}_L(l_i) + 2\mathrm{mult}_L(f) = \\ = \mathrm{codim}(L) - 1 - 2\kappa(L) + 2\mathrm{mult}_L(f) \end{aligned}$$

by Example 3.4. Hence for Φ_f^{-1} to be locally L^2-integrable at any $P \in \mathbb{R}^N$ it is enough to satisfy the inequality

$$\kappa(L) < \frac{\mathrm{codim}(L)}{2} + \mathrm{mult}_L(f)$$

for any $L \in \mathcal{L}$. The required assertion follows since

$$\mathrm{mult}_L(f) \geqslant \min_{\phi \in \mathcal{F}} \mathrm{mult}_L(\phi) = m(L).$$

□

In the case of singularities of constant order the previous corollary can be rephrased as follows.

Corollary 3.13. *In the above notations assume that l_i is not proportional to l_j for $i \neq j$ and that $c_i = c$ for all $1 \leqslant i \leqslant M$. Then for any $f \in \bar{\mathcal{F}}$ the function*

$$f \prod_{i=1}^{M} l_i^{-c}$$

is locally L^2-integrable at any point $P \in \mathbb{R}^N$ provided that

$$c < \min_{L \in \mathcal{L}} \frac{\frac{1}{2}\operatorname{codim}(L) + m(L)}{K(L)},$$

where $K(L)$ is the number of l_i vanishing along L.

4. Elementary estimates

In this section we collect a few technical lemmas which we will need in Section 5.

Consider the m-dimensional space $\mathbb{R}^m \supset \mathbb{Z}^m$. For any $\xi \in \mathbb{R}_{\geqslant 0}$ define

$$\mathfrak{S}_{m,\xi} = \{(t_1, \dots, t_m) \in \mathbb{R}^m \mid 0 \leqslant t_i \leqslant \xi, \sum t_i = \xi\} \cap \mathbb{Z}^m.$$

Having fixed $k \in \mathbb{Z}_+$, for any $q \in \mathbb{R}$ we define $0 \leqslant \rho_k(q) < k$ to satisfy $\rho_k(q) = q \bmod k$. For $\alpha \in \mathbb{R}$ we denote the integer part of α by $\lfloor \alpha \rfloor$.

Lemma 4.1. *Fix $\Lambda \in \mathbb{Z}_+$ and put $\rho = \rho_m(\Lambda)$. Consider the function*

$$C(t) = C_{m,\Lambda}(t_1, \dots, t_m) = \sum_{i=1}^{m} \frac{t_i(t_i - 1)}{2}.$$

Put

$$\mu_{m,\Lambda} = \min_{t \in \mathfrak{S}_{m,\Lambda}} C(t).$$

Then

$$\mu_{m,\Lambda} = \frac{\Lambda(\Lambda - m)}{2m} + \frac{\rho(m - \rho)}{2m}.$$

Proof. Note that for any $u, v \in \mathbb{R}$ one has

$$(u - 1)^2 + (v + 1)^2 < u^2 + v^2$$

provided that $u > v + 1$. Thus the minimum

$$\mu_{m,\Lambda} = \frac{1}{2} \min_{t \in \mathfrak{S}_{m,\Lambda}} \Big(\sum_{i=1}^{m} t_i^2 \Big) - \frac{\Lambda}{2}$$

is attained at a point $A = (a_1, \dots, a_m) \in \mathfrak{S}_{m,\Lambda}$ such that for any i and j one has $|a_i - a_j| \leqslant 1$. Since $C(t)$ is invariant under permutations of coordinates, we may assume that

$$A = \Big(\underbrace{\Big\lfloor \frac{\Lambda}{m} \Big\rfloor + 1, \dots, \Big\lfloor \frac{\Lambda}{m} \Big\rfloor + 1}_{\rho}, \underbrace{\Big\lfloor \frac{\Lambda}{m} \Big\rfloor, \dots, \Big\lfloor \frac{\Lambda}{m} \Big\rfloor}_{m-\rho} \Big).$$

Write $\Lambda = sm + \rho$ for some $s \in \mathbb{Z}_{\geqslant 0}$. One has

$$\mu_{m,\Lambda} = C(A) = \frac{1}{2}\Big(\rho\Big(\Big\lfloor\frac{\Lambda}{m}\Big\rfloor + 1\Big)\Big\lfloor\frac{\Lambda}{m}\Big\rfloor + (m-\rho)\Big(\Big\lfloor\frac{\Lambda}{m}\Big\rfloor - 1\Big)\Big\lfloor\frac{\Lambda}{m}\Big\rfloor =$$

$$= \frac{1}{2}\big(\rho(s+1)s + (m-\rho)(s-1)s\big) = \frac{m^2(s^2-s)+2\rho ms}{2m} =$$

$$= \frac{(ms+\rho)(ms+\rho-m)+\rho(m-\rho)}{2m} =$$

$$= \frac{\Lambda(\Lambda-m)}{2m} + \frac{\rho(m-\rho)}{2m}.$$

□

Lemma 4.2. *Take $a, b, z \in \mathbb{Z}_+$. Choose $\alpha \in \mathbb{R}$ such that $0 \leqslant \alpha \leqslant ab/(a+b)^2$ and put*

$$\Lambda_1 = \frac{za}{a+b} - \alpha(a+b), \quad \Lambda_2 = \frac{zb}{a+b} + \alpha(a+b).$$

Let

$$F = b\rho_a(\Lambda_1)\big(a - \rho_a(\Lambda_1)\big) + a\rho_b(\Lambda_2)\big(b - \rho_b(\Lambda_2)\big).$$

Then

$$F \geqslant \alpha(a+b)\big(ab - \alpha(a+b)^2\big).$$

Proof. Write $\Lambda_1 = sa + \rho_a(\Lambda_1)$, $s \in \mathbb{Z}_{\geqslant 0}$. Then

$$\Lambda_2 = sb + \rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a},$$

and

$$0 \leqslant \rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a} < 2b$$

by assumptions. Put

$$\mathcal{S}_1 = \Big[0, \frac{a}{b}\big(b - \alpha\frac{(a+b)^2}{a}\big)\Big) \subset [0, a) \subset \mathbb{R},$$

and

$$\mathcal{S}_2 = [0, a) \setminus \mathcal{S}_1 = \Big[\frac{a}{b}\big(b - \alpha\frac{(a+b)^2}{a}\big), a\Big) \subset \mathbb{R}.$$

Suppose first that $\rho_a(\Lambda_1) \in \mathcal{S}_1$. Then

$$\rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a} < b,$$

and

$$\rho_b(\Lambda_2) = \rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a}.$$

Note that $F = F\big(\rho_a(\Lambda_1)\big)$ is a quadratic function in $\rho_a(\Lambda_1)$ with negative coefficient at $\rho_a(\Lambda_1)^2$. Thus

$$\inf_{\rho_a(\Lambda_1)\in\mathcal{S}_1} F\big(\rho_a(\Lambda_1)\big) \geqslant \min\Big\{F\big(0\big), F\Big(\frac{a}{b}\big(b-\alpha\frac{(a+b)^2}{a}\big)\Big)\Big\} =$$

$$= \min\Big\{\alpha\frac{(a+b)^2}{a}\big(ab-\alpha(a+b)^2\big), \alpha\frac{(a+b)^2}{b}\big(ab-\alpha(a+b)^2\big)\Big\} \geqslant$$

$$\geqslant \alpha(a+b)(ab-\alpha(a+b)^2).$$

Suppose now that $\rho_a(\Lambda_1) \in \mathcal{S}_2$. Then

$$b \leqslant \rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a} < 2b,$$

and

$$\rho_b(\Lambda_2) = \rho_a(\Lambda_1)\frac{b}{a} + \alpha\frac{(a+b)^2}{a} - b.$$

Note that $F = F\big(\rho_a(\Lambda_1)\big)$ is again a quadratic function in $\rho_a(\Lambda_1)$ with negative coefficient at $\rho_a(\Lambda_1)^2$. Thus

$$\inf_{\rho_a(\Lambda_1)\in\mathcal{S}_2} F\big(\rho_a(\Lambda_1)\big) \geqslant \min\Big\{F\Big(\frac{a}{b}\big(b-\alpha\frac{(a+b)^2}{a}\big)\Big), F\big(a\big)\Big\} =$$

$$= \min\Big\{\alpha\frac{(a+b)^2}{a}\big(ab-\alpha(a+b)^2\big), \alpha\frac{(a+b)^2}{b}\big(ab-\alpha(a+b)^2\big)\Big\} \geqslant$$

$$\geqslant \alpha(a+b)(ab-\alpha(a+b)^2).$$

□

Lemma 4.3. *Fix $a, b, z \in \mathbb{Z}_+$. Consider the function*

$$\tilde{C}(t) = \tilde{C}(t_1, \ldots, t_{a+b}) = \sum_{i=1}^{a} t_i + \sum_{i=1}^{a+b} t_i(t_i - 1).$$

Put

$$\tilde{\mu} = \tilde{\mu}_{a,b,z} = \min_{t\in\mathfrak{S}_{a+b,z}} \tilde{C}(t).$$

Then

$$\tilde{\mu} \geqslant \frac{z^2}{a+b} - \frac{b}{a+b}z. \tag{4.4}$$

Proof. Denote the a-tuple $(t_1,\dots,t_a)$ by T_1, and denote the b-tuple $(t_{a+1},\dots,t_{a+b})$ by T_2. Put $\sum_{i=1}^a t_i=\Lambda_1$ and $\Lambda_2=z-\Lambda_1$. One has

$$\begin{aligned}
\tilde\mu = \min_{\Lambda_1+\Lambda_2=z}\Bigl(\min_{t\in\mathfrak{S}_{a+b,z}}\bigl(\Lambda_1+\sum_{i=1}^{a}t_i(t_i-1)+\sum_{i=a+1}^{a+b}t_i(t_i-1)\bigr)\Bigr) =\\
= \min_{\Lambda_1+\Lambda_2=z}\bigl(\Lambda_1+2\min_{T_1\in\mathfrak{S}_{a,\Lambda_1}}C_a(T_1)+2\min_{T_2\in\mathfrak{S}_{b,\Lambda_2}}C_b(T_2)\bigr) =\\
= \min_{\Lambda_1+\Lambda_2=z}\bigl(\Lambda_1+2\mu_{a,\Lambda_1}+2\mu_{b,\Lambda_2}\bigr) =\\
= \min_{\Lambda_1+\Lambda_2=z}\Bigl(\Lambda_1+\frac{\Lambda_1(\Lambda_1-a)+\rho_a(\Lambda_1)(a-\rho_a(\Lambda_1))}{a}+\\
+\frac{\Lambda_2(\Lambda_2-b)+\rho_b(\Lambda_2)(b-\rho_b(\Lambda_2))}{b}\Bigr) =\\
= \min_{\Lambda_1+\Lambda_2=z}\Bigl(\frac{b\Lambda_1^2+a\Lambda_2^2-ab\Lambda_2}{ab}+\\
+\frac{b\rho_a(\Lambda_1)(a-\rho_a(\Lambda_1))+a\rho_b(\Lambda_2)(b-\rho_b(\Lambda_2))}{ab}\Bigr).
\end{aligned}$$

Suppose that the minimum is attained for $\Lambda_1=az/(a+b)-\alpha(a+b)$, $\Lambda_2=zb/(a+b)+\alpha(a+b)$ (note that we don't assume that α is nonnegative). Then

$$\frac{b\Lambda_1^2+a\Lambda_2^2-ab\Lambda_2}{ab} = \frac{z^2}{a+b}-\frac{b}{a+b}z+\alpha\frac{a+b}{ab}\bigl(\alpha(a+b)^2-ab\bigr)$$

Thus to conclude the proof we may assume that $\alpha(\alpha(a+b)^2-ab)\leqslant 0$, i. e. $0\leqslant\alpha\leqslant ab/(a+b)^2$, and the assertion follows by Lemma 4.2. □

Choose nonnegative integers $N\geqslant 2$ and $z\leqslant N$, and let $\lambda=(\lambda_1,\dots,\lambda_{l(\lambda)})$ be a partition of $N-z$, i. e.

$$\lambda_1\geqslant\ldots\geqslant\lambda_{l(\lambda)}>0 \text{ and } \sum\lambda_i=N-z.$$

(In particular, we allow an "empty" partition when $l(\lambda)=0$ and $z=N$.) Put

$$R_k(\lambda)=\sum_{i=1}^{l(\lambda)}\rho_k(\lambda_i)(k-\rho_k(\lambda_i)). \tag{4.5}$$

Lemma 4.6. *Let $k\in\mathbb{Z}_+$, and λ be a partition of $N-z$ as above. Then*

$$\sum_{i=1}^{l(\lambda)}\frac{\lambda_i(\lambda_i-k)}{k}+\frac{R_k(\lambda)}{k}+N-l(\lambda)\geqslant\sum_{i=1}^{l(\lambda)}\frac{\lambda_i(\lambda_i-1)}{k}+z. \tag{4.7}$$

Proof. Denote by v the number of λ_i which are divisible by k. Then $N - z \geqslant kv + l(\lambda) - v$, and $R_k(\lambda) \geqslant (k-1)(l(\lambda) - v)$, hence

$$N + R_k(\lambda) \geqslant kl(\lambda) + z.$$

One has

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - k)}{k} + \frac{R_k(\lambda)}{k} + N - l(\lambda) = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i^2}{k} + \frac{R_k(\lambda)}{k} + z - l(\lambda) =$$

$$= \sum_{i=1}^{l(\lambda)} \frac{\lambda_i^2}{k} + \frac{N + R_k(\lambda) - kl(\lambda) - z}{k} - \frac{N-z}{k} + z \geqslant$$

$$\geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i^2}{k} - \frac{\sum_{i=1}^{l(\lambda)} \lambda_i}{k} + z = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{k} + z.$$

□

Lemma 4.8. *Let $k \in \mathbb{Z}_+$, and let λ be a partition of N. Assume that $\lambda_1 > 1$. Then*

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - k)}{k} + \frac{R_k(\lambda)}{k} + N - l(\lambda) > \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{k+1}.$$

Proof. By Lemma 4.6 applied for $z = 0$ one has

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - k)}{k} + \frac{R_k(\lambda)}{k} + N - l(\lambda) \geqslant$$

$$\geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{k} \geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{k+1}.$$

Moreover, the last inequality is strict if the right hand side is non-zero, which happens exactly when $\lambda_1 > 1$. □

Lemma 4.9. *Let λ be a partition of $N - z$. Assume that $z \geqslant 1$ if $\lambda_1 = 1$. Then*

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - k)}{2k} + \frac{R_k(\lambda)}{2k} + \frac{z(z-k)}{k} + \frac{\rho_k(z)(k - \rho_k(z))}{k} + \tag{4.10}$$

$$+ \max(0, \lfloor z - s \rfloor) + \frac{N - l(\lambda)}{2} >$$

$$> \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{2(k+1)} + \frac{z(z-1)}{k+1} + \frac{\frac{k+1}{2} - s}{k+1} z,$$

where $k \in \mathbb{Z}_+$, $s \in \mathbb{R}$.

Proof. It is sufficient to establish that the left-hand side of (4.10) is greater or equal than

$$\frac{1}{2k}\sum \lambda_i^2 - \frac{N-z}{2k} + \frac{z^2}{k+1} + \frac{\frac{k-1}{2}-s}{k+1}z, \tag{4.11}$$

and that it is strictly bigger than (4.11) when $\lambda_1 = 1$ or $l(\lambda) = 0$ (which are exactly the cases when the first of the three summands in the right-hand side of (4.10) vanishes).

By Lemma 4.6 our statement is implied by the inequality

$$\frac{z(z-k)}{k} + \frac{\rho_k(z)(k-\rho_k(z))}{k} + \max(0, \lfloor z-s \rfloor) > \frac{z^2}{k+1} - \frac{s+1}{k+1}z. \tag{4.12}$$

Moreover, we may assume that $z > 0$. Indeed, the case $z = 0$ leads to the equality in (4.12), but in this case $l(\lambda) > 0$ and $\lambda_1 > 1$ by assumption.

It is clear that the inequality (4.12) holds for $s \geqslant k$ as in this case

$$\frac{z(z-k)}{k} > \frac{z^2}{k+1} - \frac{s+1}{k+1}z$$

so let us suppose that $s < k$. Now consider few possible cases for the values of z. When $z < s+1$ the left-hand side of (4.12) equals 0 and the inequality holds. When $s+1 \leqslant z \leqslant k$ the left-hand side of (4.12) equals $\lfloor z-s \rfloor$ so the inequality (4.12) takes the form

$$z^2 - (s+1)z - (k+1)\lfloor z-s \rfloor < 0.$$

This inequality is correct since $-\lfloor z-s \rfloor < 1 - z + s$ and

$$z^2 - (s+1)z + (k+1)(1-z+s) = (z-s-1)(z-k-1) \leqslant 0.$$

Finally, when $z \geqslant k+1$ the left-hand side of (4.12) is bigger than

$$\frac{z^2}{k} - s - 1,$$

and (4.12) holds. □

Remark 4.13. It follows from the proof of Lemma 4.9 that its assertion remains true when $\max(0, \lfloor z-s \rfloor)$ is replaced by 0 in the case $s \geqslant k$.

5. Conclusions on unitarity for classical root systems

In this section we apply the previous estimates to establish unitarity of certain submodules in the polynomial representation. We start with the A_{N-1} case.

Proposition 5.1. *In the notations of Section 2 the function*

$$\frac{f}{\prod_{i<j}^{N}(x_i - x_j)^c}$$

is locally L^2-integrable for all $f \in I_k$ $(1 \leqslant k \leqslant N-1)$ provided that $c \leqslant 1/(k+1)$.

Proof. Assume the notation of Corollary 3.12, and consider the semilattice $\mathcal{L}$ generated by the hyperplanes $l_{ij} = x_i - x_j = 0$. By Corollary 3.13 it is enough to check that

$$\frac{1}{k+1} < \min_{L\in\mathcal{L}} \frac{\frac{1}{2}\mathrm{codim}(L) + m(L)}{K(L)},$$

where $m(L) = m_{I_k}(L)$. By S_N-symmetry it suffices to consider the linear subspaces $L = L_\lambda$ given by

$$x_1 = \ldots = x_{\lambda_1}, x_{\lambda_1+1} = \ldots = x_{\lambda_1+\lambda_2}, \ldots, \\ x_{\lambda_1+\ldots+\lambda_{l(\lambda)-1}+1} = \ldots = x_{\lambda_1+\ldots+\lambda_{l(\lambda)}}$$

for some partition $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ of N where $\lambda_1 > 1$. It is easy to see that $\mathrm{codim}(L) = N - l(\lambda)$, and

$$K(L) = \sum \frac{\lambda_i(\lambda_i - 1)}{2}.$$

To compute $m(L)$ consider $\nu = (\nu_1, \ldots, \nu_k)$ a partition of N and the corresponding polynomial $p_\nu \in I_k$ introduced in Section 2. A polynomial $\bar{p}_\nu$ from the S_N-orbit of p_ν gives rise to a presentation of each λ_i as a sum of k nonnegative summands

$$\lambda_i = \lambda_{i,1} + \ldots + \lambda_{i,k},$$

so that

$$\nu_j = \lambda_{1,j} + \ldots + \lambda_{k,j}.$$

Moreover,

$$\mathrm{mult}_L(\bar{p}_\nu) = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{k} \frac{\lambda_{ij}(\lambda_{ij} - 1)}{2}.$$

Recall that by Proposition 2.2 the S_N-orbits of the polynomials p_ν for various ν generate the ideal I_k. Hence, in the notation of Lemma 4.1 one has

$$m(L) = \min_{\sum_j \lambda_{ij} = \lambda_i} \Big(\sum_{i=1}^{l(\lambda)} \sum_{j=1}^{k} \frac{\lambda_{ij}(\lambda_{ij} - 1)}{2} \Big) = \sum_{i=1}^{l(\lambda)} \mu_{k,\lambda_i}.$$

By Lemma 4.1 one has

$$m(L) = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - k)}{2k} + \frac{R_k(\lambda)}{2k},$$

where $R_k(\lambda)$ is defined by (4.5). The desired assertion is implied by Lemma 4.8. □

The ideal I_k is an (irreducible) representation of the rational Cherednik algebra $H_c(\mathrm{S}_N)$ when $c = 1/(k+1)$. Therefore Proposition 5.1 has the following corollary which was firstly established in [9] by different arguments.

Corollary 5.2 ([9, Theorem 5.14])**.** *The representation I_k is a unitary representation of the rational Cherednik algebra $H_{1/(k+1)}(\mathrm{S}_N)$.*

Now we move to the D_N and B_N cases. We are going to establish local L^2-integrability of the relevant functions based on the polynomials from the ideal $I_k^{\pm}$. In order to do this we consider the subspaces from the intersection semi-lattice $\mathcal{L}$ of the arrangement of hyperplanes of type D_N. Namely, we say that a linear space is *of type* (λ, z) where $0 \leqslant z \leqslant N$ and $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ is a partition of $N - z$ if the space is a D_N-image of the following linear space:

$$(5.3)\quad x_1 = \ldots = x_{\lambda_1}, x_{\lambda_1+1} = \ldots = x_{\lambda_1+\lambda_2}, \ldots,$$
$$x_{\lambda_1+\ldots+\lambda_{l(\lambda)-1}+1} = \ldots = x_{\lambda_1+\ldots+\lambda_{l(\lambda)}}, x_{\lambda_1+\ldots+\lambda_{l(\lambda)}+1} = \ldots = x_N = 0.$$

For a fixed subspace L we will refer to the variables involved in the last group of equations as *z-variables*, and to the other variables as *λ-variables*.

Note that any element $L \in \mathcal{L}$ has above type with $z \neq 1$ except the case when N is even and $z = 0$. In this case $\mathcal{L}$ also contains the spaces *of type* λ^- given by the D_N-images of the linear space determined by the equations

$$-x_1 = x_2 = \ldots = x_{\lambda_1}, x_{\lambda_1+1} = \ldots = x_{\lambda_1+\lambda_2}, \ldots,$$
$$x_{\lambda_1+\ldots+\lambda_{l(\lambda)-1}+1} = \ldots = x_N,$$

where λ is a partition of N.

Recall that ideals K_k and $\mathcal{K}_{k,s}$ were defined in the end of Section 2, and by Proposition 2.20 the ideal K_{2r-1} is a representation of the algebra $H_c(D_N)$ for $c = 1/(2r)$.

Theorem 5.4. *The function*

$$\frac{f}{\prod_{i<j}^{N}(x_i^2 - x_j^2)^{\frac{1}{2r}}}$$

is locally L^2-integrable for all $f \in K_{2r-1}$ provided that $r \leqslant \frac{N}{2}$, $r \in \mathbb{Z}_+$.

Proof. Assume the notations of Corollary 3.12, and consider the semilattice $\mathcal{L}$ generated by the hyperplanes $l_{ij} = x_i - x_j = 0$ and $l'_{ij} = x_i + x_j = 0$. By Corollary 3.13 it is enough to check that

$$\frac{1}{2r} < \min_{L \in \mathcal{L}} \frac{\frac{1}{2}\mathrm{codim}(L) + m(L)}{K(L)}, \tag{5.5}$$

where $m(L) = m_{K_{2r-1}}(L)$

Choose a subspace $L \in \mathcal{L}$ of type (λ, z) where $0 \leqslant z \leqslant N$, $z \neq 1$, and $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$ is a partition of $N - z$. It is easy to see that $\mathrm{codim}(L) = N - l(\lambda)$ and

$$K(L) = \sum \frac{\lambda_i(\lambda_i - 1)}{2} + z(z-1).$$

Since $K_{2r-1} \subset \mathcal{K}_{2r-1,r-1} \subset \mathcal{K}_{2r,r}$, one has

$$m(L) \geqslant m_{\mathcal{K}_{2r-1,r-1}}(L) \geqslant m_{\mathcal{K}_{2r,r}}(L).$$

Assume first that $z > 0$. Let us estimate the value of $m_{\mathcal{K}_{2r,r}}(L)$. Consider a partition $\nu = (\nu_1, \dots, \nu_{2r})$, a set

$$T = \{\tau_1, \dots, \tau_r\} \subset \{1, \dots, 2r\}$$

and the corresponding polynomial $p_{\nu,T} \in \mathcal{K}_{2r,r}$ introduced in Section 2. A polynomial $\bar{p}_{\nu,T}$ from the S_N-orbit of $p_{\nu,T}$ gives rise to a presentation of each λ_i and z as a sum of $2r$ nonnegative summands

$$\lambda_i = \lambda_{i,1} + \ldots + \lambda_{i,2r}, \quad z = \zeta_1 + \ldots + \zeta_{2r}$$

so that

$$\nu_j = \lambda_{1,j} + \ldots + \lambda_{l(\lambda),j} + \zeta_j.$$

Moreover,

$$\mathrm{mult}_L(\bar{p}_\nu) = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{2r} \frac{\lambda_{ij}(\lambda_{ij}-1)}{2} + \sum_{j=1}^{2r} \zeta_j(\zeta_j - 1) + \sum_{\tau \in T} \zeta_\tau.$$

Hence, in the notation of Lemma 4.1 and Lemma 4.3 one has

$$m_{\mathcal{K}_{2r,r}}(L) =$$

$$= \sum_{i=1}^{l(\lambda)} \Big(\min_{\sum_j \lambda_{ij} = \lambda_i} \sum_{j=1}^{2r} \frac{\lambda_{ij}(\lambda_{ij}-1)}{2} \Big) + \min_{\sum_j \zeta_j = z} \Big(\sum_{j=1}^{2r} \zeta_j(\zeta_j - 1) + \sum_{\tau \in T} \zeta_\tau \Big) =$$

$$= \sum_{i=1}^{l(\lambda)} \mu_{2r,\lambda_i} + \tilde{\mu}_{r,r,z}.$$

By Lemmas 4.1 and 4.3 applied for $a = b = r$ one has

$$m(L) \geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 2r)}{4r} + \frac{R_{2r}(\lambda)}{4r} + \frac{z^2}{2r} - \frac{z}{2}.$$

By Lemma 4.6 applied for $k = 2r$ one has

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 2r)}{4r} + \frac{R_{2r}(\lambda)}{4r} + \frac{z^2}{2r} - \frac{z}{2} + \frac{N - l(\lambda)}{2} \geqslant$$
$$\geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{4r} + \frac{z^2}{2r} >$$
$$> \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{4r} + \frac{z(z-1)}{2r} = \frac{K(L)}{2r}$$

so (5.5) follows.

Now assume that $z = 0$ and estimate $m_{\mathcal{K}_{2r-1,r-1}}$. Arguing as in the proof of Proposition 5.1, one obtaines

$$m(L) \geqslant m_{\mathcal{K}_{2r-1,r-1}}(L) = m_{I^{\pm}_{2r-1}}(L) = m_{I_{2r-1}}(L) = \sum_{i=1}^{l(\lambda)} \mu_{2r-1,\lambda_i} =$$
$$= \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - (2r-1))}{2(2r-1)} + \frac{R_{2r-1}(\lambda)}{2(2r-1)}.$$

Thus the assertion in this case is implied by Lemma 4.8 applied for $k = 2r - 1$.

Finally, choose a subspace $L \in \mathcal{L}$ of type λ^-. It is easy to see that the values of $\mathrm{codim}(L)$, $K(L)$ and $m(L)$ are the same as for a subspace of type $(\lambda, 0)$, which completes the proof. □

Now we consider singular values $c = 1/(2r)$ with $r > N/2$. We need to use ideals J_r from Section 2.

Theorem 5.6. *The function*

$$\frac{f}{\prod_{i<j}^{N} (x_i^2 - x_j^2)^{\frac{1}{2r}}}$$

is locally L^2-integrable for all $f \in J_r$ provided that $N > r > \frac{N}{2}$, $r \in \mathbb{Z}_+$.

Proof. Let L be a subspace of type (λ, z) or λ^-. Note that the multiplicity $m(L) = \max(0, z - r)$. We need to establish that

$$(5.7)\qquad \frac{z(z-1)}{2r} + \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{4r} < \frac{N - l(\lambda)}{2} + \max(0, z - r).$$

Assume that $z > 0$. Then

$$(5.8)\qquad \frac{z(z-1)}{2r} < \frac{z}{2} + \max(0, z - r),$$

which can be easily seen by considering the cases $2r > z \geqslant r$ and $z < r$. Moreover, applying Lemma 4.6 with $k = 2r$ one obtains

$$(5.9)\qquad \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{4r} + \frac{z}{2} \leqslant \frac{N - l(\lambda)}{2}$$

since the first two summands of the left hand side of (4.7) make 0 for $k > N$. Adding up (5.8) and (5.9) one obtains (5.7).

Now assume that $z = 0$. Then (5.7) becomes

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{4r} < \frac{N - l(\lambda)}{2}.$$

Since $2r > N$ it is enough to check that

$$(5.10)\qquad \sum_{i=1}^{l(\lambda)} \lambda_i^2 \leqslant N^2 - Nl(\lambda) + N.$$

The maximum of the left hand side of (5.10) is obtained for

$$\lambda = (N - l(\lambda) + 1, 1, \dots, 1).$$

Thus (5.10) holds since

$$N^2 - Nl(\lambda) + N - (N - l(\lambda) + 1)^2 - (l(\lambda) - 1) = (N - l(\lambda))(l(\lambda) - 1) \geqslant 0.$$

$\square$

Now we move to the case of the poles supported on the B_N semi-lattice. Consider the ideal $K_{r-1,r-s-1}$ as a representation of the rational Cherednik algebra $H_c(B_N)$ where the multiplicity function $c(e_i \pm e_j) = 1/r$ and $c(e_i) = \frac{1}{2} - \frac{s}{r}$ (see Proposition 2.20). Any element from the corresponding intersection semi-lattice $\mathcal{L}(B_N)$ is the image of the space of the form (5.3) under an element of the group B_N. We say that these spaces have type (λ, z) where $z = 0, 1, \dots, N$ and λ is a partition of $N - z$.

Theorem 5.11. *The function*

$$g = \frac{f}{\prod_{i<j}^{N}(x_i^2 - x_j^2)^{\frac{1}{r}} \prod_{i=1}^{N} x_i^{\frac{1}{2}-\frac{s}{r}}}$$

is locally L^2-integrable for any $f \in K_{r-1,r-s-1}$ provided that $2 \leqslant r \leqslant N$, $0 \leqslant s \leqslant r-1$, $r, s \in \mathbb{Z}$.

Proof. By Corollary 3.12 it is sufficient to establish that

$$\kappa(L) < \frac{1}{2}\operatorname{codim}(L) + m(L) \tag{5.12}$$

where L is an arbitrary subspace from the intersection semi-lattice $\mathcal{L}(B_N)$ and $m(L) = m_{K_{r-1,r-s-1}}(L)$. Choose a subspace $L \in \mathcal{L}$ of type (λ, z). It is easy to see that $\operatorname{codim}(L) = N - l(\lambda)$ and

$$\kappa(L) = \sum \frac{\lambda_i(\lambda_i - 1)}{2r} + \frac{z(z-1)}{r} + \big(\frac{1}{2} - \frac{s}{r}\big)z.$$

Since $K_{r-1,r-s-1} \subset \mathcal{K}_{r-1,r-s-1}$, one has $m(L) \geqslant m_{\mathcal{K}_{r-1,r-s-1}}(L)$.

Assume that $z > 0$. Let us estimate the value of $m_{\mathcal{K}_{r-1,r-s-1}}(L)$. Consider a partition $\nu = (\nu_1, \ldots, \nu_{r-1})$, a set

$$T = \{\tau_1, \ldots, \tau_{r-s-1}\} \subset \{1, \ldots, r-1\}$$

and the corresponding polynomial $p_{\nu,T} \in \mathcal{K}_{r-1,r-s-1}$ introduced in Section 2. A polynomial $\bar{p}_{\nu,T}$ from the S_N-orbit of $p_{\nu,T}$ gives rise to a presentation of each λ_i and z as a sum of $r-1$ nonnegative summands

$$\lambda_i = \lambda_{i,1} + \ldots + \lambda_{i,r-1}, \quad z = \zeta_1 + \ldots + \zeta_{r-1}$$

so that

$$\nu_j = \lambda_{1,j} + \ldots + \lambda_{l(\lambda),j} + \zeta_j.$$

Moreover,

$$\operatorname{mult}_L(\bar{p}_\nu) = \sum_{i=1}^{l(\lambda)} \sum_{j=1}^{r-1} \frac{\lambda_{ij}(\lambda_{ij}-1)}{2} + \sum_{j=1}^{r-1} \zeta_j(\zeta_j - 1) + \sum_{\tau \in T} \zeta_\tau.$$

Hence, in the notation of Lemma 4.1 and Lemma 4.3 one has

$$m_{\mathcal{K}_{r-1,r-s-1}}(L) =$$

$$= \sum_{i=1}^{l(\lambda)} \Big(\min_{\sum_j \lambda_{ij} = \lambda_i} \sum_{j=1}^{r-1} \frac{\lambda_{ij}(\lambda_{ij}-1)}{2} \Big) + \min_{\sum_j \zeta_j = z} \Big(\sum_{j=1}^{r-1} \zeta_j(\zeta_j - 1) + \sum_{\tau \in T} \zeta_\tau \Big) =$$

$$= \sum_{i=1}^{l(\lambda)} \mu_{r-1,\lambda_i} + \tilde{\mu}_{r-s-1,s,z}.$$

By Lemmas 4.1 and 4.3 applied for $a = r - s - 1$ and $b = s$ one has

$$m(L) \geqslant \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - (r-1))}{2(r-1)} + \frac{R_{r-1}(\lambda)}{2(r-1)} + \frac{z^2}{r-1} - \frac{s}{r-1}z.$$

By Lemma 4.6 applied for $k = r - 1$ one has

$$\begin{aligned} m(L) + \frac{1}{2}\operatorname{codim}(L) \geqslant \\ \geqslant \sum \frac{\lambda_i(\lambda_i - 1)}{2(r-1)} + \frac{z^2}{r-1} + \left(\frac{1}{2} - \frac{s}{r-1}\right)z > \\ > \sum \frac{\lambda_i(\lambda_i - 1)}{2(r-1)} + \frac{z^2}{r} + \left(\frac{1}{2} - \frac{s+1}{r}\right)z \geqslant \\ \geqslant \sum \frac{\lambda_i(\lambda_i - 1)}{2r} + \frac{z^2}{r} + \left(\frac{1}{2} - \frac{s+1}{r}\right)z = \\ = \sum \frac{\lambda_i(\lambda_i - 1)}{2r} + \frac{z(z-1)}{r} + \left(\frac{1}{2} - \frac{s}{r}\right)z = \kappa(L) \end{aligned}$$

as required.

Now assume that $z = 0$ and estimate $m_{\mathcal{K}_{r-1,r-s-1}}$. Arguing as in the proof of Proposition 5.1, one obtaines

$$\begin{aligned} m(L) \geqslant m_{\mathcal{K}_{r-1,r-s-1}}(L) = m_{I^{\pm}_{r-1}}(L) = m_{I_{r-1}}(L) = \sum_{i=1}^{l(\lambda)} \mu_{r-1,\lambda_i} = \\ = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - (r-1))}{2(r-1)} + \frac{R_{r-1}(\lambda)}{2(r-1)}. \end{aligned}$$

Thus the assertion in this case is implied by Lemma 4.8 applied for $k = r - 1$.

Finally, choose a subspace $L \in \mathcal{L}$ of type λ^-. It is easy to see that the values of $\operatorname{codim}(L)$, $\kappa(L)$ and $m(L)$ are the same as for a subspace of type $(\lambda, 0)$, which completes the proof. □

The second statement in type B is about ideals $I_r^{\pm}$ and J_s (see Section 2).

Theorem 5.13. *The function*

$$g = \frac{f}{\prod_{i<j}^{N}(x_i^2 - x_j^2)^{\frac{1}{r}} \prod_{i=1}^{N} x_i^{\frac{1}{2} - \frac{s}{r}}}$$

is locally L^2-integrable provided that one of the following sets of conditions holds:

(i) $f \in I^{\pm}_{r-1}$, $2 \leqslant r \leqslant N$, $r \in \mathbb{Z}$, $s \geqslant r - 1$, $s \in \mathbb{R}$;

(ii) $f \in J_s$, $0 \leqslant s \leqslant N-1$, $s \in \mathbb{Z}$, $r \geqslant N+1$, $r \in \mathbb{R}$;
(iii) $f \in \mathbb{C}[x]$, $s > N-1$, $s \in \mathbb{R}$, $r \geqslant N+1$, $r \in \mathbb{R}$;
(iv) $f \in \mathbb{C}[x]$, $r < 0, s < 0, r, s \in \mathbb{R}$.

Proof. By Corollary 3.12 it is sufficient to establish that

$$\kappa(L) < \frac{1}{2}\mathrm{codim}(L) + m(L) \tag{5.14}$$

where L is an arbitrary subspace from the intersection semi-lattice $\mathcal{L}(B_N)$, that is L has type (λ, z), and $m(L) = m_{I^{\pm}_{r-1}}(L)$. Recall that $\mathrm{codim}(L) = N - l(\lambda)$.

Let $2 \leqslant r \leqslant N$, $r \in \mathbb{Z}$ and let $s > r-1$. We know that the multiplicity $m(L)$ for $f \in I^{\pm}_{r-1}$ is given by

$$m(L) = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - (r-1))}{2(r-1)} + \frac{R_{r-1}(\lambda)}{2(r-1)} + \\ + \frac{z(z-r+1)}{r-1} + \frac{\rho_{r-1}(z)(r-1-\rho_{r-1}(z))}{r-1}. \tag{5.15}$$

For the multiplicity of the denominator we have

$$\kappa(L) = \sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{2r} + \frac{z(z-1)}{r} + \frac{r-2s}{2r}z.$$

Thus the inequality (5.14) follows by Lemma 4.9 applied for $k = r-1$ and by Remark 4.13, which completes the proof in case (i).

Recall that for the ideal J_s, $s \in \mathbb{Z}_+$, one has $m(L) = \max(0, \lfloor z - s \rfloor)$. Taking $r = N+1$ we obtain $\kappa(L) < \frac{1}{2}(N - l(\lambda)) + \max(0, \lfloor z - s \rfloor)$ by Lemma 4.9 applied for $k = N$. Therefore

$$\sum_{i=1}^{l(\lambda)} \frac{\lambda_i(\lambda_i - 1)}{2r} + \frac{z^2}{r} - \frac{(s+1)z}{r} < \frac{1}{2}\big(N - z - l(\lambda)\big) + \max(0, \lfloor z - s \rfloor). \tag{5.16}$$

Moreover, the inequality (5.16) is valid for $r > N+1$ since it is valid for $r = N+1$ and its right hand side is non-negative. Therefore the statement for case (ii) is implied by Corollary 3.12. Note that the same argument applies also for $s > N-1$, $s \in \mathbb{R}$, after replacing J_s by $\mathbb{C}[x]$. Indeed, in this situation one has $m(L) = \max(0, \lfloor z - s \rfloor) = 0$. This settles case (iii).

The last case when $r, s < 0$ is obvious. □

Let $\mathbb{S}_c$ be the minimal non-zero submodule of the polynomial representation of a rational Cherednik algebra. This submodule is unique since any submodule is an ideal in $\mathbb{C}[x]$ (see also [9, Section 4.6]). For generic c the submodule $\mathbb{S}_c$ coincides with $\mathbb{C}[x]$ however for special c it becomes non-trivial. As a corollary from the previous considerations and by [9, Proposition 4.12] we have the following result on unitarity of the minimal submodule $\mathbb{S}_c$.

Theorem 5.17. (1) *The minimal submodule $\mathbb{S}_c$ for the rational Cherednik algebra $H_c(D_N)$ is unitary if $c = 1/(2r)$ where $1 \leqslant r \leqslant N-1$, $r \in \mathbb{Z}$.*
(2) *The minimal submodule $\mathbb{S}_c$ for the algebra $H_c(B_N)$ is unitary if the parameter $c = (c_1, c_2) = (\frac{1}{r}, \frac{1}{2} - \frac{s}{r})$ satisfies the restrictions stated in Theorems 5.11, 5.13. In particular, for $c_1 = c_2$ the minimal submodule is unitary for $c_1 = 1/r$ where $2 \leqslant r \leqslant 2N$, $r \in 2\mathbb{Z}$, or $r > 2N$, $r \in \mathbb{R}$.*

Theorem 5.17 in the case of constant multiplicity c establishes unitarity of the simple module $\mathbb{S}_c$ where $1/c$ has to be a degree of the corresponding Coxeter group. The following Proposition shows that this restriction is not necessary for unitarity of the simple module.

Proposition 5.18. *Let $N \geqslant 3$. Then the minimal module $\mathbb{S}_c$ is a unitary representation of $H_c(B_N)$ (resp. $H_c(D_N)$)* for $c = (1/3, a)$ and $a \leqslant 0$ (resp. $c = 1/3$).

Proof. It is sufficient to establish that

$$\frac{f}{\prod_{i<j}^N (x_i^2 - x_j^2)^{\frac{1}{3}}}$$

is locally L^2-integrable for any $f \in I_2^{\pm}$. Using Corollary 3.13 and the previous calculation of multiplicities of $f \in I_k^{\pm}$ it is sufficient to establish that

$$\sum {\lambda_i}^2 - 4\sum \lambda_i + 3R_2(\lambda) + 6\rho_2(z)(2-\rho_2(z)) + 2z^2 - 8z + 6(N - l(\lambda)) > 0$$

where $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$ is a partition of $N - z$ such that $\lambda_1 \geqslant 2$ if $z = 0$. The last inequality follows using $R_2(\lambda) + N - z \geqslant 2l(\lambda)$, $\lambda_i \geqslant 1$, $\sum \lambda_i = N - z$. □

Remark 5.19. Note that analogous considerations for $c = 1/5$ show divergence of the integral expressing the Gaussian inner product on the representation $I_4^{\pm}$ for any $N \geqslant 5$ (cf. Remark 3.11).

Remark 5.20. It would be interesting to see if the above ideals $I_k^{\pm}$, J_k, $K_{r,s}$ can be used to determine the composition series of the polynomial representation for the algebras $H_c(D_N)$, $H_c(B_N)$ in some cases. For instance in the case of the group D_N and $c = 1/(2m+1)$ for positive integer $m \leqslant (N-1)/2$ there is a natural sequence of submodules

$$0 = I^0_{2m} \subset I^1_{2m} \subset \ldots \subset I^{[\frac{N}{2m+1}]}_{2m} \subset \mathbb{C}[x],$$

where $I^1_{2m} = I^{\pm}_{2m}$ and the support of the module $\mathbb{C}[x]/I^s_{2m}$ is stabilized by the parabolic subgroup A^s_{2m} (see [10]; I^s_{2m} vanishes on the D_N orbit of the vanishing set for the corresponding ideal I^s in the composition series for the polynomial representation for $H_c(A_{N-1})$ [9, Theorem 5.10]). We note that the support of $\mathbb{C}[x]/I^{[\frac{N}{2m+1}]}_{2m}$ then coincides with the support of the irreducible factor L_c as it is determined in [7, Theorem 3.1].

6. Some more unitarity results

In this section we present a few more results on the convergence of the integrals

$$\int_{\mathbb{R}^N} |f(x)|^2 e^{-\frac{1}{2}|x|^2} \prod_{\alpha\in\mathcal{R}_+} |(\alpha, x)|^{-2c} dx \tag{6.1}$$

where $\mathcal{R} \subset \mathbb{R}^N$ is an irreducible Coxeter root system with the Coxeter group W, and $c \in \mathbb{R}$. In the case of convergence on the minimal submodule $\mathbb{S}_c$ for the corresponding rational Cherednik algebra $H_c(W)$ this integral expresses the Gaussian inner product on $\mathbb{S}_c$, the module $\mathbb{S}_c$ is then unitary (see [9]). We will be assuming without loss of generality that the rank of $\mathcal{R}$ equals N.

Proposition 6.2. *The minimal $H_{1/2}(W)$-module $\mathbb{S}_{1/2}$ is unitary.*

Proof. Let I be the ideal of polynomials divisible by

$$\Delta_W(x) = \prod_{\alpha\in\mathcal{R}_+} (\alpha, x).$$

This ideal is an $H_{1/2}(W)$-module (see [10]), therefore $\mathbb{S}_{1/2} \subset I$. But the function

$$f^2(x)\Delta_W^{-1}(x), \quad f(x) \in I$$

is locally integrable as it is regular, hence the statement follows. □

Proposition 6.3. *Let $h = h_W$ be the Coxeter number of the group W. Then the Gaussian inner product (6.1) converges on $\mathbb{S}_{1/h}$.*

Proof. We need to establish that

$$\frac{1}{h} < \frac{\frac{1}{2}\mathrm{codim}(L) + \mathrm{mult}_L(f)}{\mathrm{mult}_L(\Delta_W)}$$

for any $f \in \mathbb{S}_{1/h}$ and for arbitrary element L from the lattice generated by the reflection hyperplanes.

We note that $\mathbb{S}_{1/h}$ is contained in the $H_{1/h}(W)$-invariant ideal consisting of polynomials vanishing at 0. Therefore when $L = \{0\}$ it is sufficient to establish that $h \cdot \mathrm{rk}(\mathcal{R})$ equals the number of roots which is a well known fact. When $L \neq \{0\}$ the inequality follows from the previous fact and the property that $h_{W_0} < h$ where h_{W_0} is the Coxeter number of any proper irreducible parabolic subgroup $W_0 \subset W$. □

Proposition 6.3 gives another derivation of the following known result.

Corollary 6.4. [9, Corollary 4.2] *The minimal submodule $\mathbb{S}_{1/h}$ is unitary.*

The proof of Proposition 6.3 also provides a proof for the following statement.

Corollary 6.5. *The Gaussian inner product (6.1) converges on $\mathbb{C}[x]$ when $c < 1/h$.*

Let W_0 be a proper irreducible parabolic subgroup in the irreducible Coxeter group W. Then for the corresponding Coxeter numbers one has $h_{W_0} < h_W$. The following lemma can be checked case by case.

Lemma 6.6. *Let d be the highest degree of a Coxeter group W of type E, F or H such that $d < h_W$. Then for any proper irreducible parabolic subgroup W_0 one has $h_{W_0} < d$.*

Proposition 6.7. *Let W be of type E, F or H. Let $c = 1/d$ where degree d is defined in Lemma 6.6. Then the Gaussian inner product (6.1) converges on the minimal submodule for $H_c(W)$ hence the module is unitary.*

Proof. We check integrability condition for $L = \{0\}$ first. We need to have

$$\frac{1}{d} < \frac{\frac{1}{2}N + \mathrm{mult}_0(f)}{\mathrm{mult}_0(\Delta_W)} \tag{6.8}$$

where $f \in \mathbb{S}_c$. Notice that $\mathrm{mult}_0(f) \geqslant 2$. Indeed, if the multiplicity is 0 then $\mathbb{S}_c$ has to coincide with $\mathbb{C}[x]$ which is not the case as $c = 1/d$ is a singular value for W. Now if the multiplicity is 1 then $\mathbb{S}_c$ contains homogeneous polynomials of degree 1 and hence the whole ideal of polynomials vanishing at 0. However this ideal is H_c-invariant only if $c = 1/h_W$ which is not the case. Then since $\mathrm{mult}_0(\Delta_W) = \frac{1}{2}h_W N$, the inequality (6.8) reduces to

$$d > \frac{Nh_W}{N+4} \tag{6.9}$$

which can be checked case by case.

Take now $L \neq \{0\}$ such that its stabilizer is a parabolic subgroup $W_0 = \prod_{i=1}^k W_i$ where parabolic subgroups W_i are irreducible and stabilize L_i so $L = \bigcap_{i=1}^k L_i$. We have $\operatorname{codim}(L) = \sum_{i=1}^k \operatorname{rk}(W_i)$ and

$$h_W \operatorname{rk}(W_i) > h_{W_i} \operatorname{rk}(W_i) = \operatorname{mult}_{L_i}(\Delta_{W_i}).$$

Therefore

$$\frac{\frac{1}{2}\operatorname{codim}(L) + \operatorname{mult}_L(f)}{\operatorname{mult}_L(\Delta_W)} \geqslant \frac{\operatorname{codim}(L)}{2\operatorname{mult}_L(\Delta_{W_0})} = \tag{6.10}$$

$$= \frac{\sum_{i=1}^k \operatorname{rk}(W_i)}{2\sum_{i=1}^k \operatorname{mult}_{L_i}(\Delta_{W_i})} \geqslant \min_{1\leqslant i\leqslant k} \frac{1}{h_{W_i}}.$$

Now the statement follows by Lemma 6.6. □

More explicitly Proposition 6.7 shows that the minimal modules for $H_{1/9}(E_6)$, $H_{1/14}(E_7)$, $H_{1/24}(E_8)$, $H_{1/8}(F_4)$, $H_{1/6}(H_3)$, $H_{1/20}(H_4)$ are unitary. A few more examples are provided by the following statement.

Proposition 6.11. *The Gaussian inner products converge on the minimal submodules for $H_{1/8}(E_6)$, $H_{1/12}(E_7)$, $H_{1/5}(H_3)$ hence the modules are unitary.*

Proof. The proof is parallel to the proof of Proposition 6.7. We consider the case of $H_{1/8}(E_6)$, other cases are similar. The value $d = 8$ satisfies (6.9) hence there is convergence at $L = \{0\}$. Let now L be such that $\dim(L) = 1$ and a generic point on L is stable under the subgroup $D_5 \subset E_6$. Since $1/8 = h_{D_5}$, the minimal module $\mathbb{S}_{1/8}$ is contained in the parabolic ideal consisting of polynomials vanishing on the E_6-orbit of L which is a module for $H_{1/8}(E_6)$ (see [10]). Therefore $\operatorname{mult}_L(f) > 0$ for $f \in \mathbb{S}_{1/8}$ and the inequality (6.10) is strict as required. For L with different stabilizers the convergence follows from (6.10) straightforwardly. □

The next statement shows that the convergence of the Gaussian inner product is preserved under the restriction functor Res_b defined in [1]. Let L_b be the minimal stratum containing a point $b \in \mathbb{R}^N$, and let n be its codimension. Let W_b be the parabolic subgroup of W which stabilizes L_b.

Proposition 6.12. *Assume the Gaussian inner product converges on the minimal $H_c(W)$-module $\mathbb{S}_c \subset \mathbb{C}[x_1, \dots, x_N]$. Then the Gaussian inner product converges on the $H_c(W_b)$-module* $\mathrm{Res}_b(\mathbb{S}_c)$.

Proof. Let M be the affine plane orthogonal to L_b such that $b \in M$. Let $L \subset M$ be the element of the intersection lattice of W_b acting in M. Note that $L = \tilde{L} \bigcap M$ where $\tilde{L}$ is an element of the intersection lattice for W such that $\tilde{L} \supset L_b$. Let

$$\delta(W, c) = \prod_{\alpha \in \mathcal{R}_+} (\alpha, x)^c, \qquad \delta(W_b, c) = \prod_{\substack{\alpha \in \mathcal{R}_+ \\ (\alpha, L_b) = 0}} (\alpha, x)^c$$

Due to convergence of the initial Gaussian product we have

$$\mathrm{mult}_{\tilde{L}}\big(\delta(W, c)\big) < \frac{1}{2}\mathrm{codim}(\tilde{L}) + m_{\mathbb{S}_c}(\tilde{L}), \tag{6.13}$$

where $m_{\mathbb{S}_c}(\tilde{L})$, as usual, denotes the minimal multiplicity of the elements of $\mathbb{S}_c$ on $\tilde{L}$.

The module $\mathrm{Res}_b(\mathbb{S}_c)$ is obtained by completion $\widehat{(\mathbb{S}_c)}_b$ at b with subsequent extraction of the polynomial part such that the polynomials are constant in the direction of the stratum L_b. Under this process we have

$$m_{\mathbb{S}_c}(\tilde{L}) \leqslant m_{\mathrm{Res}_b \mathbb{S}_c}(\tilde{L}) = m_{\mathrm{Res}_b \mathbb{S}_c}(L).$$

Since $\mathrm{codim}(L) = \mathrm{codim}(\tilde{L})$ and $\mathrm{mult}_{\tilde{L}}\big(\delta(W, c)\big) = \mathrm{mult}_L\big(\delta(W_b, c)\big)$, the inequality (6.13) implies

$$\mathrm{mult}_L\big(\delta(W_b, c)\big) < \frac{1}{2}\mathrm{codim}(L) + m_{\mathrm{Res}_b \mathbb{S}_c}(L),$$

and the statement follows by Corollary 3.12. □

Note that the module $\mathrm{Res}_b(\mathbb{S}_c)$ is non-trivial for any $b \in \mathbb{R}^N$ so the convergence of the Gaussian inner product on the minimal submodule for $H_c(W_b)$ also follows. Note also that the proof of Proposition 6.12 works also in the case of non-constant W-invariant c.

7. A few negative results

In this section[2] we explain that the minimal submodule $\mathbb{S}_c$ is not unitary in the case of the groups D_N, B_N, $c = 1/N$, N is odd, and present a few more examples when the integral (6.1) diverges on the minimal submodule (cf. Remark 5.19).

We are indebted to S. Griffeth for explanations leading to the following result.

[2]This section is largely based on the comments which P. Etingof and S. Griffeth kindly provided to us on the preliminary version of the paper.

Proposition 7.1. [16] *The minimal submodule $\mathbb{S}_c$ for $H_c(D_N)$ is not unitary when $N \geqslant 5$ is odd and $c = 1/N$.*

We start with the following statement.

Lemma 7.2. *Let $c = 1/N$ where N is odd. Then $\mathbb{C}[x]/\mathbb{S}_c$ is a non-trivial irreducible $H_c(D_N)$-module. Also $\mathbb{S}_c \cong L_\tau$ where L_τ is irreducible $H_c(D_N)$-module corresponding to D_N-module τ given by the reflection representation of $\mathrm{S}_N \subset D_N$. More specifically, the lowest homogeneous component of $\mathbb{S}_c$ is generated by the polynomials $x_1^2 - x_i^2$ for $2 \leqslant i \leqslant N$.*

Proof. Consider the polynomial representation $\mathbb{C}[x]$ for the rational Cherednik algebra $H_c(B_N)$ where N is odd, $c(e_i) = 0$ and $c(e_i \pm e_j) = 1/N$. It follows from [11, Theorem 7.5] that this representation has unique non-trivial submodule. On the other hand we know that $I_{N-1}^{\pm}$ is a submodule in $\mathbb{C}[x]$. Therefore the only submodule for $H_c(B_N)$ is the minimal submodule $\mathbb{S}_c^{B_N} = I_{N-1}^{\pm}$. By Proposition 2.4 the elements in its lowest homogeneous component are linearly generated by $x_1^2 - x_i^2$ where $1 \leqslant i \leqslant N$.

Consider now the polynomial representation for the algebra $H_c(D_N)$, let M be a non-trivial submodule. It is clear that the minimal degree of the homogeneous elements in M is 2. Indeed the degree cannot be 1 as otherwise $M = J_{N-1}$ which is not possible for $c = 1/N$. Also the degree cannot be bigger than 2 as in this case there are singular polynomials in this degree for $H_c(B_N)$-module $\mathbb{S}_c^{B_N}$, so that it is not a simple module which is a contradiction.

Since the span $\langle x_1^2 - x_i^2 \rangle$, $1 \leqslant i \leqslant N$, is irreducible D_N-module it follows that the lowest homogeneous component of M coincides with the lowest homogeneous component of $I_{N-1}^{\pm}$, therefore $M = I_{N-1}^{\pm}$. □

Now we prove Proposition 7.1.

Proof. Let $f = x_2^2 - x_3^2 \in \mathbb{S}_c$. It is easy to check straightforwardly that

$$\nabla_{e_1}(x_1 f) = \lambda f,$$

where $\lambda = \frac{4}{N} - 1$. Let $(\cdot, \cdot)_\tau$ be the contravariant form on L_τ. We have

$$(x_1 f, x_1 f)_\tau = \lambda (f, f)_\tau.$$

Since $\lambda < 0$ when $N \geqslant 5$, the module $\mathbb{S}_c \cong L_\tau$ is not unitary. □

Proposition 7.1 shows that in general the minimal $H_c(W)$-module $\mathbb{S}_c$ is not unitary when $c = 1/d_i$, with d_i a degree of the Coxeter group W, thus providing negative answer to the Cherednik's question [2, 9]. However the exceptions are rare namely the only exception for the classical root systems and constant parameter c is given by $W = D_N$ with odd N, $c = 1/N$.

As we saw in Propositions 5.18, 6.11 there are also examples when the Gaussian inner product converges on the minimal $H_c(W)$-module $\mathbb{S}_c$ hence the module is unitary however $c \neq 1/d_i$ for any degree d_i. The examples found above are $H_{1/3}(D_N)$ with $N \geqslant 4$ and $H_{1/5}(H_3)$. It would be interesting to investigate when exactly $\mathbb{S}_c$ is unitary.

Below we give a few more examples when Gaussian product diverges on $\mathbb{S}_c$. First we present some analysis in type B which is similar to the Proposition 7.1 above on type D.

Proposition 7.3. *Consider the rational Cherednik algebra $H_c(B_N)$, $N \geqslant 3$ with the parameters $c(e_i \pm e_j) = 1/N$, $c(e_i) = a$ such that $2a + 2j/N$ is not a positive odd number for any $0 \leqslant j \leqslant N-1$. Suppose also that $N(2a+1) > 4$. Then the minimal submodule $\mathbb{S}_c$ is not unitary.*

Proof. The $H_c(B_N)$ module $\mathbb{C}[x]$ has unique non-trivial submodule if the first stated restriction for a holds [11, Theorem 7.5]. Hence we have $\mathbb{S}_c = L_\tau = I^{\pm}_{N-1}$. The direct norm calculation similar to the proof of Proposition 7.1 gives

$$(x_1 f, x_1 f)_\tau = \lambda (f, f)_\tau,$$

where $f = x_2^2 - x_3^2$ and $\lambda = (4-N)/N - 2a$. Under the second stated restriction for a we have $\lambda < 0$ hence the module is not unitary. $\square$

As a corollary we have the following statement on non-unitarity in the case of equal parameters.

Corollary 7.4. *Let $N \geqslant 3$ be odd, let $c = 1/N$. Then the minimal submodule $\mathbb{S}_c$ for $H_c(B_N)$ is not unitary.*

Using Proposition 6.12 we get further corollary on divergence of the integral for the Gaussian product for the equal parameter cases.

Corollary 7.5. *The integral (6.1) is not convergent on the minimal submodules for $H_c(B_N)$ when $c = 1/k$ with $3 \leqslant k \leqslant N$, k is odd, and for $H_{1/3}(F_4)$.*

The following statement was explained to us by P. Etingof.

Proposition 7.6. [17] *The integral (6.1) is not convergent on the minimal submodule $\mathbb{S}_c$ for $H_c(E_7)$ when $c = 1/10$.*

Proof. Let $b \in \mathbb{R}^7$ be a point such that its stabilizer is isomorphic to the subgroup $E_6 \subset E_7$. Note first that there are elements $p \in \mathbb{S}_c$ such that $p(b) \neq 0$. Indeed, otherwise $b \in \mathrm{supp}(\mathbb{C}[x]/\mathbb{S}_c)$ and $\mathrm{Res}_b(\mathbb{C}[x]/\mathbb{S}_c)$ is a non-trivial factor of the polynomial representation for $H_c(E_6)$. But this is not possible since $c = 1/10$ is not a singular value for $H_c(E_6)$.

Let L be the one-dimensional linear space containing b, so that one has $\operatorname{codim}(L) = 6$. By above there are elements $p \in \mathbb{S}_c$ such that $p(b) \neq 0$ and thus $\operatorname{mult}_L(p) = 0$. For the convergence of the Gaussian product on p we need to have

$$\frac{1}{10} < \frac{3}{K(L)}, \tag{7.7}$$

where $K(L)$ is then equal to the number of positive roots in E_6, so $K(L) = 36$. Thus (7.7) fails. □

Proposition 7.8. *The integral (6.1) is not convergent on the minimal submodules for $H_{1/9}(E_7)$, $H_{1/9}(E_8)$, $H_{1/7}(E_7)$, $H_{1/15}(E_8)$.*

The proof is parallel to the proof of Proposition 7.6 where one takes L of codimension 6 stabilized by the parabolic subgroup $D_6 \subset E_7 \subset E_8$ for the first two cases. One can take L stabilized by the parabolic $D_5 \subset E_7$ for the case of $H_{1/7}(E_7)$, and one can take L stabilized by the parabolic $E_7 \subset E_8$ in the last case.

The following statement follows from the Proposition 6.12 and from Propositions 7.1, 7.6.

Proposition 7.9. *The integral (6.1) is not convergent on the minimal submodules for $H_{1/m}(D_N)$ where $5 \leqslant m \leqslant N$, m is odd, for $H_{1/10}(E_8)$, $H_{1/7}(E_8)$, $H_{1/5}(E_8)$, $H_{1/5}(E_7)$, $H_{1/5}(E_6)$.*

Funding. M.F. was partially supported by the Engineering and Physical Sciences Research Council (grant number EP/F032889/1). M.F. also acknowledges support of the British Council (PMI2 Research Cooperation award). C.S. was partially supported by the Russian Foundation for Basic Research (grant numbers 08-01-00395-a, 11-01-00185-a, 11-01-00336-a), grant N.Sh.-4713.2010.1 and by AG Laboratory GU-HSE, RF government grant 11 11.G34.31.0023.

Acknowledgements. We are very grateful to P. Etingof for attracting our attention to the problem and for helpful discussions at different stages of the work. We are also very grateful to S. Griffeth for stimulating discussions and explanations, and to I. Cheltsov, J. Kollár and Yu. Prokhorov for useful advice.

References

[1] R. Bezrukavnikov, P. Etingof, *Parabolic induction and restriction functors for rational Cherednik algebras*, Selecta Math., New Series, 14, No. 3–4, 397–425 (2009).

[2] I. Cherednik, Towards harmonic analysis on DAHA, http://www-math.mit.edu/∼etingof/hadaha.pdf

[3] T. Chmutova, P. Etingof, *On some representations of the rational Cherednik algeba*, Represent. Theory, 7 (2003), pp. 641–650.

[4] C. F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. 311, No. 1 (1989), pp. 167–183.

[5] C. F. Dunkl, M. F. E. De Jeu, E. M. Opdam, *Singular polynomials for finite reflection groups*, Transactions of the AMS, V.346, No. 1 (1994), pp. 237–256.

[6] C. F. Dunkl, E. M. Opdam, *Dunkl operators for complex reflection groups*, Proc. London Math. Soc. (3) 86 (2003), no. 1, 70–108.

[7] P. Etingof, *Supports of irreducible spherical representations of rational Cherednik algebras of finite Coxeter groups*, arXiv:0911.3208

[8] P. Etingof, V. Ginzburg, *Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism*, Invent. Math., V. 147, No. 2 (2002), pp.243–348.

[9] P. Etingof, E. Stoica, S. Griffeth, *Unitary representations of rational Cherednik algebras*, Represent. Theory 13 (2009), pp. 349–370.

[10] M. Feigin, *Generalized Calogero–Moser systems from rational Cherednik algebras*, arXiv:0809.3487

[11] S. Griffeth, *Orthogonal functions generalizing Jack polynomials*, Trans. Amer. Math. Soc. 362 (2010), no. 11, pp. 6131 - 6157.

[12] J. E. Humphreys, *Reflection groups and Coxeter groups*, CUP, 1990.

[13] J. Kollár, *Singularities of pairs*, Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995. Providence, RI: American Mathematical Society. Proc. Symp. Pure Math. 62(pt.1), 221–287 (1997).

[14] J. Kollár, *Which powers of holomorphic functions are integrable?*, arXiv:0805.0756

[15] M. Saito, *On real log canonical thresholds*, arXiv:0707.2308

[16] S. Griffeth, *Private communication*, 2010

[17] P. Etingof, *Private communication*, 2010

School of Mathematics and Statistics, University of Glasgow, 15 University Gardens, Glasgow G12 8QW, UK

E-mail address: misha.feigin@glasgow.ac.uk

Steklov Mathematical Institute, Gubkina str., 8, Moscow, 119991, Russia

E-mail address: shramov@mccme.ru