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Generalized Calogero-Moser systems from rational Cherednik algebras

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Abstract

We consider ideals of polynomials vanishing on the W -orbits of the intersections of mirrors of a finite reflection group W . We determine all such ideals which are invariant under the action of the corresponding rational Cherednik algebra hence form submodules in the polynomial module. We show that a quantum integrable system can be defined for every such ideal for a real reflection group W . This leads to known and new integrable systems of Calogero-Moser type which we explicitly specify. In the case of classical Coxeter groups we also obtain generalized Calogero-Moser systems with added quadratic potential.

Keywords: quantum integrable systems, rational Cherednik algebra, polynomial representation

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1 Introduction

The usual Calogero-Moser (CM) system describes a pairwise interaction of n particles on the line with the inverse square potential. This system appeared and was studied in [1], [2], [3]. Olshanetsky and Perelomov introduced CM systems related to the root systems of finite Coxeter groups [4]. An elegant and uniform proof of integrability was proposed by Heckman [5] who used Dunkl operators [6]. After that quantum CM systems related to the root systems became ultimately related to the rational Cherednik algebras $H_{1,c}$ [7]. More exactly, CM operators and their quantum integrals can be thought

of as elements of the spherical subalgebra of the rational Cherednik algebra. We refer to the book [8] for the exposition of this and other developments.

Generalized CM systems related to non-symmetric arrangements of hyperplanes appeared in the work of Chalykh, Veselov and the author [9]. These systems were studied in [10], [11] where integrability was established with the help of Baker-Akhiezer functions. Two families of operators were found corresponding to the deformations of the root systems $\mathcal{A}_n(m)$, $\mathcal{C}_n(m, l)$. The $\mathcal{A}_n(m)$ system describes Calogero-Moser type pairwise interaction where one particle has different mass from the mass of all other particles.

Sergeev considered the generalization of $\mathcal{A}_n(m)$ CM system with arbitrary number of particles of each of the two types [12]. He obtained the corresponding Calogero-Moser-Sutherland (CMS) operator at $m = -1/2$ as the radial part of the Laplace-Beltrami operator on a symmetric superspace [12, 13]. Sergeev and Veselov introduced further generalization of the families of CMS operators by defining special deformations of the root systems of contragredient superalgebras Lie [14]. Then integrability for the classical series was established in [14] by rather involved computations.

In the works [15, 16] Sergeev and Veselov gave another proof of integrability of CMS systems related to deformations of classical root systems. They showed that the generalized CMS operators are restrictions to generalized discriminants of the non-deformed CMS operators of A or BC type acting in the space of symmetric functions in infinite number of variables. Special eigenfunctions of these systems were studied by Hallnäs and Langmann in [17, 18].

In the present paper we approach integrability of generalized CM systems making use of Dunkl operators and special representations of rational Cherednik algebras. We show that for special values of parameters the Dunkl operators can be restricted to certain parabolic strata which are the Coxeter orbits of intersections of the mirrors. Equivalently, the ideals of polynomials vanishing on these strata are submodules in the polynomial representation of the corresponding rational Cherednik algebra. Then sum of squares of base Dunkl operators takes the form of a generalized CM operator when acting on the invariants. In this way we recover CM systems from [9]-[16] at the special values of parameters and also obtain new integrable generalized CM systems. It is simpler to work in the non-symmetric settings and we give complete description of the CM systems which can be obtained in this way. The corresponding parabolic strata are easy to describe. Namely, in the case of constant multiplicity the Coxeter subgraph for the parabolic sub-

group defining the stratum should have the same Coxeter number for all its connected components.

We note that the interest in the submodules of the polynomial representation of the rational Cherednik algebras goes back to the pioneering work by Dunkl, de Jeu and Opdam [19] where the set of singular parameters when the representation is reducible was completely determined. The singular set for the trigonometric degeneration of the Cherednik algebra was found by Etingof in [20], the non-degenerate case was studied by Cherednik in [21]. The actual submodules for the non-degenerate Cherednik algebras were under investigation, in particular, by Kasatani for A and C cases in [22], [23], and by Cherednik in [21]. The representations of the Coxeter group on the singular vectors were determined by Dunkl in the rational A case [24].

The paper is organized as follows. In Section 2 we determine parabolic strata of the finite real reflection groups such that the ideals of polynomials vanishing on these strata are submodules for the polynomial representation of the corresponding rational Cherednik algebra. These strata are rational Coxeter versions of Kasatani's nonsymmetric vanishing conditions ([22], see also [25, 26]). It follows from [27], [28] that it is necessary to impose vanishing conditions on the parabolic strata in order to get non-trivial invariant ideals.

In Section 3 we show that parabolic strata defining invariant ideals also determine quantum integrable systems of CM type corresponding to the sets of vectors which are obtained by projections of the original Coxeter root system. In Section 4 we explicitly specify these invariant parabolic strata and the corresponding CM systems in the case of classical Coxeter groups. The corresponding generalized CM systems are known to be integrable by [11], [14]. In Section 5 we show that systems found in Section 4 remain integrable if a quadratic term is added to the Hamiltonian. For that we review the proof of integrability of the CM systems for classical series in the external quadratic field through the Dunkl operators [29]. Then our restriction procedure can be applied in this case as well. In Section 6 we explicitly determine generalized CM systems corresponding to the invariant parabolic ideals for the exceptional Coxeter groups. In Section 7 we determine invariant parabolic ideals for complex reflection groups.

2 Invariant parabolic ideals

Let W be a finite real reflection group acting by orthogonal transformations in its complexified reflection representation $V = \mathbb{C}^N$. Let \mathcal{R} be the corresponding Coxeter root system, and let Γ be the corresponding Coxeter graph (see, e.g., [30]). We assume that a positive subsystem $\mathcal{R}_+ \subset \mathcal{R}$ is chosen so the vertices of the graph Γ can be identified with the simple roots. Similarly, for a subgraph $\Gamma_0 \subset \Gamma$ we will denote by Γ_0^v the set of roots corresponding to the vertices of Γ_0 .

Let $c(\alpha) = c_\alpha$ be a W -invariant function on the set of roots \mathcal{R} . The *rational Cherednik algebra* $H_c = H_c^{\mathcal{R}}$ is associated with the root system \mathcal{R} and multiplicity c (see [7]; in this paper we assume that parameter $t = 1$). Also in this paper we will need only the faithful representation of H_c in the space of polynomials $\mathbb{C}[x] = \mathbb{C}[x_1, \dots, x_N]$. Any element of H_c acts on $p \in \mathbb{C}[x]$ as a linear combination of the compositions $r(\nabla)wq(x)p$ where $q(x) \in \mathbb{C}[x]$, $w \in W$, and $r(\nabla) = r(\nabla_1, \dots, \nabla_N)$ is a polynomial in Dunkl operators corresponding to the bases directions e_1, \dots, e_N . For any direction $\xi \in \mathbb{C}^N$ the Dunkl operator ∇_ξ is defined as

$$\nabla_\xi = \partial_\xi - \sum_{\alpha \in \mathcal{R}_+} \frac{c_\alpha(\alpha, \xi)}{(\alpha, x)} (1 - s_\alpha), \quad (1)$$

where (\cdot, \cdot) is the standard scalar product in V , and s_α is the orthogonal reflection with respect to the hyperplane $(\alpha, x) = 0$. Note that the Dunkl operators satisfy commutativity $[\nabla_\xi, \nabla_\eta] = 0$ ([6]) and it is clear that $\nabla_\xi \mathbb{C}[x] \subset \mathbb{C}[x]$.

Let Γ_0 be a subgraph of the Coxeter graph Γ obtained by specifying some vertices of Γ and preserving all the edges between these vertices. The subgraph Γ_0 defines the plane π obtained as the intersection of the corresponding mirrors

$$\pi = \{x \in V \mid (\beta, x) = 0 \quad \forall \beta \in \Gamma_0^v\}. \quad (2)$$

The associated *parabolic stratum* is defined as

$$D_{\Gamma_0} = \bigcup_{w \in W} w(\pi).$$

We define the corresponding *parabolic ideal* I_{Γ_0} as a set of polynomials vanishing on the stratum, that is $I_{\Gamma_0} = \{p \in \mathbb{C}[x] \mid p|_{D_{\Gamma_0}} = 0\}$. It is obvious that I_{Γ_0} is an ideal in $\mathbb{C}[x]$ and that it is W -invariant. We are going to determine

the parabolic strata D_{Γ_0} which define ideals I_{Γ_0} invariant under the whole rational Cherednik algebra H_c .

Theorem 1 *Let $\Gamma_0 = \coprod_{i=1}^k \Gamma_i$ be the decomposition of the subgraph $\Gamma_0 \subset \Gamma$ into the connected components. Then the parabolic ideal I_{Γ_0} is invariant under the algebra H_c if and only if for any $i = 1, \dots, k$ we have*

$$\sum_{\alpha \in V_i \cap \mathcal{R}} \frac{c_\alpha(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = (u, v) \quad (3)$$

for any $u, v \in V_i$, where V_i is a linear space spanned by the roots Γ_i^v .

Proof. Denote by $V_0 = \bigoplus_{i=1}^k V_i$. Let $f \in I_{\Gamma_0}$. We are going to analyze the submodule condition $\nabla_\xi f|_{D_{\Gamma_0}} = 0$ where ∇_ξ is the Dunkl operator (1). At first we consider the condition $\nabla_\xi f|_\pi = 0$. For that we recall that $f|_\pi = 0$ and we can represent polynomial f in the form

$$f = \sum_{\beta \in \Gamma_0^v} f_\beta(x)(\beta, x),$$

where f_β are some polynomials. Since $s_\alpha f|_\pi = f|_{s_\alpha \pi} = 0$ we note that $\frac{1-s_\alpha}{(\alpha, x)} f|_\pi = 0$ if $(\alpha, x)|_\pi \neq 0$. Therefore we rearrange

$$\nabla_\xi f|_\pi = \sum_{\beta \in \Gamma_0^v} \left((\beta, \xi) f_\beta|_\pi - \sum_{\alpha \in \mathcal{R}_+ \cap V_0} \frac{2c_\alpha(\alpha, \xi)(\alpha, \beta)}{(\alpha, \alpha)} f_\beta|_\pi \right).$$

By collecting the coefficients at f_β it follows that

$$(\beta, \xi) - \sum_{\alpha \in \mathcal{R}_+ \cap V_0} \frac{2c_\alpha(\alpha, \xi)(\alpha, \beta)}{(\alpha, \alpha)} = 0,$$

which is equivalent to the property (3) as V_i are pairwise orthogonal for $i > 0$.

Conditions $\nabla_\xi f|_{w(\pi)} = 0$ for non-trivial $w \in W$ are obtained from conditions (3) by the W -action, they are equivalent to the properties (3) hence the theorem is proven.

Theorem 2 *Assume that the multiplicity function c is constant on the roots Γ_0^v of the subgraph $\Gamma_0 \subset \Gamma$. Then the ideal I_{Γ_0} is H_c -invariant if and only if all the connected components of Γ_0 have same Coxeter number $h = 1/c$.*

This theorem is a direct corollary of Theorem 1 and of the following lemma.

Lemma 1 [31, Chapter 5, §6.2, Theorem 1, Corollary]

For any irreducible Coxeter root system \mathcal{R} in a Euclidean space V , for any $u, v \in V$

$$\sum_{\alpha \in \mathcal{R}} \frac{(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = h(u, v),$$

where h is the Coxeter number of \mathcal{R} .

Submodules appearing in Theorem 2 correspond to the values $c = 1/m$ where m is the Coxeter number of a parabolic subgroup of W . It follows from the description of the singular multiplicities that is the multiplicities when the polynomial representation is reducible [19] that the multiplicity $c = k/m$ is then singular too if $k \in \mathbb{N}$ is coprime to m (see also [32]). Note however that not all the singular multiplicities have the latter form in general. For example, $c = 1/9$ is singular for \mathcal{E}_6 but any parabolic subgroup of \mathcal{E}_6 has the Coxeter number at most 8. More generally the singular values $c = 1/d$ where $d \in \mathbb{N}$ is not the Coxeter number of any parabolic subgroup of W correspond to the cuspidal numbers d of W (see [28]). In this case any quotient of the polynomial representation of H_c over its non-trivial submodule is finite dimensional [28].

For the case of different multiplicities we define the generalized Coxeter number $h^c = h_{\mathcal{R}}^c$ for the irreducible Coxeter root system \mathcal{R} as the coefficient of proportionality between the following two W -invariant inner products

$$\sum_{\alpha \in \mathcal{R}} \frac{c_{\alpha}(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = h^c(u, v).$$

In the case $c = 1$ we have $h^1 = h$ is the usual Coxeter number. Then Theorem 1 has the following reformulation.

Theorem 3 Let $\Gamma_0 = \coprod_{i=1}^k \Gamma_i$ be the decomposition of the subgraph $\Gamma_0 \subset \Gamma$ into the connected components. Then the parabolic ideal I_{Γ_0} is invariant under the algebra H_c if and only if the generalized Coxeter numbers h_i^c for the Coxeter root systems determined by subgraphs Γ_i satisfy $h_i^c = 1$ for all $i = 1, \dots, k$.

Remark 1 *It would be interesting to see if Theorem 3 can be established using induction and restriction functors from [28]. Let \mathcal{R}_0 be the root system with the Coxeter graph Γ_0 . At the values of c under consideration the corresponding rational Cherednik algebra $H_c^{\mathcal{R}_0}$ has trivial one-dimensional module L . The induced module $\text{Ind}(L)$ for $H_c^{\mathcal{R}}$ is not generally contained in the polynomial module $\mathbb{C}[x]$. However we note that the modules $\text{Ind}(L)$ and $\mathbb{C}[x]/I_{\Gamma_0}$ have same support D_{Γ_0} .*

Our considerations allow to determine all radical ideals I which are submodules for the algebra H_c in the polynomial representation. Indeed, it is shown in [28], [27] that any radical ideal I must consist of polynomials vanishing on the union of some parabolic strata D_{Γ_s} , $s = 1, \dots, L$:

$$I = \{p \in \mathbb{C}[x] \mid p|_{\bigcup_{s=1}^L D_{\Gamma_s}} = 0\}. \quad (4)$$

We assume that all the strata included in the union of parabolic strata $\bigcup D_{\Gamma_s}$ are essential in the sense that

$$\bigcup_{s \neq j} D_{\Gamma_s} \subsetneq \bigcup_{s=1}^L D_{\Gamma_s}$$

for any $1 \leq j \leq L$.

Let $\Gamma_s = \Gamma_s^1 \sqcup \dots \sqcup \Gamma_s^{k_s}$ be decomposition of the corresponding Coxeter graphs into the connected components.

Theorem 4 *The radical ideal (4) is H_c -invariant if and only if $h^c(\Gamma_s^i) = 1 \forall s = 1, \dots, L; i = 1, \dots, k_s$.*

The proof is similar to the proof of Theorem 1. In order to derive the conditions $h^c(\Gamma_s^i) = 1$ we take a polynomial $p \in I$ in the form $p = \prod_{\pi} f_{\pi}$ where π runs over planes forming our union of strata: $\bigcup_{s=1}^L D_{\Gamma_s} = \bigcup \pi$ and f_{π} is a generic polynomial vanishing on π . When π is given as

$$\pi = \{x \mid (\beta, x) = 0, \forall \beta \in \Gamma_s^j\}$$

for $1 \leq s \leq L, 1 \leq j \leq k_s$ we have $f_{\pi} = \sum_{\beta \in \Gamma_s^j} (x, \beta) f_{\beta}$ where f_{β} are generic polynomials, in particular, $f_{\beta}|_{\pi}$ are linearly independent. We can also assume that $f_{\tilde{\pi}}|_{\pi} \neq 0$ if $\tilde{\pi}$ is different from π . Then like in the proof of Theorem 1 the calculation of $\nabla_{\xi} f|_{\pi}$ leads to the property $h^c(\Gamma_s^i) = 1$ and hence to the Theorem.

The submodules in the polynomial representation appearing in Theorems 1, 2 correspond to the radical ideals and to the particular singular values only. The description of all singular values of parameters [19] gives, for instance, that all the values $c = (2m - 1)/2$ are singular when $m \in \mathbb{Z}_+$. The next proposition describes submodules corresponding to these singular values. When $m = 1$ the proposition is a particular case of Theorem 2 when the subgraph Γ_0 is a Coxeter graph A_1 , that is consists of one vertex, and the proof is different.

Proposition 1 *Let \mathcal{R} be a Coxeter root system, let c be invariant multiplicity function on \mathcal{R} . Let S_1 be an orbit of the corresponding Coxeter group acting on \mathcal{R} with the multiplicity $c_1 = c(S_1)$. Let I be the ideal of polynomials having zero of order $2m - 1$ on the hyperplanes $(\alpha, x) = 0$ for any $\alpha \in S_1$. Then I is H_c -invariant if and only if $c_1 = (2m - 1)/2$.*

Proof. We denote by D_{S_1} the parabolic stratum $\cup_{\alpha \in S_1} \{x : (\alpha, x) = 0\}$. An arbitrary polynomial $p(x)$ vanishing on D_{S_1} with order $2m - 1$ has the form

$$p(x) = \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x)$$

for some polynomial f . Let $S_2 = \mathcal{R} \setminus S_1$ be (possibly empty) set of roots not contained in S_1 .

Using invariance of $\prod_{\alpha \in S_1} (\alpha, x)^{2m-1}$ with respect to reflections s_β for $\beta \in S_2$ and its anti-invariance with respect to s_β for $\beta \in S_1$ we rearrange $\nabla_\xi p(x)$ as

$$\begin{aligned} \sum_{\beta \in S_1} \left(\frac{(2m-1)(\beta, \xi)}{(\beta, x)} \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x) - 2c_1 \frac{(\beta, \xi)}{(\beta, x)} \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} f(x) \right) \\ - \sum_{\beta \in S_2} c_\beta(\beta, \xi) \prod_{\alpha \in S_1} (\alpha, x)^{2m-1} \frac{f(x) - s_\beta f(x)}{(\beta, x)} \quad (5) \end{aligned}$$

modulo elements of the ideal I . The last sum in (5) belongs to the ideal I . The first sum in (5) belongs to the ideal if and only if $c_1 = (2m - 1)/2$. Proposition is proven.

3 Restricted Calogero-Moser systems

In this section we explain how H_c -invariant parabolic ideals lead to the quantum integrable systems of Calogero-Moser type. We say that a differential operator L acting in N -dimensional space is *quantum integrable* if there exist N pairwise commuting differential operators $L_1 = L, \dots, L_N$ so that L_i are algebraically independent.

Consider the parabolic ideal I consisting of polynomials vanishing on the parabolic stratum D which is the W -orbit of the subspace π . Assume that ideal I is a submodule for the rational Cherednik algebra H_c . Let $p = \bar{p}|_D$ be a restriction to the stratum of a polynomial \bar{p} defined in the whole space V . It follows from the invariance of the ideal I that for any Dunkl operator ∇_ξ the result of restriction $\nabla_\xi \bar{p}|_D$ does not depend on the extension \bar{p} but depends on p only. Therefore the restricted Dunkl operators $\nabla_\xi|_D$ are correctly defined.

Moreover, the analysis of the property of the ideal I to be invariant in the proof of Theorem 1 implies that the restricted operators $\nabla_\xi|_D$ are defined correctly in the locally analytic settings. Namely, let point $x_0 \in \pi$ be generic, let $U \ni x_0$ be its small neighborhood, $U \subset \pi$. Consider its W -orbit $U^W = \cup_{w \in W} w(U)$. Let f be a union of analytic germs defined in the neighborhoods U^W , let \bar{f} be analytic extension of these germs to $\tilde{U}^W = \cup_{w \in W} w(\tilde{U})$ where $\tilde{U} \supset U$ is a small neighborhood of x_0 in the space V . Then the result of the restriction $\nabla_\xi \bar{f}|_D$ does not depend on the locally analytic extension \bar{f} but depends on f only.

Consider now the space \mathcal{L} of W -invariant union of germs f defined in U^W . So f is determined by its values $f_\pi = f|_U$ in the neighborhood $U \subset \pi$. The invariant combinations of Dunkl operators $\sigma(\nabla) = \sigma(\nabla_1, \dots, \nabla_N)$, where $\sigma(x_1, \dots, x_N) \in \mathbb{C}[x_1, \dots, x_N]^W$, act in the space \mathcal{L} . We denote by $\sigma(\nabla)^{Res_\pi}$ the operator which maps f_π to the result of the restriction $\sigma(\nabla) \bar{f}|_U$ on to the neighborhood $U \subset \pi$ of the application of the operator $\sigma(\nabla)$ to any W -invariant extension \bar{f} of f from U^W to \tilde{U}^W (c.f. [5]).

Theorem 5 *Assume that a stratum D defines H_c -invariant parabolic ideal. Then the operator $\sum_{i=1}^N \nabla_i^2$ restricted to the W -invariant functions on D has the generalized Calogero-Moser form*

$$\left(\sum_{i=1}^N \nabla_i^2 \right)^{Res_\pi} = \Delta_y - \sum_{\substack{\alpha \in \mathcal{R}_+ \\ \hat{\alpha} \neq 0}} \frac{2c_\alpha}{(\hat{\alpha}, y)} \partial_{\hat{\alpha}}, \quad (6)$$

where $y = (y_1, \dots, y_n)$ are orthonormal coordinates on the plane π , $\Delta_y = \frac{\partial^2}{\partial y_1^2} + \dots + \frac{\partial^2}{\partial y_n^2}$, vector $\hat{\alpha}$ is orthogonal projection of vector α onto π .

For any polynomials $\sigma, \tau \in \mathbb{C}[x]^W$ the restrictions $\sigma(\nabla)^{Res\pi}$, $\tau(\nabla)^{Res\pi}$ are commuting differential operators in the space π , in particular, operator (6) is quantum integrable.

Proof. The operator

$$H = \sum_{i=1}^N \nabla_i^2$$

can be expanded as

$$H = \Delta - \sum_{\alpha \in \mathcal{R}_+} \frac{2c_\alpha}{(\alpha, x)} \partial_\alpha + \sum_{\alpha \in \mathcal{R}_+} \frac{c_\alpha(\alpha, \alpha)(1 - s_\alpha)}{(\alpha, x)^2}. \quad (7)$$

Consider f which is a W -invariant analytic function defined in the neighborhoods $w(U) \subset w(\pi)$ of the W -orbit of the generic point $x_0 \in \pi$. Consider now invariant analytic extension \bar{f} of the function f to the union of neighborhoods $w(\tilde{U})$ where $\tilde{U} \supset U$ is a neighborhood of x_0 in \mathbb{C}^N , $w \in W$. We are going to apply the operator H to the function \bar{f} . The assumption of the theorem says that the result of the restriction of $H\bar{f}$ onto π does not depend on the extension \bar{f} and it is determined by $f_\pi = \bar{f}|_U$ only. So we may choose \bar{f} to be constant along the normal directions to π . Then

$$\partial_\alpha \bar{f} = \partial_{\hat{\alpha}} \bar{f}, \quad \Delta \bar{f} = \Delta_y \bar{f},$$

and also $(\alpha, x) = (\hat{\alpha}, x)$ when $x \in \pi$. Since function \bar{f} is W -invariant, the last sum in (7) disappears and the Calogero-Moser operator (7) takes the form (6). This proves the first part of the theorem.

The second statement follows from the commutativity of the Dunkl operators $[\nabla_\xi, \nabla_\eta] = 0$ and from the fact that the operators $\sigma(\nabla)$ preserve the space \mathcal{L} of W -invariant germs. The highest terms of the restrictions $\sigma(\nabla)^{Res\pi}$ are obtained by the restriction of the highest term of the operator $\sigma(\nabla)^{Res}$ onto the plane π . Since the stratum D considered in the space of orbits \mathbb{C}^N/W is a (singular) variety of dimension n there exist polynomials $\sigma_1 = x_1^2 + \dots + x_N^2, \sigma_2, \dots, \sigma_n \in \mathbb{C}[x]^W$ such that their restrictions on D are algebraically independent. The corresponding differential operators are algebraically independent as well, hence the operator (7) is quantum integrable. This completes the proof of the theorem.

We note that the CM system itself can also be restricted to the stratum D considered inside the orbit space if the corresponding parabolic ideal is H_c -invariant. Also it is known that in the orbit space the CM system becomes algebraic [34] so we have restriction of the non-singular differential operator to a subvariety.

Specific choice of invariants in Theorem 5 leading to a collection of algebraically independent differential operators on π depends of course on the particular Coxeter group W and the stratum. In the case of classical Coxeter groups one can always take the Newton sums as such invariant polynomials. More exactly let $\sigma_k = \sum_{i=1}^N x_i^k$. Let π be an intersection of mirrors of the group W of dimension n . In the case $W = A_{N-1}$ the polynomials $\sigma_1|_\pi, \dots, \sigma_n|_\pi$ and the corresponding differential operators on π are algebraically independent. This follows from the explicit form of the H_c -invariant strata and the fact that deformed Newton sums $\hat{\sigma}_k = \sum_{i=1}^{n_1} y_i^k + \kappa \sum_{i=1}^{n_2} z_i^k$ are algebraically independent for $k = 1, \dots, n = n_1 + n_2$ if $\kappa \in \mathbb{N}$ (this in turn follows from [14, Proposition 4]). In the case $W = B_N$ or $W = D_N$ the polynomials $\sigma_{2i}|_\pi, i = 1, \dots, n$ and the corresponding differential operators are algebraically independent by similar reasons.

So far we were using the “radial normalization” (6) of the generalized CM systems. The restricted operators are also gauge equivalent to the operators in the “potential normalization”. More exactly, we have the following property of the generalized CM systems related to arbitrary parabolic stratum.

Proposition 2 *Let $\pi \subset V$ be an intersection of mirrors*

$$\pi = \{x \in V \mid (\beta, x) = 0 \quad \forall \beta \in \Gamma_0^v\}, \quad (8)$$

corresponding to a Coxeter subgraph $\Gamma_0 \subset \Gamma$. Let \hat{u} denote the orthogonal projection of a vector $u \in V$ onto the space π . Consider the decomposition $\hat{\mathcal{R}}_+ = R_1 \sqcup \dots \sqcup R_k$ such that for any two vectors $u, v \in R_i$ we have collinearity $u \sim v$, and for any two non-zero vectors $u \in R_i, v \in R_j$ one has $u \not\sim v$ when $i \neq j$. Choose a nonzero element $\hat{\gamma}_i \in R_i$ for any $i = 1, \dots, k$. Define $c_i = \sum_{\substack{\alpha \in \mathcal{R}_+ \\ \hat{\alpha} \in R_i \setminus 0}} c_\alpha$.

Then

$$f^{-1}(\Delta - \sum_{\substack{\alpha \in \mathcal{R}_+ \\ \hat{\alpha} \neq 0}} \frac{2c_\alpha}{(\hat{\alpha}, x)} \partial_{\hat{\alpha}})f = \Delta - \sum_{i=1}^k \frac{c_i(c_i + 1)(\hat{\gamma}_i, \hat{\gamma}_i)}{(\hat{\gamma}_i, x)^2} \quad (9)$$

where $f = \prod_{\substack{\alpha \in \mathcal{R}_+ \\ \hat{\alpha} \neq 0}} (\hat{\alpha}, x)^{c_\alpha}$, and Δ is Laplacian on π .

Proof. The gauge property (9) is equivalent to the following series of identities for all $\alpha \in \mathcal{R}$:

$$\sum_{\substack{\beta \in \mathcal{R} \\ \widehat{\beta} \sim \widehat{\alpha}}} \frac{c_\beta(\widehat{\alpha}, \widehat{\beta})}{(\widehat{\beta}, x)} = 0 \quad (10)$$

when $x \in \pi$ and $(\widehat{\alpha}, x) = 0$. To establish identity (10) consider the set $S \subset \mathcal{R}$ of the roots β such that $\widehat{\beta}$ is not proportional to $\widehat{\alpha}$, we also assume that $\widehat{\alpha} \neq 0$. Consider the action on S of the group W_0 which is generated by reflections at the simple roots Γ_0^v and by the reflection s_α . We have decomposition of the set S as the union of W_0 -orbits $S = \mathcal{O}_1 \cup \dots \cup \mathcal{O}_k$. For each $i = 1, \dots, k$ we claim that

$$\sum_{\beta \in \mathcal{O}_i} \frac{(\widehat{\alpha}, \widehat{\beta})}{(\widehat{\beta}, x)} = 0 \quad (11)$$

when $x \in \pi$ and $(\widehat{\alpha}, x) = 0$. Indeed, for vectors x under consideration, we have $(\widehat{\beta}_1, x) = (\widehat{\beta}_2, x)$ for any roots $\beta_1, \beta_2 \in \mathcal{O}_i$. Also vector $b := \sum_{\beta \in \mathcal{O}_i} \beta$ satisfies

$$(\alpha, b) = (\gamma, b) = 0 \quad (12)$$

for any $\gamma \in \Gamma_0^v$ because of invariance $s_\alpha b = b = s_\gamma b$. Property (12) implies $\widehat{b} = b$ and then the identities (11) hold which in turn imply (10). The Proposition is proven.

4 Parabolic ideals and CM systems from classical Coxeter groups

We are going now to apply the described method of restriction to derive particular integrable systems of Calogero-Moser type. In this section we deal with the rational Cherednik algebras for the classical root systems, exceptional Coxeter root systems are dealt with in Section 6.

4.1 \mathcal{A} -series

Consider the root system $\mathcal{A}_{N-1} \subset \mathbb{C}^N$ given by the collection of vectors $e_i - e_j$, $1 \leq i < j \leq N$. Theorems 1-3 remain true when the rank of the root system is less than the dimension of the ambient space V . As it follows from

Theorem 2 the parabolic strata which define invariant ideals for the rational Cherednik algebra for the \mathcal{A}_{N-1} root system must have the form

$$D_{m,k} = \bigcup_{w \in S_N} w(\pi_{m,k}) \quad (13)$$

where the plane $\pi_{m,k}$ is given by the equations

$$\begin{aligned} x_1 = x_2 = \dots = x_k, \quad x_{k+1} = x_{k+2} = \dots = x_{2k}, \\ \dots \\ x_{(m-1)k+1} = x_{(m-1)k+2} = \dots = x_{mk}. \end{aligned} \quad (14)$$

Here $m \geq 1, k > 1$ are integer such that $mk \leq N$, the corresponding parameter $c = 1/k$.

Proposition 3 *For the root system \mathcal{A}_{N-1} , the parabolic stratum (13), (14) and the multiplicity $c = 1/k$ the restricted Calogero-Moser operator takes the form*

$$\begin{aligned} H = \left(\sum_{i=1}^N \nabla_i^2 \right)^{\text{Res}_{\pi_{m,k}}} &= \Delta - 2k \sum_{1 \leq i < j \leq m} \frac{\partial_i - \partial_j}{y_i - y_j} - \\ &\frac{2}{k} \sum_{m+1 \leq i < j \leq m+n} \frac{\partial_i - \partial_j}{y_i - y_j} - 2 \sum_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \frac{\partial_i - \sqrt{k}\partial_j}{y_i - \sqrt{k}y_j}, \end{aligned} \quad (15)$$

where $\Delta = \sum_{i=1}^{m+n} \partial_i^2$, $\partial_i = \frac{\partial}{\partial y_i}$, and $n = N - mk$.

Proof. One way to deduce this statement is to consider the projection of the root system \mathcal{A}_{N-1} to the plane $\pi_{m,k}$ as Theorem 5 directs. Alternatively, we may introduce the following change of coordinates. For every block of colliding coordinates $x_{jk+1} = \dots = x_{jk+k}$ we define new coordinates $z_{jk+1}, \dots, z_{jk+k}$ such that

$$\begin{aligned} z_{jk+1} = \frac{x_{jk+1} + \dots + x_{jk+k}}{\sqrt{k}}, \quad z_{jk+2} = \frac{x_{jk+1} - x_{jk+2}}{\sqrt{k}}, \quad z_{jk+3} = \frac{x_{jk+2} - x_{jk+3}}{\sqrt{k}}, \dots, \\ z_{jk+k} = \frac{x_{jk+k-1} - x_{jk+k}}{\sqrt{k}} \end{aligned}$$

where $j = 0, \dots, m-1$. The remaining coordinates are not changed: $z_i = x_i$ for $i > mk$. Then on the plane $\pi_{m,k}$ we have $x_{jk+s} = \frac{1}{\sqrt{k}} z_{jk+1}$, $s = 1, \dots, k$. Also $\frac{\partial}{\partial x_{jk+s}} = \frac{1}{\sqrt{k}} \frac{\partial}{\partial z_{jk+1}}$ when acting on functions \bar{f} which are constant along the directions orthogonal to the plane $\pi_{m,k}$. One gets operator (15) after renaming surviving z -coordinates into y -coordinates.

The operator (15) in the case $n = 1$ appeared first in the work [9] where its (algebraic) integrability was established. For arbitrary m, n and k the integrability of the trigonometric version of the operator (15) was established in [14] using explicit calculations. In the work [15] integrability for arbitrary m, n, k was established by obtaining the operator as a restriction of the trigonometric Calogero-Moser operator acting in the space of symmetric functions of infinitely many variables to the generalized discriminants.

4.2 \mathcal{B} -series

Consider the group \mathcal{B}_N generated by reflections at $x_i = \pm x_j$ for $1 \leq i < j \leq N$, and at $x_i = 0$ for $i = 1, \dots, N$. Let c_2 be the multiplicity of the roots e_i , and let c_1 be the multiplicity of the roots $e_i \pm e_j$. There are parabolic strata corresponding to the Coxeter subgraphs of the type $\mathcal{A}_{k-1} \times \dots \times \mathcal{A}_{k-1}$, $k > 1$, where the number of subsystems \mathcal{A}_{k-1} is m , $mk \leq N$. The corresponding plane $\pi_{m,k} \subset \mathbb{C}^N$ is given by the equations

$$\begin{aligned} x_1 = x_2 = \dots = x_k, \quad x_{k+1} = x_{k+2} = \dots = x_{2k}, \\ \dots \\ x_{(m-1)k+1} = x_{(m-1)k+2} = \dots = x_{mk}. \end{aligned} \quad (16)$$

The corresponding parabolic stratum $D_{m,k} \subset \mathbb{C}^N$ is the orbit of the plane $\pi_{m,k}$ under the group \mathcal{B}_N :

$$D_{m,k} = \bigcup_{w \in \mathcal{B}_N} w(\pi_{m,k}). \quad (17)$$

Theorem 2 and easy calculations imply the following

Proposition 4 *For the root system \mathcal{B}_N with the multiplicity $c_1 = 1/k$ the parabolic ideal*

$$I_{m,k} = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_{m,k}} = 0\}$$

corresponding to the stratum (16), (17) is invariant under the rational Cherednik algebra $H_c^{\mathbb{B}^N}$. The restricted Calogero-Moser operator takes the form

$$\begin{aligned}
H = & \Delta - \sum_{\substack{i,j=1 \\ i \neq j}}^m \left(\frac{2k(\partial_i - \partial_j)}{y_i - y_j} + \frac{2k(\partial_i + \partial_j)}{y_i + y_j} \right) - \sum_{i=1}^m \sum_{j=m+1}^{m+n} \left(\frac{2(\partial_i - \sqrt{k}\partial_j)}{y_i - \sqrt{k}y_j} + \frac{2(\partial_i + \sqrt{k}\partial_j)}{y_i + \sqrt{k}y_j} \right) \\
& - \sum_{\substack{i,j=m+1 \\ i \neq j}}^{m+n} \left(\frac{2k^{-1}(\partial_i - \partial_j)}{y_i - y_j} + \frac{2k^{-1}(\partial_i + \partial_j)}{y_i + y_j} \right) - \sum_{i=1}^m \frac{(kq + k - 1)\partial_i}{y_i} - \sum_{i=m+1}^{m+n} \frac{q\partial_i}{y_i},
\end{aligned} \tag{18}$$

where $\Delta = \sum_{i=1}^{m+n} \partial_i^2$, $q = 2c_2 \in \mathbb{C}$, and $n = N - mk$.

For $n = 1$ the first sum in the second line in (18) disappears and the integrability of this operator was established in [11] where it corresponded to the configuration $C_{m+1}(\frac{kq+k-1}{2}, \frac{q}{2})$. In the case $n > 1$ the integrability of the trigonometric version of (18) was shown in [14],[16].

Consider the plane $\pi_l \subset \mathbb{C}^N$ given by the equations

$$x_1 = x_2 = \dots = x_l = 0.$$

Consider the corresponding parabolic stratum D_l which is the orbit of the plane π_l under the group B_N :

$$D_l = \bigcup_{i_1 < \dots < i_l} \{x \mid x_{i_1} = \dots = x_{i_l} = 0\}. \tag{19}$$

Define the ideal $I_l = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_l} = 0\}$.

Proposition 5 *The Dunkl operators ∇_i preserve the ideal I_l if and only if $2(l-1)c_1 + 2c_2 = 1$. The corresponding restricted Calogero-Moser operator is the CM operator constructed by the root system B_{N-l} .*

Proof. It is easy to check that the generalized Coxeter number h_c for the root system \mathcal{B}_l with the multiplicities $c(e_i \pm e_j) = c_1$, $c(e_i) = c_2$ equals $h_c = 2c_1(l-1) + 2c_2$. The statement follows from Theorem 3.

Theorem 6 *All possible parabolic strata defining $H_c^{\mathbb{B}^N}$ -invariant ideals are either the strata $D_{m,k}$ defined by (17) (with $c_1 = 1/k$) or strata D_l defined by*

(19) (with $2(l-1)c_1 + 2c_2 = 1$) or their intersection. In the latter case the stratum is the B_N -orbit of the subspace

$$\begin{aligned} x_1 = x_2 = \dots = x_k, \quad x_{k+1} = x_{k+2} = \dots = x_{2k}, \\ \dots \\ x_{(m-1)k+1} = x_{(m-1)k+2} = \dots = x_{mk}, \\ x_{mk+1} = \dots = x_{mk+l} = 0, \end{aligned} \quad (20)$$

with $c_1 = 1/k$, $c_2 = \frac{1}{2} - \frac{l-1}{k}$, and $k > 1$, $l \geq 1$, $mk + l \leq N$. In this case the corresponding restricted Calogero-Moser operator is operator (18) where $q = \frac{k+2l+2}{k}$.

The proof follows from the general structure of the parabolic strata for the B_N group, Theorem 3 and Propositions 4, 5.

4.3 \mathcal{D} -series

Consider the group \mathcal{D}_N generated by reflections at $x_i = \pm x_j$ for $1 \leq i < j \leq N$. Take two integer parameters $m > 0$, $k > 1$ such that $mk \leq N$. Consider the plane $\pi_{m,k}^\varepsilon \subset \mathbb{C}^N$ given by the equations

$$\begin{aligned} x_1 = x_2 = \dots = x_k, \quad x_{k+1} = x_{k+2} = \dots = x_{2k}, \\ \dots \\ x_{(m-1)k+1} = x_{(m-1)k+2} = \dots = \varepsilon x_{mk}. \end{aligned} \quad (21)$$

Here $\varepsilon = 1$ except the case when k is even and $N = mk$. In the latter case $\varepsilon = \pm 1$. Consider the corresponding parabolic stratum $D_{m,k}^\varepsilon \subset \mathbb{C}^N$ which is the orbit of the plane $\pi_{m,k}^\varepsilon$ under the group \mathcal{D}_N :

$$D_{m,k}^\varepsilon = \bigcup_{w \in \mathcal{D}_N} w(\pi_{m,k}^\varepsilon). \quad (22)$$

This describes all possible strata in \mathcal{D}_N with the Coxeter graphs $\mathcal{A}_{k-1} \times \dots \times \mathcal{A}_{k-1}$ (see [33]).

Theorem 2 and easy calculations imply the following

Proposition 6 *For the root system \mathcal{D}_N , the parabolic stratum (21), (22) and the multiplicity $c = 1/k$ the corresponding ideal*

$$I_{m,k}^\varepsilon = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_{m,k}^\varepsilon} = 0\}$$

is invariant under the corresponding rational Cherednik algebra H_c . The restricted Calogero-Moser operator has the form (18) with $q = 0$.

Another type of the parabolic strata for the \mathcal{D}_N group has the Coxeter graph of type \mathcal{D}_p , $1 < p < N$. Consider the plane $\pi_p \subset \mathbb{C}^N$ given by the equations

$$x_1 = x_2 = \dots = x_p = 0.$$

The corresponding parabolic stratum D_p is the orbit of the plane π_p under the group \mathcal{D}_N :

$$D_p = \bigcup_{i_1 < \dots < i_p} \{x \mid x_{i_1} = \dots = x_{i_p} = 0\}. \quad (23)$$

Theorem 2 implies that the ideal $I_p = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_p} = 0\}$ is invariant under the rational Cherednik algebra if and only if $c = \frac{1}{2(p-1)}$. The corresponding restricted Calogero-Moser operator is the CM operator constructed by the root system \mathcal{B}_{N-p} .

The next theorem is a corollary of Theorem 2, previous considerations and calculations of the restricted root systems.

Theorem 7 *For the root system \mathcal{D}_N all possible parabolic strata defining $H_c^{\mathcal{D}_N}$ -invariant ideals are either the strata $D_{m,k}^\varepsilon$ defined by (22) (with $c = 1/k$) or strata D_p defined by (23) (with $c = \frac{1}{2(p-1)}$) or their intersection. In the latter case the stratum is the \mathcal{D}_N -orbit of the plane*

$$\begin{aligned} x_1 = x_2 = \dots = x_k, \quad x_{k+1} = x_{k+2} = \dots = x_{2k}, \\ \dots \\ x_{(m-1)k+1} = x_{(m-1)k+2} = \dots = x_{mk}, \\ x_{mk+1} = \dots = x_{mk+\frac{k}{2}+1} = 0, \end{aligned} \quad (24)$$

where $k \geq 2$ is even, and $c = 1/k$. In this case the restricted Calogero-Moser operator is the operator (18) with $q = \frac{2(k+2)}{k}$.

5 Generalized CM systems with quadratic potential

In this section we show that generalized CM operators (15), (18) obtained from the parabolic strata for the classical Coxeter groups remain integrable when the term $\omega^2 \sum_{i=1}^{m+n} y_i^2$ is added to the Hamiltonians H .

First we recall integrability of the Calogero-Moser systems with square potential following [29]. Let ∇_i be Dunkl operator in the basis direction e_i for the symmetric group S_N ,

$$\nabla_i = \partial_i - c \sum_{\substack{j=1 \\ j \neq i}}^N \frac{1}{x_i - x_j} (1 - s_{ij}).$$

Define now the operators ∇_i^+, ∇_i^- as

$$\nabla_i^\pm = \nabla_i \pm \omega x_i,$$

where $\omega \in \mathbb{C}$ is a parameter. A combination of operators ∇_i^\pm gives CM system with quadratic potential. Namely, because of commutation relations

$$[x_i, \nabla_i] = -1 + c \sum_{\substack{j=1 \\ j \neq i}}^N s_{ij}, \quad (25)$$

we have

$$\sum_{i=1}^N \nabla_i^+ \nabla_i^- = \sum_{i=1}^N \nabla_i^2 - \omega^2 \sum_{i=1}^N x_i^2 - \omega N + 2\omega c \sum_{i < j}^N s_{ij}.$$

The last expression becomes CM Hamiltonian with the added quadratic terms and constant when restricted to the space of symmetric functions. Denoting the operation of restriction to invariants by Res (c.f. [5]) we recall the following result.

Theorem 8 [29] *The CM Hamiltonian with quadratic potential*

$$H = \Delta - \sum_{i < j}^N \frac{2c(\partial_i - \partial_j)}{(x_i - x_j)} - \omega^2 \sum_{i=1}^N x_i^2$$

can be obtained as

$$H = \left(\sum_{i=1}^N \nabla_i^+ \nabla_i^- \right)^{Res} + \omega N - \omega c N(N-1).$$

The differential operators $\left(\sum_{i=1}^N (\nabla_i^+ \nabla_i^-)^k \right)^{Res}$ for $k \in \mathbb{Z}_+$ pairwise commute, so H is quantum integrable.

Proof. Commutativity of the differential operators follows from the fact that the combinations $\sum_{i=1}^N (\nabla_i^+ \nabla_i^-)^k$ preserve the space of symmetric polynomials and from the following commutativity of the combinations of Dunkl operators

$$\left[\sum_{i=1}^N h_i^k, \sum_{i=1}^N h_i^l \right] = 0, \quad (26)$$

where $h_i = \nabla_i^+ \nabla_i^-$ and $k, l \in \mathbb{Z}_+$. To establish (26) we note first that from the relations $[x_i, \nabla_j] = -cs_{ij}$ valid for $i \neq j$ it follows that $[h_i, h_j] = 2\omega c(h_i - h_j)s_{ij}$. Then by induction in k it is easy to deduce that

$$[h_i^k, h_j] = 2\omega c(h_i^k - h_j^k)s_{ij},$$

and then by induction in l one obtains that

$$[h_i^k, h_j^l] = 2\omega c \sum_{t=1}^l (h_j^{t-1} h_i^{k+l-t} - h_j^{k+l-t} h_i^{t-1}) s_{ij}.$$

The last formula implies by induction in l that $[h_i^k, h_j^l]$ is antisymmetric with respect to permutation h_i and h_j . Therefore $[h_i^k, h_j^l] + [h_j^k, h_i^l] = 0$, hence commutativity of the operators (26) holds, and the theorem follows.

Similar arguments allow to establish quantum integrability of the Calogero-Moser systems with quadratic potential in the case of other classical Coxeter root systems [35].

Indeed, let now $\nabla_i^\pm = \nabla_i \pm \omega x_i$, where ∇_i is the Dunkl operator in the direction e_i for the root system \mathcal{B}_N or \mathcal{D}_N , $i = 1, \dots, N$. The operators $\sum_{i=1}^N h_i^k$ where $h_i = \nabla_i^+ \nabla_i^-$ preserve the spaces of the corresponding \mathcal{B}_N or \mathcal{D}_N invariants. Commutativity of these combinations for different k can be established similar to the proof of commutativity (26) in the proof of Theorem 8. Indeed, the commutation relations need to be modified as follows

$$[x_i, \nabla_j] = -c(s_{ij} - s_{ij}^+), [h_i, h_j] = 2\omega c(h_i - h_j)(s_{ij} + s_{ij}^+), [h_i^k, h_j] = 2\omega c(h_i^k - h_j^k)(s_{ij} + s_{ij}^+),$$

where $1 \leq i < j \leq N$, and then

$$[h_i^k, h_j^l] = 2\omega c \sum_{t=1}^l (h_j^{t-1} h_i^{k+l-t} - h_j^{k+l-t} h_i^{t-1})(s_{ij} + s_{ij}^+),$$

where s_{ij}^+ is the reflection at the hyperplane $x_i + x_j = 0$.

Finally, the differential operators $(\sum_{i=1}^N \nabla_i^+ \nabla_i^-)^{Res}$ coincides with the corresponding \mathcal{B}_N or \mathcal{D}_N CM operator with quadratic potential up to a constant, if Res denotes the restriction to \mathcal{B}_N or \mathcal{D}_N -invariants respectively (c.f. [5]).

Now we are ready to obtain generalized quantum Calogero-Moser systems with the additional quadratic potential.

Theorem 9 *The operators $H - \omega^2 \sum_{i=1}^{m+n} y_i^2$ are quantum integrable if H is given either by the formula (15) or by the formula (18).*

In this theorem we assume that $k \in \mathbb{Z}$ as it also happens in Propositions 3, 4, although we expect Theorem 9 to be true for any k . When H is given by (15) with $n = 1$ Theorem 9 is already established in [10] for any $k \in \mathbb{C}$. We also note that some eigenfunctions of the operators from the theorem were already investigated in [17, 18].

The proof of the theorem follows from the fact that if a parabolic ideal I corresponding to the Coxeter orbit of a linear subspace π is preserved under the action of the Dunkl operators ∇_i , then I is also preserved under the action of the operators ∇_i^\pm . Indeed, consider the combinations

$$L_k = \sum_{i=1}^N h_i^k$$

where $h_i = \nabla_i^+ \nabla_i^-$ is a product of Dunkl operators for the Coxeter groups \mathcal{A}_{N-1} or \mathcal{B}_N . The operators L_k are invariant under the action of the corresponding classical Coxeter group. The restrictions $L_k^{res\pi}$ are commuting differential operators on the subspace π . For the Coxeter group \mathcal{A}_{N-1} with multiplicity $c = k^{-1}$ and the subspace π given by the equations (14) the operator

$$L_1^{Res\pi} = H - \omega^2 \sum_{i=1}^{m+n} y_i^2 - \omega N + \omega k^{-1} N(N-1),$$

where H is given by (15). For the Coxeter group \mathcal{B}_N with multiplicities $c(e_i \pm e_j) = k^{-1}$, $c(e_i) = q/2$ and the subspace π given by the equations (16) the operator

$$L_1^{Res\pi} = H - \omega^2 \sum_{i=1}^{m+n} y_i^2 - \omega N + 2\omega k^{-1} N(N-1) + \omega q N,$$

where H is given by (18).

6 Generalized CM systems from exceptional Coxeter groups

First we consider invariant ideals for the Coxeter root system \mathcal{F}_4 . We use Theorem 3 and the fact that the parabolic strata are given by non-isomorphic Coxeter subgraphs and by isomorphic subgraphs if they are different as Dynkin subgraphs of the root system \mathcal{F}_4 ([33]). We also use that for the root system \mathcal{A}_n with multiplicity c we have $h^c = (n+1)c$, and for the root system \mathcal{B}_n with multiplicities $c(e_i \pm e_j) = c_1$, $c(e_i) = c_2$ we have $h^c = 2(n-1)c_1 + 2c_2$.

The action of the group on the root system \mathcal{F}_4 has two orbits with multiplicities c_1, c_2 . There are two strata corresponding to the subgraphs of type \mathcal{A}_1 , the corresponding ideals are invariant when the corresponding multiplicity c_1 or c_2 equals $1/2$. There are two strata corresponding to subgraphs of type \mathcal{A}_2 . They are invariant iff the corresponding multiplicity $c|_{\mathcal{A}_2} = 1/3$. There is one stratum of the type $\mathcal{A}_1 \times \mathcal{A}_1$, it is invariant iff $c_1 = c_2 = 1/2$. The stratum of type \mathcal{B}_2 is invariant iff $c_1 + c_2 = 1/2$. There are two strata of type \mathcal{B}_3 , they are invariant iff the corresponding generalized Coxeter number $h_{\mathcal{B}_3}^c = 1$ which gives $4c_1 + 2c_2 = 1$ or $4c_2 + 2c_1 = 1$ respectively. There are two strata of type $\mathcal{A}_1 \times \mathcal{A}_2$, they are invariant iff $c|_{\mathcal{A}_2} = 1/3$ and $c|_{\mathcal{A}_1} = 1/2$.

The restricted Calogero-Moser systems correspond to the root system \mathcal{G}_2 for the strata \mathcal{A}_2 , and the restricted system corresponds to the root system \mathcal{B}_2 for the stratum \mathcal{B}_2 .

The restricted Calogero-Moser systems corresponding to the strata \mathcal{A}_1 give equivalent new non-Coxeter one-parametric families of integrable systems in dimension 3.

Theorem 10 *The restricted \mathcal{F}_4 CM system for the stratum \mathcal{A}_1 has the form*

$$H = \Delta - \sum_{i=1}^3 \frac{(4c+1)\partial_i}{x_i} - \sum_{\substack{i,j=1 \\ i < j}}^3 \frac{2c(\partial_i \pm \partial_j)}{x_i \pm x_j} - \sum \frac{2(\partial_1 \pm \partial_2 \pm \partial_3)}{x_1 \pm x_2 \pm x_3}, \quad (27)$$

where $c \in \mathbb{C}$ and summations run over arbitrary choices of signs. In particular, operator (27) is quantum integrable.

In the case of the $\mathcal{A}_1 \times \mathcal{A}_1$ strata the restricted integrable CM system has

the Hamiltonian

$$H = \partial_x^2 + \partial_y^2 - \frac{2m}{x}\partial_x - \frac{2m}{y}\partial_y - \frac{2n}{x+y}(\partial_x + \partial_y) - \frac{2n}{x-y}(\partial_x - \partial_y) - \frac{2(\partial_x + \alpha\partial_y)}{x + \alpha y} - \frac{2(\partial_x - \alpha\partial_y)}{x - \alpha y} - \frac{2(\partial_x + \alpha^{-1}\partial_y)}{x + \alpha^{-1}y} - \frac{2(\partial_x - \alpha^{-1}\partial_y)}{x - \alpha^{-1}y} \quad (28)$$

where $m = 7/2$, $n = 0$ and $\alpha = \sqrt{2}$.

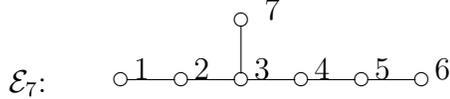
Proposition 7 *The system (28) is quantum integrable for any m, n and $\alpha = \pm(\sqrt{2n+1} \pm \sqrt{2(m+n+1)})/\sqrt{2m+1}$.*

Proof follows from the fact that operator (28) in the potential form satisfies the locus conditions ([11]) when $m, n \in \mathbb{Z}$. There is a commuting operator to (28) with the highest symbol $(\alpha^2\xi_1^2 - \xi_2^2)^3(\xi_1^2 - \alpha^2\xi_2^2)^3$.

Now we give a complete list of the sets A of vectors α with multiplicities m_α for the generalized CM systems, which are obtained as restrictions of the exceptional Coxeter root systems of \mathcal{E} and \mathcal{H} type. The corresponding operators

$$\Delta - \sum_{\alpha \in A} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha, \quad \Delta - \sum_{\alpha \in A} \frac{m_\alpha(m_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2}$$

are quantum integrable. These systems are labeled by a pair (Γ, Γ_0) , where Γ is a Coxeter graph of \mathcal{E} or \mathcal{H} type, and Γ_0 is its subgraph satisfying conditions in Theorem 3 so the parabolic ideal I_{Γ_0} is H_c -invariant. We will assume that there are at least two vertices in $\Gamma \setminus \Gamma_0$ so that the restricted CM system (6) is at least two-dimensional hence non-trivial. Then in all the cases below the parabolic strata are in one to one correspondence with the isomorphism classes of the Coxeter subgraphs Γ_0 except the following cases for the \mathcal{E}_7 root system ([33]). Namely, there are two different \mathcal{A}_1^3 strata denoted as $(\mathcal{E}_7, \mathcal{A}_1^3)_{1,2}$ and two different \mathcal{A}_5 strata denoted as $(\mathcal{E}_7, \mathcal{A}_5)_{1,2}$.



The stratum $(\mathcal{E}_7, \mathcal{A}_1^3)_2$ corresponds to the choice of subgraph Γ_0 corresponding to the roots numbered by 4,6,7 in the diagram. The stratum $(\mathcal{E}_7, \mathcal{A}_1^3)_1$ corresponds to any other choice of the subgraph of type \mathcal{A}_1^3 . Also the stratum $(\mathcal{E}_7, \mathcal{A}_5)_2$ corresponds to the choice of the subgraph with the vertices 3,4,5,6,7. And the stratum $(\mathcal{E}_7, \mathcal{A}_5)_1$ corresponds to any of the two remaining embeddings of the subgraph \mathcal{A}_5 into \mathcal{E}_7 .

In the following table for every system (Γ, Γ_0) we specify vectors A in this system, their multiplicities, dimension of the linear space spanned by A , and the number of vectors in A .

	(Γ, Γ_0)	Vectors A of the restricted CM system	Mult	Dim	$ A $
1	$(\mathcal{E}_8, \mathcal{A}_1)$	$\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 + \sqrt{2}e_7$ (even \natural of minuses); $e_i \pm e_j (1 \leq i < j \leq 6)$; e_7 $e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6$ (odd \natural of minuses); $\sqrt{2}(e_i \pm e_7) (1 \leq i \leq 6)$	1/2 1	7	91
2	$(\bar{\mathcal{E}}_8, \mathcal{A}_1^2)$	$e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6$ $e_1, e_2, e_3, e_4, e_5, e_6$ $e_i \pm e_j (1 \leq i < j \leq 6)$	1 2 1/2	6	68
3	$(\mathcal{E}_8, \mathcal{A}_2)$	$e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \sqrt{3}e_6$ (even \natural of minuses); $e_i \pm e_j (1 \leq i < j \leq 5)$ $e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm \frac{1}{\sqrt{3}}e_6$ (odd \natural of minuses); $\sqrt{3}e_i \pm e_6 (1 \leq i \leq 5)$; e_6	1/3 1	6	63
4	$(\mathcal{E}_8, \mathcal{A}_1^3)$	$e_1 \pm e_2 \pm e_3 \pm e_4 \pm \sqrt{2}e_5$; $\sqrt{2}e_i \pm e_5 (1 \leq i \leq 4)$ $e_1 \pm e_2 \pm e_3 \pm e_4$; e_1, e_2, e_3, e_4 $e_i \pm e_j (1 \leq i < j \leq 4)$ e_5	1 2 1/2 9/2	5	49
5	$(\mathcal{E}_8, \mathcal{A}_3)$	$e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5$ $e_i \pm e_j (1 \leq i < j \leq 5)$ e_1, e_2, e_3, e_4, e_5	1 1/4 3/2	5	41
6	$(\mathcal{E}_8, \mathcal{A}_1^4)$	$e_1 \pm e_2 \pm e_3$; $e_1 \pm e_2 \pm e_4$; $e_1 \pm e_3 \pm e_4$; $e_2 \pm e_3 \pm e_4$ $e_1 \pm e_2$; $e_1 \pm e_3$; $e_1 \pm e_4$; $e_2 \pm e_3$; $e_2 \pm e_4$; $e_3 \pm e_4$ e_1, e_2, e_3, e_4	1 2 9/2	4	32
7	$(\mathcal{E}_8, \mathcal{A}_2^2)$	$\pm e_1 \pm e_2 + \frac{1}{\sqrt{3}}e_3 \pm \sqrt{3}e_4$ (odd \natural of minuses); $\pm e_1 \pm e_2 \pm \sqrt{3}e_3 + \frac{1}{\sqrt{3}}e_4$ (odd \natural of minuses); $\sqrt{3}e_1 \pm e_3$; $\sqrt{3}e_2 \pm e_3$; $\sqrt{3}e_1 \pm e_4$; $\sqrt{3}e_2 \pm e_4$; e_3, e_4 $e_1 \pm e_2 \pm \sqrt{3}e_3 \pm \sqrt{3}e_4$ (even \natural of minuses); $e_1 \pm e_2$ $e_1 + e_2 \pm \frac{1}{\sqrt{3}}(e_3 + e_4)$; $e_1 - e_2 \pm \frac{1}{\sqrt{3}}(e_3 - e_4)$; $e_3 \pm e_4$	1 1/3 3	4	30
8	$(\mathcal{E}_8, \mathcal{A}_4)$	$e_1 \pm e_2 \pm e_3 \pm \sqrt{5}e_4$ (even \natural of minuses); $e_1 \pm e_2$; $e_1 \pm e_3$; $e_2 \pm e_3$ $e_1 \pm e_2 \pm e_3 \pm \frac{1}{\sqrt{5}}e_4$ (even \natural of minuses); e_5 $e_1 \pm e_2 \pm e_3 \pm \frac{3}{\sqrt{5}}e_4$ (odd \natural of minuses); $\sqrt{5}e_1 \pm e_4$; $\sqrt{5}e_2 \pm e_4$; $\sqrt{5}e_3 \pm e_4$	1/5 2 1	4	25
9	$(\mathcal{E}_8, \mathcal{D}_4)$	\mathcal{F}_4	4/3, 1/6	4	24

	(Γ, Γ_0)	Vectors A of the restricted CM system	Mult	Dim	$ A $
10	$(\mathcal{E}_8, \mathcal{A}_5)$	$e_1 + e_2 \pm \sqrt{6}e_3; e_1 + e_2$ $e_1 - e_2 \pm \frac{2\sqrt{6}}{3}e_3; \sqrt{6}e_1 \pm e_3; \sqrt{6}e_2 \pm e_3$ $e_1 + e_2 \pm \frac{\sqrt{6}}{3}e_3; e_3$ $e_1 - e_2$	1/6 1 5/2 7/2	3	13
11	$(\mathcal{E}_8, \mathcal{D}_5)$	$e_1 \pm e_2 \pm e_3$ $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3$ e_1, e_2, e_3	2 1/8 5/4	3	13
12	$(\mathcal{E}_8, \mathcal{A}_3^2)$	e_1, e_2 $e_1 \pm 2e_2; e_1 \pm \frac{1}{2}e_2$ $e_1 \pm e_2$	15/2 1 4	2	8
13	$(\mathcal{E}_8, \mathcal{A}_6)$	e_1 e_2 $\sqrt{7}e_1 \pm 3e_2$ $\sqrt{7}e_1 \pm e_2$	1/7 6 1 3	2	6
14	$(\mathcal{E}_8, \mathcal{D}_6)$	\mathcal{B}_2	33/10, 6/5	2	4
15	$(\mathcal{E}_8, \mathcal{E}_6)$	\mathcal{G}_2	23/12, 1/12	2	6
16	$(\mathcal{E}_7, \mathcal{A}_1)$	$\sqrt{2}e_6 \pm \sqrt{2}e_5 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ (odd \natural of minuses in the last four terms); $e_i \pm e_j$ ($1 \leq i < j \leq 4$); e_5, e_6 $\sqrt{2}e_6 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ (even \natural of minuses); $\sqrt{2}e_i \pm e_5$ ($1 \leq i \leq 4$)	1/2 1	6	46
17	$(\mathcal{E}_7, \mathcal{A}_1^2)$	$\sqrt{2}e_5 \pm e_1 \pm e_2 \pm e_3 \pm e_4$ $e_i \pm e_j$ ($1 \leq i < j \leq 4$); e_5 e_1, e_2, e_3, e_4	1 1/2 2	5	33
18	$(\mathcal{E}_7, \mathcal{A}_2)$	$\pm e_1 \pm e_2 \pm e_3 \pm \sqrt{3}e_4 + \sqrt{2}e_5$ (odd \natural of minuses); $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3; e_5$ $\pm e_1 \pm e_2 \pm e_3 \pm \frac{1}{\sqrt{3}}e_4 + \sqrt{2}e_5$ (even \natural of minuses); $\sqrt{3}e_1 \pm e_4; \sqrt{3}e_2 \pm e_4; \sqrt{3}e_3 \pm e_4; e_4$	1/3 1	5	30
19	$(\mathcal{E}_7, \mathcal{A}_1^3)_1$	\mathcal{F}_4	2, 1/2	4	24

	(Γ, Γ_0)	Vectors A of the restricted CM system	Mult	Dim	$ A $
20	$(\mathcal{E}_7, \mathcal{A}_1^3)_2$	$e_1 \pm e_3 \pm e_4; e_1 \pm e_2 \pm e_4; e_2 \pm e_3 \pm e_4$ $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3$ e_1, e_2, e_3 e_4	1 2 1/2 9/2	4	22
21	$(\mathcal{E}_7, \mathcal{A}_3)$	$e_1 \pm e_2 \pm e_3 \pm \sqrt{2}e_4$ $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3; e_4$ e_1, e_2, e_3	1 1/4 3/2	4	18
22	$(\mathcal{E}_7, \mathcal{A}_1^4)$	$e_1 \pm e_2 \pm e_3$ $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3$ e_1, e_2, e_3	1 2 9/2	3	13
23	$(\mathcal{E}_7, \mathcal{A}_2^2)$	$\sqrt{3}(e_1 - e_2) \pm \sqrt{2}e_3; e_3$ $e_1 + 3e_2 \pm \sqrt{6}e_3, 3e_1 + e_2 \pm \sqrt{6}e_3, e_1, e_2$ $e_1 - e_2 \pm \sqrt{6}e_3, e_1 \pm e_2$	1/3 1 3	3	13
24	$(\mathcal{E}_7, \mathcal{A}_4)$	$e_1 - \sqrt{5}e_2 \pm \sqrt{2}e_3; e_3$ $e_1 + \frac{3}{\sqrt{5}}e_2 \pm \sqrt{2}e_3; \sqrt{5}e_1 \pm e_2$ $e_1 - \frac{1}{\sqrt{5}}e_2 \pm \sqrt{2}e_3; e_2$	1/5 1 2	3	10
25	$(\mathcal{E}_7, \mathcal{D}_4)$	\mathcal{B}_3	4/3, 1/6	3	9
26	$(\mathcal{E}_7, \mathcal{D}_5)$	$e_1 \pm \sqrt{2}e_2$ e_1 e_2	2 5/4 1/8	2	4
27	$(\mathcal{E}_7, \mathcal{A}_5)_1$	$\mathcal{C}_2(\frac{5}{2}, \frac{7}{2})$	1, 5/2, 7/2	2	4
28	$(\mathcal{E}_7, \mathcal{A}_5)_2$	\mathcal{G}_2	5/2, 1/6	2	6
29	$(\mathcal{E}_6, \mathcal{A}_1)$	$\pm e_1 \pm e_2 \pm e_3 + \sqrt{2}e_4 \pm \sqrt{3}e_5$ (even \sharp of minuses); $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3; e_4$ $\pm e_1 \pm e_2 \pm e_3 + \sqrt{3}e_5$ (odd \sharp of minuses); $\sqrt{2}e_1 \pm e_4;$ $\sqrt{2}e_2 \pm e_4; \sqrt{2}e_3 \pm e_4$	1/2 1	5	25
30	$(\mathcal{E}_6, \mathcal{A}_1^2)$	$e_1 \pm e_2 \pm e_3 \pm \sqrt{3}e_4$ $e_1 \pm e_2; e_1 \pm e_3; e_2 \pm e_3$ e_1, e_2, e_3	1 1/2 2	4	17

	(Γ, Γ_0)	Vectors A of the restricted CM system	Mult	Dim	$ A $
31	$(\mathcal{E}_6, \mathcal{A}_2)$	$\pm e_1 \pm e_2 \pm \sqrt{3}e_3 + \sqrt{3}e_4$ (even \natural of minuses); $e_1 \pm e_2$ $\pm e_1 \pm e_2 \pm \frac{1}{\sqrt{3}}e_3 + \sqrt{3}e_4$ (odd \natural of minuses); $\sqrt{3}e_1 \pm e_3; \sqrt{3}e_2 \pm e_3; e_3$	1/3 1	4	15
32	$(\mathcal{E}_6, \mathcal{A}_1^3)$	$e_1 \pm \sqrt{2}e_2 \pm \sqrt{3}e_3; \sqrt{2}e_1 \pm e_2$ $e_1 \pm \sqrt{3}e_3; e_1$ e_2	1 2 9/2	3	10
33	$(\mathcal{E}_6, \mathcal{A}_3)$	$\mathcal{C}_3(1/4, 0)$	1, 1/4, 3/2	3	8
34	$(\mathcal{E}_6, \mathcal{A}_2^2)$	\mathcal{G}_2	3, 1/3	2	6
35	$(\mathcal{E}_6, \mathcal{D}_4)$	\mathcal{A}_2	4/3	2	3
36	$(\mathcal{E}_6, \mathcal{A}_4)$	e_1 e_2 $\sqrt{5}e_1 \pm \sqrt{3}e_2$	1/5 1 2	2	4
37	$(\mathcal{H}_4, \mathcal{A}_1)$	$(\sqrt{5}+1)e_1 \pm 2e_2 \pm (\sqrt{5}-1)e_3; 2e_1 \pm (\sqrt{5}-1)e_2 \pm$ $(\sqrt{5}+1)e_3; (\sqrt{5}-1)e_1 \pm (\sqrt{5}+1)e_2 \pm 2e_3; e_1,$ e_2, e_3 $2e_1 \pm (\sqrt{5}+3)e_3; (\sqrt{5}+3)e_2 \pm 2e_3; (\sqrt{5}+3)e_1 \pm$ $2e_2; e_1 \pm e_2 \pm e_3$ $2e_1 \pm (\sqrt{5}+1)e_2; (\sqrt{5}+1)e_1 \pm 2e_3; 2e_2 \pm (\sqrt{5}+$ $1)e_3$	1/2 1 2	3	31
38	$(\mathcal{H}_4, \mathcal{A}_2)$	$(\sqrt{5} \pm 2)e_1 + \sqrt{3}e_2, \sqrt{5}e_1 - \sqrt{3}e_2$ $\sqrt{3}e_1 + \sqrt{5}e_2, \sqrt{3}e_1 - (\sqrt{5} \pm 2)e_2$ $\sqrt{3}e_1 + (\sqrt{5} \pm 4)e_2, \sqrt{3}e_1 \pm e_2, \sqrt{15}e_1 + e_2, e_2$	1/3 4 1	2	12
39	$(\mathcal{H}_4, \mathcal{I}_2(5))$	$I_2(10)$	1/5, 2	2	10
40	$(\mathcal{H}_4, \mathcal{A}_1^2)$	e_1, e_2 $e_1 \pm e_2; (\sqrt{5} \pm 1)e_1 \pm 2e_2$ $(\sqrt{5} \pm 3)e_1 \pm 2e_2$	13/2 2 1	2	12
41	$(\mathcal{H}_3, \mathcal{A}_1)$	e_1, e_2 $(\sqrt{5}+1)e_1 \pm 2e_2$ $(\sqrt{5}+3)e_1 \pm 2e_2$	1/2 2 1	2	6

The restrictions of Coxeter root systems appear also, in particular, in the context of \vee -systems [36]. We note that the number of vectors in the

\vee -system $(\mathcal{H}_4, \mathcal{A}_1)$ is stated inaccurately in [36] as some of the vectors listed there are actually proportional. We also refer to [38] where, in particular, bases of the restricted Weyl root systems are discussed.

The CM systems corresponding to the Coxeter restrictions $(\mathcal{E}_7, \mathcal{D}_5)$, $(\mathcal{E}_6, \mathcal{A}_4)$ belong to the one-parametric family of two-dimensional integrable CM systems introduced and studied in [37]. The system 12 belongs to the family from Proposition 7 and the system 22 belongs to the family from Theorem 10.

We note that in the table above in all the cases there are non-integer multiplicities. So all integrable systems with integer multiplicities which are restrictions of the CM systems with Coxeter root systems already appeared in [11]. Also according to the result from [39] the group generated by reflections along the hyperplanes with non-integer multiplicities is finite. This agrees with the table above.

7 Invariant parabolic ideals for the complex reflection groups

Let W be an irreducible finite complex reflection group. Let V be its reflection representation. Let \mathcal{A} be the set of reflection hyperplanes. For any hyperplane $H \in \mathcal{A} \subset V$ let m_H be the order of the stabilizer of H in the group W , and let α_H be a covector vanishing on H . Let $s_{H,i}$, $i = 1, \dots, m_H - 1$ be the set of reflections in W which fix H . We numerate these reflections so that $s_{H,i} = s_{H,1}^i$, and we suppose that $\det s_{H,1} = \xi_H = e^{2\pi i/m_H}$. Put $s_{H,0} = e \in W$. Let $a(s_{H,i}) = a_{H,i}$ be a W -invariant function on the set of reflections.

For any reflection $s_{H,i}$ we choose a pair $\alpha_{H,i} \in V^*$, $\alpha_{H,i}^\vee \in V$ such that $s_{H,i}(f) = f - f(\alpha_{H,i}^\vee)\alpha_{H,i}$ for any $f \in V^*$. Note that these pairs are not uniquely defined by reflections but the elements $\alpha_{H,i} \otimes \alpha_{H,i}^\vee \in V^* \otimes V$ are. Define the bilinear form $B : V^* \otimes V \rightarrow \mathbb{C}$ by the formula

$$B(f, v) = \sum_{H \in \mathcal{A}} \sum_{i=1}^{m_H-1} a_{H,i} \alpha_{H,i}(v) f(\alpha_{H,i}^\vee).$$

Because of W -invariance one has $B(f, v) = h_{W,a} f(v)$ for some constant $h_{W,a}$. More directly this coefficient is defined as

$$h_{W,a} = \sum_{H \in \mathcal{A}} \sum_{i=1}^{m_H-1} a_{H,i} (1 - s_{H,i}), \quad (29)$$

where the reflections $s_{H,i}$ act in V^* or, equivalently, in V .

Remark 2 *It is noted in [40] that $h_W = \sum_{H \in \mathcal{A}} \sum_{i=1}^{m_H-1} (1 - s_{H,i}) = \frac{1}{\dim V} \sum_{H \in \mathcal{A}} m_H$, so in particular h_W is a generalization of the Coxeter number for the complex reflection group W . In the case $a_{H,i} = \text{const} = a$ we have $h_{W,a} = ah_W$.*

Recall that the Dunkl operators are defined for any $\xi \in V$ as [41]

$$\nabla_\xi = \partial_\xi - \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=1}^{m_H-1} a_{H,i} (1 - s_{H,i}). \quad (30)$$

More exactly the definition of the Dunkl operator in [41] is

$$\nabla_\xi = \partial_\xi - \sum_{H \in \mathcal{A}} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{t=1}^{m_H-1} b_{H,t} \sum_{i=0}^{m_H-1} \xi_H^{it} s_{H,i}, \quad (31)$$

and the formulas (30), (31) coincide if the parameters are related by

$$a_{H,s} = - \sum_{i=1}^{m_H-1} b_{H,i} \xi^{si}, \quad s = 1, \dots, m_H - 1$$

for any $H \in \mathcal{A}$.

Let now W_0 be a parabolic subgroup of W that is W_0 is the stabilizer of an intersection L of the reflection hyperplanes. The corresponding *parabolic stratum* is the orbit

$$D_{W_0} = \bigcup_{w \in W} w(L).$$

The associated *parabolic ideal* I_{W_0} is defined as the set of polynomials vanishing on the stratum, that is $I_{W_0} = \{p \in \mathbb{C}[x] \mid p|_{D_{W_0}} = 0\}$.

Remark 3 *In the case of a real reflection group (Section 2) the ideal I_{Γ_0} coincides with the ideal I_{W_0} where W_0 is the parabolic subgroup generated by simple reflections corresponding to the vertices of Γ_0 .*

We are going to determine the parabolic strata D_{W_0} such that ideals I_{W_0} are invariant under the rational Cherednik algebra associated to W . Equivalently, the ideals are invariant under the Dunkl operators (30).

Let $W_0 = W_1 \times \dots \times W_k$ be the decomposition of W_0 into the irreducible parabolic subgroups so that $V = \bigoplus_{i=1}^k V_i \oplus L$ where each V_i ($1 \leq i \leq k$) is the reflection representation for W_i and W_i acts trivially in V_j , $j \neq i$ and in L .

Theorem 11 *The parabolic ideal I_{W_0} is invariant under the Dunkl operators (30) for any $\xi \in V$ if and only if $h_{W_i, a} = 1$ for any $i = 1, \dots, k$.*

The proof is similar to the proof of Theorem 1. One can check that invariance of the ideal under ∇_ξ is equivalent to the property that for any $\gamma \in V^*$ such that $\gamma|_L = 0$ one has

$$\gamma(\xi) = \sum_{H \in \mathcal{B}} \sum_{i=1}^{m_H-1} a_{H,i} \alpha_{H,i}(\xi) \gamma(\alpha_{H,i}^\vee),$$

where $\mathcal{B} \subset \mathcal{A}$ is the collection of the reflection hyperplanes containing the subspace L . Then Theorem 11 follows.

Remark 4 *Similar to the real case (Theorem 4) any $H_c(W)$ -invariant radical ideal corresponds to the union of parabolic strata so that the parabolic subgroups W_i defining each stratum satisfy the property $h_{W_i, a} = 1$.*

Consider now the complex reflection group $W = G(m, p, N)$ and its natural action in \mathbb{C}^N . Recall that the group $G(m, p, N)$ defined when $p|m$ is generated by the elements s_{ij}^k for $1 \leq i < j \leq N$, $k = 0, \dots, m-1$ and the elements τ_i for $i = 1, \dots, N$. The element τ_i acts on the basis vectors as $\tau_i(e_i) = \eta^{-1}e_i$, where $\eta = e^{2\pi i p/m}$ and $\tau_i(e_j) = e_j$ for $j \neq i$. The elements s_{ij}^k defined for $i \neq j$ act as $s_{ij}^k(e_j) = \xi^k e_i$, $s_{ij}^k(e_i) = \xi^{-k} e_j$, where $\xi = e^{2\pi i/m}$, and $s_{ij}^k(e_s) = e_s$ for $s \neq i, j$. The complex reflections s_{ij}^k are reflections of order 2 at the hyperplanes $x_i - \xi^k x_j = 0$. The complex reflections τ_i are reflections of order m/p at the hyperplanes $x_i = 0$. We are going to specify invariant parabolic ideals explicitly.

The Dunkl operators for the complex reflection group $G(m, p, N)$ depend on m/p complex parameters $c_0, \dots, c_{\frac{m}{p}-1}$ and have the form [41]

$$\nabla_i = \partial_i - c_0 \sum_{\substack{j=1 \\ j \neq i}}^N \sum_{k=0}^{m-1} \frac{1 - s_{ij}^k}{x_i - \xi^k x_j} - \sum_{t=1}^{\frac{m}{p}-1} c_t \sum_{s=0}^{\frac{m}{p}-1} \frac{\eta^{-st} \tau_i^s}{x_i}, \quad (32)$$

$i = 1, \dots, N$. The commutativity $[\nabla_i, \nabla_j] = 0$ holds.

The parabolic strata are the $G(m, p, N)$ -orbits of the intersection of the reflection hyperplanes $x_i - \xi^k x_j = 0$, $x_s = 0$. Consider the plane $\pi_{q,r}^\varepsilon \subset \mathbb{C}^N$

given by the equations

$$\begin{aligned} x_1 = x_2 = \dots = x_r, \quad x_{r+1} = x_{r+2} = \dots = x_{2r}, \\ \dots \\ \varepsilon x_{(q-1)r+1} = x_{(q-1)r+2} = \dots = x_{qr}, \end{aligned} \quad (33)$$

where $\varepsilon^m = 1$. We may assume that $\varepsilon = 1$ unless $qr = N$. The corresponding parabolic stratum $D_{q,r}^\varepsilon \subset \mathbb{C}^N$ is the orbit

$$D_{q,r}^\varepsilon = \bigcup_{w \in G(m,p,N)} w(\pi_{q,r}^\varepsilon). \quad (34)$$

Proposition 8 *The parabolic ideal*

$$I_{q,r}^\varepsilon = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_{q,r}^\varepsilon} = 0\}$$

corresponding to the stratum (33), (34) is invariant under the $G(m, p, N)$ Dunkl operators (32) if and only if $c_0 = 1/r$.

Proof. The stratum (34) corresponds to the parabolic subgroup of type $W_0 \cong A_{r-1}^q$. The Coxeter number of any irreducible component of W_0 equals r . The statement follows from Theorem 11.

Consider now the plane $\pi_l \subset \mathbb{C}^N$ given by the equations $x_1 = x_2 = \dots = x_l = 0$. Consider the corresponding parabolic stratum D_l which is the orbit of the plane π_l under the group $G(m, p, N)$:

$$D_l = \bigcup_{i_1 < \dots < i_l} \{x|x_{i_1} = \dots = x_{i_l} = 0\}. \quad (35)$$

Define the ideal $I_l = \{f \in \mathbb{C}[x_1, \dots, x_N] \mid f|_{D_l} = 0\}$.

Proposition 9 *The Dunkl operators (32) for the group $G(m, m, N)$ preserve the ideal I_l if and only if $c_0 = \frac{1}{m(l-1)}$. The Dunkl operators (32) for the group $G(m, p, N)$ with $p < m$ preserve the ideal I_l if and only if $(l-1)c_0 + c_1 p^{-1} = m^{-1}$.*

Proof. Let $f \in I_l$. We analyze first the condition $\nabla_1 f \in I_l$. Since polynomial f vanishes on the plane $\pi_l : x_1 = x_2 = \dots = x_l = 0$ we can represent f as

$f = \sum_{i=1}^l x_i f_i$ for some polynomials f_i . In order to compute $\nabla_1 f$ we note at first that for $2 \leq j \leq l$

$$\left(\frac{1 - s_{1j}^k}{x_1 - \xi^k x_j} \sum_{i=1}^l x_i f_i \right) |_{\pi_l} = f_1 |_{\pi_l} - \xi^{-k} f_j |_{\pi_l},$$

and that $\sum_{k=0}^{m-1} \xi^{-k} = 0$, $\sum_{s=0}^{\frac{m}{p}-1} \eta^{-st} = 0$ for $\frac{m}{p} - 1 \geq t \geq 1$. Then it follows that

$$\nabla_1 f |_{\pi_l} = (1 - c_0 m(l-1) - c_1 \frac{m}{p}) f_1 |_{\pi_l},$$

where we assume that $c_1 = 0$ for the group $G(m, m, N)$. Therefore $\nabla_1 f |_{\pi_l} = 0$ iff $m^{-1} = c_0(l-1) + c_1 p^{-1}$. The property that $\nabla_1 f$ vanishes on other subspaces π from D_l is either satisfied for all values of parameters (when $e_1 \in \pi$) or is satisfied under the same relation among the parameters as for the plane π_l (when the vector e_1 is orthogonal to the subspace π). This shows that invariance of the ideal I_l under the Dunkl operator ∇_1 , as well as under all operators ∇_i , is equivalent to the condition $m^{-1} = c_0(l-1) + c_1 p^{-1}$ as stated.

Remark 5 *It follows from Proposition 9 and Theorem 11 that in the case $W = G(m, p, N)$ the generalized Coxeter number (29) can be rearranged as*

$$h_{W,a} = m(N-1)c_0 + \frac{m}{p}c_1,$$

where $c_0 = a(s_{ij}^k)$ and $c_1 = \frac{p}{m} \sum_{k=1}^{\frac{m}{p}-1} a(\tau_i^k)(1 - \eta^{-k})$, and we assume $c_1 = 0$ in the case $m = p$.

Propositions 8, 9 and their proofs imply the following

Theorem 12 *All possible parabolic strata defining parabolic ideals invariant under the $G(m, p, N)$ Dunkl operators (32) are either the strata $D_{q,r}^\varepsilon$ defined by (17) with $c_0 = 1/r$ or strata D_l defined by (19) with parameters values specified in Proposition 9 or the strata $D_{q,r,l}$. The latter stratum is the $G(m, p, N)$ -orbit of the subspace*

$$\begin{aligned} x_1 = x_2 = \dots = x_r, \quad x_{r+1} = x_{r+2} = \dots = x_{2r}, \\ \dots \\ x_{(q-1)r+1} = x_{(q-1)r+2} = \dots = x_{qr}, \\ x_{qr+1} = \dots = x_{qr+l} = 0, \end{aligned} \quad (36)$$

where $r > 1$, $l \geq 1$, $qr + l \leq N$. The corresponding parameters satisfy $c_0 = 1/r$, $(l - 1)c_0 + \frac{c_1}{p} = 1/m$, where it is assumed that $c_1 = 0$ in the case of the group $G(m, m, N)$.

To conclude the consideration of the invariant parabolic ideals for the complex reflection group $G(m, p, N)$ we note that when $N = 2$ and p is even the roots $e_1 - \xi^s e_2$, $s = 0, \dots, m - 1$ form two $G(m, p, N)$ -orbits. This adds extra parameter \tilde{c}_0 to the associated Dunkl operators comparing to the general case (32). Namely the operators take the form

$$\begin{aligned} \nabla_1 &= \partial_1 - c_0 \sum_{k=0}^{\frac{m}{2}-1} \frac{1 - s_{12}^{2k}}{x_1 - \xi^{2k} x_2} - \tilde{c}_0 \sum_{k=0}^{\frac{m}{2}-1} \frac{1 - s_{12}^{2k+1}}{x_1 - \xi^{2k+1} x_2} - \sum_{t=1}^{\frac{m}{p}-1} c_t \sum_{s=0}^{\frac{m}{p}-1} \frac{\eta^{-st} \tau_1^s}{x_1}, \\ \nabla_2 &= \partial_2 - c_0 \sum_{k=0}^{\frac{m}{2}-1} \frac{1 - s_{12}^{2k}}{x_2 - \xi^{2k} x_1} - \tilde{c}_0 \sum_{k=0}^{\frac{m}{2}-1} \frac{1 - s_{12}^{2k+1}}{x_2 - \xi^{2k+1} x_1} - \sum_{t=1}^{\frac{m}{p}-1} c_t \sum_{s=0}^{\frac{m}{p}-1} \frac{\eta^{-st} \tau_1^s}{x_2}. \end{aligned} \quad (37)$$

Then we have the following four parabolic ideals

$$I_1 = \{f \in \mathbb{C}[x_1, x_2] \mid f = 0 \text{ if } x_1 = \xi^{2k} x_2 \quad \forall k = 0, \dots, \frac{m}{2} - 1\},$$

$$I_2 = \{f \in \mathbb{C}[x_1, x_2] \mid f = 0 \text{ if } x_1 = \xi^{2k+1} x_2 \quad \forall k = 0, \dots, \frac{m}{2} - 1\},$$

$$I_3 = \{f \in \mathbb{C}[x_1, x_2] \mid f = 0 \text{ if } x_1 x_2 = 0\}, \quad I_4 = \{f \in \mathbb{C}[x_1, x_2] \mid f = 0 \text{ if } x_1 = x_2 = 0\}.$$

The following Proposition can be established similarly to the previous results of this Section.

Proposition 10 *The parabolic ideal I_1 is invariant under the Dunkl operators (37) iff $c_0 = 1/2$, the parabolic ideal I_2 is invariant under the Dunkl operators (37) iff $\tilde{c}_0 = 1/2$, the parabolic ideal I_3 is invariant under the Dunkl operators (37) iff $c_1 = p/m$, the parabolic ideal I_4 is invariant under the Dunkl operators (37) iff $\frac{1}{2}(c_0 + \tilde{c}_0) + c_1/p = 1/m$.*

Concluding Remarks

In the paper we were systematically deriving generalized CM systems from the special subrepresentations in the polynomial representation of the rational Cherednik algebra. The natural development is to extend this approach

to obtain integrable generalizations of the Calogero-Moser-Sutherland and Macdonald-Ruijsenaars systems starting from less degenerate Cherednik algebras as well as to obtain elliptic generalized Calogero-Moser systems. Some integrable generalizations of these systems are known from [10], [14], [42], [43], [16], [48]. We refer to the recent development in [49] where generalized Macdonald-Ruijsenaars systems were derived from the special submodules of the polynomial representations of double affine Hecke algebras.

It would also be interesting to analyze if the approach can be extended to cover matrix integrable systems of the generalized Calogero-Moser type [44]. A close direction is to investigate possible generalizations of matrix and scalar CM systems associated to special complex reflection groups (see [45] and [50] respectively).

Regarding representations of the rational Cherednik algebras, it is clear by Theorem 1 how to form the chains of submodules in the polynomial module of the rational Cherednik algebras for any Coxeter group. In contrast to the A_N case in general there may be non-isomorphic Coxeter subgroups with equal Coxeter numbers. Therefore there are non-trivial intersections of the corresponding parabolic ideals. It would be interesting to investigate if the subsequent quotients in the natural chains are irreducible like in the A_N case ([22], [46], [47]). Also in the paper we consider submodules corresponding to special singular values only, it would also be interesting to see if the submodules for other singular values can be described in a natural way (c.f. [47] and Proposition 1 above).

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