



Steiner, R. (2012) *The algebraic structure of the universal complicial sets*. Journal of Pure and Applied Algebra. ISSN 0022-4049

<http://eprints.gla.ac.uk/63718>

Deposited on: 16 May 2012

The algebraic structure of the universal complicial sets

Richard Steiner

ABSTRACT. The nerve of a strict omega-category is a simplicial set with additional structure, making it into a so-called complicial set, and strict omega-categories are in fact equivalent to complicial sets. The nerve functor is represented by a sequence of strict omega-categories, called orientals, which are associated to simplexes. In this paper we give a detailed algebraic description of the morphisms between orientals. The aim is to describe complicial sets algebraically, by operators and equational axioms.

1. Introduction

The orientals or oriented simplexes are a sequence of strict ω -categories

$$\mathcal{O}_0, \mathcal{O}_1, \dots$$

associated to simplexes. They were discovered by Street, who described them as fundamental objects in nature [1]. A strict ω -category X has a *nerve*, consisting of the sequence of morphism sets

$$\mathrm{Hom}(\mathcal{O}_0, X), \mathrm{Hom}(\mathcal{O}_1, X), \dots;$$

Verity [2] has shown that the nerve functor makes the category of strict ω -categories equivalent to a category of simplicial sets with additional structure, called *complicial sets*.

By definition, a complicial set is a simplicial set with a distinguished class of elements, called *thin elements*, subject to certain axioms; Verity's theorem therefore amounts to a description of strict ω -categories in combinatorial terms. There is an analogous cubical theory [3] which gives a more algebraic description, in terms of cubical sets with additional operations and equational axioms. This paper is part of a programme aimed at producing a similar algebraic description for complicial sets, using operations and equational axioms rather than distinguished subsets.

In this paper we consider the universal examples; in other words, we consider the nerves of the orientals themselves. This is in fact a purely algebraic problem. The category of orientals can be embedded in the category of chain complexes and chain maps [4]: the objects are the chain complexes of the standard simplexes; the morphisms are the augmentation-preserving chain maps taking standard basis elements to sums of standard basis elements. This gives a simple algebraic description of the nerves of orientals, as subsets of graded abelian groups, but we really need

2010 *Mathematics Subject Classification*. 18D05.

Key words and phrases. complicial set, oriental, omega-category.

an internal description independent of any supersets. We will therefore solve the following problem: find an algebraic structure on graded sets, consisting of internal operations and equational axioms, such that the nerve of \mathcal{O}_n is freely generated by its identity endomorphism ι_n . The equational axioms which solve this problem will be called *complicial identities*; they are listed in Definition 6.1.

The structure of the paper is as follows. In Section 2 we recall the description of orientals in terms of chain maps. In Section 3 we describe the additional operations (they were introduced with different notation and terminology in [4]). In Section 4 we show that the nerve of \mathcal{O}_n is generated by ι_n (see Theorem 4.12); this was also done in [4], but here we give more precise details and in effect obtain canonical forms for the elements of the nerve (see Theorem 4.11). In Section 5 we give some additional properties of the nerves, for later use. In Section 6 we describe the complicial identities, and we show that they are satisfied in the nerves of orientals. In Section 7 we describe certain consequences of the complicial identities, and in Section 8 we prove the main theorem (Theorem 8.7), showing that the nerve of \mathcal{O}_n is the set with complicial identities freely generated by ι_n .

2. Orientals and chain complexes

From [4] we recall the description of the category of orientals in terms of chain maps. For $n = 0, 1, \dots$ let $\mathbf{Z}\Delta(n)$ be the cellular chain complex of the standard n -simplex. We regard $\mathbf{Z}\Delta(n)$ as a free graded abelian group with a prescribed basis. The basis elements, written in the form $[a_0, \dots, a_q]$, correspond to the $(q+1)$ -tuples of integers a_0, \dots, a_q such that

$$0 \leq a_0 < a_1 < \dots < a_q \leq n;$$

a basis element $[a_0, \dots, a_q]$ is homogeneous of degree q . The boundary homomorphism $\partial: \mathbf{Z}\Delta(n) \rightarrow \mathbf{Z}\Delta(n)$, which lowers degrees by 1, is given on basis elements of positive degree by

$$\partial[a_0, \dots, a_q] = \sum_{i=0}^q (-1)^i [a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_q].$$

There is also an augmentation homomorphism $\epsilon: \mathbf{Z}\Delta(n) \rightarrow \mathbf{Z}$, which is given on basis elements of degree 0 by

$$\epsilon[a_0] = 1,$$

and which vanishes on basis elements of positive degree.

We will write $\mathbf{Z}\Delta(m, n)$ for the abelian group consisting of the chain maps from $\mathbf{Z}\Delta(m)$ to $\mathbf{Z}\Delta(n)$; thus a member of $\mathbf{Z}\Delta(m, n)$ is a degree-preserving abelian group homomorphism $f: \mathbf{Z}\Delta(m) \rightarrow \mathbf{Z}\Delta(n)$ such that $\partial f = f\partial$. Ordinary function composition makes the groups $\mathbf{Z}\Delta(m, n)$ into the morphism sets of a category $\mathbf{Z}\Delta$ with objects $0, 1, 2, \dots$. For each fixed target n , it is convenient to regard the sequence

$$\mathbf{Z}\Delta(0, n), \mathbf{Z}\Delta(1, n), \dots$$

as a graded abelian group $\mathbf{Z}\Delta(-, n)$.

We will also write $\mathcal{O}(m, n)$ for the subset of $\mathbf{Z}\Delta(m, n)$ consisting of the chain maps f which are augmentation-preserving ($\epsilon f = \epsilon$) and which take basis elements to sums of basis elements (if a is a basis element then $f(a) = b_1 + \dots + b_k$ for some $k \geq 0$ and for some basis elements b_1, \dots, b_k). As before, we get a category \mathcal{O} with

morphism sets $\mathcal{O}(m, n)$, and we get graded sets $\mathcal{O}(-, n)$. It follows from [4] that there are natural bijections

$$\mathcal{O}(m, n) \cong \text{Hom}(\mathcal{O}_m, \mathcal{O}_n),$$

where $\text{Hom}(\mathcal{O}_m, \mathcal{O}_n)$ is the set of strict ω -category morphisms from the oriental \mathcal{O}_m to the oriental \mathcal{O}_n ; the category \mathcal{O} will therefore be called the *category of orientals*. Because of the bijections, the nerves of the orientals \mathcal{O}_n can be identified with the graded sets $\mathcal{O}(-, n)$. We will describe the nerves by using these identifications.

3. Operations in the nerves of orientals

In this section we construct three families of operations in $\mathcal{O}(-, n)$: *face operations*, *degeneracy operations* and *wedge operations*. They will be restrictions of operations in $\mathbf{Z}\Delta(-, n)$. The face and degeneracy operations come from the standard simplicial set structure on $\mathbf{Z}\Delta(-, n)$, and we begin by recalling the definition of a simplicial set.

DEFINITION 3.1. A *simplicial set* X is a sequence of sets X_0, X_1, \dots together with face operations

$$\partial_i: X_m \rightarrow X_{m-1} \quad (m > 0, 0 \leq i \leq m)$$

and degeneracy operations

$$\epsilon_i: X_m \rightarrow X_{m+1} \quad (0 \leq i \leq m)$$

such that

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad (i < j), \\ \partial_i \epsilon_j &= \epsilon_{j-1} \partial_i \quad (i < j), \\ \partial_i \epsilon_i &= \partial_{i+1} \epsilon_i = \text{id}, \\ \partial_i \epsilon_j &= \epsilon_j \partial_{i-1} \quad (i > j + 1), \\ \epsilon_i \epsilon_j &= \epsilon_{j+1} \epsilon_i \quad (i \leq j). \end{aligned}$$

It is well known that simplicial set operations $X_m \rightarrow X_p$ correspond to non-decreasing functions

$$\{0, 1, \dots, p\} \rightarrow \{0, 1, \dots, m\}$$

between sets of integers; a face operation $\partial_i: X_m \rightarrow X_{m-1}$ corresponds to the function ∂_i^\vee with

$$\partial_i^\vee(0) = 0, \dots, \partial_i^\vee(i-1) = i-1, \partial_i^\vee(i) = i+1, \dots, \partial_i^\vee(m-1) = m,$$

and a degeneracy $\epsilon_i: X_m \rightarrow X_{m+1}$ corresponds to the function ϵ_i^\vee with

$$\epsilon_i^\vee(0) = 0, \dots, \epsilon_i^\vee(i) = \epsilon_i^\vee(i+1) = i, \dots, \epsilon_i^\vee(m+1) = m.$$

We will now describe the corresponding operations in $\mathbf{Z}\Delta(-, n)$ in terms of the basis elements $[a_0, \dots, a_q]$ for the chain groups $\mathbf{Z}\Delta(p)$. For these basis elements we will use notations such as $[\mathbf{b}, \mathbf{c}]$ or $[\mathbf{b}, i, \mathbf{c}]$, where \mathbf{b} and \mathbf{c} are suitable sequences of integers. The outcome is as follows.

DEFINITION 3.2. Let x be a chain map in $\mathbf{Z}\Delta(m, n)$.

For $m > 0$ and $0 \leq i \leq m$, the *face* $\partial_i x$ is the chain map in $\mathbf{Z}\Delta(m-1, n)$ given on basis elements by

$$(\partial_i x)[\mathbf{b}, \mathbf{c}] = x[\mathbf{b}, \mathbf{c}'],$$

where the terms of \mathbf{b} are less than i , the terms of \mathbf{c} are greater than or equal to i , and the terms of \mathbf{c}' are obtained from those of \mathbf{c} by adding 1.

For $0 \leq i \leq m$ the *degeneracy* $\epsilon_i x$ is the chain map in $\mathbf{Z}\Delta(m+1, n)$ given on basis elements by

$$\begin{aligned} (\epsilon_i x)[\mathbf{b}, \mathbf{c}] &= x[\mathbf{b}, \mathbf{c}''], \\ (\epsilon_i x)[\mathbf{b}, i, \mathbf{c}] &= (\epsilon_i x)[\mathbf{b}, i+1, \mathbf{c}] = x[\mathbf{b}, i, \mathbf{c}''], \\ (\epsilon_i x)[\mathbf{b}, i, i+1, \mathbf{c}] &= 0, \end{aligned}$$

where the terms of \mathbf{b} are less than i , the terms of \mathbf{c} are greater than $i+1$, and the terms of \mathbf{c}'' are obtained from those of \mathbf{c} by subtracting 1.

We get the following result.

PROPOSITION 3.3. *The face and degeneracy operations*

$$\partial_i: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m-1, n), \quad \epsilon_i: \mathbf{Z}\Delta(m, n) \rightarrow \mathbf{Z}\Delta(m+1, n)$$

are group homomorphisms making $\mathbf{Z}\Delta(-, n)$ into a simplicial set. They restrict to operations

$$\partial_i: \mathcal{O}(m, n) \rightarrow \mathcal{O}(m-1, n), \quad \epsilon_i: \mathcal{O}(m, n) \rightarrow \mathcal{O}(m+1, n)$$

making $\mathcal{O}(-, n)$ into a simplicial set.

PROOF. It is clear that the operations in $\mathbf{Z}\Delta(-, n)$ are homomorphisms satisfying the simplicial identities. If $x \in \mathcal{O}(-, n)$, so that x is augmentation-preserving and takes basis elements to sums of basis elements, then $\partial_i x$ and $\epsilon_i x$ clearly belong to $\mathcal{O}(-, n)$ as well. \square

Degeneracies can be characterised as follows.

PROPOSITION 3.4. *Let x be a morphism in $\mathcal{O}(-, n)$. Then x is in the image of ϵ_i if and only if $xa = 0$ for every basis element a including i and $i+1$.*

PROOF. By definition, if x is in the image of ϵ_i then x vanishes on every basis element including i and $i+1$.

Conversely, suppose that x vanishes on every basis element including i and $i+1$. Since x is a chain map,

$$x[\mathbf{b}, i, \mathbf{c}] = x[\mathbf{b}, i+1, \mathbf{c}]$$

for all basis elements of the form $[\mathbf{b}, i, i+1, \mathbf{c}]$, and it follows that $x = \epsilon_i \partial_i x$. \square

Next we define the wedge operations.

DEFINITION 3.5. Let m and i be integers with $0 \leq i \leq m-1$. If x and y are chain maps in $\mathbf{Z}\Delta(m, n)$ such that $\partial_i x = \partial_{i+1} y$, then the *wedge* $x \wedge_i y$ is the chain map in $\mathbf{Z}\Delta(m+1, n)$ given by

$$x \wedge_i y = \epsilon_{i+1} x - \epsilon_i^2 \partial_{i+1} y + \epsilon_i y.$$

One can think of the partial binary operation

$$(x, y) \mapsto \partial_{i+1}(x \wedge_i y)$$

as a pasting composition, gluing two m -simplexes along a common $(m-1)$ -dimensional face to form a larger m -simplex. The wedge $x \wedge_i y$ itself is a higher-dimensional structure containing this composite. It also contains the factors, because of the following obvious result.

PROPOSITION 3.6. *If $x \wedge_i y$ is defined in $\mathbf{Z}\Delta(-, n)$ then*

$$\partial_i(x \wedge_i y) = y, \quad \partial_{i+2}(x \wedge_i y) = x.$$

We also get the following results.

PROPOSITION 3.7. *Let x and y be morphisms in $\mathcal{O}(m, n)$ such that $\partial_i x = \partial_{i+1} y$. Then $x \wedge_i y$ is a morphism in $\mathcal{O}(m+1, n)$.*

PROOF. Given that x and y are augmentation-preserving and that they take basis elements to sums of basis elements, we must show that $x \wedge_i y$ has the same properties. This is straightforward. \square

PROPOSITION 3.8. *Let z be a morphism in $\mathcal{O}(m+1, n)$, and let i be an integer with $0 \leq i \leq m-1$. Then the following are equivalent.*

- (1) *There are morphisms x, y in $\mathcal{O}(m, n)$ with $\partial_i x = \partial_{i+1} y$ such that $z = x \wedge_i y$.*
- (2) *There are chain maps u, v in $\mathbf{Z}\Delta(m, n)$ such that $z = \epsilon_i u + \epsilon_{i+1} v$.*
- (3) *One has $za = 0$ for every basis element a including $i, i+1, i+2$.*

PROOF. We show that (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1).

If $z = x \wedge_i y$ then z has the form $\epsilon_i u + \epsilon_{i+1} v$ by definition.

If $z = \epsilon_i u + \epsilon_{i+1} v$ then clearly $za = 0$ for every basis element a including $i, i+1, i+2$.

Suppose that $za = 0$ for every basis element a including $i, i+1, i+2$. Let

$$x = \partial_{i+2} z, \quad y = \partial_i z.$$

It follows from Proposition 3.3 that x and y are morphisms in $\mathcal{O}(m, n)$ and that $\partial_i x = \partial_{i+1} y$; the wedge $x \wedge_i y$ therefore exists. The morphisms z and $x \wedge_i y$ then agree on basis elements not including i , on basis elements not including $i+2$, and on basis elements including $i, i+1, i+2$. Since z and $x \wedge_i y$ are chain maps, they must also agree on basis elements including i and $i+2$ but not $i+1$. Therefore $z = x \wedge_i y$.

This completes the proof. \square

4. Canonical forms

Let ι_n be the identity morphism in $\mathcal{O}(n, n)$. In this section we show that the elements of $\mathcal{O}(-, n)$ can be expressed in terms of ι_n by using the face, degeneracy and wedge operations. In effect we find canonical forms for the morphisms in $\mathcal{O}(-, n)$ (see Theorem 4.11). The argument is based on the following result.

THEOREM 4.1. *Let x be a morphism in $\mathcal{O}(m, n)$. If $[i]$ is a zero-dimensional basis element in $\mathbf{Z}\Delta(m)$, then $x[i]$ is a zero-dimensional basis element $[x(i)]$ in $\mathbf{Z}\Delta(n)$. The integers $x(0), \dots, x(m)$ are such that*

$$0 \leq x(0) \leq x(1) \leq \dots \leq x(m) \leq n.$$

If $a = [a_0, \dots, a_q]$ is a basis element in $\mathbf{Z}\Delta(m)$, then xa is a sum of basis elements $[b_0, \dots, b_q]$ with

$$x(a_0) \leq b_0 < b_1 < \dots < b_q \leq x(a_q).$$

PROOF. Let $[i]$ be a zero-dimensional basis element in $\mathbf{Z}\Delta(m)$. Then $x[i]$ is a sum of zero-dimensional basis elements in $\mathbf{Z}\Delta(n)$, and this sum has exactly one term because x is augmentation-preserving. Therefore $x[i] = [x(i)]$ for some integer $x(i)$ with $0 \leq x(i) \leq n$.

For $0 < i \leq m$ we have

$$\partial x[i-1, i] = x\partial[i-1, i] = x([i] - [i-1]) = [x(i)] - [x(i-1)].$$

But $x[i-1, i]$ is a sum of basis elements $[j, k]$, so $\partial x[i-1, i]$ is a sum of expressions $[k] - [j]$ with $j < k$. Therefore $x(i-1) \leq x(i)$.

Now let $a = [a_0, \dots, a_q]$ be a basis element for $\mathbf{Z}\Delta(m)$. We will show by induction on q that xa is a sum of basis elements $[b_0, \dots, b_q]$ with $x(a_0) \leq b_0$. The result is clear when $q = 0$, and is trivial when $xa = 0$. From now on, suppose that $q > 0$ and $xa \neq 0$, so that xa is a non-empty sum of basis elements of positive dimension. Let $[k_0, \dots, k_q]$ be a term in this sum such that k_0 is as small as possible and, subject to this condition, such that k_1 is as large as possible. Then the basis element $[k_0, k_2, \dots, k_q]$ has a negative coefficient in ∂xa because there is no possibility of cancellation. Since $\partial xa = x\partial a$, it follows that $[k_0, k_2, \dots, k_q]$ is a term in $x[a_0, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_q]$ for some odd value of i , and it then follows from the inductive hypothesis that $x(a_0) \leq k_0$. Since k_0 is minimal, xa is a sum of basis elements $[b_0, \dots, b_q]$ with $x(a_0) \leq b_0$.

A similar argument shows that xa is a sum of basis elements $[b_0, \dots, b_q]$ with $b_q \leq x(a_q)$.

This completes the proof. \square

Let x be a morphism in $\mathcal{O}(m, n)$, let $x[m] = [t]$, and let r be minimal such that $x[r] = [t]$. The images of the zero-dimensional basis elements $[0], \dots, [m]$ in $\mathbf{Z}\Delta(m)$ can then be listed in the form

$$[x(0)], \dots, [x(r-1)], [t], \dots, [t],$$

with $0 \leq r \leq m$ and with $x(0) \leq \dots \leq x(r-1) < t$. The canonical form for x depends on the parameters t , r and $m-r$, for which we will use the following terminology.

DEFINITION 4.2. Let x be a morphism in $\mathcal{O}(m, n)$. Then the *terminus* of x , denoted *terminus* x , is the integer t such that $x[m] = [t]$; the *rank* of x , denoted *rank* x , is the number of zero-dimensional basis elements $[a]$ in $\mathbf{Z}\Delta(m)$ such that $x[a] \neq x[m]$; the *corank* of x , denoted *corank* x , is the number of zero-dimensional basis elements $[a]$ in $\mathbf{Z}\Delta(m)$ such that $a < m$ and $x[a] = x[m]$.

Thus the terminus of a morphism is a measure of the size of its image, the rank is a measure of its non-degeneracy, and the corank is a measure of its degeneracy.

Using Theorem 4.1, we immediately draw the following conclusions.

PROPOSITION 4.3. *Let x be a morphism in $\mathcal{O}(m, n)$ with terminus t , rank r and corank s . Then t , r and s are nonnegative integers such that $t \leq n$ and $r + s = m$.*

We will find the canonical forms by an induction on terminus; to be more precise, for $0 \leq t \leq n$ we will express the morphisms in $\mathcal{O}(-, n)$ with terminus at most t in terms of the morphism $\partial_{t+1}^{n-t} \iota_n$. The set of morphisms of this kind is obviously isomorphic to $\mathcal{O}(-, t)$, with $\partial_{t+1}^{n-t} \iota_n$ corresponding to ι_t , so the induction is equivalent to an induction on n , but it is easier to work inside the single graded set $\mathcal{O}(-, n)$.

The passage to morphisms of terminus t from morphisms of lower terminus is based on a cone construction ρ_t : if x is a morphism in $\mathcal{O}(m, n)$ with $x[m] = [t']$ such that $t' < t$, then $\rho_t x$ is the obvious extension of x to a morphism in $\mathcal{O}(m+1, n)$ with $(\rho_t x)[m+1] = [t]$. The precise definition is as follows.

DEFINITION 4.4. Let x be a morphism in $\mathcal{O}(m, n)$ with terminus less than t . Then the t -cone on x is the morphism $\rho_t x$ in $\mathcal{O}(m+1, n)$ given on basis elements as follows: $(\rho_t x)[m+1] = [t]$; if

$$x[\mathbf{a}] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}]$$

then

$$(\rho_t x)[\mathbf{a}] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}], \quad (\rho_t x)[\mathbf{a}, m+1] = \sum_{\mathbf{b}} x_{\mathbf{ab}}[\mathbf{b}, t].$$

It is easy to see that $\rho_t x$ as defined here really is a morphism in $\mathcal{O}(m+1, n)$, with terminus t , rank m and corank zero. It is also easy to verify the following result.

PROPOSITION 4.5. For $t > 0$, let S_{t-1} be the subset of $\mathcal{O}(-, n)$ consisting of the morphisms with terminus less than t . Then S_{t-1} is closed under the face, degeneracy and wedge operations. The t -cone construction

$$\rho_t: S_{t-1} \rightarrow \mathcal{O}(-, n)$$

commutes with the operations in S_{t-1} , and

$$\rho_t \partial_t^{n-t+1} \iota_n = \partial_{t+1}^{n-t} \iota_n.$$

The canonical form for a morphism x of rank r will be expressed in terms of simpler morphisms

$$\alpha_{r-1} x, \dots, \alpha_0 x, \gamma x.$$

We will now describe these simpler morphisms, beginning with γx .

Let x be a morphism in $\mathcal{O}(m, n)$ with terminus t and rank r ; then γx is the morphism in $\mathcal{O}(r, n)$ constructed as follows. For each basis element of the form $[\mathbf{a}, r]$ with $[\mathbf{a}]$ a basis element in $\mathbf{Z}\Delta(r-1)$, we can write $x[\mathbf{a}, r]$ in the form

$$x[\mathbf{a}, r] = \sum_{\mathbf{b}} (x'_{\mathbf{ab}}[\mathbf{b}] + x''_{\mathbf{ab}}[\mathbf{b}, t])$$

with the sum running over basis elements $[\mathbf{b}]$ for $\mathbf{Z}\Delta(t-1)$; then

$$\gamma x: \mathbf{Z}\Delta(r) \rightarrow \mathbf{Z}\Delta(n)$$

is to be the group homomorphism given on basis elements by $(\gamma x)[r] = [t]$ and by

$$(\gamma x)[\mathbf{a}] = \sum_{\mathbf{b}} x''_{\mathbf{ab}}[\mathbf{b}], \quad (\gamma x)[\mathbf{a}, r] = \sum_{\mathbf{b}} x''_{\mathbf{ab}}[\mathbf{b}, t].$$

Essentially, γx is obtained by selecting the terms of the form $[\mathbf{b}, t]$ in the images $x[\mathbf{a}, r]$.

We find that γx has the following properties.

PROPOSITION 4.6. Let x be a morphism in $\mathcal{O}(-, n)$ with terminus t and rank r . Then γx is a morphism in $\mathcal{O}(r, n)$ with terminus t , rank r and corank zero. If $r = 0$ then $\gamma x = \partial_0^t \partial_{t+1}^{n-t} \iota_n$; if $r > 0$ then γx is a t -cone.

PROOF. If $a = [a_0, \dots, a_q]$ is a basis element for $\mathbf{Z}\Delta(m)$ with $a_q < r$, then it follows from Theorem 4.1 that xa is a sum of basis elements $[b_0, \dots, b_q]$ with $b_q < t$. From this one can show that γx is a chain map. The rest follows easily. \square

Now let x be a morphism in $\mathcal{O}(r+s, n)$ with terminus t , rank r and corank s , so that $\gamma x \in \mathcal{O}(r, n)$. It is convenient to define a morphism $\beta x \in \mathcal{O}(r+s, n)$ by the formula

$$\beta x = \epsilon_r^s \gamma x,$$

so that βx has the same domain as x and accounts for the same terms as γx . For $0 \leq p < r$ we then define chain maps

$$\alpha_p x: \mathbf{Z}\Delta(p+s+1) \rightarrow \mathbf{Z}\Delta(n)$$

by the formula

$$\alpha_p x = \partial_{p+1}^{r-p-1}(x - \beta x) + \epsilon_p \partial_{p+1}^{r-p} \beta x.$$

Essentially, $\alpha_p x$ accounts for the terms $[b_0, \dots, b_q]$ with $b_q < t$ in the images $x[a_0, \dots, a_{q-1}, r]$ for $a_{q-1} \leq p$, but in practice it is easier to work with $\alpha_p x$ by using algebraic formulae. There is one particularly simple case, as follows.

EXAMPLE 4.7. Let x be a morphism in $\mathcal{O}(r, n)$ which is equal to $\partial_0^t \partial_{t+1}^{n-t} \iota_n$ or is a t -cone. Then rank $x = r$, corank $x = 0$, $\beta x = \gamma x = x$, and

$$\alpha_p x = \epsilon_p \partial_{p+1}^{r-p} x \quad (0 \leq p < r).$$

In general we get the following result.

PROPOSITION 4.8. *Let x be a morphism in $\mathcal{O}(-, n)$ with terminus t , rank r and corank s . If $s = 0$ and $0 \leq p < r$ then $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1, n)$ with terminus less than t . If $s > 0$ and $0 \leq p < r$ then $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1, n)$ with terminus t , rank $p+2$ and corank $s-1$.*

PROOF. It is clear that $\alpha_p x$ is an augmentation-preserving chain map from $\mathbf{Z}\Delta(p+s+1)$ to $\mathbf{Z}\Delta(n)$. To show that $\alpha_p x$ is a morphism in \mathcal{O} , we must show that $\alpha_p x$ takes each basis element a to a sum of basis elements.

Suppose that a does not have a term $p+1$. Then $(\alpha_p x)a$ is a sum of basis elements because $\partial_{p+1} \alpha_p x = \partial_{p+1}^{r-p} x$ and because x is a morphism in \mathcal{O} .

Suppose that a has a term $p+1$ and a term greater than $p+1$. Then $(\partial_{p+1}^{r-p-1} \beta x)a = 0$, because βx has the form $\epsilon_r^s \gamma x$, so that

$$(\alpha_p x)a = (\partial_{p+1}^{r-p-1} x)a + (\epsilon_p \partial_{p+1}^{r-p} \beta x)a.$$

It now follows that $(\alpha_p x)a$ is a sum of basis elements because x and βx are morphisms in \mathcal{O} .

Suppose that $a = [\mathbf{a}, p+1]$. Then $(\partial_{p+1}^{r-p-1} \beta x)a$ is the sum of the terms of the form $[\mathbf{b}, t]$ in $(\partial_{p+1}^{r-p-1} x)a$, so $\{\partial_{p+1}^{r-p-1}(x - \beta x)\}a$ is the sum of the remaining terms in $(\partial_{p+1}^{r-p-1} x)a$, and it again follows that $(\alpha_p x)a$ is a sum of basis elements.

This shows that $(\alpha_p x)a$ is a sum of basis elements in all cases; therefore $\alpha_p x$ is a morphism in $\mathcal{O}(p+s+1, n)$.

We will now consider the terminus and corank of $\alpha_p x$. For $p+2 \leq i \leq p+s+1$ it follows from the calculations above and from Proposition 4.3 that

$$(\alpha_p x)[i] = (\partial_{p+1}^{r-p-1} x)[i] = x[r+i-p-1] = [t];$$

on the other hand,

$$(\alpha_p x)[p+1] = (x - \beta x)[r] + (\beta x)[p] = [t] - [t] + (\beta x)[p] = (\gamma x)[p] \neq [t].$$

If $s = 0$ it now follows that $\alpha_p x$ has terminus less than t ; if $s > 0$ it follows that $\alpha_p x$ has terminus t , rank $p+2$ and corank $s-1$.

This completes the proof. \square

We will express a morphism x in terms of the morphisms $\alpha_p x$ and γx by using the following notation.

NOTATION 4.9. In a set with operations ∂_i and \wedge_i we write

$$u(\wedge_k v) = u \wedge_k v,$$

regarding $(\wedge_k v)$ as an operator which acts on the right, and for $l \geq 1$ we write

$$u \wedge_k^l v = u(\wedge_k \partial_{k+1}^{l-1} v)(\wedge_k \partial_{k+1}^{l-2} v) \dots (\wedge_k \partial_{k+1} v)(\wedge_k v).$$

We also write

$$(u \wedge_k^l) v = u \wedge_k^l v,$$

regarding $(u \wedge_k^l)$ as an operator which acts on the left, and for $r \geq 0$ we write

$$\Lambda^r(u_{r-1}, \dots, u_0, v) = \partial_r(u_{r-1} \wedge_{r-1}^1) \partial_{r-1}(u_{r-2} \wedge_{r-2}^2) \dots \partial_1(u_0 \wedge_0^r) v.$$

Note in particular that $\Lambda^0(v) = v$.

In $\mathcal{O}(-, n)$ an induction based on Definition 3.5 gives the following result.

PROPOSITION 4.10. *If u and v are morphisms in $\mathcal{O}(-, n)$ and if $\partial_k u = \partial_{k+1}^l v$, then $u \wedge_k^l v$ is defined and*

$$u \wedge_k^l v = \epsilon_{k+1}^l u - \epsilon_k^{l+1} \partial_{k+1}^l v + \epsilon_k v.$$

For an iterated wedge $z = u \wedge_k^l v$ in $\mathcal{O}(-, n)$ as in Proposition 4.10, one finds that $\partial_{k+2}^l z = u$, $\partial_k z = v$, and

$$\partial_{k+1} z = \epsilon_{k+1}^{l-1} u - \epsilon_k \partial_{k+1}^l v + v,$$

so that $\partial_{k+1}(u \wedge_k^l)$ is a pasting composite of u and v and so that $u \wedge_k^l v$ joins this composite to its factors. The operation Λ^r is an iterated pasting composite.

The desired canonical form is now as follows.

THEOREM 4.11. *Let x be a morphism in $\mathcal{O}(-, n)$, with rank r and corank s . Then*

$$x = \Lambda^r(\alpha_{r-1} x, \dots, \alpha_0 x, \epsilon_r^s \gamma x).$$

PROOF. Recall that

$$\alpha_p x = \partial_{p+1}^{r-p-1}(x - \beta x) + \epsilon_p \partial_{p+1}^{r-p} \beta x,$$

where $\beta x = \epsilon_r^s \gamma x$. For $0 \leq p \leq r$, let

$$v_p = \epsilon_p^{r-p} \partial_p^{r-p}(x - \beta x) + \beta x,$$

so that, in particular, $v_r = x$. It suffices to show that

$$v_0 = \epsilon_r^s \gamma x,$$

$$v_{p+1} = \partial_{p+1}(\alpha_p x \wedge_p^{r-p} v_p) \quad (0 \leq p < r);$$

we proceed as follows.

The morphisms $\partial_0^r x$ and $\partial_0^r \beta x$ are morphisms in $\mathcal{O}(s, n)$ such that

$$(\partial_0^r x)[i] = (\partial_0^r \beta x)[i] = [t]$$

for $0 \leq i \leq s$. By Theorem 4.1, if a is a basis element for $\mathbf{Z}\Delta(s)$ with positive dimension then

$$(\partial_0^r x)a = (\partial_0^r \beta x)a = 0;$$

therefore $\partial_0^r x = \partial_0^r \beta x$. It follows that $v_0 = \beta x = \epsilon_r^s \gamma x$.

For $0 \leq p < r$ it is straightforward to verify that $\partial_p \alpha_p x = \partial_{p+1}^{r-p} v_p$ and that

$$v_{p+1} = \partial_{p+1}(\epsilon_{p+1}^{r-p} \alpha_p x - \epsilon_p^{r-p+1} \partial_{p+1}^{r-p} v_p + \epsilon_p v_p);$$

therefore $v_{p+1} = \partial_{p+1}(\alpha_p x \wedge_p^{r-p} v_p)$.

This completes the proof. \square

We can now prove the main result.

THEOREM 4.12. *Every morphism in $\mathcal{O}(-, n)$ can be expressed in terms of ι_n by using the face, degeneracy and wedge operations.*

PROOF. Let S_t be the set of morphisms with terminus at most t . We will prove by induction on t that S_t is generated by $\partial_{t+1}^{n-t} \iota_n$; the case $t = n$ will then give the result.

Let x be a morphism in $S_t \setminus S_{t-1}$, so that x has terminus t . By Theorem 4.11, x has an expression in terms of the morphisms γx and $\alpha_p x$. By Proposition 4.6, $\gamma x = \partial_0^t \partial_{t+1}^{n-t} \iota_n$ or γx is a t -cone. By Proposition 4.8, $\alpha_p x \in S_t$, and if $\alpha_p x \in S_t \setminus S_{t-1}$ then $\alpha_p x$ has lower corank than x . By a recursion on corank, it follows that S_t has a set of generators consisting of the morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$, the t -cones, and the morphisms in S_{t-1} . It therefore suffices to show that t -cones and members of S_{t-1} can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$; equivalently, it suffices to show that $\rho_t y$ and y can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$ for every morphism y in S_{t-1} .

For $t = 0$ there is nothing to prove, because S_{-1} is empty.

Now suppose that $t > 0$. By the inductive hypothesis, y can be expressed in terms of $\partial_t^{n-t+1} \iota_n$. It now follows from Proposition 4.5 that $\rho_t y$ can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$. It also follows that y itself can be expressed in terms of $\partial_{t+1}^{n-t} \iota_n$, because $\partial_t^{n-t+1} \iota_n = \partial_t \partial_{t+1}^{n-t} \iota_n$.

This completes the proof. \square

5. Further properties of the nerves of orientals

We have shown in Theorem 4.12 that $\mathcal{O}(-, n)$ is generated by ι_n , and in Theorem 4.11 we have found canonical forms for the members of $\mathcal{O}(-, n)$. In the later sections of this paper we will find identities determining the structure of $\mathcal{O}(-, n)$ completely. In the present section we give some auxiliary results on the canonical forms, for later use.

First we describe the canonical forms for certain faces.

PROPOSITION 5.1. *Let x be a morphism in $\mathcal{O}(-, n)$ with rank r , and let $\partial_i x$ be a face with $0 \leq i < r$. Then $\partial_i x$ has rank $r - 1$ and corank s , and*

$$\begin{aligned} \gamma \partial_i x &= \partial_i \gamma x, \\ \alpha_p \partial_i x &= \alpha_p x \quad (0 \leq p < i), \\ \alpha_p \partial_i x &= \partial_i \alpha_{p+1} x \quad (i \leq p < r - 1). \end{aligned}$$

PROOF. Let the terminus of x be t and the corank be s . It follows from Proposition 4.3 that $\partial_i x$ has terminus t , rank $r - 1$ and corank s . It is clear from the construction that $\gamma \partial_i x = \partial_i \gamma x$, it follows that

$$\beta \partial_i x = \epsilon_{r-1}^s \gamma \partial_i x = \epsilon_{r-1}^s \partial_i \gamma x = \partial_i \epsilon_r^s \gamma x = \partial_i \beta x,$$

and for $0 \leq p < r - 1$ it follows that

$$\alpha_p \partial_i x = \partial_{p+1}^{r-p-2} (\partial_i x - \partial_i \beta x) + \epsilon_p \partial_{p+1}^{r-p-1} \partial_i \beta x.$$

If now $0 \leq p < i$ then

$$\alpha_p \partial_i x = \partial_{p+1}^{r-p-1}(x - \beta x) + \epsilon_p \partial_{p+1}^{r-p} \beta x = \alpha_p x;$$

if $i \leq p < r - 1$ then

$$\alpha_p \partial_i x = \partial_i \partial_{p+2}^{r-p-2}(x - \beta x) + \partial_i \epsilon_{p+1} \partial_{p+2}^{r-p-1} \beta x = \partial_i \alpha_{p+1} x.$$

□

Next we consider wedges. If x is a wedge, say $x = u \wedge_i v$, then $u = \partial_{i+2} x$ and $v = \partial_i x$ by Proposition 3.6. A wedge operation \wedge_i is therefore entirely determined by its image set, which will be denoted $\text{im } \wedge_i$. We will now describe the canonical forms of wedges in terms of these image sets.

PROPOSITION 5.2. *Let x be a morphism of rank r in $\mathcal{O}(-, n)$.*

If $x \in \text{im } \wedge_i$ with $i \leq r - 3$ then $\gamma x \in \text{im } \wedge_i$, and $\alpha_p x \in \text{im } \wedge_i$ for $i + 2 \leq p < r$.

If $x \in \text{im } \wedge_{r-2}$ then $\gamma x \in \text{im } \epsilon_{r-2}$ and $\alpha_{r-1} x \in \text{im } \wedge_{r-2}$.

If $x \in \text{im } \wedge_{r-1}$ then $\alpha_{r-1} x \in \text{im } \wedge_{r-1}$.

If $x \in \text{im } \wedge_i$ with $r \leq i$ then $\alpha_p x \in \text{im } \wedge_{i-r+p+1}$ for $0 \leq p < r$.

PROOF. Let the corank of x be s , so that

$$\alpha_p x = \partial_{p+1}^{r-p-1} x - \partial_{p+1}^{r-p-1} \epsilon_r^s \gamma x + \epsilon_p \partial_{p+1}^{r-p} \epsilon_r^s \gamma x,$$

and recall from Proposition 3.8 that a morphism is in $\text{im } \wedge_i$ if and only if it annihilates all basis elements including $i, i + 1, i + 2$.

Suppose that $x \in \text{im } \wedge_i$ with $i \leq r - 3$. Then γx annihilates basis elements including $i, i + 1, i + 2$ because x annihilates basis elements including $i, i + 1, i + 2, r$; therefore $\gamma x \in \text{im } \wedge_i$. For $i + 2 \leq p < r$ it follows that $\partial_{p+1}^{r-p-1} x$, $\partial_{p+1}^{r-p-1} \epsilon_r^s \gamma x$ and $\epsilon_p \partial_{p+1}^{r-p} \epsilon_r^s \gamma x$ annihilate all basis elements including $i, i + 1, i + 2$, and it then follows that $\alpha_p x \in \text{im } \wedge_i$.

Suppose that $x \in \text{im } \wedge_{r-2}$. In this case γx annihilates all basis elements including $r - 2, r - 1$ because x annihilates all basis elements including $r - 2, r - 1, r$; hence, by Proposition 3.4, $\gamma x \in \text{im } \epsilon_{r-2}$. As in the previous case, $\alpha_{r-1} x \in \text{im } \wedge_{r-2}$.

Suppose that $x \in \text{im } \wedge_{r-1}$. Then $s \geq 1$, so that $\epsilon_r^s \gamma x$ and $\epsilon_{r-1} \partial_r \epsilon_r^s \gamma x$ are in $\text{im } \wedge_{r-1}$. As before, it follows that $\alpha_{r-1} x \in \text{im } \wedge_{r-1}$.

Finally, suppose that $x \in \text{im } \wedge_i$ with $r \leq i$, and suppose that $0 \leq p < r$. Then $i \leq r + s - 2$, so $\partial_{p+1}^{r-p-1} x$, $\partial_{p+1}^{r-p-1} \epsilon_r^s \gamma x$ and $\epsilon_p \partial_{p+1}^{r-p} \epsilon_r^s \gamma x$ are all in $\text{im } \wedge_{i-r+p+1}$; therefore $\alpha_p x \in \text{im } \wedge_{i-r+p+1}$. □

To finish this section, we describe the termini and coranks that appear in the evaluation of an expression of the form $\Lambda^r(u_{r-1}, \dots, u_0, v)$ (see Notation 4.9).

PROPOSITION 5.3. *Let $\Lambda^r(u_{r-1}, \dots, u_0, v)$ be an expression defined in $\mathcal{O}(-, n)$, let $v_0 = v$, and for $1 \leq p \leq r$ let*

$$w_p = u_{p-1} \wedge_{p-1}^{r-p+1} v_{p-1}, \quad v_p = \partial_p w_p,$$

so that $\Lambda^r(u_{r-1}, \dots, u_0, v) = v_r$. Then the morphisms v_p and w_p all have the same terminus as v . If $\text{rank } v \geq r$ then the morphisms v_p and w_p all have the same corank as v .

PROOF. Let $v \in \mathcal{O}(m, n)$, so that $m \geq r$, and so that $v_p \in \mathcal{O}(m, n)$ and $w_p \in \mathcal{O}(m+1, n)$ for all relevant p . Since $v_p = \partial_p w_p$ by construction, and since $\partial_{p-1} w_p = v_{p-1}$ by Proposition 3.6, for $r-1 \leq i \leq m$ we get

$$v_r[i] = w_r[i+1] = v_{r-1}[i] = \dots = w_1[i+1] = v_0[i] = v[i].$$

Let the terminus of v be t , so that $v[m] = t$; then $v_p[m] = t$ and $w_p[m+1] = t$, so the morphisms v_p and w_p also have terminus t .

Now suppose that $\text{rank } v \geq r$. Let the corank of v be s , so that $m-s \geq r$. By the definition of corank, we have $v[i] = [t]$ if and only if $m-s \leq i \leq m$. It follows that $v_p[i] = [t]$ if and only if $m-s \leq i \leq m$ and that $w_p[j] = [t]$ if and only if $m+1-s \leq j \leq m+1$. Therefore v_p and w_p also have corank s . \square

6. Complicial identities

In this section we define sets with complicial identities; they will be simplicial sets with wedge operations subject to certain axioms. We will then show that the simplicial sets $\mathcal{O}(-, n)$ satisfy these axioms.

DEFINITION 6.1. A *set with complicial identities* is a simplicial set X , together with *wedges*

$$x \wedge_i y \in X_{m+1},$$

defined when $x, y \in X_m$ and $\partial_i x = \partial_{i+1} y$, such that the following axioms hold.

(1) If $x \wedge_i y$ is defined with $x, y \in X_m$, then

$$\partial_j(x \wedge_i y) = \partial_j x \wedge_{i-1} \partial_j y \quad (0 \leq j < i),$$

$$\partial_i(x \wedge_i y) = y,$$

$$\partial_{i+2}(x \wedge_i y) = x,$$

$$\partial_j(x \wedge_i y) = \partial_{j-1} x \wedge_i \partial_{j-1} y \quad (i+3 \leq j \leq m+1).$$

(2) If $x \in X_m$ and $0 \leq i < m$ then

$$\epsilon_i x = \epsilon_i \partial_{i+1} x \wedge_i x, \quad \epsilon_{i+1} x = x \wedge_i \epsilon_i \partial_i x.$$

(3) If A is of the form $b \wedge_i (y \wedge_i z)$ then

$$A = (\partial_{i+2} b \wedge_i y) \wedge_{i+1} \partial_{i+1} A.$$

(4) If A is of the form $(x \wedge_i y) \wedge_{i+1} c$ then

$$A = \partial_{i+2} A \wedge_i (y \wedge_i \partial_i c).$$

(5) The equality

$$[x \wedge_i \partial_{i+1}(y \wedge_i z)] \wedge_i (y \wedge_i z) = (x \wedge_i y) \wedge_{i+1} [\partial_{i+1}(x \wedge_i y) \wedge_i z]$$

holds whenever either side is defined.

(6) If A is of the form $\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)]$, then the equality

$$A \wedge_i (w \wedge_{i+1} \partial_i A) = (\partial_{i+3} A \wedge_i w) \wedge_{i+2} A$$

holds whenever either side is defined.

(7) If $i \leq j-3$ then the equality

$$(x \wedge_i y) \wedge_j (z \wedge_i w) = (x \wedge_{j-1} z) \wedge_i (y \wedge_{j-1} w)$$

holds whenever either side is defined.

A *morphism of sets with complicial identities* $f: X \rightarrow Y$ is a sequence of functions $f: X_m \rightarrow Y_m$ commuting with the face, degeneracy and wedge operations.

REMARK 6.2. There is some redundancy in these axioms. Indeed, the degeneracy operations on elements of positive dimension are redundant because of axiom (2). For similar reasons, one can omit some of the simplicial identities, retaining only $\partial_i \partial_j = \partial_{j-1} \partial_i$ (for $i < j$) and $\partial_i \epsilon_i = \partial_{i+1} \epsilon_i = \text{id}$.

REMARK 6.3. Axiom (3) could be written simply as an equality

$$b \wedge_i (y \wedge_i z) = (\partial_{i+2} b \wedge_i y) \wedge_{i+1} \partial_{i+1} [b \wedge_i (y \wedge_i z)],$$

required when either side is defined. A similar remark applies to axioms (4) and (6).

REMARK 6.4. Note that axiom (1) does not give a formula for $\partial_{i+1}(x \wedge_i y)$. In a sense, the operation \wedge_i exists in order to construct the operation

$$(x, y) \mapsto \partial_{i+1}(x \wedge_i y).$$

By using axioms (1), (2) and (5), one can show that this binary operation is associative and that it makes the m -dimensional elements into the morphisms of a category. The objects are the $(m-1)$ -dimensional elements, the source and target of a morphism x are $\partial_i x$ and $\partial_{i+1} x$, and the identity of an object a is $\epsilon_i a$.

REMARK 6.5. I intend to show in a future paper that sets with complicial identities are equivalent to the complicial sets used by Verity in [2].

We will now show that the axioms of Definition 6.1 apply to orientals.

PROPOSITION 6.6. *The graded sets $\mathcal{O}(-, n)$ are sets with complicial identities.*

PROOF. From Section 3 we know that $\mathcal{O}(-, n)$ is a simplicial set and that it has wedge operations with the correct domains and codomains.

It is straightforward to verify axiom (1).

Next we verify axiom (5). The existence of the expression on one side is equivalent to the existence of the expression on the other, because the existence of either expression is equivalent to the truth of the equalities

$$\partial_i x = \partial_{i+1} y, \quad \partial_i y = \partial_{i+1} z.$$

Also, if the two expressions do exist, they have the same value, namely

$$\epsilon_{i+3} \epsilon_{i+2} x - \epsilon_i^3 \partial_{i+1} y + \epsilon_{i+3} \epsilon_i y - \epsilon_i^3 \partial_{i+1} z + \epsilon_{i+1} \epsilon_i z.$$

The remaining axioms all say that expressions of certain forms are equal to wedges. They can be proved by using Proposition 3.8, which says that an element of $\mathcal{O}(-, n)$ is in the image of \wedge_i if and only if it is in $\text{im } \epsilon_i + \text{im } \epsilon_{i+1}$. For example, suppose that the expression on the left of axiom (6) is defined, and let

$$B = A \wedge_i (w \wedge_{i+1} \partial_i A).$$

We find that $B \in \text{im } \epsilon_{i+2} + \text{im } \epsilon_{i+3}$; therefore B has the form $B' \wedge_{i+2} B''$. Using axiom (1) we then find that

$$\begin{aligned} B' &= \partial_{i+4} B = \partial_{i+3} A \wedge_i \partial_{i+3} (w \wedge_{i+1} \partial_i A) = \partial_{i+3} A \wedge_i w, \\ B'' &= \partial_{i+2} B = A; \end{aligned}$$

therefore $B = (\partial_{i+3} A \wedge_i w) \wedge_{i+2} A$.

This completes the proof. \square

7. Consequences of the complicial identities

We have shown in Theorem 4.12 that $\mathcal{O}(-, n)$ is generated by the identity morphism ι_n . In Section 8 we will show that $\mathcal{O}(-, n)$ is freely generated by ι_n subject to the complicial identities of Definition 6.1; in other words, we will show that the identities determine the entire structure. In order to do this, we now consider consequences of the identities in general.

There are three parts to this section. In the first part (Propositions 7.1–7.3) we find the domains of definition and most of the faces for the expressions $u \wedge_k^l v$ and $\Lambda^r(u_{r-1}, \dots, u_0, v)$ of Notation 4.9. In the second part (Proposition 7.4) we find a sufficient condition for an expression $\Lambda^r(u_{r-1}, \dots, u_0, v)$ to collapse to v . The third part, concluding with Proposition 7.11, is designed to find conditions implying that an expression of the form $\Lambda^r(u_{r-1}, \dots, u_0, \epsilon_r^s w)$ is a wedge.

First we find the domains of definition and most of the faces for the expressions $u \wedge_k^l v$.

PROPOSITION 7.1. *An expression $u \wedge_k^l v$ in a set with complicial identities exists if and only if $\partial_k u = \partial_{k+1}^l v$. If the expression does exist, then*

$$\begin{aligned} \partial_i(u \wedge_k^l v) &= \partial_i u \wedge_{k-1}^l \partial_i v \quad (i < k), \\ \partial_k(u \wedge_k^l v) &= v, \\ \partial_{k+2}(u \wedge_k^l v) &= u, \\ \partial_{i+1}(u \wedge_k^l v) &= u \wedge_k^{l-1} \partial_i v \quad (l > 1, k < i \leq k+l), \\ \partial_{i+1}(u \wedge_k^l v) &= \partial_{i-l+1} u \wedge_k^l \partial_i v \quad (k+l < i). \end{aligned}$$

PROOF. Recall from Notation 4.9 that $u \wedge_k^1 v = u \wedge_k v$ and that

$$u \wedge_k^l v = (u \wedge_k^{l-1} v) \wedge_k v$$

for $l > 1$. Recall also from Definition 6.1 that a wedge $x \wedge_k y$ is defined if and only if $\partial_k x = \partial_{k+1} y$, and that $\partial_k(x \wedge_k y) = y$ whenever $x \wedge_k y$ is defined. It follows from this that $u \wedge_k^l v$ is defined if and only if $\partial_k u = \partial_{k+1}^l v$. The formulae for $\partial_i(u \wedge_k^l v)$ follow by induction on l from Definition 6.1(1). \square

The domains of definition for the expressions $\Lambda^r(u_{r-1}, \dots, u_0, v)$ are as follows.

PROPOSITION 7.2. *An expression $\Lambda^r(u_{r-1}, \dots, u_0, v)$ exists in a set with complicial identities if and only*

$$\partial_p u_p = \Lambda^p(u_{p-1}, \dots, u_0, \partial_{p+1}^{r-p} v)$$

for $0 \leq p < r$.

PROOF. For $0 \leq p \leq r$ let

$$v_p = \partial_p(u_{p-1} \wedge_{p-1}^{r-p+1}) \partial_{p-1}(u_{p-2} \wedge_{p-2}^{r-p+2}) \dots \partial_1(u_0 \wedge_0^r) v,$$

so that $\Lambda^r(u_{r-1}, \dots, u_0, v) = v_r$. For $0 \leq p < r$, by Proposition 7.1, v_{p+1} exists if and only if $\partial_p u_p = \partial_{p+1}^{r-p} v_p$; it therefore suffices to show that

$$\partial_{p+1}^{r-p} v_p = \Lambda^p(u_{p-1}, \dots, u_0, \partial_{p+1}^{r-p} v).$$

But for $0 < q \leq p$ it follows from Proposition 7.1 that

$$\partial_{p+1}^{r-p} \partial_q(u_{q-1} \wedge_{q-1}^{r-q+1}) = \partial_q \partial_{p+2}^{r-p}(u_{q-1} \wedge_{q-1}^{r-q+1}) = \partial_q(u_{q-1} \wedge_{q-1}^{p-q+1}) \partial_{p+1}^{r-p},$$

and this gives the result. \square

The faces of the expressions $\Lambda^r(u_{r-1}, \dots, u_0, v)$ are as follows.

PROPOSITION 7.3. *In a set with complicial identities the following equalities are valid whenever their left sides are defined:*

$$\partial_i \Lambda^r(u_{r-1}, \dots, u_0, v) = \Lambda^{r-1}(\partial_i u_{r-1}, \dots, \partial_i u_{i+1}, u_{i-1}, \dots, u_0, \partial_i v) \quad (i < r),$$

$$\partial_0 \Lambda^0(v) = \partial_0 v,$$

$$\partial_r \Lambda^r(u_{r-1}, \dots, u_0, v) = \partial_r u_{r-1} \quad (r > 0),$$

$$\partial_i \Lambda^r(u_{r-1}, \dots, u_0, v) = \Lambda^r(\partial_i u_{r-1}, \partial_{i-1} u_{r-2}, \dots, \partial_{i-r+1} u_0, \partial_i v) \quad (i > r).$$

PROOF. Suppose first that $i < r$. Using Proposition 7.1, for $r \geq p > i + 1$ we get

$$\partial_i \partial_p(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_{p-1} \partial_i(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_{p-1}(\partial_i u_{p-1} \wedge_{p-2}^{r-p+1}) \partial_i,$$

we then get

$$\partial_i \partial_{i+1}(u_i \wedge_i^{r-i}) = \partial_i \partial_i(u_i \wedge_i^{r-i}) = \partial_i,$$

and for $i \geq p > 0$ we get

$$\partial_i \partial_p(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_p \partial_{i+1}(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_p(u_{p-1} \wedge_{p-1}^{r-p}) \partial_i;$$

therefore

$$\partial_i \Lambda^r(u_{r-1}, \dots, u_0, v) = \Lambda^{r-1}(\partial_i u_{r-1}, \dots, \partial_i u_{i+1}, u_{i-1}, \dots, u_0, \partial_i v).$$

It is obvious that $\partial_0 \Lambda^0(v) = \partial_0 v$, because $\Lambda^0(v) = v$.

For $r > 0$ the equality $\partial_r \Lambda^r(u_{r-1}, \dots, u_0, v) = \partial_r u_{r-1}$ is immediate from Proposition 7.1.

Finally, suppose that $i > r$. For $r \geq p > 0$ it follows from Proposition 7.1 that

$$\partial_i \partial_p(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_p \partial_{i+1}(u_{p-1} \wedge_{p-1}^{r-p+1}) = \partial_p(\partial_{i-r+p} u_{p-1} \wedge_{p-1}^{r-p+1}) \partial_i;$$

therefore

$$\partial_i \Lambda^r(u_{r-1}, \dots, u_0, v) = \Lambda^r(\partial_i u_{r-1}, \partial_{i-1} u_{r-2}, \dots, \partial_{i-r+1} u_0, \partial_i v).$$

□

Next we give the collapsing result.

PROPOSITION 7.4. *If x is an element of dimension at least r in a set with complicial identities, then*

$$\Lambda^r(\epsilon_{r-1} \partial_r x, \epsilon_{r-2} \partial_{r-1}^2 x, \dots, \epsilon_0 \partial_1^r x, x) = x.$$

PROOF. Repeated applications of Definition 6.1(2) show that for $0 \leq p < r$ we have

$$\partial_{p+1}(\epsilon_p \partial_{p+1}^{r-p} x \wedge_p^{r-p} x) = \partial_{p+1} \epsilon_p x = x.$$

The result follows. □

The rest of this section is designed to prove Proposition 7.11, giving conditions for an expression of the form $\Lambda^r(u_{r-1}, \dots, u_0, \epsilon_r^s w)$ to be a wedge. We use a sequence of lemmas showing that various constituent expressions are wedges.

LEMMA 7.5. *Let $u \wedge_k v$ be a wedge in a set with complicial identities. If $i = k - 1$ or $i = k$ and if $u, v \in \text{im } \wedge_i$ then $\partial_{k+1}(u \wedge_k v) \in \text{im } \wedge_i$.*

PROOF. Suppose that $u = u' \wedge_{k-1} u''$ and $v = v' \wedge_{k-1} v''$. The existence of the wedge $u \wedge_k v$ implies that $v' = \partial_{k+1} v = \partial_k u = \partial_k(u' \wedge_{k-1} u'')$; therefore

$$u \wedge_k v = (u' \wedge_{k-1} u'') \wedge_k [\partial_k(u' \wedge_{k-1} u'') \wedge_{k-1} v''].$$

It now follows from Definition 6.1(5) that

$$\partial_{k+1}(u \wedge_k v) = u' \wedge_{k-1} \partial_k(u'' \wedge_{k-1} v'').$$

If $u = u' \wedge_k u''$ and $v = v' \wedge_k v''$, then it similarly follows that $u'' = \partial_{k+1}(v' \wedge_k v'')$ and that

$$\partial_{k+1}(u \wedge_k v) = \partial_{k+1}(u' \wedge_k v') \wedge_k v''.$$

□

LEMMA 7.6. *In a set with complicial identities, let A be an element of the form $x \wedge_{i+1} y$ or $y \wedge_i z$ or $\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y' \wedge_i z)]$. Then elements of the form $A \wedge_i (w \wedge_{i+1} v)$ are in $\text{im } \wedge_{i+2}$, and elements of the form $(u \wedge_i w) \wedge_{i+2} A$ are in $\text{im } \wedge_i$.*

PROOF. Note first that A must be m -dimensional with $0 \leq i \leq m-3$, because of the existence of $A \wedge_i (w \wedge_{i+1} v)$ or of $(u \wedge_i w) \wedge_{i+2} A$.

Suppose that B has the form $A \wedge_i (w \wedge_{i+1} v)$; then $\partial_i A = \partial_{i+1}(w \wedge_{i+1} v) = v$, so

$$B = A \wedge_i (w \wedge_{i+1} \partial_i A).$$

Because of Definition 6.1(6), to show that $B \in \text{im } \wedge_{i+2}$ it suffices to show that A is of the form

$$\partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)],$$

and we do this as follows: if $A = x \wedge_{i+1} y$ then

$$\begin{aligned} A &= \partial_{i+2} \epsilon_{i+2} A \\ &= \partial_{i+2}(A \wedge_{i+1} \epsilon_{i+1} \partial_{i+1} A) \\ &= \partial_{i+2}(A \wedge_{i+1} \epsilon_{i+1} y) \\ &= \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i \epsilon_i \partial_i y)]; \end{aligned}$$

if $A = y \wedge_i z$ then similarly

$$A = \partial_{i+2}[(\epsilon_{i+1} \partial_{i+2} y \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)];$$

if $A = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y' \wedge_i z)]$ then $y = \partial_{i+1}(x \wedge_{i+1} y) = \partial_{i+2}(y' \wedge_i z) = y'$, so

$$A = \partial_{i+2}[(x \wedge_{i+1} y) \wedge_{i+1} (y \wedge_i z)].$$

A similar argument shows that elements of the form $(u \wedge_i w) \wedge_{i+2} A$ are in $\text{im } \wedge_i$. □

LEMMA 7.7. *In a set with complicial identities, if $r \geq i+3$ then*

$$\epsilon_r(x \wedge_i y) = \epsilon_{r-1} x \wedge_i \epsilon_{r-1} y$$

whenever $x \wedge_i y$ is defined.

PROOF. The proof is by induction on r . We use Definition 6.1(1) and (2) repeatedly.

In the case $r = i + 3$ we have

$$\begin{aligned}\epsilon_{i+3}(x \wedge_i y) &= (x \wedge_i y) \wedge_{i+2} \epsilon_{i+2} \partial_{i+2}(x \wedge_i y) \\ &= (x \wedge_i y) \wedge_{i+2} \epsilon_{i+2} x \\ &= (x \wedge_i y) \wedge_{i+2} (x \wedge_{i+1} \epsilon_{i+1} \partial_{i+1} x).\end{aligned}$$

It follows from Lemma 7.6 that $\epsilon_{i+3}(x \wedge_i y)$ is in $\text{im } \wedge_i$, and it then follows that

$$\begin{aligned}\epsilon_{i+3}(x \wedge_i y) &= \partial_{i+2} \epsilon_{i+3}(x \wedge_i y) \wedge_i \partial_i \epsilon_{i+3}(x \wedge_i y) \\ &= \epsilon_{i+2} \partial_{i+2}(x \wedge_i y) \wedge_i \epsilon_{i+2} \partial_i(x \wedge_i y) \\ &= \epsilon_{i+2} x \wedge_i \epsilon_{i+2} y.\end{aligned}$$

For $r > i + 3$ it follows from the inductive hypothesis and Definition 6.1(7) that

$$\begin{aligned}\epsilon_r(x \wedge_i y) &= (x \wedge_i y) \wedge_{r-1} \epsilon_{r-1} \partial_{r-1}(x \wedge_i y) \\ &= (x \wedge_i y) \wedge_{r-1} \epsilon_{r-1} (\partial_{r-2} x \wedge_i \partial_{r-2} y) \\ &= (x \wedge_i y) \wedge_{r-1} (\epsilon_{r-2} \partial_{r-2} x \wedge_i \epsilon_{r-2} \partial_{r-2} y) \\ &= (x \wedge_{r-2} \epsilon_{r-2} \partial_{r-2} x) \wedge_i (y \wedge_{r-2} \epsilon_{r-2} \partial_{r-2} y) \\ &= \epsilon_{r-1} x \wedge_i \epsilon_{r-1} y.\end{aligned}$$

This completes the proof. \square

LEMMA 7.8. *In a set with complicial identities, an element of the form*

$$(u \wedge_i u') \wedge_k (v \wedge_i v')$$

is in $\text{im } \wedge_i$ for $i \leq k$ and is in $\text{im } \wedge_{i+1}$ for $i \geq k - 1$.

PROOF. For $i = k - 1$ and for $i = k$ the results are trivial or are contained in Definition 6.1(3) and (4); for $i = k - 2$ and for $i = k + 1$ the results follow from Lemma 7.6; for $i \leq k - 3$ and for $i \geq k + 2$ the results are contained in Definition 6.1(7). \square

LEMMA 7.9. *In a set with complicial identities, an element of the form*

$$\partial_{i+3}[u \wedge_{i+2}^l \partial_{i+2}(u' \wedge_{i+1}^{l+1} v)]$$

is in $\text{im } \wedge_i$ if u and v are in $\text{im } \wedge_i$.

PROOF. Let

$$z = u \wedge_{i+2}^l \partial_{i+2}(u' \wedge_{i+1}^{l+1} v).$$

Because of Definition 6.1(1), it suffices to show that $z \in \text{im } \wedge_i$. We will do this by induction on l .

Let

$$A = \partial_{i+2}(u' \wedge_{i+1}^{l+1} v)$$

and let

$$u'' = \begin{cases} u & (l = 1), \\ u \wedge_{i+2}^{l-1} \partial_{i+3} A & (l > 1), \end{cases}$$

so that

$$z = u'' \wedge_{i+2} A.$$

Note that

$$A = \partial_{i+2}[(u' \wedge_{i+1}^l \partial_{i+2} v) \wedge_{i+1} v]$$

with $u' \wedge_{i+1}^l \partial_{i+2} v \in \text{im } \wedge_{i+1}$ by the definition of \wedge_{i+1}^l and with $v \in \text{im } \wedge_i$ by hypothesis. By Lemma 7.6, to show that $z \in \text{im } \wedge_i$ it suffices to show that $u'' \in \text{im } \wedge_i$.

Suppose that $l = 1$. Then $u'' = u$, so $u'' \in \text{im } \wedge_i$ by hypothesis.

Suppose that $l > 1$. Using Proposition 7.1 we get

$$\begin{aligned} u'' &= u \wedge_{i+2}^{l-1} \partial_{i+3} A \\ &= u \wedge_{i+2}^{l-1} \partial_{i+2} \partial_{i+4} (u' \wedge_{i+1}^{l+1} v) \\ &= u \wedge_{i+2}^{l-1} \partial_{i+2} (u' \wedge_{i+1}^l \partial_{i+3} v). \end{aligned}$$

Since $v \in \text{im } \wedge_i$, it follows from Definition 6.1(1) that $\partial_{i+3} v \in \text{im } \wedge_i$, and it then follows from the inductive hypothesis that $u'' \in \text{im } \wedge_i$.

This completes the proof. \square

LEMMA 7.10. *In a set with complicial identities, let v' be an element of the form $\partial_{k+1}(u \wedge_k^l v)$ with $v \in \text{im } \wedge_i$. Then $v' \in \text{im } \wedge_i$ whenever one of the following sets of conditions is satisfied:*

- (1) $i \leq k - 2$ and $u \in \text{im } \wedge_i$;
- (2) $k \leq i < k + l$ and $l \geq 2$;
- (3) $k + l \leq i$ and $u \in \text{im } \wedge_{i-l+1}$.

PROOF. Let $u_0 = u$, and for $0 < j \leq l$ let

$$u_j = u_{j-1} \wedge_k \partial_{k+1}^{l-j} v;$$

we must show in each case that $\partial_{k+1} u_l \in \text{im } \wedge_i$.

(1) Since $v \in \text{im } \wedge_i$ with $i \leq k - 2$, it follows from Definition 6.1(1) that $\partial_{k+1}^{l-j} v \in \text{im } \wedge_i$ for $0 \leq j < l$. Since $u_0 \in \text{im } \wedge_i$, it follows from Lemma 7.8 that $u_1, \dots, u_l \in \text{im } \wedge_i$, and it then follows from Definition 6.1(1) that $\partial_{k+1} u_l \in \text{im } \wedge_i$.

(2) Suppose first that $i = k$; we must show that

$$\partial_{k+1}(u_{l-1} \wedge_k v) \in \text{im } \wedge_k.$$

But $u_{l-1} \in \text{im } \wedge_k$ because $l \geq 2$, and $v \in \text{im } \wedge_k$ because $i = k$, so the result follows from Lemma 7.5.

Now suppose that $k < i < k + l$. We have $u_{k+l-i} \in \text{im } \wedge_k$ because $i < k + l$, and we have $\partial_{k+1}^{i-k-1} v \in \text{im } \wedge_{k+1}$ by Definition 6.1(1); therefore $u_{k+l-i+1} \in \text{im } \wedge_{k+2}$ by Lemma 7.6. We also have $\partial_{k+1}^{i-k-2} v \in \text{im } \wedge_{k+2}$ by Definition 6.1(1), so $u_{k+l-i+2} \in \text{im } \wedge_{k+3}$ by Lemma 7.8. By repeating the use of Lemma 7.8, we eventually get $u_l \in \text{im } \wedge_{i+1}$. Definition 6.1(1) now gives us $\partial_{k+1} u_l \in \text{im } \wedge_i$.

(3) In this case $u_0 \in \text{im } \wedge_{i-l+1}$ by hypothesis and $\partial_{k+1}^{l-1} v \in \text{im } \wedge_{i-l+1}$ by Definition 6.1(1); therefore $u_1 \in \text{im } \wedge_{i-l+2}$ by Lemma 7.8. In the same way $u_2 \in \text{im } \wedge_{i-l+3}$, and so on. Eventually we get $u_l \in \text{im } \wedge_{i+1}$. As before, Definition 6.1(1) now gives us $\partial_{k+1} u_l \in \text{im } \wedge_i$. \square

The final result is a converse to Proposition 5.2.

PROPOSITION 7.11. *Let x be an element of the form*

$$x = \Lambda^r(u_{r-1}, \dots, u_0, \epsilon_r^s w)$$

in a set with complicial identities.

If $i \leq r - 3$, if $w \in \text{im } \wedge_i$, and if $u_p \in \text{im } \wedge_i$ for $i + 2 \leq p < r$, then $x \in \text{im } \wedge_i$.

If $w \in \text{im } \epsilon_{r-2}$ and $u_{r-1} \in \text{im } \wedge_{r-2}$ then $x \in \text{im } \wedge_{r-2}$.

If $s \geq 1$ and $u_{r-1} \in \text{im } \wedge_{r-1}$ then $x \in \text{im } \wedge_{r-1}$.
 If $r \leq i \leq r + s - 2$, and if $u_p \in \text{im } \wedge_{i-r+p+1}$ for $0 \leq p < r$, then $x \in \text{im } \wedge_i$.

PROOF. Let $v_0 = \epsilon_r^s w$, and for $0 \leq p < r$ let

$$v_{p+1} = \partial_{p+1}(u_p \wedge_p^{r-p} v_p);$$

thus $x = v_r$. We claim that $v_p \in \text{im } \wedge_i$ for $0 \leq p \leq r$, except possibly for cases with $i \leq r - 3$ and $p = i + 2$.

First we suppose that $i \leq r - 3$, that $w \in \text{im } \wedge_i$, and that $u_p \in \text{im } \wedge_i$ for $i + 2 \leq p < r$. We have $v_0 \in \text{im } \wedge_i$ by Lemma 7.7, and we get

$$v_1 \in \text{im } \wedge_i, \dots, v_{i+1} \in \text{im } \wedge_i$$

by repeated applications of Lemma 7.10(2). We now have

$$v_{i+3} = \partial_{i+3}[u_{i+2} \wedge_{i+2}^{r-i-2} \partial_{i+2}(u_{i+1} \wedge_{i+1}^{r-i-1} v_{i+1})]$$

with u_{i+2} and v_{i+1} in $\text{im } \wedge_i$, so $v_{i+3} \in \text{im } \wedge_i$ by Lemma 7.9. Since $u_p \in \text{im } \wedge_i$ for $i + 3 \leq p < r$, repeated applications of Lemma 7.10(1) now show that the elements

$$v_{i+4}, v_{i+5}, \dots, v_r$$

are in $\text{im } \wedge_i$.

Next we suppose that $w \in \text{im } \epsilon_{r-2}$ and $u_{r-1} \in \text{im } \wedge_{r-2}$. We have $v_0 \in \text{im } \epsilon_{r-2}$ because $\epsilon_r^s \epsilon_{r-2} = \epsilon_{r-2} \epsilon_r^s$; hence, by Definition 6.1(2), $v_0 \in \text{im } \wedge_{r-2}$. As in the previous case, we get

$$v_1 \in \text{im } \wedge_{r-2}, \dots, v_{r-1} \in \text{im } \wedge_{r-2}.$$

Since $v_r = \partial_r(u_{r-1} \wedge_{r-1} v_{r-1})$ and since $u_{r-1} \in \text{im } \wedge_{r-2}$, it follows from Lemma 7.5 that $v_r \in \text{im } \wedge_{r-2}$.

Next we suppose that $s \geq 1$ and $u_{r-1} \in \text{im } \wedge_{r-1}$. Then $v_0 \in \text{im } \epsilon_r$, and it follows from Definition 6.1(2) that $v_0 \in \text{im } \wedge_{r-1}$. The rest of the argument is as in the previous case.

Finally we suppose that $r \leq i \leq r + s - 2$ and that $u_p \in \text{im } \wedge_{i-r+p+1}$ for $0 \leq p < r$. We get $v_0 = \epsilon_{i+1} \epsilon_r^{s-1} w$; therefore $v_0 \in \text{im } \wedge_i$ by Definition 6.1(2). We then get

$$v_1 \in \text{im } \wedge_i, \dots, v_r \in \text{im } \wedge_i$$

by repeated applications of Lemma 7.10(3).

This completes the proof. \square

8. Freeness

We have shown that $\mathcal{O}(-, n)$ is a set with complicial identities generated by the identity morphism ι_n (see Proposition 6.6 and Theorem 4.12). We will now show that $\mathcal{O}(-, n)$ is freely generated by ι_n ; that is to say, given an n -dimensional element u in a set with complicial identities U , we will show that there is a unique morphism $f: \mathcal{O}(-, n) \rightarrow U$ with $f\iota_n = u$.

We will construct morphisms on $\mathcal{O}(-, n)$ by combining suitable functions, called partial morphisms, which are defined on subsets of sets with complicial identities and take their values in sets with complicial identities. The domain of a partial morphism is closed under face operations, but not necessarily under degeneracy or wedge operations. A partial morphism preserves the operations so far as can be expected, and is allowed to increase degrees.

DEFINITION 8.1. Let S be a subset of a set with complicial identities, let U be a set with complicial identities, and let k be a nonnegative integer. Then a *partial morphism of degree k* from S to U is a function $f: S \rightarrow U$ which increases degrees by k and which satisfies the following conditions.

- (1) If x is an m -dimensional member of S with $m > 0$ and if $0 \leq i \leq m$, then $\partial_i x \in S$ and $f\partial_i x = \partial_i f x$.
- (2) If x is a 1-dimensional member of S and $x \in \text{im } \epsilon_0$ then $f x \in \text{im } \epsilon_0$.
- (3) If $x \in S$ and $x \in \text{im } \wedge_i$ then $f x \in \text{im } \wedge_i$.

Partial morphisms preserve degeneracies and wedges as much as possible.

PROPOSITION 8.2. *Let $f: S \rightarrow U$ be a partial morphism and let x be a member of S .*

- (1) *If $x = \epsilon_i y$ for some y , then $y \in S$ and $f x = \epsilon_i f y$.*
- (2) *If $x = y \wedge_i z$ for some y and z , then $y, z \in S$ and $f x = f y \wedge_i f z$.*

PROOF. We prove these statements in reverse order.

(2) Suppose that $x = y \wedge_i z$. Then $y = \partial_{i+2} x$ and $z = \partial_i x$ by Definition 6.1(1); therefore $y, z \in S$. Since also $f x \in \text{im } \wedge_i$, it then follows that

$$f x = \partial_{i+2} f x \wedge_i \partial_i f x = f \partial_{i+2} x \wedge_i f \partial_i x = f y \wedge_i f z.$$

(1) Suppose that $x = \epsilon_i y$. Then $y = \partial_i x$, so that $y \in S$. We will now prove that $f x = \epsilon_i f y$ by induction on the dimension of x .

Suppose that $x = \epsilon_0 y$ and x is 1-dimensional. Then $f x \in \text{im } \epsilon_0$, and it follows that

$$f x = \epsilon_0 \partial_0 f x = \epsilon_0 f \partial_0 x = \epsilon_0 f y.$$

Suppose that $x = \epsilon_0 y$ and that the dimension of x is greater than 1. Then $x = \epsilon_0 \partial_1 y \wedge_0 y$ by Definition 6.1(2). From the inductive hypothesis and from what we have already proved, it now follows that

$$f x = f \epsilon_0 \partial_1 y \wedge_0 f y = \epsilon_0 f \partial_1 y \wedge_0 y = \epsilon_0 \partial_1 f y \wedge_0 f y = \epsilon_0 f y.$$

Finally suppose that $x = \epsilon_i y$ with $i > 0$. Then $x = y \wedge_{i-1} \epsilon_{i-1} \partial_{i-1} y$ by Definition 6.1(2), and it follows as in the previous case that $f x = \epsilon_i f y$.

This completes the proof. \square

In certain circumstances, it follows that partial morphisms also preserve expressions of the form $\Lambda^r(u_{r-1}, \dots, u_0, v)$.

PROPOSITION 8.3. *Let f be a partial morphism on a subset S of $\mathcal{O}(-, n)$, let*

$$x = \Lambda^r(u_{r-1}, \dots, u_0, v)$$

be defined in $\mathcal{O}(-, n)$, let the terminus of v be t , and let the corank of v be s . Suppose that S contains all morphisms with terminus t ; alternatively, suppose that $\text{rank } v \geq r$ and that S contains all morphisms with terminus t and corank s . Then the morphisms $x, u_{r-1}, \dots, u_0, v$ are members of S , and

$$f x = \Lambda^r(f u_{r-1}, \dots, f u_0, f v).$$

PROOF. This follows from Propositions 5.3 and 8.2. \square

Given an n -dimensional element u in a set with complicial identities, we will now show how to construct a morphism on $\mathcal{O}(-, n)$ sending ι_n to u . We use a recursion on terminus and corank. For $t \geq 0$ let S_t be the subset of $\mathcal{O}(-, n)$

consisting of the morphisms of terminus at most t . For $t \geq 0$ and $s \geq -1$ let S_t^s be the subset of S_t consisting of the following morphisms: the morphisms with terminus less than t ; the morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$; the t -cones, in the case that $t > 0$; the morphisms with terminus t and with corank at most s . We now proceed as follows. We construct a partial morphism on S_0^{-1} sending $\partial_1^n \iota_n$ to u , we extend it to S_0^0, S_0^1, \dots , and eventually to the whole of S_0 , we then construct a partial morphism on S_1^{-1} sending $\partial_2^{n-1} \iota_n$ to u , and so on, finishing with a partial morphism on $S_n = \mathcal{O}(-, n)$ (actually a genuine morphism) sending ι_n to u . The steps in this process are given by Lemmas 8.4–8.6.

LEMMA 8.4. *Let u be an element in a set with complicial identities U . Then there is a partial morphism $f: S_0^{-1} \rightarrow U$ such that $f\partial_1^n \iota_n = u$.*

PROOF. The only morphism in S_0^{-1} is the zero-dimensional morphism $\partial_1^n \iota_n$. The condition $f\partial_1^n \iota_n = u$ therefore defines a function $f: S_0^{-1} \rightarrow U$, and this function is trivially a partial morphism. \square

LEMMA 8.5. *For $t > 0$, let $F: S_{t-1} \rightarrow U$ be a partial morphism of degree $k > 0$. Then there is a partial morphism $f: S_t^{-1} \rightarrow U$ of degree $k - 1$ such that $f\partial_{t+1}^{n-t} \iota_n = F\partial_t^{n-t+1} \iota_n$.*

PROOF. We define a function $f: S_t^{-1} \rightarrow U$ as follows: if x is a member of $\mathcal{O}(m, n)$ with terminus less than t , then

$$fx = \partial_{m+1} Fx;$$

if $x = \partial_0^t \partial_{t+1}^{n-t} \iota_n$ then

$$fx = \partial_0^t F\partial_t^{n-t+1} \iota_n;$$

if x is a t -cone in $\mathcal{O}(m+1, n)$ then $\partial_{m+1} x$ has terminus less than t , and we make the definition

$$fx = F\partial_{m+1} x.$$

It is clear that f increases degrees by $k - 1$ and that

$$f\partial_{t+1}^{n-t} \iota_n = F\partial_t \partial_{t+1}^{n-t} \iota_n = F\partial_t^{n-t+1} \iota_n;$$

it therefore remains to verify conditions (1)–(3) of Definition 8.1.

(1) Let $\partial_i x$ be a face of an m -dimensional member of S_{t-1} , so that $m > 0$ and $0 \leq i \leq m$. Then $\partial_i x$ is an $(m-1)$ -dimensional member of S_{t-1} , so that $\partial_i x \in S_{t-1}^{-1}$, and

$$f\partial_i x = \partial_m F\partial_i x = \partial_m \partial_i Fx = \partial_i \partial_{m+1} Fx = \partial_i fx.$$

The zero-dimensional morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$ does not have any faces.

Let x be a 1-dimensional t -cone, and consider the face $\partial_0 x$. This is the zero-dimensional morphism $\partial_0^t \partial_{t+1}^{n-1} \iota_n$; therefore $\partial_0 x \in S_t^{-1}$ and $f\partial_0 x = \partial_0^t F\partial_t^{n-t+1} \iota_n$. The zero-dimensional morphism $\partial_1 x$ must have the form $[0] \mapsto [j]$ with $0 \leq j \leq t-1$, and it can be expressed as $\partial_0^j \partial_{j+1}^{t-j-1} \partial_t^{n-t+1} \iota_n$. Therefore

$$\partial_0 fx = \partial_0 F\partial_1 x = \partial_0 \partial_0^j \partial_{j+1}^{t-j-1} F\partial_t^{n-t+1} \iota_n = \partial_0^t F\partial_t^{n-t+1} \iota_n,$$

and it follows that $f\partial_0 x = \partial_0 fx$.

Let $\partial_i x$ be a face of an $(m+1)$ -dimensional t -cone such that $m > 0$ and $0 \leq i \leq m$. Then $\partial_i x$ is an m -dimensional t -cone, so that $\partial_i x \in S_t^{-1}$ and

$$f\partial_i x = F\partial_m \partial_i x = F\partial_i \partial_{m+1} x = \partial_i F\partial_{m+1} x = \partial_i fx.$$

Finally, let x be an $(m + 1)$ -dimensional t -cone with $m \geq 0$ and consider the face $\partial_{m+1}x$. This is an m -dimensional member of S_{t-1} ; therefore $\partial_{m+1}x \in S_t^{-1}$ and

$$f\partial_{m+1}x = \partial_{m+1}F\partial_{m+1}x = \partial_{m+1}fx.$$

(2) Let x be a 1-dimensional member of S_t^{-1} lying in the image of ϵ_0 . Then $x \in S_{t-1}$, so that $fx = \partial_2Fx$, and we also have $x = \epsilon_0\partial_0x$. It now follows from Proposition 8.2 that $fx \in \text{im } \epsilon_0$, because

$$fx = \partial_2F\epsilon_0\partial_0x = \partial_2\epsilon_0F\partial_0x = \epsilon_0\partial_1F\partial_0x.$$

(3) Let x be an m -dimensional member of S_{t-1} lying in the image of \wedge_i . Then $fx = \partial_{m+1}Fx$. We also have $x = \partial_{i+2}x \wedge_i \partial_i x$ by Definition 6.1(1), and we must have $0 \leq i \leq m - 2$. It now follows from Proposition 8.2 and Definition 6.1(1) that $fx \in \text{im } \wedge_i$, because

$$fx = \partial_{m+1}F(\partial_{i+2}x \wedge_i \partial_i x) = \partial_{m+1}(F\partial_{i+2}x \wedge_i F\partial_i x) = \partial_m F\partial_{i+2}x \wedge_i \partial_m F\partial_i x.$$

The zero-dimensional morphism $\partial_0^t \partial_{t+1}^{n-t} \iota_n$ cannot belong to the image of a wedge operation.

Let x be an $(m + 1)$ -dimensional t -cone, and suppose that $x \in \text{im } \wedge_i$ with $0 \leq i \leq m - 2$. Then $fx \in \text{im } \wedge_i$ as before, because

$$fx = F\partial_{m+1}(\partial_{i+2}x \wedge_i \partial_i x) = F(\partial_m \partial_{i+2}x \wedge_i \partial_m \partial_i x) = F\partial_m \partial_{i+2}x \wedge_i F\partial_m \partial_i x.$$

Finally, suppose that x is an $(m + 1)$ -dimensional t -cone lying in the image of \wedge_{m-1} . We must have $x = \epsilon_{m-1}\partial_{m-1}x$ (see Propositions 3.8 and 3.4). It now follows from Proposition 8.2 and Definition 6.1(2) that $fx \in \text{im } \wedge_{m-1}$, because

$$fx = F\partial_{m+1}\epsilon_{m-1}\partial_{m-1}x = F\epsilon_{m-1}\partial_m \partial_{m-1}x = \epsilon_{m-1}F\partial_m \partial_{m-1}x.$$

This completes the proof. \square

LEMMA 8.6. *For $t \geq 0$ and $s \geq 0$, if f is a partial morphism on S_t^{s-1} , then there is a partial morphism f' on S_t^s which is an extension of f .*

PROOF. Let x be a morphism in S_t^s with terminus less than t or with terminus t and corank less than s ; then x belongs to S_t^{s-1} and we make the definition

$$f'x = fx.$$

Now let x be a morphism with terminus t and corank s , and let the rank of x be r . According to Theorem 4.11, x has a canonical form given by

$$x = \Lambda^r(\alpha_{r-1}x, \dots, \alpha_0x, \epsilon_r^s \gamma x).$$

According to Propositions 4.8 and 4.6, the morphisms $\alpha_p x$ and γx are members of S_t^{s-1} , and we will obtain $f'x$ by applying f to each of these morphisms.

In order to show that this is possible, we use Proposition 7.2. For $0 \leq p < r$ it follows from this proposition and from the existence of the canonical form that

$$\partial_p \alpha_p x = \Lambda^p(\alpha_{p-1}x, \dots, \alpha_0x, \partial_{p+1}^{r-p} \epsilon_r^s \gamma x).$$

If $s = 0$ then the argument $\partial_{p+1}^{r-p} \epsilon_r^s \gamma x$ has terminus less than t ; if $s > 0$ then it has terminus t , rank $p+1$ and corank $s-1$; in both cases, it follows from Proposition 8.3 that

$$\partial_p f \alpha_p x = f \partial_p \alpha_p x = \Lambda^p(f \alpha_{p-1}x, \dots, f \alpha_0x, f \partial_{p+1}^{r-p} \epsilon_r^s \gamma x).$$

If now $s = 0$ then

$$f \partial_{p+1}^{r-p} \epsilon_r^s \gamma x = f \partial_{p+1}^{r-p} \gamma x = \partial_{p+1}^{r-p} f \gamma x = \partial_{p+1}^{r-p} \epsilon_r^s \gamma x;$$

if $s > 0$ then $\epsilon_r^{s-1}\gamma x \in S_t^{s-1}$ and we get

$$f\partial_{p+1}^{r-p}\epsilon_r^s\gamma x = f\partial_{p+1}^{r-p-1}\epsilon_r^{s-1}\gamma x = \partial_{p+1}^{r-p-1}\epsilon_r^{s-1}f\gamma x = \partial_{p+1}^{r-p}\epsilon_r^s f\gamma x;$$

in any case we get

$$\partial_p f\alpha_p x = \Lambda^p(f\alpha_{p-1}x, \dots, f\alpha_0x, \partial_{p+1}^{r-p}\epsilon_r^s f\gamma x).$$

Because of Proposition 7.2, we can now define $f'x$ by the formula

$$f'x = \Lambda^r(f\alpha_{r-1}x, \dots, f\alpha_0x, \epsilon_r^s f\gamma x).$$

We will now show that f' is an extension of f ; that is to say, we will show that $f'x = fx$ when x is a morphism with terminus t and corank s belonging to S_t^{s-1} . This situation can occur only when $s = 0$, with $x = \partial_0^t \partial_{t+1}^{n-t+1} \iota_n$ or with x a t -cone. In these cases, let r be the rank of x ; then it follows from Example 4.7 that

$$f'x = \Lambda^r(f\epsilon_{r-1}\partial_r x, \dots, f\epsilon_0\partial_1^r x, fx),$$

it follows from Proposition 8.2 that

$$f'x = \Lambda^r(\epsilon_{r-1}\partial_r fx, \dots, \epsilon_0\partial_1^r fx, fx),$$

and it follows from Proposition 7.4 that $f'x = fx$ as required.

It remains to show that f' is a partial morphism by verifying the conditions of Definition 8.1. It clearly suffices to consider morphisms with terminus t and corank s , and we argue as follows.

(1) Let x be a morphism with terminus t , with corank s and with rank r , and consider a face $\partial_i x$ with $0 \leq i < r$. In this case $\partial_i x$ has terminus t , corank s and rank $r - 1$, so that $\partial_i x \in S_t^s$ and it follows from Proposition 5.1 that

$$f'\partial_i x = \Lambda^{r-1}(f\partial_i\alpha_{r-1}x, \dots, f\partial_i\alpha_{i+1}x, f\alpha_{i-1}x, \dots, f\alpha_0x, \epsilon_{r-1}^s f\partial_i\gamma x).$$

Since $f\partial_i\alpha_p x = \partial_i f\alpha_p x$ for $i < p < r$ and since

$$\epsilon_{r-1}^s f\partial_i\gamma x = \epsilon_{r-1}^s \partial_i f\gamma x = \partial_i \epsilon_r^s f\gamma x,$$

it now follows from Proposition 7.3 that

$$f'\partial_i x = \partial_i \Lambda^r(f\alpha_{r-1}x, \dots, f\alpha_0x, \epsilon_r^s f\gamma x) = \partial_i f'x.$$

Now let x be a morphism with terminus t , with corank s and with rank zero, and consider the face $\partial_0 x$. This exists only in cases with $s > 0$. It has terminus t , corank $s - 1$ and rank zero, and it therefore belongs to S_t^{s-1} . It follows that $\partial_0 x \in S_t^s$. It also follows from Theorem 4.11 and Proposition 7.3 that

$$f'\partial_0 x = f\partial_0 x = f\partial_0 \Lambda^0(\epsilon_0^s \gamma x) = f\partial_0 \epsilon_0^s \gamma x = f\epsilon_0^{s-1} \gamma x = \epsilon_0^{s-1} f\gamma x$$

and that

$$\partial_0 f'x = \partial_0 \Lambda^0(\epsilon_0^s f\gamma x) = \partial_0 \epsilon_0^s f\gamma x = \epsilon_0^{s-1} f\gamma x;$$

therefore $f'\partial_0 x = \partial_0 f'x$.

Now let x be a morphism with terminus t , with corank s , and with rank $r > 0$, and consider the face $\partial_r x$. If $s = 0$ then $\partial_r x$ has terminus less than t ; if $s > 0$ then $\partial_r x$ has terminus t and corank $s - 1$; in both cases, $\partial_r x \in S_t^{s-1}$. It follows that $\partial_r x \in S_t^s$. It also follows from Theorem 4.11 and Proposition 7.3 that

$$f'\partial_r x = f\partial_r x = f\partial_r \Lambda^r(\alpha_{r-1}x, \dots, \alpha_0x, \epsilon_r^s \gamma x) = f\partial_r \alpha_{r-1}x = \partial_r f\alpha_{r-1}x$$

and

$$\partial_r f'x = \partial_r \Lambda^r(f\alpha_{r-1}x, \dots, f\alpha_0x, \epsilon_r^s f\gamma x) = \partial_r f\alpha_{r-1}x;$$

therefore $f'\partial_r x = \partial_r f'x$.

Finally, let x be a morphism with terminus t , with corank s , and with rank r , and consider $\partial_i x$ for $r < i \leq r + s$. In this case $\partial_i x$ has terminus t and corank $s - 1$, so that $\partial_i x \in S_i^{s-1}$; therefore $\partial_i x \in S_i^s$. It now follows from Theorem 4.11 and Proposition 7.3 that

$$\begin{aligned} f' \partial_i x &= f \partial_i x \\ &= f \partial_i \Lambda^r(\alpha_{r-1} x, \dots, \alpha_0 x, \epsilon_r^s \gamma x) \\ &= f \Lambda^r(\partial_i \alpha_{r-1} x, \dots, \partial_{i-r+1} \alpha_0 x, \partial_i \epsilon_r^s \gamma x) \\ &= f \Lambda^r(\partial_i \alpha_{r-1} x, \dots, \partial_{i-r+1} \alpha_0 x, \epsilon_r^{s-1} \gamma x). \end{aligned}$$

Here $\epsilon_r^{s-1} \gamma x$ has terminus t , rank r and corank $s - 1$, so it follows from Propositions 8.3 and 8.2 that

$$\begin{aligned} f' \partial_i x &= \Lambda^r(f \partial_i \alpha_{r-1} x, \dots, f \partial_{i-r+1} \alpha_0 x, f \epsilon_r^{s-1} \gamma x) \\ &= \Lambda^r(\partial_i f \alpha_{r-1} x, \dots, \partial_{i-r+1} f \alpha_0 x, \epsilon_r^{s-1} f \gamma x). \end{aligned}$$

On the other hand, it follows from Proposition 7.3 that

$$\begin{aligned} \partial_i f' x &= \Lambda^r(\partial_i f \alpha_{r-1} x, \dots, \partial_{i-r+1} f \alpha_0 x, \partial_i \epsilon_r^s f \gamma x) \\ &= \Lambda^r(\partial_i f \alpha_{r-1} x, \dots, \partial_{i-r+1} f \alpha_0 x, \epsilon_r^{s-1} f \gamma x); \end{aligned}$$

therefore $f' \partial_i x = \partial_i f' x$.

(2) Suppose that x is a 1-dimensional morphism with terminus t and positive corank s lying in the image of ϵ_0 . This can occur only in the case that $s = 1$, and the rank of x must be zero; therefore $f' x = \Lambda^0(\epsilon_0 f \gamma x) = \epsilon_0 f \gamma x$. It follows that $f' x$ is in the image of ϵ_0 .

(3) Suppose that x is a morphism with terminus t and corank s lying in the image of \wedge_i . Let the rank of x be r , so that $0 \leq i \leq r + s - 2$. It follows from Proposition 5.2 that certain morphisms $\alpha_p x$ are in certain sets $\text{im } \wedge_j$, and it may also follow that γx is in $\text{im } \wedge_i$ or $\text{im } \epsilon_i$. The images $f \alpha_p x$ and $f \gamma x$ then satisfy the same conditions, and it follows from Proposition 7.11 that $f' x \in \text{im } \wedge_i$.

This completes the proof. \square

From Lemmas 8.4–8.6 we get the main result.

THEOREM 8.7. *For $n \geq 0$, the graded set $\mathcal{O}(-, n)$ is the set with complicial identities freely generated by the identity morphism ι_n in $\mathcal{O}(n, n)$.*

PROOF. First, by Proposition 6.6, $\mathcal{O}(-, n)$ is a set with complicial identities.

Now let u be an n -dimensional element in a set with complicial identities U . We must show that there is a unique morphism from $\mathcal{O}(-, n)$ to U sending ι_n to u .

We construct a suitable morphism as follows. By Lemma 8.4 there is a partial morphism on S_0^{-1} sending $\partial_1^n \iota_n$ to u ; by repeated applications of Lemma 8.6, there is a partial morphism on the entire set S_0 sending $\partial_1^n \iota_n$ to u ; by Lemma 8.5, there is a partial morphism on S_1^{-1} sending $\partial_2^{n-1} \iota_n$ to u ; by Lemma 8.6 there is a partial morphism on the entire set S_1 sending $\partial_2^{n-1} \iota_n$ to u ; etc. Since $\mathcal{O}(-, n) = S_n$, we eventually obtain a partial morphism $f: \mathcal{O}(-, n) \rightarrow U$ such that $f \iota_n = u$. By Proposition 8.2, f is in fact a morphism of sets with complicial identities. Since, according to Theorem 4.12, $\mathcal{O}(-, n)$ is generated by ι_n , it follows that f is the only morphism from $\mathcal{O}(-, n)$ to U with $f \iota_n = u$.

This completes the proof. \square

References

- [1] R. Street, The algebra of oriented simplexes, *J. Pure Appl. Algebra* 49 (1987) 283–335.
- [2] D. Verity, Complicial sets characterising the simplicial nerves of strict ω -categories, *Mem. Amer. Math. Soc.* 193 (2008) no. 905.
- [3] F. A. Al-Agl, R. Brown and R. Steiner, Multiple categories: the equivalence of a globular and a cubical approach, *Adv. Math.* 170 (2002) 71–118.
- [4] R. Steiner, Orientals, in: *Categories in algebra, geometry and mathematical physics*, *Contemp. Math.* vol. 431, Amer. Math. Soc. (2007) 427–439.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, UNIVERSITY GARDENS,
GLASGOW, GREAT BRITAIN G12 8QW

E-mail address: `Richard.Steiner@glasgow.ac.uk`