GALOIS THEORY AND LUBIN-TATE COCHAINS ON CLASSIFYING SPACES

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ABSTRACT. We consider brave new cochain extensions $F(BG_+, R) \longrightarrow F(EG_+, R)$, where R is either a Lubin-Tate spectrum E_n or the related 2-periodic Morava K-theory K_n, and G is a finite group. When R is an Eilenberg-Mac Lane spectrum, in some good cases such an extension is a G-Galois extension in the sense of John Rognes, but not always faithful. We prove that for E_n and K_n these extensions are always faithful in the K_n local category. However, for a cyclic p-group C_{p^r}, the cochain extension $F(BC_{p^r}{}_+, E_n) \longrightarrow F(EC_{p^r}{}_+, E_n)$ is not a Galois extension because it ramifies. As a consequence, it follows that the E_n-theory Eilenberg-Moore spectral sequence for G and BG does not always converge to its expected target.

1. INTRODUCTION

In the algebraic Galois theory of commutative rings [5], faithful flatness is a property implied by separability. However, in the topological analogue, the brave new Galois theory of Rognes [16], this is not true. The simplest counterexample, due to Ben Wieland [17], is provided by the C_2-Galois extension

$$F(BC_{2+}, H\mathbb{F}_2) \longrightarrow F(EC_{2+}, H\mathbb{F}_2) \sim H\mathbb{F}_2 \tag{1.1}$$

which is not faithful. This example relies on the algebraic fact that

$$\pi_*(F(BC_{2+}, H\mathbb{F}_2)) = H^{-*}(BC_2; \mathbb{F}_2)$$

is a polynomial algebra and so has finite global dimension.

In this note we consider this question for a Lubin-Tate spectrum E_n and the related Morava K-theory K_n, and show that for any finite group G, the extension

$$E_n^{BG} = F(BG_+, E_n) \longrightarrow F(EG_+, E_n) \sim E_n \tag{1.2}$$

is faithful as an E_n-module. We also show that the non-commutative extension

$$F(BG_+, K_n) \longrightarrow F(EG_+, K_n) \sim K_n \tag{1.3}$$

is faithful and $F(BG_+, K_n)$ is a faithful E_n-module. A crucial difference from $F(BG_+, H\mathbb{F}_p)$ is that $K_n^*(BG_+)$ is always an Artinian algebra over $(K_n)_*$, and so if $K_n^*(BG_+) \neq K_n^*$ then it has infinite global dimension by Proposition 2.2.

Our approach to this involves introducing an analogue of the algebraic socle series for a module over an Artinian ring, and we show that this behaves well enough to prove our result.

We show in Section 5 that for a cyclic p-group C_{p^r}, the cochain extension $F(BC_{p^r}{}_+, E_n) \longrightarrow F(EC_{p^r}{}_+, E_n)$ is ramified and hence it is not a Galois extension. As a consequence it follows that the E_n-theory Eilenberg-Moore spectral sequence for such groups does not converge to its expected target, whereas work of Tilman Bauer indicates that this is not the case for Morava K-theory.

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Notation, etc. In discussing purely algebraic notions we will often use boldface symbols $\boldsymbol{A}, \boldsymbol{M}, \ldots$ to denote rings, modules, etc, while for topological objects such as S-algebras and their modules we will use italic symbols $A, M, \ldots$, thereby hopefully reducing the possibility of confusion between the two settings. For an associative S-algebra A, we denote by $\mathscr{D}_A$ the derived category of A-module spectra defined in [6, chapter III, construction 2.11].

We follow Lam [11, theorem 19.1] in using the phrase *local ring* to indicate a ring with a unique maximal left ideal (necessarily 2-sided and equal to its Jacobson radical); the quotient of such a ring by its Jacobson radical is a division ring. For non-commutative rings other terminology is often encountered such as *scalar local ring*.

Brave new Galois extensions. The following definition of a Galois extension is due to John Rognes [16]. Let A be a commutative S-algebra and let B be a commutative cofibrant A-algebra. Let G be a finite (discrete) group and suppose that there is an action of G on B by commutative A-algebra morphisms. Then B/A is a *G-Galois extension* if it satisfies the following two conditions:

- The natural map
$$A \longrightarrow B^{hG} = F(EG_+, B)^G$$
is a weak equivalence of A-algebras.
- There is a natural equivalence of B-algebras
$$\Theta\colon B \wedge_A B \xrightarrow{\sim} F(G_+, B)$$
induced from the action of G on the right hand factor of B.

Furthermore, B/A is a *faithful G-Galois extension* if it also satisfies

- B is faithful as an A-module, *i.e.*, for any A-module M, $B \wedge_A M \sim *$ implies that $M \sim *$.

Examples like (1.1) show that not every Galois extension is faithful.

2. Recollections on modules over Artinian algebras

In this section we review some standard algebraic background material; good sources for this are [1, 11].

Let $\boldsymbol{D}$ be a division ring. A ring $\boldsymbol{A}$ equipped with homomorphisms of rings $\eta\colon \boldsymbol{D} \longrightarrow \boldsymbol{A}$ and $\varepsilon\colon \boldsymbol{A} \longrightarrow \boldsymbol{D}$ is an *augmented $\boldsymbol{D}$-algebra* if the following diagram commutes.

$$\begin{array}{ccc} \boldsymbol{D} & \xrightarrow{=} & \boldsymbol{D} \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\varepsilon} & \\ & \boldsymbol{A} & \end{array}$$

The augmentation ε splits the unit η. We will also say that $\boldsymbol{A}$ is an *Artinian local* $\boldsymbol{D}$-algebra if it is Artinian and local.

If $\boldsymbol{A}$ is an Artinian local augmented $\boldsymbol{D}$-algebra, then the Jacobson radical of $\boldsymbol{A}$ is
$$\boldsymbol{J} = \operatorname{rad}(\boldsymbol{A}) = \ker \varepsilon.$$

By [11, theorem 4.12], $\boldsymbol{J}$ is nilpotent, say $\boldsymbol{J}^e = 0$ and $\boldsymbol{J}^{e-1} \neq 0$.

Lemma 2.1. *Let $\boldsymbol{A}$ be as above and let $\boldsymbol{M}$ be a left $\boldsymbol{A}$-module. If $\boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} = 0$, then $\boldsymbol{M} = 0$.*

Proof. Comparing the two horizontal exact sequences

$$\begin{array}{ccccccccc} & & \boldsymbol{J} \otimes_{\boldsymbol{A}} \boldsymbol{M} & \longrightarrow & \boldsymbol{A} \otimes_{\boldsymbol{A}} \boldsymbol{M} & \longrightarrow & \boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \boldsymbol{J}\boldsymbol{M} & \longrightarrow & \boldsymbol{M} & \longrightarrow & \boldsymbol{M}/\boldsymbol{J}\boldsymbol{M} & \longrightarrow & 0 \end{array}$$

we see that if $\boldsymbol{D} \otimes_{\boldsymbol{A}} \boldsymbol{M} = 0$ then
$$\boldsymbol{M} = \boldsymbol{J}\boldsymbol{M} = \ldots = \boldsymbol{J}^e\boldsymbol{M} = 0. \qquad \square$$

Let $\boldsymbol{M}$ be a left $\boldsymbol{A}$-module. The *socle* of $\boldsymbol{M}$ is the submodule

$$\operatorname{soc}^1 \boldsymbol{M} = \operatorname{soc} \boldsymbol{M} = \{x \in \boldsymbol{M} : \boldsymbol{J}x = 0\},$$

which can also be characterized as the sum of all the simple $\boldsymbol{A}$-submodules of $\boldsymbol{M}$. The *socle series* of $\boldsymbol{M}$ is the increasing sequence of submodules

$$0 = \operatorname{soc}^0 \boldsymbol{M} \subseteq \operatorname{soc}^1 \boldsymbol{M} \subseteq \ldots \subseteq \operatorname{soc}^k \boldsymbol{M} \subseteq \operatorname{soc}^{k+1} \boldsymbol{M} \subseteq \ldots \subseteq \boldsymbol{M},$$

where for each k the following is a pullback square

$$\begin{array}{ccc} \operatorname{soc}^{k+1} \boldsymbol{M} & \longrightarrow & \operatorname{soc}(\boldsymbol{M}/\operatorname{soc}^k \boldsymbol{M}) \\ \downarrow & & \downarrow \\ \boldsymbol{M} & \longrightarrow & \boldsymbol{M}/\operatorname{soc}^k \boldsymbol{M} \end{array}$$

so we have

$$\operatorname{soc}^k \boldsymbol{M} = \{x \in \boldsymbol{M} : \boldsymbol{J}^k x = 0\},$$

and

$$\operatorname{soc}^e \boldsymbol{M} = \boldsymbol{M}.$$

In fact, for small k

$$\operatorname{soc}^k \boldsymbol{M} \subset \operatorname{soc}^{k+1} \boldsymbol{M},$$

until we reach a value $k = k_0 \leqslant e$ for which $\operatorname{soc}^{k_0} \boldsymbol{M} = \boldsymbol{M}$.

It is also clear that given a homomorphism $\varphi \colon \boldsymbol{M} \longrightarrow \boldsymbol{N}$ of $\boldsymbol{A}$-modules there are compatible homomorphisms

$$\operatorname{soc}^k \boldsymbol{M} \longrightarrow \operatorname{soc}^k \boldsymbol{N}.$$

For details on the socle series see [11], especially Ex. 4.18, and [1, chapter I, section 1].

We end this section with a result that supplies an algebraic backdrop for some of our later work. We give a proof suggested by K. Brown.

Proposition 2.2. *Let $\boldsymbol{A}$ be a local left-Artinian ring which is not a division ring. Then*

$$\text{proj dim}(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \text{gl dim}\, \boldsymbol{A} = \infty,$$

where $\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})$ is the unique simple left $\boldsymbol{A}$-module.

Proof. Since $\boldsymbol{A}$ is local, it has only one simple module and therefore

$$\text{proj dim}(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \text{gl dim}\, \boldsymbol{A}.$$

Also, since $\boldsymbol{A}$ is Artinian it has a left ideal $\boldsymbol{I}$ isomorphic to $\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})$. The corresponding exact sequence

$$0 \to \boldsymbol{I} \longrightarrow \boldsymbol{A} \longrightarrow \boldsymbol{A}/\boldsymbol{I} \to 0 \tag{2.1}$$

cannot split since $\boldsymbol{A}$ is local and therefore it has no non-trivial idempotents.

If

$$\text{proj dim}(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) = \text{gl dim}\, \boldsymbol{A} < \infty,$$

then (2.1) would give

$$\text{proj dim}(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})) + 1 = \text{proj dim}(\boldsymbol{A}/\boldsymbol{I}) \leqslant \text{gl dim}\, \boldsymbol{A} = \text{proj dim}(\boldsymbol{A}/\operatorname{rad}(\boldsymbol{A})),$$

which is impossible. □

Remark 2.3. We end this section by noting that the above discussion works as well if we assume that $\boldsymbol{A}$ is graded, provided this is suitably interpreted. In our work below we are interested in $\mathbb{Z}$-gradings which are also 2-periodic, *i.e.*, for all $n \in \mathbb{Z}$, $(-)_{n+2} = (-)_n$. This can be interpreted as a $\mathbb{Z}/2$-grading.

3. Socle series in topology

Let D be an S-algebra for which $\pi_0 D$ is a non-trivial division ring, $\pi_1 D = 0$, and the graded ring $\pi_* D = \boldsymbol{D}$ has period two. Suppose that A is an S-algebra both under and over D, giving the following diagram of morphisms of S-algebras.

(3.1)
$$\begin{array}{ccc} D & \xrightarrow{=} & D \\ & {\scriptstyle\eta}\searrow \quad \nearrow{\scriptstyle\varepsilon} & \\ & A & \end{array}$$

We assume that $\boldsymbol{A} = \pi_* A$ is an Artinian local augmented $\boldsymbol{D}$-algebra, so that the augmentation ideal $\ker \varepsilon$ is the Jacobson radical of $\boldsymbol{A}$, $\operatorname{rad}(\boldsymbol{A})$, and also $\operatorname{rad}(\boldsymbol{A})^e = 0$ and $\operatorname{rad}(\boldsymbol{A})^{e-1} \neq 0$.

Remark 3.1. Let M be a left A-module. Then $\boldsymbol{M} = \pi_* M$ is a left $\boldsymbol{A}$-module and its socle $\operatorname{soc} \boldsymbol{M}$ is a $\boldsymbol{D}$-module through both the unit η and the augmentation ε, and these module structures agree since $\operatorname{rad}(\boldsymbol{A}) = \ker \varepsilon$.

Theorem 3.2. *There are functors* $\operatorname{soc}^k \colon \mathscr{D}_A \longrightarrow \mathscr{D}_A$ *for* $0 \leqslant k \leqslant e$ *such that*
(a) *for each* k, $\pi_*(\operatorname{soc}^k M) = \operatorname{soc}^k \boldsymbol{M}$*;*
(b) *there are natural transformations* $\operatorname{soc}^k M \longrightarrow \operatorname{soc}^{k+1} M$ *giving a commutative diagram*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_* \operatorname{soc}^1 M & \longrightarrow & \pi_* \operatorname{soc}^2 M & \longrightarrow & \cdots & \longrightarrow & \pi_* \operatorname{soc}^e M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longrightarrow & \operatorname{soc}^1 \boldsymbol{M} & \longrightarrow & \operatorname{soc}^2 \boldsymbol{M} & \longrightarrow & \cdots & \longrightarrow & \operatorname{soc}^e \boldsymbol{M} & \longrightarrow & 0 \end{array}$$

which is natural with respect to morphisms of A-modules.

Proof. As $\boldsymbol{D}$ is a graded division ring, $\operatorname{soc} \boldsymbol{M}$ is a $\boldsymbol{D}$-vector space. Since M is a D-module via the unit we can find a morphism of D-modules

(3.2)
$$\bigvee_j \Sigma^{s(j)} D \longrightarrow M$$

to realize an algebraic isomorphism

$$\bigoplus_j \boldsymbol{D}_{*-s(j)} \xrightarrow{\cong} \operatorname{soc} \boldsymbol{M} \subseteq \boldsymbol{M}.$$

Now Remark 3.1 implies that the morphism of (3.2) is actually one of A-modules. We set $\operatorname{soc} M = \bigvee_j \Sigma^{s(j)} D$.

Now we can repeat this on the cofibre $M/\operatorname{soc} M$ of the map $\operatorname{soc} M \longrightarrow M$, obtaining $\operatorname{soc}(M/\operatorname{soc} M) \longrightarrow M/\operatorname{soc} M$. We then define $\operatorname{soc}^2 M$ using the right hand pullback square in the diagram

$$\begin{array}{ccccc} \operatorname{soc} M & \longrightarrow & \operatorname{soc}^2 M & \longrightarrow & \operatorname{soc}(M/\operatorname{soc} M) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ \operatorname{soc} M & \longrightarrow & M & \longrightarrow & M/\operatorname{soc} M \end{array}$$

from which we see by a standard diagram chase that $\pi_*(\operatorname{soc}^2 M) \cong \operatorname{soc}^2 \boldsymbol{M}$. Continuing in this way we inductively build the socle tower

$$* \to \operatorname{soc}^1 M \longrightarrow \operatorname{soc}^2 M \longrightarrow \ldots \longrightarrow \operatorname{soc}^{e-1} M \longrightarrow \operatorname{soc}^e M = M,$$

using pullback squares

$$\begin{array}{ccc} \operatorname{soc}^{k+1} M & \longrightarrow & \operatorname{soc}(M/\operatorname{soc}^k M) \\ \downarrow & & \downarrow \\ M & \longrightarrow & M/\operatorname{soc}^k M \end{array}$$

for each k. These satisfy

$$\pi_*(\operatorname{soc}^k M) = \operatorname{soc}^k \boldsymbol{M}. \qquad \square$$

An important consequence of this construction is that there is a minimal k_0 for which $\operatorname{soc}^{k_0} M = M$, so since $\operatorname{soc}^{k_0-1} \boldsymbol{M} \neq \boldsymbol{M}$, using the fibre sequence

$$\operatorname{soc}^{k_0-1} M \longrightarrow M \longrightarrow M/\operatorname{soc}^{k_0-1} M, \tag{3.3}$$

we obtain $\pi_*(M/\operatorname{soc}^{k_0-1} M) \neq 0$.

Lemma 3.3. *The A-module D satisfies $\pi_*(D \wedge_A D) \neq 0$.*

Proof. There is a diagram of left D-modules induced from (3.1)

$$\begin{array}{ccc} D \wedge_D D & \xrightarrow{\quad = \quad} & D \wedge_D D \\ & \searrow \quad \nearrow & \\ & D \wedge_A D & \end{array}$$

in which $D \wedge_D D \cong D$. On applying $\pi_*(-)$ we see that $\pi_*(D \wedge_A D) \neq 0$. $\square$

Theorem 3.4. *Let M be an A-module for which $\pi_* M \neq 0$. Then $\pi_*(D \wedge_A M) \neq 0$,* i.e.*, D is a faithful A-module.*

Proof. Using the socle series we can find a fibration sequence as in (3.3),

$$M' \longrightarrow M \longrightarrow M'', \tag{3.4}$$

where $\boldsymbol{M}'' = \pi_* M'' \neq 0$, $J\boldsymbol{M}'' = 0$ and there is a short exact sequence

$$0 \to \pi_*(M') \longrightarrow \pi_*(M) \longrightarrow \pi_*(M'') \to 0. \tag{3.5}$$

As remarked in the proof of Theorem 3.2, M'' is weakly equivalent to a wedge of copies of suspensions of the A-module D. So $\pi_*(M'')$ is a direct sum of copies of suspensions of $\pi_*(D)$, hence by Lemma 3.3, $\pi_*(M'') \neq 0$. The fibre sequence (3.4) induces a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \pi_*(D \wedge_D M') & \longrightarrow & \pi_*(D \wedge_D M) & \twoheadrightarrow & \pi_*(D \wedge_D M'') & \\ & \downarrow & & \downarrow & & \downarrow & \\ & \pi_*(D \wedge_A M') & \longrightarrow & \pi_*(D \wedge_A M) & \twoheadrightarrow & \pi_*(D \wedge_A M'') & \Big) = \\ & & & & & \downarrow & \\ & & & & & \pi_*(D \wedge_D M'') & \end{array}$$

in which a non-zero element $x \in \pi_*(D \wedge_D M'')$ lifts to $\pi_*(D \wedge_D M)$ and so is in the image of composition passing through $\pi_*(D \wedge_A M)$. Therefore $\pi_*(D \wedge_A M) \neq 0$. $\square$

4. Lubin-Tate cohomology of classifying spaces

We will denote by E any Lubin-Tate spectrum such as E_n or E_n^{nr}, and then K will denote the corresponding version of Morava K-theory see [2] for details. The spectrum E is a commutative S-algebra, while K is an E-algebra in the sense of [6]. The homotopy groups $\pi_* E$ and $\pi_* K$ are 2-periodic and $\pi_0 E$ is Noetherian; $\pi_0 K$ is a field, although K is only homotopy commutative if p is an odd prime, while when $p = 2$ it is not even that. Nevertheless, we will view K as a kind of 'topological division ring'.

The following lemma will allows us in certain circumstances to relate modules over $E^{BG} = F(BG_+, E)$ to modules over $K^{BG} = F(BG_+, K)$.

Lemma 4.1. *For any E^{BG}-module M, there is isomorphism of K-modules*

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

In particular, there is an isomorphism of K-modules

$$K \wedge_{E^{BG}} E \cong K \wedge_{K^{BG}} K.$$

Proof. This follows from an obvious generalization of [6, proposition III.3.10]. Since there are isomorphisms of E-algebras $K \cong K \wedge_E E$ and $K^{BG} \cong K \wedge_E E^{BG}$, for any E^{BG}-module M,

$$\begin{aligned} K \wedge_{E^{BG}} M &\cong K \wedge_E (E \wedge_{E^{BG}} M) \\ &\cong (K \wedge_K K) \wedge_E (E \wedge_{E^{BG}} M) \\ &\cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M). \end{aligned}$$

□

Remark 4.2. By a standard argument making use of the Becker-Gottlieb transfer [4], after p-localization, $\Sigma^\infty BG_+$ is a retract of $\Sigma^\infty BG'_+$ where G' is any p-Sylow subgroup of G. In particular, when $p \nmid |G|$ we have

$$F(BG_+, E) \sim E, \quad F(BG_+, K) \sim K.$$

Theorem 4.3. *Let G be a finite group.*
(a) *The K-cohomology $K^*(BG_+)$ is a finite dimensional K^*-vector space and the E-cohomology $E^*(BG_+)$ is a finitely generated E^*-module.*
(b) *If $K^*(BG_+)$ is concentrated in even degrees, then $E^*(BG_+)$ is a free E^*-module of finite rank and*

$$K^*(BG_+) = K^* \otimes_{E^*} E^*(BG_+) = E^*(BG_+)/\mathfrak{m}E^*(BG_+).$$

(c) *$K^*(BG_+)$ is an augmented Artinian local K^*-algebra whose maximal ideal is nilpotent. Hence $E^*(BG_+)$ is an augmented pro-Artinian local E^*-algebra,*

$$E^*(BG_+) = \lim_r E^*(BG_+)/\mathfrak{m}^r E^*(BG_+).$$

Proof. (a) See [7, 8] for example.
(b) See [9, proposition 2.5].
(c) Following Remark 4.2, we can reduce to the case where G is a p-group using the transfer associated with a p-Sylow subgroup $G' \leqslant G$. The case of a cyclic p-group C_{p^r} is well known and

$$K^*(BC_{p^r +}) = K^*[y]/(y^{p^r}).$$

The case of a general p-group G of order p^m follows by induction on m since there is always a normal subgroup $N \lhd G$ of index p and this permits an argument with the Serre spectral sequence associated with the fibration

$$BN \longrightarrow BG \longrightarrow BC_p$$

as used in [13] to calculate $K^*(BG_+)$ from knowledge of $K^*(BN_+)$ as input. □

It is known that $K^*(BG_+)$ need not be concentrated in even degrees [10].

We are interested in the E-algebras $E^{BG} = F(BG_+, E)$ and $K^{BG} = F(BG_+, K)$, each of which is K-local. Of course the diagonal $BG \longrightarrow BG \times BG$ induces the product on each of these, but only E^{BG} is strictly commutative, while K^{BG} is homotopy commutative when $p \neq 2$ and merely associative when $p = 2$. At the level of homotopy groups, $E^*(BG_+) = \pi_*(E^{BG})$ and $K^*(BG_+) = \pi_*(K^{BG})$ are both graded commutative.

Now we can apply our earlier results to give

Theorem 4.4. *For any finite group G, E and K are faithful E^{BG}-modules in the K-local category.*

Proof. It suffices to show that K is faithful. By Lemma 4.1, for any E^{BG}-module there is an isomorphism

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

The natural morphism of E-algebras

$$K \wedge_E F(BG_+, E) \longrightarrow F(BG_+, K \wedge_E E)$$

is a weak equivalence since K is a finite cell E-module, so by [6, theorem III.4.2] it is enough to know that

$$(K \wedge_E E) \wedge_{K^{BG}} (K \wedge_E M) \cong K \wedge_{K^{BG}} (K \wedge_E M) \nsim *.$$

If M is K-local and non-trivial, then $K \wedge_{K^{BG}} (K \wedge_E M) \nsim *$, because we know from Theorem 3.4 that K is faithful as a K^{BG}-module. □

5. Galois theory and E^{BG}

In this section we will consider extensions of the form

$$E^{BG} = F(BG_+, E) \longrightarrow F(EG_+, E) \sim E$$

with G a finite group and consider whether or not they are Galois. Since we know they are faithful, the issue is whether such an extension satisfies the unramified condition that the map

$$\Theta \colon F(BG_+, E) \wedge_{E^{BG}} F(BG_+, E) \longrightarrow F(G_+, E)$$

is weak equivalence, and therefore there is a weak equivalence

$$E \wedge_{E^{BG}} E \xrightarrow{\sim} \prod_G E. \tag{5.1}$$

In particular, this condition implies that $\pi_*(E \wedge_{E^{BG}} E)$ is concentrated in even degrees.

We begin by considering the case of cyclic p-groups C_{p^r}.

Theorem 5.1. *For each $r \geqslant 1$, the extension*

$$E^{BC_{p^r}} = F(BC_{p^r +}, E) \longrightarrow F(EC_{p^r +}, E)$$

is ramified and hence it is not C_{p^r}-Galois.

Proof. We recall (see for example [8, lemma 5.1]) that

$$(E^{BC_{p^r}})_* = E^*[[y]]/([p^r]y),$$

where $y \in (E^{BC_{p^r}})_0 = E^0(BC_{p^r +})$ and the p-series $[p]y$ has the form

$$[p]y \equiv y^{p^n} \mod \mathfrak{m},$$

so for each $r \geqslant 1$ the p^r-series is inductively defined by

$$\begin{aligned}[p^r]y = [p]([p^{r-1}]y) &= p^r y + \cdots + y^{p^{rn}} + \cdots \\ &\equiv y^{p^{rn}} \mod \mathfrak{m}.\end{aligned}$$

By the Weierstrass preparation theorem, there is a polynomial

$$\langle p^r \rangle y = p^r + \cdots + y^{p^{rn}-1} \equiv y^{p^{rn}-1} \mod \mathfrak{m}$$

for which

$$[p^r]y = y\langle p^r \rangle y(1 + y f_r(y)),$$

where $f_r(y) \in E^*[[y]]$. Then we have

$$(E^{BC_{p^r}})_* = E^*[[y]]/(y\langle p^r \rangle y).$$

The $(E^{BC_{p^r}})_*$-module E_* admits the periodic minimal free resolution

(5.2)

$$0 \leftarrow E_* \longleftarrow (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \longleftarrow \ldots,$$

so $\mathrm{Tor}_{*,*}^{(E^{BC_{p^r}})_*}(E_*,E_*)$ is the homology of the complex

$$0 \leftarrow E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I\otimes y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I\otimes\langle p^r\rangle y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_*$$
$$\xleftarrow{I\otimes y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{I\otimes\langle p^r\rangle y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \longleftarrow \cdots,$$

which is equivalent to

(5.3) $$0 \leftarrow E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \longleftarrow \cdots.$$

Since E_* is torsion-free, for $s \geqslant 0$ this gives

(5.4) $$\mathrm{Tor}_{s,*}^{(E^{BC_{p^r}})_*}(E_*,E_*) = \begin{cases} E_* & \text{if } s=0,\\ E_*/p^rE_* & \text{if } s \text{ is odd},\\ 0 & \text{otherwise.}\end{cases}$$

Thus in the Künneth spectral sequence

(5.5) $$\mathrm{E}^2_{s,t} = \mathrm{Tor}_{s,t}^{(E^{BC_{p^r}})_*}(E_*,E_*) \Longrightarrow \pi_{s+t}(E \wedge_{E^{BC_{p^r}}} E)$$

there can be no non-trivial differentials since for degree reasons the only possibilities involve E_*-module homomorphisms of the form

$$d^{2k-1}\colon \mathrm{E}^2_{2k-1,t} = E_t/p^rE_t \longrightarrow \mathrm{E}^2_{0,t+2k-2} = E_{t+2k-2},$$

with torsion-free target. This shows that the odd degree terms in $\pi_*(E \wedge_{E^{BC_{p^r}}} E)$ are not zero, contradicting the unramified condition 5.1 for a Galois extension. □

Remark 5.2. If we work rationally, then the Künneth spectral sequence

$$\mathrm{E}^2_{s,t}(C_{p^r};\mathbb{Q}) = \mathrm{Tor}_{s,t}^{((E^{BC_{p^r}})\mathbb{Q})_*}(E_*\mathbb{Q},E_*\mathbb{Q}) \Longrightarrow \pi_{s+t}(E\mathbb{Q} \wedge_{(E^{BC_{p^r}})\mathbb{Q}} E\mathbb{Q})$$

has $\mathrm{E}^2_{s,*}(C_p^r;\mathbb{Q}) = 0$ except when $s=0$, giving

$$\pi_*(E\mathbb{Q} \wedge_{(E^{BC_{p^r}})\mathbb{Q}} E\mathbb{Q}) = E_*\mathbb{Q} \otimes_{(E^{BC_{p^r}})_*\mathbb{Q}} E_*\mathbb{Q}.$$

This shows that higher filtration terms in the Künneth spectral sequence 5.5 contribute p-torsion.

Now we extend Theorem 5.1 to arbitrary p-groups.

Theorem 5.3. *Let G be a non-trivial p-group. Then the extension*

$$F(BG_+,E) \longrightarrow F(EG_+,E)$$

is not G-Galois. More precisely, this extension is ramified:

$$F(EG_+,E) \wedge_{F(BG_+,E)} F(EG_+,E) \nsim \prod_G F(EG_+,E).$$

Proof. Choose a non-trivial epimorphism $G \longrightarrow C_p$; then for some $k \geqslant 1$ there is a factorization

(5.6) $$C_{p^k} \rightarrowtail G \twoheadrightarrow C_p$$

inducing morphisms between the associated Künneth spectral sequences

(5.7) $$\mathrm{E}^r_{**}(C_p) \longrightarrow \mathrm{E}^r_{**}(G) \longrightarrow \mathrm{E}^r_{**}(C_{p^k}).$$

As we saw in the proof of Theorem 5.1, the two outer spectral sequences have trivial differentials. We will analyze the composite morphism $\mathrm{E}^2_{**}(C_p) \longrightarrow \mathrm{E}^2_{**}(C_{p^k})$.

On choosing generators appropriately, the canonical epimorphism $C_{p^k} \longrightarrow C_p$ induces the E_*-algebra monomorphism

$$(E^{BC_p})_* = E_*[[y]]/([p]y) \longrightarrow (E^{BC_{p^k}})_* = E_*[[y]]/([p^k]y); \quad y \mapsto [p^{k-1}]y,$$

hence the induced map between the two resolutions of the form (5.2) is

$$\begin{array}{ccccccccc}
0 \longleftarrow & E_* & \longleftarrow & (E^{BC_p})_* & \xleftarrow{\;y\;} & (E^{BC_p})_* & \xleftarrow{\;\langle p\rangle y\;} & (E^{BC_p})_* & \xleftarrow{\;y\;} \cdots \\
 & \downarrow{=} & & \downarrow{\rho_0} & & \downarrow{\rho_1} & & \downarrow{\rho_2} & \\
0 \longleftarrow & E_* & \longleftarrow & (E^{BC_{p^k}})_* & \xleftarrow{\;y\;} & (E^{BC_{p^k}})_* & \xleftarrow{\;\langle p^k\rangle y\;} & (E^{BC_{p^k}})_* & \xleftarrow{\;y\;} \cdots
\end{array}$$

where the vertical maps are given by

$$\rho_{2s}\colon g(y) \mapsto g([p^{k-1}]y), \quad \rho_{2s-1}\colon h(y) \mapsto h([p^{k-1}]y)\langle p^{k-1}\rangle y.$$

Applying $E_* \otimes_{(E^{BC_{p^r}})_*} (-)$ to the first and second rows with $r = 1$ and k respectively, we obtain a map of chain complexes

$$\begin{array}{ccccccc}
0 \longleftarrow & E_* & \xleftarrow{\;0\;} & E_* & \xleftarrow{\;p\;} & E_* & \xleftarrow{\;0\;} \cdots \\
 & \downarrow{=\,\rho'_0} & & \downarrow{\rho'_1 = p^{k-1}\cdot} & & \downarrow{=\,\rho'_2} & \\
0 \longleftarrow & E_* & \xleftarrow{\;0\;} & E_* & \xleftarrow{\;p^k\;} & E_* & \xleftarrow{\;0\;} \cdots
\end{array}$$

where

$$\rho'_{2s} = \mathrm{id}, \quad \rho'_{2s-1} = p^{k-1} \cdot \ .$$

Applying this to the odd degree terms given in (5.4) we see that the induced map

$$E_*/pE_* \xrightarrow{p^{k-1}\cdot} E_*/p^kE_*$$

is always a monomorphism. Therefore in (5.7), the first of the induced morphisms

$$\mathrm{E}^2_{**}(C_p) \longrightarrow \mathrm{E}^r_{**}(G) \longrightarrow \mathrm{E}^r_{**}(C_{p^k})$$

is a monomorphism. There can be no higher differentials killing elements in its image because they map to non-trivial elements of $\mathrm{E}^2_{**}(C_{p^k})$ which survive the right hand spectral sequence. This shows that $\mathrm{E}^\infty_{**}(G)$ contains elements of odd degree, and as in the cyclic group case this is incompatible with the unramified condition. □

We can extend this result to the class of p-nilpotent groups. A finite group G is p-nilpotent if one and hence each p-Sylow subgroup $P \leqslant G$ has a normal p-complement, *i.e.*, there is a normal subgroup $N \lhd G$ with $p \nmid |N|$ and $G = PN = P \ltimes N$. A convenient summary of the properties of such groups can be found in [12, section 7], see also [15].

Corollary 5.4. *If G is a p-nilpotent group for which p divides $|G|$, then the extension*

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

is ramified and so is not G-Galois.

Proof. By a result of Tate [18], G being p-nilpotent is equivalent to the restriction homomorphism giving an isomorphism

$$\mathrm{res}^G_P\colon H^*(BG; \mathbb{F}_p) \xrightarrow{\cong} H^*(BP; \mathbb{F}_p),$$

and in fact it is sufficient that this holds in degree 1. Comparison of the Serre spectral sequences for $K^*(BG_+)$ and $K^*(BP_+)$ shows that

$$K^*(BG_+) \xrightarrow{\cong} K^*(BP_+).$$

It now follows that

$$E^*(BG_+) \xrightarrow{\cong} E^*(BP_+).$$

and the result can be deduced from Theorem 5.3. □

Remark 5.5. The condition of G being a p-nilpotent group should not be confused with the condition that the conjugation action of G on $\mathbb{F}_p[G]$ is nilpotent. The latter is used in [16, proposition 5.6.3] to ensure convergence of the Eilenberg-Moore spectral sequence and so to prove that for such groups

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p)$$

is a G-Galois extension. The example of $G = \Sigma_3$, the third symmetric group, for the prime $p = 2$ illustrates this. For each of the Sylow 2-subgroups

$$\{\mathrm{id}, (1,2)\},\ \{\mathrm{id}, (1,3)\},\ \{\mathrm{id}, (2,3)\}$$

has as normal complement

$$N = \{\mathrm{id}, (1,2,3), (1,3,2)\},$$

therefore Σ_3 is 2-nilpotent. However, the Σ_3-module $\mathbb{F}_2[\Sigma_3]$ contains the 2-dimensional non-trivial simple submodule

$$V = \{x(1,2) + y(1,3) + z(2,3) : x + y + z = 0\},$$

so by Jordan-Hölder theory every composition series for $\mathbb{F}_2[\Sigma_3]$ must have this as a composition factor. Hence the action of Σ_3 on $\mathbb{F}_2[\Sigma_3]$ cannot be nilpotent.

6. Some observations on the Eilenberg-Moore spectral sequence

In [16, section 5.6], it is shown that for a finite p-group G, the Eilenberg-Moore spectral sequence with

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}^{H^*(BG_+;\mathbb{F}_p)}_{s,t}(\mathbb{F}_p, \mathbb{F}_p) \tag{6.1}$$

converges to $\pi_*(F(G_+, H\mathbb{F}_p)) = \pi_*(\prod_G \mathbb{F}_p)$. By comparing it with the Künneth spectral sequence for $\pi_*(H\mathbb{F}_p \wedge_{F(BG_+,H\mathbb{F}_p)} H\mathbb{F}_p)$, it is also shown that

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p)$$

is a G-Galois extension.

Let us consider in detail the case $G = C_p$ for p an odd prime. The case when $p = 2$ is similar. First we write

$$H^*(BC_p) = H^*(BC_{p+}; \mathbb{F}_p) = \mathbb{F}_p[y] \otimes \Lambda(z),$$

where $y \in H^2(BC_p)$ and $z \in H^1(BC_p)$. Then (6.1) becomes

$$\mathrm{E}^2_{**} = \Gamma(\sigma z) \otimes \Lambda(\sigma y),$$

where $\sigma y \in \mathrm{E}^2_{1,-2}$ and $\sigma z \in \mathrm{E}^2_{1,-1}$ are the suspensions of y and z, see [14]. Writing $\gamma_r = \gamma_r(\sigma z)$. The first non-trivial differential is

$$d^{p-1}\gamma_p = \sigma y,$$

and we have

$$\mathrm{E}^p_{**} = \mathbb{F}_p[\zeta]/(\zeta^p) \otimes \Gamma(\gamma_{p^2}) \otimes \Lambda(\gamma_p \sigma y),$$

where ζ represents the class of σz. The remaining differentials are determined by the formulae

$$d^{p^s - p^{s-1} - 1}\gamma_{p^s} = \gamma_{p^{s-1}}\sigma y$$

in

$$\mathrm{E}^{p^s - p^{s-1} - 1}_{**} = \mathbb{F}_p[\zeta]/(\zeta^p) \otimes \Gamma(\gamma_{p^s}) \otimes \Lambda(\gamma_{p^{s-1}}\sigma y).$$

Finally we have

$$\mathrm{E}^\infty_{**} = \mathbb{F}_p[\zeta]/(\zeta^p),$$

which is an avatar of $\prod_{C_p} \mathbb{F}_p$. These differentials are forced by the known answer and multiplicativity, and are also related to the discussion of [14, section 6]. For Lubin-Tate theory $(E^{BC_{p^r}})_*$ is free over E_* and the comparison of the Eilenberg-Moore with the Künneth spectral sequence together with our Theorems 5.1 and 5.3 has the following consequence.

Proposition 6.1. *For the cyclic p-group C_{p^r} the E-theory Eilenberg-Moore spectral sequence for BC_{p^r} with*

$$^{\text{L-T}}\mathrm{E}^2_{s,t} = \mathrm{Tor}^{(E^{BC_{p^r}})_*}(E_*, E_*)$$

does not converge to $\pi_*(\prod_{C_{p^r}} E)$.

Just as in the $H\mathbb{F}_p$ case, we can compare the Morava K-theory based Eilenberg-Moore spectral sequence with the Künneth spectral sequence. Work of Bauer [3] on the convergence of the Cotor-version of this Eilenberg-Moore spectral sequence shows that the corresponding spectral sequence converges for $G = C_p$ and odd primes p, and therefore

$$K \wedge_{K^{BC_p}} K \sim \prod_{C_p} K.$$

The extension of S-algebras $K^{BC_p} \longrightarrow K^{EC_p}$ can be interpreted as a Galois extension of non-commutative S-algebras.

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