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A continuous path of singular masas in the hyperfinite II_1 factor

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Abstract

Using methods of R.J.Tauer [13] we exhibit an uncountable family of singular masas in the hyperfinite II_1 factor R all with Pukánszky invariant $\{1\}$, no pair of which are conjugate by an automorphism of R. This is done by introducing an invariant $\Gamma(A)$ for a masa A in a II_1 factor N as the maximal size of a projection $e \in A$ for which Ae contains non-trivial centralising sequences for eNe. The masas produced give rise to a continuous map from the interval $[0,1]$ into the singular masas in R equiped with the $d_{\infty,2}$-metric.

A result is also given showing that the Pukánszky invariant [11] is $d_{\infty,2}$-upper semi-continuous. As a consequence, the sets of masas with Pukánszky invariant $\{n\}$ are all closed.

1 Introduction

The study of maximal abelian self-adjoint von Neumann subalgebras (masas) in II_1 factors dates back to J.Dixmier [5] in 1954, who classified them using normalisers. Given a masa A in a II_1 factor N, the normaliser group $\mathcal{N}(A)$ consists of all the unitaries $u \in N$ with $uAu^* = A$. The masa A is *Cartan* if this normaliser group generates N as a von Neumann algebra whereas at the other end of the spectrum, A is called *singular* if $\mathcal{N}(A) \subset A$.

Given two Cartan masas A and B in the hyperfinite II_1 factor R, there is an automorphism θ of R with $\theta(A) = B$ ([3]). We say that masas A and B with this last property are conjugate via an automorphism of R. The most sucessful invariant for distinguishing between non-conjugate singular masas is that of L.Pukánszky [11], which he used to give countably many

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pairwise non-conjugate singular masas in R. More recently, E.Størmer and S.Neshveyev [8] have used the Pukánszky invariant to exhibit uncountably many pairwise non-conjugate singular masas in R and they also give two non-conjugate singular masas in R with the same Pukánskzy invariant. One of our objectives here is to produce uncountably main non-conjugate singular masas in the hyperfinite II_1 factor with the same Pukánszky invariant. This result, stated formally as Theorem 1.1 below, follows directly from Theorem 5.1.

Theorem 1.1. *There exist uncountably many singular masas in the hyperfinite* II_1 *factor R, each with Pukánszky invariant* $\{1\}$*, such that no pair of these masas is conjugate by an automorphism of R.*

To show the non-conjugacy of pairs of masas we look for non-trivial centralising sequences for R lying in these masas — the idea used by Størmer and Neshveyev in [8] to distinguish between two singular masas with Pukánszky invariant $\{1\}$. The presence of non-trivial centralising sequences inside masas has also been used by A.Connes and V.Jones [4] to give a factor containing two non-conjugate Cartan masas, and by V.Jones and S.Popa [6] in the context of non-conjugate semi-regular masas whose normalisers generate the same irreducible subfactor of R.

There is a natural metric, $d_{\infty,2}$, on the space of all masas of a II_1 factor, [10]. The uncountably many masas we shall produce for Theorem 1.1, will actually give us a continuous map from the unit interval, $[0,1]$ into this metric space — a continuous path of pairwise non-conjugate singular masas.

In the next section we state some background, defining the metric $d_{\infty,2}$, the Pukánszky invariant and Tauer masas. In section 3 we discuss the behaviour of the Pukánskzy invariant on limits of sequences of masas, showing that it is upper semicontinous and that the sets of masas with invariant $\{n\}$ are all closed (Theorem 3.2, Corollary 3.3). Next, in section 4, we define a Γ-invariant for masas using centralising sequences and establish some basic properties for later use. It is this invariant we use in section 5 to show the non-conjugacy of the masas we construct to establish Theorem 5.1, the main result of the paper. The work in this paper forms part of sections 3.1 and 3.3 of the second authors PhD thesis [15].

2 Preliminaries

Let N be a II_1 factor. Write tr for the faithful normal trace on N, and let $\|x\|_2 = \mathrm{tr}(x^*x)^{1/2}$ be the Hilbert space norm induced on N by tr. Write

$L^2(N)$ for the completion of N in this norm. Given a linear map $\Phi : N_1 \to N_2$ between two II_1 factors write $\|\Phi\|_{\infty,2}$ for the norm of Φ regarded as a map from N_1 into $L^2(N_2)$ [12], that is

$$\|\Phi\|_{\infty,2} = \sup \{ \|\Phi(x)\|_2 \mid x \in N_1, \|x\| \leq 1 \} .$$

Given a von Neumann subalgebra M of N, let $\mathbb{E}_M$ be the unique trace-preserving normal conditional expectation from N onto M. This conditional expectation is obtained by restricting to N the orthogonal projection e_M from $L^2(N)$ onto $L^2(M)$. In [10] a metric, $d_{\infty,2}$, is introduced on the set of all von Neumann subalgebras of N, by

$$d_{\infty,2}(M_1, M_2) = \|\mathbb{E}_{M_1} - \mathbb{E}_{M_2}\|_{\infty,2} .$$

This metric is equivalent to an older metric of E.Christensen defined in [2]. As a consequence the set of all von Neumann subalgebras equipped with $d_{\infty,2}$ is a complete metric space, and the subsets of all masas, all singular masas, all subfactors and all irreduicble subfactors are closed, [2].

To define the Pukánskzy invariant [11] of a masa in the separable II_1 factor N, we form the standard representation of N acting by left multplication on $L^2(N)$. Let J denote the modular conjugation operator on $L^2(N)$ given by extending $x \mapsto x^*$ from N. For each $x \in N$, JxJ is the operator of right mutiplication by x^* and $x \mapsto JxJ$ is a conjugate linear anti-isomorphism of N onto N'. Given a masa A in N, let $\mathcal{A} = (A \cup JAJ)''$ — an abelian von Neumann subalgebra of $\mathbb{B}\left(L^2(N)\right)$, so that $\mathcal{A}'$ is type I. The orthogonal projection e_A from $L^2(N)$ onto $L^2(A)$ lies in $\mathcal{A}$ and $\mathcal{A}'e_A = \mathcal{A}e_A = Ae_A$ — an abelian algebra. The Pukánszky invariant is obtained by taking the type decomposition of $\mathcal{A}'(1 - e_A)$. More formally, $\mathrm{Puk}\,(A)$ is the subset of $\mathbb{N} \cup \{\infty\}$ consisting of all those n for which there is a non-zero projection $p \leq 1 - e_A$ in $\mathcal{A}$ such that $\mathcal{A}'p$ is type I_n [11].

We shall use the methods of R.J.Tauer [13] to construct masas in the hyperfinite II_1 factor R. The second author introduced the concept of a Tauer masa in R in [14, 15]. A masa A in R is said to be a *Tauer masa* if there exists an increasing chain $(N_n)_{n=1}^{\infty}$ of matrix algebras with $(\bigcup_{n=1}^{\infty} N_n)'' = R$, such that $A \cap N_n$ is a masa in N_n for each n. In this case we write A_n for $A \cap N_n$ and say for emphasis that A is *Tauer with respect to* $(N_n)_{n=1}^{\infty}$. Tauer masas have Pukánszky invariant $\{1\}$, [14, Theorem 4.1]. Chains $(N_n)_{n=1}^{\infty}$ of matrix algebras in R can always be realised as a tensor products. More formally, there are finite dimensional subfactors $(M_m)_{m=1}^{\infty}$ of R such that we have $N_n = \bigotimes_{m=1}^{n} M_m$, for each n. We use the notation of [14, 15] to consider the inclusions $A_{n_1} \subset A_{n_2}$ of approximates of a Tauer masa A with respect to the chain $(N_n)_{n=1}^{\infty}$. Let $\mathcal{P}_{\min}(A_{n_1})$ denote the set of minimal projections of A_{n_1}. The finite dimensional approximation A_{n_2} can then be written as

$$A_{n_2} = \bigoplus_{e \in \mathcal{P}_{\min}(A_{n_1})} e \otimes A_{n_2,n_1}^{(e)}, \tag{2.1}$$

for some masas $A_{n_2,n_1}^{(e)}$ in $\bigotimes_{m=n_1+1}^{n_2} M_m$.

In [14, Theorem 3.2] a technical criteron was given for a Tauer masa to be singular in terms of these $A_{n_2,n_1}^{(e)}$. We use part of this calculation, which is essentially Proposition 3.5 of [14]; the exact statement given can be found as Proposition 2.2.2 of [15].

Proposition 2.1. *Let A be a Tauer masa in R with respect to the subfactors $(N_n)_{n=1}^{\infty}$. If for infinitely many $n_1 \in \mathbb{N}$, each minimal projection e of A_{n_1} and $\epsilon > 0$, there is an $n_2 > n_1$ and a unitary $w_e \in A_{n_2,n_1}^{(e)}$ with*

$$\left\| \mathbb{E}_{A_{n_2,n_1}^{(f)}} (w_e) \right\|_2 \leq \epsilon,$$

for every minimal projection $f \neq e$ in A_{n_1}, then A is singular.

3 Semi-continuity of the Pukánszky invariant

The key tool in determining the limiting behaviour of the Pukánskzy invariant on sequences of masas is a perturbation theorem for subalgebras of a II_1 factor [10, Theorem 6.5], which we state below for the convenience of the reader.

Theorem 3.1 ([10, Theorem 6.5 (ii)])**.** *If A and B are masas in a separable II_1 factor N with $d_{\infty,2}(A,B) \leq \epsilon$, then there are projections $p \in A$ and $q \in B$, and a unitary $u \in N$ satisfying*

- $u(Bq)u^* = Ap$;
- $\|u - \mathbb{E}_B(u)\|_2 \leq 45\epsilon$;
- $tr(p) = tr(q) \geq 1 - (15\epsilon)^2$.

Theorem 3.2. *Let A_n be a sequence of masas in a separable II_1 factor N converging in the $d_{\infty,2}$-metric to a von Neumann subalgebra B of N. This B is a masa in N, and*

$$Puk(B) \subset \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} Puk(A_n). \tag{3.1}$$

Proof. That the set of masas is $d_{\infty,2}$-closed is due to E.Christensen in [2]. For each n, we apply Theorem 3.1 to the pair (A_n, B) to obtain projections $p_n \in A_n$, $q_n \in B$ and a unitary $u_n \in N$ satisfying the conditions of the theorem. Take $B_n = u_n^* A_n u_n$ — a masa in N which has $B_n q_n = B q_n$, by the first property of Theorem 3.1.

As A_n converges to B in $d_{\infty,2}$, the last property of Theorem 3.1 ensures that

$$\lim_{n\to\infty} \|1 - q_n\|_2 = 0.$$

For any $x \in N$,

$$\begin{aligned}
\|q_n J q_n J x - x\|_2 = \|q_n x q_n - x\|_2 &\leq \|q_n x - x\|_2 + \|q_n(x q_n - x)\|_2 \\
&\leq \|q_n - 1\|_2 \left(\|x\| + \|q_n x\|\right) \\
&\leq 2\,\|x\|\,\|q_n - 1\|_2\,,
\end{aligned}$$

so that the projections $q_n J q_n J$ in $\mathcal{B}_n \cap \mathcal{B}$ converge strongly to 1, by density of N in $L^2(N)$.

Given some $m \in \mathrm{Puk}\,(B)$, there must be a central projection $f \in \mathcal{B} = \mathcal{B}' \cap \mathcal{B}$ with $f \leq 1 - e_B$, such that $\mathcal{B}' f$ is of type I_m. As $q_n J q_n J f$ converges strongly to f we must have $q_n J q_n J f \neq 0$ for sufficiently large n, those with $n \geq n_1$ say. Now

$$\mathcal{B}_n' q_n J q_n J = \mathcal{B}' q_n J q_n J,$$

a type I von Neumann algebra with centre $\mathcal{B}_n q_n J q_n J = \mathcal{B} q_n J q_n J$. For $n \geq n_1$, $q_n J q_n J f$ is a non-zero projection in this centre, and $\mathcal{B}_n' q_n J q_n J f$ is then a central cutdown of $\mathcal{B}' f$, so a type I_m von Neumann algebra.

Observe that q_n and $J q_n J$ commute with both e_B and e_{B_n}, as $q_n \in B \cap B_n$. We also have $q_n e_{B_n} = q_n e_B$ and $J q_n J e_{B_n} = J q_n J e_B$, as $B_n q_n = B q_n$. In this way, $q_n J q_n J f \leq 1 - e_{B_n}$, so that $m \in \mathrm{Puk}\,(B_n)$, for $n \geq n_1$. As B_n and A_n are unitarily equivalent, $m \in \mathrm{Puk}\,(A_n)$ for all $n \geq n_1$, exactly as required. ☐

In the special case when the Pukánszky invariant of each A_n is $\{n\}$, the only possibility for the Pukánszky invariant of the limit masa B is also $\{n\}$.

Corollary 3.3. *Let N be a separable II_1 factor. For each $n \in \mathbb{N} \cup \{\infty\}$, the set of all masas with Pukánszky invariant $\{n\}$ is $d_{\infty,2}$-closed.*

In general we do not have equality in (3.1).

Example 3.4. Let A be a masa in the hyperfinite II_1 factor R with Pukánszky invariant $\{1\}$. Take projections $p_n \neq 1$ in A with $p_n \to 1$ strongly.

For each n, let B_n be a masa in the hyperfinite II_1 factor $(1-p_n)R(1-p_n)$ with Pukánszky invariant $\{2\}$. The existance of such masas dates back to Pukánszky's original examples in [11]. Define

$$A_n = \{\, ap_n + b \mid a \in A, b \in B_n \,\}\,,$$

which is a masa in R. It is then immediate that $d_{\infty,2}(A_n, A) \to 0$ as $n \to \infty$ and that both 1 and 2 lie in $\operatorname{Puk}(A_n)$, for each n. It should be noted that we do not know the exact Pukánskzy invariant of these A_n, only that 1 and 2 are members of $\operatorname{Puk}(A_n)$.

We can also use Theorem 3.2, to show that the Pukánszky invariant can not be used to give a continuous path of non-conjugate singular masas even though the cardinality of the set of non-conjugate singular masas is large enough. The proof is omited, it can be found in [15, Corollary 3.1.8].

Corollary 3.5. *Let N be a separable II_1 factor. There is no continuous map $t \mapsto A(t)$ from $[0,1]$ into the set of all masas in N equiped with the $d_{\infty,2}$-metric such that $t \mapsto \mathit{Puk}(A(t))$ is injective.*

4 A Γ-invariant for masas

To show that all the uncountably many masas we shall produce are pairwise non-conjugate via automorphisms of the underlying II_1 factor, we introduce a conjugacy invariant.

Definition 4.1. Let A be a masa in a II_1 factor N. Define $\Gamma(A)$ to be the supremum of $\mathrm{tr}(p)$ over all projections $p \in A$ such that Ap contains non-trivial centralising sequences for pNp. If $\Gamma(A) = 0$, then we say that A is *totally non-*Γ.

Recall that a centralising sequence in a non-empty subset B of a II_1 factor N is a sequence $\{x_n\} \subset B$ with

$$\|x_n y - y x_n\|_2 \to 0 \quad \text{for all} \quad y \in N.$$

The centralising sequence $\{x_n\} \subset B$ is *trivial* if there is a sequence $\{\lambda_n\} \subset \mathbb{C}$ with $\|x_n - \lambda_n\|_2 \to 0$.

It is immediate that $\Gamma(A)$ is a conjugacy invariant of A, in the sense that for an automorphism θ of N, we have $\Gamma(\theta(A)) = \Gamma(A)$.

We shall produce masas in a similar fashion to Example 3.4, taking a 'direct sum' of a Γ-masa, that is one containing non-trivial centralising sequences for its underlying II_1 factor, and a totally non-Γ masa. The next lemma is the tool that allows us to do this.

Lemma 4.2. *Let A be a masa in a II_1 factor N. Suppose that there is a projection $p \in A$ such that*

- *Ap contains non-trivial centralising sequences for pNp;*
- *$A(1-p)$ is totally non-Γ in $(1-p)N(1-p)$.*

Then $\Gamma(A) = tr(p)$.

Proof. Take a projection $r \in A$ such that Ar contains non-trivial centralising sequences for rNr. To obtain a contradiction, suppose that $r \not\leq p$. Let $(x_n)_{n=1}^{\infty}$ be a non-trivial centralising sequence for rNr in Ar, write $y_n = x_n pr = x_n rp$ and $z_n = x_n r(1-p)$ so that $x_n = y_n + z_n$ for all n. The sequence $(z_n)_{n=1}^{\infty}$ is a centralising sequence of $r(1-p)Nr(1-p)$ and so is trivial by hypothesis. Without losing generality, we may assume that $z_n = r(1-p)$ for all n.

Take a partial isometry $v \in N$ with $v^*v \leq r(1-p)$ and $vv^* = p_0 \leq pr$, so that $y_n v = x_n v$ and $v = vz_n = vx_n$. Now

$$\|(y_n - 1)p_0\|_2 = \|(y_n - 1)v\|_2 = \|x_n v - v x_n\|_2 \to 0, \tag{4.1}$$

as $n \to \infty$.

By Kadison's Theorem on projections in a masa ([7]) choose orthogonal projections $(p_m)_{m=1}^{m_0}$ in A, with $p_m \leq pr$ and $\mathrm{tr}(p_m) \leq \mathrm{tr}(r(1-p))$, for each m, so that $\sum_{m=1}^{m_0} p_m = pr$. Then, by (4.1),

$$\|y_n - pr\|_2 = \|(y_n - 1)pr\|_2 \leq \sum_{m=1}^{m_0} \|(y_n - 1)p_m\|_2 \to 0,$$

so that $(x_n)_{n=1}^{\infty}$ is a trivial centralising sequence. This contradiction ensures that $r \leq p$ and so $\Gamma(A) = \mathrm{tr}(p)$, as required. □

The Γ-invariant is uniformally continuous with respect to the $d_{\infty,2}$-metric on masas in separable II_1 factors.

Lemma 4.3. *For masas A and B in a separable II_1 factor N, we have*

$$|\Gamma(A) - \Gamma(B)| \leq 15 d_{\infty,2}(A, B).$$

Proof. Suppose that A and B are masas in N with $d_{\infty,2}(A,B) \leq \epsilon$. Let u, p and q be as in Theorem 3.1, so that

$$\|1 - p\|_2 = \|1 - q\|_2 \leq 15\epsilon.$$

Given a projection $e \in A$ such that Ae has non-trivial central sequences for eNe, take $f = uepu^*$ — a projection in Bq with $f \leq q$. Since $uAepu^* = Bf$, we can use u to conjugate the centralising sequences for $epRep$ lying in Ae into centralising sequences for fRf lying in Bf. Therefore,

$$\Gamma(B) \geq \mathrm{tr}(ep) = \mathrm{tr}(e) - \mathrm{tr}(e(1-p)) \geq \mathrm{tr}(e) - \|e\|_2 \|1-p\|_2 \geq \mathrm{tr}(e) - 15\epsilon$$

for every such projection $e \in A$. Hence,

$$\Gamma(B) \geq \Gamma(A) - 15\epsilon.$$

By interchanging the roles of A and B we have

$$\Gamma(A) \geq \Gamma(B) - 15\epsilon,$$

and these two inequalities combine to give the result. □

One might attempt to produce uncountably many non-conjugate singular masas in the hyperfinite II_1 factor R with Pukánszky invariant $\{1\}$ by taking projections $e \in R$ and singular masas B_1 in eRe and B_2 in $(1-e)R(1-e)$ both with Pukánszky invariant $\{1\}$, such that B_1 is Γ in eRe and B_2 is totally non-Γ in $(1-e)R(1-e)$. The 'direct-sum' $A = \{\, b_1 + b_2 \mid b_1 \in B_1, b_2 \in B_2 \,\}$ will be a masa in R with $\Gamma(A) = \mathrm{tr}(e)$ by Lemma 4.2. Unfortunately, we do not have control over the exact Pukánszky invariant of such a masa A, all we can say is that $1 \in \mathrm{Puk}\,(A)$. Indeed, there is a masa A in R with $\mathrm{Puk}\,(A) = \{\, 1, 2 \,\}$ for which there is a projection $e \in A$ with $\mathrm{tr}(e) = 1/2$ such that

$$\mathrm{Puk}\,(Ae \subset eRe) = \mathrm{Puk}\,(A(1-e) \subset (1-e)R(1-e)) = \{1\}.$$

Examples to this effect will be given in subsequent work by the second author. In the next section, we get round this problem using Tauer masas to control the Pukánskzy invariant of these direct sums.

5 A continuous path of singular masas

Here is the main result of this paper, from which Theorem 1.1 follows immediately.

Theorem 5.1. *There is a map $t \mapsto A(t)$, taking each $t \in [0,1]$ to a masa $A(t)$ in R such that*

(i) $d_{\infty,2}(A(s), A(t)) \to 0$ *as* $|s-t| \to 0$.

(ii) Every $A(t)$ has Pukánszky invariant $\{1\}$.

(iii) Each $A(t)$ is singular.

(iv) $\Gamma(A(t)) = t$, for each t.

We shall construct Tauer masas, $A(t)$, for a dense set of t in $[0,1]$ with the required properties, then use continuity to produce the required path. The construction in the dense set of t is based on a rapidly increasing sequence of primes and adjusting the definition of the approximately finite dimensional approximating algebras according to t being in suitable ranges of rationals.

Notation 5.2. Let $k_1 = 2$, and for each $r \geq 2$ take k_r to be a prime exceeding $k_1 \dots k_{r-1}$. Let M_r to be the algebra of $k_r \times k_r$ matrices. By [9, Theorem 3.2], there is a family $({}^rD^{(m)})_{m=0}^{k_1 \dots k_{r-1}}$ of pairwise orthogonal masas in M_r. Write ${}^re_l^{(m)}$ for the minimal projections of ${}^rD^{(m)}$ indexed by $l = 0, 1 \dots, k_r - 1$. Let N_n be the tensor product $\bigotimes_{r=1}^n M_r$. We have the natural unital inclusion $x \mapsto x \otimes 1$ of N_n inside N_{n+1} and we work in the hyperfinite II_1 factor R, obtained as the direct limit of these N_n with respect to normalised trace.

For each $n \in \mathbb{N}$ write

$$I_n = \left\{ \frac{m}{k_1 \dots k_n} \;\middle|\; m = 0, 1, 2, \dots, k_1 \dots k_n \right\},$$

so that $I_n \subset I_{n+1}$, for each n. Let $I = \bigcup_{n=1}^\infty I_n$ — a dense set of rationals in $[0,1]$. For each $t \in I$, we will define a Tauer masa $A(t)$ in R with respect to the chain $(N_n)_{n=n_0(t)}^\infty$, where $n_0(t)$ is the minimal n for which $t \in I_n$. For each $n \geq n_0(t)$, we denote the n-th approximate of $A(t)$ by $A_n(t)$, and enumerate the minimal projections of $A_n(t)$ as ${}^nf_m(t)$ for $0 \leq m < k_1 \dots k_n$.

Construction 5.3. The process begins by defining $A_0(0) = A_0(1/2) = A_0(1) = {}^1D^{(0)}$ with the minimal projections ${}^1f_m(0) = {}^1f_m(1/2) = {}^1f_m(1) = {}^1e_m^{(0)}$ coinciding for $m = 0, 1$. For some n_1, suppose that we have defined $A_n(t)$ and enumerated the minimal projections ${}^nf_m(t)$, for all $t \in I_{n_1}$ and $n_0(t) \leq n \leq n_1$. For $t \in I_{n_1}$, the definition of $A_{n_1+1}(t)$ is split into two cases, depending on whether n_1 is even or odd.

1. n_1 is *even*: Set

$$A_{n_1+1}(t) = \bigoplus_{m=0}^{k_1 \dots k_{n_1} - 1} {}^{n_1}f_m(t) \otimes {}^{n_1+1}D^{(m)}. \tag{5.1}$$

Enumerate the minimal projections ${}^{n_1+1}f_{m'}(t)$ by dividing m' by k_{n_1+1} to obtain $m' = k_{n_1+1}m + l$ for some $0 \le l < k_{n_1+1}$. Now take

$$ {}^{n_1+1}f_{m'}(t) = {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(m)}. \tag{5.2} $$

2. n_1 is *odd*: Here we take

$$ \begin{aligned} A_{n_1+1}(t) = &\bigoplus_{m=0}^{tk_1\dots k_{n_1}-1} {}^{n_1}f_m(t) \otimes {}^{n_1+1}D^{(k_1\dots k_{n_1})} \\ &\oplus \bigoplus_{m=tk_1\dots k_{n_1}}^{k_1\dots k_{n_1}-1} {}^{n_1}f_m(t) \otimes {}^{n_1+1}D^{(m)}. \end{aligned} \tag{5.3} $$

The enumeration of the minimal projections happens in the same way as the even n_1 case. Namely, given $0 \le m' < k_1 \dots k_{n_1+1}$ write $m' = mk_{n_1+1} + l$ for some $0 \le l < k_{n_1+1}$ and set

$$ {}^{n_1+1}f_{m'}(t) = \begin{cases} {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(k_1\dots k_{n_1})} & 0 \le m < tk_1 \dots k_{n_1} \\ {}^{n_1}f_m(t) \otimes {}^{n_1+1}e_l^{(m)} & tk_1 \dots k_{n_1} \le m < k_1 \dots k_{n_1} \end{cases}. \tag{5.4} $$

It remains to define $A_{n_1+1}(t)$ when $t \in I_{n_1+1} \setminus I_{n_1}$. In this case this is the first approximate of the Tauer masa $A(t)$. Write $m_0 = \lfloor tk_1 \dots k_{n_1} \rfloor$ and define the minimal projections of $A_{n_1+1}(t)$ by

$$ {}^{n_1+1}f_m(t) = \begin{cases} {}^{n_1+1}f_m((m_0+1)/k_1 \dots k_{n_1}) & 0 \le m < tk_1 \dots k_{n_1+1} \\ {}^{n_1+1}f_m(m_0/k_1 \dots k_{n_1}) & tk_1 \dots k_{n_1+1} \le m < k_1 \dots k_{n_1+1} \end{cases}. \tag{5.5} $$

Theorem 4.1 of [14] shows that the Tauer masas constructed above have $\mathrm{Puk}\,(A(t)) = \{1\}$, which is condition (ii) of Theorem 5.1. We now check that these masas satisfy conditions (iii) and (iv) of Theorem 5.1.

Lemma 5.4. *The Tauer masas $A(t)$ of Construction 5.3 are singular.*

Proof. Fix $t \in I$ and let $n \ge n_0(t)$ be even. In the notation of (2.1), the even stage of Construction 5.3 gives

$$ A_{n+1,n}^{({}^n f_m(t))}(t) = {}^{n+1}D^{(m)}. $$

Take a unitary $w \in {}^{n+1}D^{(m)}$ with $\mathrm{tr}(w) = 0$. When $m' \ne m$, the orthogonality of ${}^{n+1}D^{(m)}$ and ${}^{n+1}D^{(m')}$ gives $\mathbb{E}_{{}^{n+1}D^{(m')}}(w) = 0$. The singularity of $A(t)$ then follows from Proposition 2.1. $\square$

The next Lemma verifies the hypothesis of Lemma 4.2, so the masas of Construction 5.3 have $\Gamma(A(t)) = t$.

Lemma 5.5. *Fix $t \in I$ and write n_0 for $n_0(t)$. Let*

$$p = \sum_{m=0}^{tk_1\dots k_{n_0}-1} {}^{n_0}f_m(t), \tag{5.6}$$

a projection in $A(t)$. Then

1. *$A(t)p$ contains non-trivial centralising sequences for pRp;*

2. *$A(t)(1-p)$ is totally non Γ in $(1-p)R(1-p)$.*

Proof of 1: Note that

$$p = \sum_{m=0}^{tk_1\dots k_n-1} {}^{n}f_m(t),$$

for all $n \geq n_0$. Fix $n \geq n_0$ odd and consider $x_1, \dots, x_r \in N_n$. Let $v \in {}^{n+1}D^{(k_1\dots k_n)}$ be a unitary with $\mathrm{tr}(v) = 0$. Examining the odd n form of Construction 5.3, we see that

$$u = \sum_{m=0}^{tk_1\dots k_n-1} {}^{n}f_m(t) \otimes v = p \otimes v \in N_n \otimes M_{n+1} = N_{n+1}$$

is a trace free unitary in $A_{n+1}(t)p$. It is then immediate that u commutes with each px_ip, and so $A(t)p$ contains non-trivial centralising sequences for pRp by the $\|.\|_2$-density of $\cup_{n=1}^{\infty} N_n$ in R. $\square$

We prove part 2 of Lemma 5.5 in two stages. We first establish an orthogonality condition which suffices to establish that no Ae can contain centralising sequences for eRe, when $e \leq 1-p$ is a minimal projection of some $A_n(t)$. A density argument, which contains the proof of an observation of Popa ([9, Remark 5.4.2], also found in [1, Lemma 2.1]), then completes the proof of Lemma 5.5.

Lemma 5.6. *Fix $t \in I$, $n \geq n_0(t)$ and m, m' with $tk_1 \dots k_n \leq m < m' < k_1 \dots k_n$. Let v be a partial isometry in N_n with $vv^* = {}^{n}f_m(t)$ and $v^*v = {}^{n}f_{m'}(t)$. Then $v(A(t)\,{}^{n}f_{m'}(t))v^*$ is orthogonal to $A(t)\,{}^{n}f_m(t)$ in ${}^{n}f_m(t)R\,{}^{n}f_m(t)$.*

Proof. Fix $n \geq n_0$ and regard R as $N_n \otimes R_1$, where R_1 is generated as the infinite von Neumann tensor product $(\bigotimes_{r=n+1}^{\infty} M_r)''$ with respect to the unique normalised trace. Using the notation of (2.1), for $n_1 > n$ we have

$$A_{n_1}(t) = \bigoplus_{m=0}^{k_1 \ldots k_n - 1} {}^n f_m(t) \otimes A_{n_1,n}^{({}^n f_m(t))}(t),$$

for masas $A_{n_1,n}^{({}^n f_m(t))}(t)$ in $\bigotimes_{r=n+1}^{n_1} M_r$. In this way we obtain Tauer masas

$$A_{\infty,n}^{({}^n f_m(t))}(t) = \left(\bigcup_{n_1=n+1}^{\infty} A_{n_1,n}^{({}^n f_m(t))}(t) \right)''$$

in R_1, so that

$$A(t) = \bigoplus_{m=0}^{k_1 \ldots k_n - 1} {}^n f_m(t) \otimes A_{\infty,n}^{({}^n f_m(t))}(t).$$

Now take m, m' and v as in the statement, and note that

$$vA(t)^n f_{m'}(t) v^* = {}^n f_m(t) \otimes A_{\infty,n}^{({}^n f_{m'}(t))}(t),$$

so that it suffices to show that $A_{\infty,n}^{({}^n f_m(t))}(t)$ and $A_{\infty,n}^{({}^n f_{m'}(t))}(t)$ are orthogonal masas in R_1. We shall show that $A_{n_1,n}^{({}^n f_m(t))}(t)$ and $A_{n_1,n}^{({}^n f_{m'}(t))}(t)$ are orthogonal in $\bigotimes_{r=n+1}^{n_1} M_r$, for all $n_1 > n$, from which the result immediately follows by density.

To this end, note that Construction 5.3 gives $A_{n+1,n}^{({}^n f_m(t))}(t) = {}^{n+1}D^{(m)}$ and $A_{n+1,n}^{({}^n f_{m'}(t))}(t) = {}^{n+1}D^{(m')}$, from (5.1) when n is even and from (5.3) when n is odd. In the latter case, we use the hypothesis that $tk_1 \ldots k_n \leq m < m'$. As $D^{(m)}$ and $D^{(m')}$ are orthogonal masas in M_{n+1}, the claim holds when $n_1 = n + 1$.

Suppose inductively that the claim holds for some $n_1 > n$. Write

$$A_{n_1+1,n}^{({}^n f_m(t))}(t) = \bigoplus_{g \in \mathcal{P}_{\min}\left(A_{n_1,n}^{({}^n f_m(t))}(t)\right)} g \otimes B^{(g,m)},$$

and

$$A_{n_1+1,n}^{({}^n f_{m'}(t))}(t) = \bigoplus_{h \in \mathcal{P}_{\min}\left(A_{n_1,n}^{({}^n f_{m'}(t))}(t)\right)} h \otimes B^{(h,m')},$$

for masas $B^{(g,m)}$ and $B^{(h,m')}$ in M_{n_1+1}. Again, Construction 5.3 ensures that all these masas are pairwise orthogonal. This is immediate from (5.1) for even n_1; when n_1 is odd we again use the hypothesis $tk_1 \dots k_n \leq m < m'$ in our examination of (5.3). The orthogonality of $A_{n_1+1,n}^{({}^n f_m(t))}(t)$ and $A_{n_1+1,n}^{({}^n f_{m'}(t))}(t)$ follows immediately, yielding the result. $\square$

Proof of part 2 of Lemma 5.5: Take $t \in I$ and fix some projection $0 \neq e \leq 1-p$ in $A(t)$. For each $n \in \mathbb{N}$, find $l_n \geq n_0(t)$ and a family $P_n \subset \mathcal{P}_{\min}(A_{l_n}(t))$ of minimal projections in $A_{l_n}(t)$ lying under $1-p$, such that upon writing $q_n = \sum_{q \in P_n} q$, we have

$$\|q_n - e\|_2^2 < 1/n.$$

For each n, take a permutation σ_n of P_n with no fixed points. Take partial isometries $v_{\sigma_n(q),q}$ in N_{l_n} with $v_{\sigma_n(q),q} v_{\sigma_n(q),q}{}^* = \sigma_n(q)$ and $v_{\sigma_n(q),q}{}^* v_{\sigma_n(q),q} = q$. Define

$$x_n = \sum_{q \in P_n} v_{\sigma_n(q),q} + (1 - q_n),$$

a unitary in N_{l_n} which has $x_n q x_n^* = \sigma_n(q)$, for every $q \in P_n$. Observe that

$$x_n(Aq_n)x_n^* = \bigoplus_{q \in P_n} x_n(Aq)x_n^* = \bigoplus_{q \in P_n} v_{\sigma_n(q),q}(Aq)v_{\sigma_n(q),q}{}^* = \bigoplus_{q \in P_n} v_{\sigma_n(q),q} A v_{\sigma_n(q),q}{}^*$$

is orthogonal to $\bigoplus_{q \in P_n} A\sigma_n(q) = Aq_n$ in $q_n R q_n$ by Lemma 5.6.

Suppose that Ae contains non-trivial centralising sequences for eRe. Find a sequence of unitaries $u_n \in A$, with $\mathrm{tr}(u_n e) = 0$ for each n, and such that

$$\|eu_n e x_n e - e x_n e u_n e\|_2 < \|e - q_n\|_2 \,. \tag{5.7}$$

We have the following simple estimate, showing that $u_n q_n$ asymptotically commutes with the $q_n x_n q_n$:

$$\begin{aligned}
&\|q_n u_n q_n x_n q_n - q_n x_n q_n u_n q_n\|_2 \\
\leq\ &\|(q_n - e)u_n q_n x_n q_n\|_2 + \|eu_n(q_n - e)x_n q_n\|_2 + \|eu_n e x_n(q_n - e)\|_2 \\
&+ \|eu_n e x_n e - e x_n e u_n e\|_2 + \|e x_n e u_n(e - q_n)\|_2 + \|e x_n(e - q_n)u_n q_n m\|_2 \\
&+ \|(e - q_n)x_n q_n u_n q_n\|_2 \\
\leq\ &7\,\|e - q_n\|_2 \to 0.
\end{aligned}$$

On the other hand, using $x_nq_n = q_nx_n$ we have

$$\begin{aligned}
&\left\|q_nx_nq_nu_nq_n - q_nu_nq_nx_nq_n\right\|_2^2 \\
=\ &\left\|q_nx_nu_nq_nx_n^*q_n - u_nq_n\right\|_2^2 \\
=\ &\left\|q_nx_nu_nq_nx_n^*q_n\right\|_2^2 + \left\|u_nq_n\right\|_2^2 - 2\Re\mathrm{tr}(x_nu_nq_nx_n^*u_nq_n) \\
=\ &2\left\|q_n\right\|_2^2 - 2\Re\mathrm{tr}(x_nu_nq_nx_n^*)\mathrm{tr}(u_nq_n)/\mathrm{tr}(q_n) \to 2\left\|e\right\|_2^2 \neq 0,
\end{aligned}$$

where the last line comes from the orthogonality of $x_n(Aq_n)x_n^*$ and Aq_n in q_nRq_n — the quotient of $\mathrm{tr}(q_n)$ appearing as a normalisation constant. The convergence is a simple calculation, as

$$|\mathrm{tr}(u_nq_n)| \leq |\mathrm{tr}(u_ne)| + |\mathrm{tr}(u_n(q_n - e))| \leq 0 + \left\|u_n\right\|_2 \left\|q_n - e\right\|_2 \to 0.$$

This contradiction completes the proof. □

For t in the dense subset I of $[0,1]$, we have singular Tauer masas $A(t)$ with $\Gamma(A(t)) = t$. We wish to use completeness to define $A(t)$ for $t \in [0,1] \setminus I$ and so we need to control the distance between the $A(t)$'s we have already defined. It is here that the form of $A_{n_0(t)}(t)$ specified in Construction 5.3 becomes relevant.

Lemma 5.7. *Fix $s, t \in I$ with $s < t$. Let n_0 be the maximum of $n_0(s)$ and $n_0(t)$ and take*

$$q = \sum_{m=0}^{sk_1\dots k_{n_0}-1} {}^{n_0}f_m(s) + \sum_{m=tk_1\dots k_{n_0}}^{k_1\dots k_{n_0}-1} {}^{n_0}f_m(s)$$

a projection of trace $1-(t-s)$. This q lies in $A(s) \cap A(t)$ and $A(s)q = A(t)q$.

Proof. We shall demonstrate that Construction 5.3 ensures that whenever we have $s, t \in I_n$, then

$$ {}^nf_m(s) = {}^nf_m(t), \tag{5.8}$$

for all m with

$$0 \leq m < sk_1 \dots k_n \text{ or } tk_1 \dots k_n \leq m < k_1 \dots k_n. \tag{5.9}$$

This will immediately show that q lies in $A(t)$, as well as $A(s)$. Furthermore, as $A(s)q$ and $A(t)q$ are generated by all ${}^nf_m(s)$ and ${}^nf_m(t)$ respectively, with $n \geq \max\{n_0(s), n_0(t)\}$ and m satisfying (5.9), the claim also implies that $A(s)q = A(t)q$, as required.

We proceed by induction on n. When $n = 1$, the result is certainly true, as Construction 5.3 began by defining ${}^1f_m(0) = {}^1f_m(1/2) = {}^1f_m(1)$ for $m = 0, 1$. Suppose that we have established the claim for all $n \leq n_1$. We investigate the $n_1 + 1$ situation, starting with the case when s and t both lie in I_{n_1}.

Take $s, t \in I_{n_1}$ with $s < t$. Take m' with either $0 \leq m' < sk_1 \dots k_{n_1+1}$ or $tk_1 \dots k_{n_1+1} \leq m' < k_1 \dots k_{n_1+1}$, and divide by k_{n_1+1} to obtain $m' = mk_{n_1+1} + l$ with $0 \leq l < k_{n_1+1}$. This m must have $0 \leq m < sk_1 \dots k_{n_1}$ in the first case or $tk_1 \dots k_{n_1} \leq m < k_1 \dots k_{n_1}$ in the second. In any event, the inductive hypothesis ensures that ${}^{n_1}f_m(s) = {}^{n_1}f_m(t)$. When n_1 is even, the definition (5.2) of ${}^{n_1+1}f_{m'}(s)$ and ${}^{n_1+1}f_{m'}(t)$ immediately gives ${}^{(n_1+1)}f_{m'}(s) = {}^{n_1+1}f_{m'}(t)$. When n_1 is odd, this is also true, as we have excluded the possibility that $sk_1 \dots k_{n_1} \leq m < tk_1 \dots k_{n_1}$, so both these minimal projections must come from the same case of equation (5.4). Therefore, the minimal projections ${}^{n_1+1}f_{m'}(s)$ and ${}^{n_1+1}f_{m'}(t)$ coincide whenever they are required to do so.

We now examine what happens when precisely one of s and t lies in $I_{n_1+1} \setminus I_{n_1}$. Take s in I_{n_1} and $t \in I_{n_1+1} \setminus I_{n_1}$ with $s < t$. As in the definition of $A_{n_1+1}(t)$, we write $m_0 = \lfloor tk_1 \dots k_n \rfloor$ so that $s \leq m_0/k_1 \dots k_{n_1}$. For $0 \leq m < sk_1 \dots k_{n_1+1}$, we have

$$ {}^{n_1+1}f_m(s) = {}^{n_1+1}f_m((m_0 + 1)/k_1 \dots k_{n_1}) = {}^{n_1+1}f_m(t), $$

where the second equality is the definition, (5.5), of ${}^{n_1+1}f_m(t)$, and the first follows as the m-th minimal projections for $A_{n_1+1}(s)$ and $A_{n_1+1}((m_0 + 1)k_1 \dots k_{n_1})$ coincide by the case we analysed in the previous paragraph. When $tk_1 \dots k_{n_1+1} \leq m < k_1 \dots k_{n_1+1}$, we have

$$ {}^{n_1+1}f_m(t) = {}^{n_1+1}f_m(m_0/k_1 \dots k_{n_1}) = {}^{n_1+1}f_m(s), $$

the first equality being (5.5) – the definition of ${}^{n_1+1}f_m(t)$, and the second equality is (5.8) for appropriate minimal projections of $A_{n_1+1}(s)$ and $A_{n_1+1}(m_0/k_1 \dots k_{n_1})$, as $m \geq m_0 k_{n_1+1}$. These last two algebras may turn out to be the same, but then the minimal projections will certainly coincide. Interchanging the roles of s and t above ensures that the claim holds for $n_1 + 1$ whenever either s or t lies in I_{n_1}.

We complete the proof by examining the situation when $s, t \in I_{n_1+1} \setminus I_{n_1}$. Take $s < t$ with $s, t \in I_{n_1+1} \setminus I_{n_1}$. Suppose first that $\lfloor sk_1 \dots k_{n_1} \rfloor = \lfloor tk_1 \dots k_{n_1} \rfloor = m_0$. In this instance the definition, (5.5), of the minimal projections ${}^{n_1+1}f_m(s)$ and ${}^{n_1+1}f_m(t)$ ensures that these projections conincide for all m with $0 \leq m < sk_1 \dots k_{n_1+1}$ or $tk_1 \dots k_{n_1+1} \leq m < k_1 \dots k_{n_1+1}$.

Finally, suppose that $\lfloor sk_1 \dots k_{n_1} \rfloor = m_0 < m_1 = \lfloor tk_1 \dots k_{n_1} \rfloor$. Given m with $0 \le m < sk_1 \dots k_{n_1+1}$, (5.5) ensures that ${}^{n_1+1}f_m(s) = {}^{n_1+1}f_m(m_0+1)$ and ${}^{n_1+1}f_m(t) = {}^{n_1+1}f_m(m_1)$. Since $m < sk_1 \dots k_{n_1+1} < (m_0+1)k_1 \dots k_{n_1+1}$, the case when $s, t \in I_n$ (with $s = m_0 + 1$ and $t = m_1$) ensures that ${}^{n_1+1}f_m(m_0+1) = {}^{n_1+1}f_m(m_1)$. In conclusion, we have ${}^{n_1+1}f_m(s) = {}^{n_1+1}f_m(t)$ as required. The case when $tk_1 \dots k_{n_1+1} \le m < k_1 \dots k_{n_1+1}$ is similar, and this completes the proof. □

Corollary 5.8. *For $s, t \in I$ we have*

$$\left\|\mathbb{E}_{A(s)} - \mathbb{E}_{A(t)}\right\|_{\infty,2} \le 2\sqrt{|s-t|}$$

Proof. We may assume that $s < t$. Let n_0 be the maximum of $n_0(s)$ and $n_0(t)$. Let q be the projection of Proposition 5.7, so that $A(s)q = A(t)q$. The simple estimate

$$\left\|\mathbb{E}_{A(s)} - \mathbb{E}_{A(t)}\right\|_{\infty,2} \le 2\left\|1-q\right\|_2,$$

can be found in part (i) of Theorem 6.5 in [10]. As $\mathrm{tr}(1-q) = t - s$, this is exactly what was claimed. □

We can now combine the results of this section to esablish Theorem 5.1.

Proof of Theorem 5.1. For $t \in I$ we take $A(t)$ to be the Tauer masa produced in Construction 5.3. When $t \in [0,1] \setminus I$, we define $A(t)$ by taking a sequence $t_n \to t$ with each t_n in the dense set of rationals I. The resulting sequence of masas $(A(t_n))_{n=1}^{\infty}$ is then $d_{\infty,2}$-Cauchy by Corollary 5.8, and so converges to a masa $A(t)$ in R. Recall that the set of all von Neumann subalgebras of R is a $d_{\infty,2}$-complete metric space, and the subset of masas is closed, [2]. This masa is well defined, in that $A(t)$ is independent of the choice of sequence t_n in I converging to t. An approximation argument extends Corollary 5.8 to show that $d_{\infty,2}(A_s, A_t) \to 0$ whenever $|s-t| \to 0$.

Furthermore each $A(t)$ is singular, as this holds for $t \in I$ (Lemma 5.4) and the set of all singular masas is closed; again this can be found in [2]. All the $A(t)$ have Pukánszky invariant $\{1\}$; for $t \in I$ this is Theorem 4.1 of [14] and Corollary 3.3 then gives the result for general t. That $\Gamma(A(t)) = t$ for every $t \in [0,1]$ follows first by observing that Lemma 5.5 combines with Lemma 4.2 to give the result for $t \in I$. Continuity gives the result for all t, this time in the form of Lemma 4.3. □

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