Packing $K_r$s in bounded degree graphs

Michael McKay *, David Manlove
School of Computing Science, University of Glasgow, UK

A R T I C L E I N F O

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A B S T R A C T

We study the problem of finding a maximum-cardinality set of $r$-cliques in an undirected graph of fixed maximum degree $\Delta$, subject to the cliques in that set being either vertex disjoint or edge disjoint. It is known for $r = 3$ that the vertex-disjoint (edge-disjoint) problem is solvable in linear time if $\Delta = 3$ ($\Delta = 4$) but APX-hard if $\Delta \geq 4$ ($\Delta \geq 5$).

We generalise these results to an arbitrary but fixed $r \geq 3$, and provide a complete complexity classification for both the vertex- and edge-disjoint variants in graphs of maximum degree $\Delta$.

Specifically, we show that the vertex-disjoint problem is solvable in linear time if $\Delta < 3r/2 - 1$, solvable in polynomial time if $\Delta < 5r/3 - 1$, and APX-hard if $\Delta \geq \lceil 5r/3 \rceil - 1$. We also show that if $r \geq 6$ then the above implications also hold for the edge-disjoint problem. If $r \leq 5$, then the edge-disjoint problem is solvable in linear time if $\Delta < 3r/2 - 1$, solvable in polynomial time if $\Delta \leq 2r - 2$, and APX-hard if $\Delta > 2r - 2$.

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1. Introduction

1.1. Background

In this paper we consider two problems related to clique packings in undirected graphs. Both problems involve finding a maximum-cardinality set of $r$-cliques, in a given undirected graph, where $r$ is a fixed constant. We call such a set a $K_r$-packing. In the first problem, which we call the Vertex-Disjoint $K_r$-Packing Problem (VDK$_r$), the cliques in the $K_r$-packing must be pairwise vertex disjoint. In the second problem, which we call the Edge-Disjoint $K_r$-Packing Problem (EDK$_r$), the cliques in the $K_r$-packing must be pairwise edge disjoint. Note that in both problems $r$ is a fixed constant and does not form part of the problem input. If $r$ is not fixed then both problems generalise the well-studied problem of finding a clique of a given size [11]. Note also that if a vertex-disjoint $K_r$-packing has cardinality $|V|/r$ then we refer to it as a $K_r$-factor [12].

Most existing research concerning vertex- and edge-disjoint $K_r$-packings relates to either special or more general cases. For example, a special case of VDK$_3$ is VDK$_2$, also known as Maximum Cardinality Matching. Maximum Cardinality Matching is a central problem of graph theory and algorithms [22]. A classical result of Edmonds [9] is that a maximum-cardinality matching can be found in polynomial time. Conversely, EDK$_3$ is trivial.

VDK$_3$ and EDK$_3$ have been the subject of much research. In particular, VDK$_3$ is closely associated with the decision problem known as Partition Into Triangles (PIT) [11], which asks whether a given undirected graph contains a $K_3$-factor. Karp [17] noted in 1975 that PIT is NP-complete. In 2002, Caprara and Rizzi [4] considered VDK$_3$ and EDK$_3$ in graphs of

* Corresponding author.

E-mail addresses: mikemckay2203@gmail.com (M. McKay), david.manlove@glasgow.ac.uk (D. Manlove).
a fixed maximum degree $\Delta$. They showed that $\text{VDK}_3$ is solvable in polynomial time if $\Delta \leq 3$ and APX-hard even when $\Delta = 4$, and $\text{EDK}_3$ is solvable in polynomial time if $\Delta \leq 4$ and APX-hard even when $\Delta = 5$. They also showed that $\text{VDK}_3$ is NP-hard for planar graphs even when $\Delta = 4$ and $\text{EDK}_3$ is NP-hard for planar graphs even when $\Delta = 5$. In their paper, Caprara and Rizzi [4] referenced a well-known approximation algorithm of Hurkens and Schrijver [15] for a more general type of packing problem. They noted that this algorithm leads to, for any fixed constant $\varepsilon > 0$, a $(3/(2+\varepsilon))$-approximation algorithm for $\text{VDK}_3$ and $\text{EDK}_3$. In 2013, van Rooij et al. [27] established an equivalence between $\text{VDK}_3$ when $\Delta = 4$ and Exact 3-Satisfiability (XSAT). They used this equivalence to devise an $O(1.02220^\Delta)$-time algorithm for PIT when $\Delta = 4$.

A well-studied generalisation of $\text{VDK}_r$ involves finding in a given undirected graph $G$ a maximum-cardinality set of vertex-disjoint subgraphs where each subgraph is isomorphic to some fixed graph $H$. Such a set is known as a $G$-packing. A $G$-packing is a $G$-factor if the packing has cardinality $|V(H)|/|V(G)|$, where $V(H)$ is the set of vertices in $H$ and $V(G)$ is the set of vertices in $G$. In 1978, Kirkpatrick and Hell [19] showed that if $G$ contains a component with three or more vertices then it is NP-complete to decide whether a given undirected graph contains a $G$-factor. In 1983, Kirkpatrick and Hell [20] surveyed previous work on $G$-packing, and also consider a further generalisation to so-called $\mathcal{G}$-packing, where $\mathcal{G}$ is a fixed set of graphs and any subgraph in the $\mathcal{G}$-packing must be isomorphic to some element of $\mathcal{G}$. A survey of research involving $G$-packing can be found in a paper of Vusler [31] published in 2007.

Another interesting generalisation of $\text{VDK}_r$ involves $\mathcal{G}$- packings such that $\mathcal{G}$ is a set of cliques. Of course, any $K_r$-packing problem can be seen as a $\mathcal{G}$-packing problem in which $\mathcal{G} = \{K_r\}$. In 1984, Kirkpatrick and Hell [21] presented polynomial-time algorithms for any case of the vertex-disjoint $\mathcal{G}$-packing problem in which $\mathcal{G} \subseteq \{K_4, K_2^2, \ldots \}$ and $\mathcal{G}$ contains $K_2$. In the same paper they also showed that the decision version of the $\mathcal{G}$-packing problem is NP-complete for any set $\mathcal{G}$ where $\mathcal{G} \subseteq \{K_4, K_2^2, \ldots \}$. In 2008, Chataigner et al. [5] studied a related optimisation problem in which $\mathcal{G} = \{K_2, K_3, \ldots, K_r\}$ and the goal is to maximise the number of edges covered by such a $\mathcal{G}$-packing. They showed that if $\mathcal{G}$ is an APX-complete, even when the input graph has fixed maximum degree 4. They also presented new approximation algorithms, which in some cases improve on approximation ratios obtained via the previously-mentioned result of Hurkens and Schrijver [15].

Compared to their various special and more general cases, $\text{VDK}_r$ and $\text{EDK}_r$, as we have described them here appear to have received less attention in the literature. In 1998, Dahlhaus and Karpinski [7] considered $\text{VDK}_3$ and $\text{VDK}_4$ in chordal and strongly chordal graphs. They showed that a $K_r$-factor can be found in polynomial time in a given chordal or strongly chordal graph, if it exists. They remarked that if $r \geq 4$ then the decision version of $\text{VDK}_r$ is NP-complete for split graphs (a subset of chordal graphs), but left open the complexity for split graphs and chordal graphs when $r = 3$. Later, in 2001, Guruswami et al. [12] also considered $\text{VDK}_r$ in relation to restricted classes of graphs, and resolved the open question of Dahlhaus and Karpinski. Guruswami et al. showed that if $r \geq 3$ then the decision version of $\text{VDK}_r$ is NP-complete for chordal graphs, planar graphs (only for $r = 3$ and $r = 4$), line graphs, and total graphs. They also described polynomial-time algorithms for $\text{VDK}_3$ in split graphs, the $K_r$-factor decision problem in split graphs, and $\text{VDK}_3$ in cographs (also known as $P_4$-free graphs). They noted that this completely characterised the complexity of $\text{VDK}_3$ for split graphs. The algorithm of Guruswami et al. for cographs was later extended by Pedrotti and de Mello [24] for so-called $P_4$-sparse graphs.

The approximability of $\text{VDK}_r$ and $\text{EDK}_r$ has also been studied. It is straightforward to apply the previously-discussed result of Hurkens and Schrijver [15] to show that there exists a polynomial-time $(\Delta/2 + \varepsilon)$-approximation algorithm for $\text{VDK}_r$ and $\text{EDK}_r$, for any fixed constant $\varepsilon > 0$. In 2005, Manić and Wakabayashi [23] described approximation algorithms that improve on this approximation ratio in the restricted cases of $\text{VDK}_3$ when $\Delta = 4$, and $\text{EDK}_3$ when $\Delta = 5$. They also presented a linear-time algorithm for $\text{VDK}_3$ on indifference graphs.

From the converse perspective of graphs with a fixed minimum degree, a classical result of Hajnal and Szemerédi [13] is that if a given undirected graph $G = (V, E)$ contains a $K_r$-factor if it has a minimum degree greater than or equal to $(1 − 1/r)|V|$. Kierstead and Kostochka [18] later generalised this result to show that, in this case, such a packing can be constructed in polynomial time. Subsequent research has explored more general conditions for the existence of $K_r$-factors and G-factors [2,29].

1.2. Our contribution

Caprara and Rizzi [4] showed that $\text{VDK}_3$ is solvable in polynomial time if $\Delta = 3$ and APX-hard if $\Delta = 4$; and $\text{EDK}_3$ is solvable in polynomial time if $\Delta = 4$ and APX-hard if $\Delta = 5$. In this paper we generalise their results and provide a full classification of the complexity of $\text{VDK}_3$ and $\text{EDK}_3$ for any $\Delta \geq 1$ and any fixed $r \geq 3$. This classification is shown in Table 1.

In the next section, Section 1.3, we define some additional notation and make an observation on the coincidence of vertex- and edge-disjoint $K_r$-packings. In Section 2 we consider the case when $\Delta < 3r/2 − 1$. We show that in this case any maximal vertex- or edge-disjoint $K_r$-packing is also maximum, and devise a linear-time algorithm for both $\text{VDK}_r$ and $\text{EDK}_r$ in this setting. In Section 3 we present our algorithmic results, which show that $\text{VDK}_r$ can be solved in polynomial time if $\Delta < 5r/3 − 1$; and $\text{EDK}_r$ can be solved in polynomial time if either $3 \leq r \leq 5$ and $\Delta \leq 2r − 2$, or $r \geq 6$ and $\Delta < 5r/3 − 1$. In Section 4 we show that our algorithmic results are in a sense best possible, unless $\text{P} \neq \text{NP}$. Specifically, we show that $\text{VDK}_r$ is APX-hard if $\Delta \geq 5r/3 − 1$; and $\text{EDK}_r$ is APX-hard if either $3 \leq r \leq 5$ and $\Delta > 2r − 2$, or $r \geq 6$ and $\Delta \geq 5r/3 − 1$. In other words, we prove that there exist fixed constants $\varepsilon > 1$ and $\varepsilon' > 1$ such that no polynomial-time $\varepsilon$-approximation algorithm exists for $\text{VDK}_r$, if $\Delta \geq 5r/3 − 1$; and no polynomial-time $\varepsilon'$-approximation algorithm exists for $\text{EDK}_r$, if either $3 \leq r \leq 5$ and $\Delta > 2r − 2$, or $r \geq 6$ and $\Delta \geq 5r/3 − 1$. In Section 5 we recap our results and consider directions for future work.
Lemma 2. Suppose $T$ is a maximal vertex-disjoint $K_r$-packing in $G$, which by definition corresponds to a maximal independent set in $K^G_r$. Since $K^G_r$ is the disjoint union of cliques (by Lemma 2), any two maximal independent sets in $K^G_r$ have the same cardinality, so $T$ is also maximum. \[\square\]
We have shown in Theorem 1 that any maximal vertex-disjoint $K_r$-packing is also a maximum vertex-disjoint $K_r$-packing. It follows immediately that VDK can be solved in $O(|V'|^2)$ time by constructing the $K_r$-vertex intersection graph $K_r^G$ and greedily selecting an independent set. In fact, the explicit construction of $K_r^G$ can be avoided by exploring $G$ and greedily selecting $K_r$s. We present Algorithm greedyCliques, shown in Algorithm 1, and show that it requires $O(|V|)$ time.

Algorithm 1: Algorithm greedyCliques

\begin{algorithm}
\begin{algorithmic}
  \State {Input: a fixed integer $r \geq 1$ and a simple undirected graph $G = (V, E)$ where $\Delta(G) < 3r/2 - 1$}
  \State {Output: a maximum $K_r$-packing $T$}
  \State $T \leftarrow \emptyset$
  \While {$|V| > 0$}
    \State $v \leftarrow$ any vertex in $V$
    \If {$\deg_G(v) \geq r - 1$}
      \State $K \leftarrow \emptyset$
      \For {each subset $W$ of size $r - 1$ of $N_G(v)$}
        \If {$G[W]$ has $\binom{|W|}{r-1}$ edges}
          \State $W$ must be a clique of size $r - 1$ in $G$
          \State $K \leftarrow W \cup \{v\}$
        \EndIf
      \EndFor
      \If {$K \neq \emptyset$}
        \State $G \leftarrow G[V \setminus K]$
        \State $T \leftarrow T \cup \{|K|\}$
      \Else
        \State $G \leftarrow G[V \setminus \{v\}]$
      \EndIf
    \EndIf
  \EndWhile
  \State \Return $T$
\end{algorithmic}
\end{algorithm}

Lemma 3. Algorithm greedyCliques requires $O(|V|)$ time.

Proof. In any iteration of the outermost while loop, either a single vertex $v$ or a non-empty set of vertices $K$ is removed from $G$. It follows that the algorithm terminates after at most $|V|$ iterations of this loop. It remains to show that one iteration of this loop can be performed in constant time.

In each iteration, either $\deg_G(v) \geq r - 1$ or $\deg_G(v) < r - 1$. Computing $\deg_G(v)$ requires $O(r)$ time, since $\Delta < 3r/2 - 1$. Consider the first branch of the outermost if statement. There are $\binom{|N_G(v)|}{r-1} \leq \binom{\Delta}{r-1} < \binom{3r/2-1}{r-1} = O(2^r)$ iterations of the loop. In each iteration, the algorithm tests if $G[W]$ contains $\binom{|W|}{r-1}$ edges. This can be performed in $O(r^2)$ time. Removing $K$ from $G$ and adding $K$ to $T$, if $K \neq \emptyset$, can be done in $O(r^2)$ time. In both the else branch in which $K = \emptyset$ and the second branch of the outermost if statement, $v$ can be removed from $G$ in $O(r)$ time. \hfill $\square$

Theorem 2. If $\Delta(G) < 3r/2 - 1$ then VDK can be solved in linear time.

Proof. By Lemma 3, Algorithm greedyCliques terminates in $O(2^r|V|)$ time. By Theorem 1, it suffices to show that this algorithm returns a set $T$ that is a maximal vertex-disjoint $K_r$-packing in $G$. Suppose $K^r$ is an arbitrary $K_r$ in $G$. We show that either $K^r$ is added to $T$ or at least one vertex in $K^r$ belongs to some other $K_r$ in $T$. By the pseudocode, the algorithm removes at least one vertex in each iteration of the while loop, which ends once there are no remaining vertices. Consider the first iteration of the while loop in which any vertex $v$ in $K^r$ is identified and removed. Let $G'$ be the value of $G$ at the beginning of this iteration. By definition, at this point every vertex in $K^r$ is present in $G'$, including $v$. Since $\deg_{G'}(v) \geq r - 1$, it must be that $v$ was not deleted from $G'$ by the second branch of the outermost if statement. Similarly, $v$ cannot have been deleted from $G'$ by the second branch of the innermost if statement, since $v$ belongs to $K^r$, which is a clique of size $r$ in $G'$. The only possibility is that $v$ was deleted from $G'$ as a result of $v$ being part of some $K_r$ in $G'$, which was at some point added to $T$. \hfill $\square$

Corollary 1. If $\Delta(G) < 3r/2 - 1$ then EDK can be solved in linear time.
Proof. If \( r \leq 2 \) then EDK_6 is trivial. If \( r \geq 3 \) then it must be that \( \Delta < 3r/2 - 1 < 2r - 2 \), so by Observation 1 any edge-disjoint \( K_r \)-packing is also vertex disjoint. It follows that any maximum vertex-disjoint \( K_r \)-packing returned by Algorithm greedyClique is also a maximum edge-disjoint \( K_r \)-packing. \( \square \)

3. Polynomial-time solvability

3.1. Vertex-disjoint \( K_r \)-packing

In this section we consider VDK_\( r \). We show that VDK_\( r \) is solvable in polynomial time if \( \Delta < 5r/3 - 1 \). The proof involves finding an independent set in the \( K_r \)-vertex intersection graph \( \mathcal{K}^G_r \). We build on the technique of Caprara and Rizzi [4] and first show that if \( \Delta(G) < 5r/3 - 1 \) then \( \mathcal{K}^G_r \) is claw-free. It follows that a maximum independent set in \( \mathcal{K}^G_r \) can be found in polynomial time [24,28], which corresponds directly to a maximum vertex-disjoint \( K_r \)-packing.

There is an evident relationship between packing problems and independent sets in intersection graphs [14,16,23]. In his paper on claw-free graphs, Minty [24] remarked that an algorithm to find a maximum cardinality matching (i.e. solve VDK_\( 2 \)) can be used to find a maximum independent set in a line graph (i.e. a \( K_2 \)-vertex intersection graph). Here, like Caprara and Rizzi [4], we make use of the converse relationship and show that if the corresponding intersection graph is claw-free then VDK_\( r \) and EDK_\( r \) can be solved in polynomial time.

In what follows, suppose \( G = (V, E) \) is an undirected graph where \( \Delta(G) < 5r/3 - 1 \). In Lemma 4 we place a lower bound on the size of the intersection of any two \( K_r \)'s in \( G \) that intersect by at least one vertex.

**Lemma 4.** \( |U_i \cap U_j| > r/3 \) for any \( (U_i, U_j) \in E_{\mathcal{K}^G_r} \).

**Proof.** Consider some \((U_i, U_j) \in E_{\mathcal{K}^G_r}\) and an arbitrary vertex \( u_t \in |U_i \cap U_j| \). Now \( 5r/3 - 1 > \Delta(G) \geq \deg_G(u_t) \geq |U_t \cap U_j| - 1 = 2r - |U_t \cap U_j| - 1 \). Rearranging gives \( |U_t \cap U_j| > r/3 \). \( \square \)

**Lemma 5.** \( \mathcal{K}^G_r \) is claw-free.

**Proof.** Consider some \( U_1, U_2, U_3 \subseteq E_{\mathcal{K}^G_r} \) where \( (U_a, U_a) \in E_{\mathcal{K}^G_r} \) for each \( a \in \{1, 2, 3\} \). By Lemma 4, it must be that \( |U_1 \cap U_2| > r/3, |U_1 \cap U_3| > r/3, \) and \( |U_2 \cap U_3| > r/3 \). Since \( |U| = r \) it follows by the pigeonhole principle that either \( U_1 \) intersects \( U_2, U_3 \) intersects \( U_2 \), or \( U_3 \) intersects \( U_3 \), so the subgraph induced by \( \{U_i, U_j, U_k, U_3\} \) is not a claw in \( \mathcal{K}^G_r \). \( \square \)

**Theorem 3.** If \( \Delta(G) < 5r/3 - 1 \) then VDK_\( r \) can be solved in polynomial time.

**Proof.** First, construct the \( K_r \)-vertex intersection graph \( \mathcal{K}^G_r = (\mathcal{K}^G_r, E_{\mathcal{K}^G_r}) \). The set \( K^G_r \) can be constructed in \( O(|V|^2) \) by considering every possible set of \( r \) vertices in \( V \). The set \( E_{\mathcal{K}^G_r} \) can then be constructed in \( O(|V|^2) \) time. Next, find a maximum independent set in \( \mathcal{K}^G_r \), which can be done in polynomial time since \( \mathcal{K}^G_r \) is claw-free (by Lemma 5) [24,28]. \( \square \)

Given a claw-free graph in which each vertex has a real weight, there exist polynomial-time algorithms that can find an independent set of maximum total weight [10,25]. We remark that the result shown in Theorem 3 can be generalised to a version of VDK_\( r \) in which vertices or edges have weights, and the goal is to find a \( K_r \)-packing of maximum total weight.

3.2. Edge-disjoint \( K_r \)-packing

In this section we consider EDK_\( r \). Using Theorem 3 and Observation 1, it is straightforward to show that if \( \Delta < 5r/3 - 1 \) then EDK_\( r \) can be solved in polynomial time. We state this result as Theorem 4. In what follows, suppose \( G \) is an undirected graph.

**Theorem 4.** If \( \Delta(G) < 5r/3 - 1 \) then EDK_\( r \) can be solved in polynomial time.

**Proof.** If \( \Delta(G) < 5r/3 - 1 \) then we can find a maximum vertex-disjoint \( K_r \)-packing in polynomial time by Theorem 3. Such a packing is also a maximum edge-disjoint \( K_r \)-packing, by Observation 1. \( \square \)

We now show that this upper bound on \( \Delta(G) \) can be increased if \( r \in \{4, 5\} \). The key insight in this case is that if \( r \in \{4, 5\} \) and \( \Delta \leq 2r - 2 \) then the \( K_r \)-edge intersection graph \( \mathcal{K}^G_{\mathcal{E}^G_r} \) is claw-free. This is the same technique used by Caprara and Rizzi [4] to show that EDK_\( 3 \) is solvable in polynomial time when \( \Delta(G) \leq 4 \).

**Lemma 6.** If \( r \in \{4, 5\} \) and \( \Delta(G) \leq 2r - 2 \) then the \( K_r \)-edge intersection graph \( \mathcal{K}^G_{\mathcal{E}^G_r} \) is claw-free.
Proof. Consider some $U_1, U_2, U_3, U_j \in K_r^G$ where $\{U_i, U_j\} \in E_{K_r^G}$ for each $a \in \{1, 2, 3\}$. Suppose for a contradiction that the subgraph induced by $\{U_i, U_j, U_k, U_a\} \in K_r^G$ is a claw. By the definition of $K_r^G$, it must be that $|U_i \cap U_j| < 2$ for any $a$ and $b$ where $a \neq b$ and $a, b \in \{1, 2, 3\}$. It must also be that $|U_i \cap U_j| > 2$ for each $a \in \{1, 2, 3\}$. Since $|U_i| = r \leq 5$, assume without loss of generality that $|U_i \cap U_j \cap U_k| < 1$. Furthermore, it must be that $|U_i \cap U_j \cap U_k| = 1$, otherwise $|U_i \cap U_j \cap U_k| > 1$ which is a contradiction. Let $v_i$ be the single vertex in $U_i \cap U_j \cap U_k$. Since $U_i$ and $U_j$ are $K_r$'s in $G$ and $\deg_G(v_i) \leq 5r - 2$ it must be that $N_G[v_i] = U_i \cup U_j$. Since $U_i$ is also a $K_r$ it follows that $U_i \subseteq (U_i \cup U_j)$.

Now consider $U_i \cap U_j$ and $U_i \cap U_k$. If $|U_i \cap U_j| + |U_i \cap U_k| \geq r + 2$ then since $|U_i| = r$ it follows that $|U_i \cap U_j \cap U_k| > 1$ which is a contradiction. It follows that $|U_i \cap U_j| + |U_i \cap U_k| \leq r + 1$ and either $|U_i \cap U_j| \leq (r + 1)/2$ or $|U_i \cap U_j| \leq (r + 1)/2$. Assume without loss of generality that $|U_i \cap U_j| \leq (r + 1)/2$.

Now consider $U_i$. Since the subgraph induced by $\{U_i, U_j, U_k, U_j\}$ in $K_r^G$ is a claw, $|U_i \cap U_j| \

Recall that since the subgraph induced by $\{U_i, U_j, U_k, U_j\}$ in $K_r^G$ is a claw, $|U_i \cap U_j| \leq 1$. We deduced earlier that $|U_i \cap U_j| \leq (r + 1)/2, |U_i \cap U_j| = 2$, and $|U_j \cap U_i \cap U_j| = 1$. Since $r \geq 4$ it follows that $\deg_G(v_i) \geq (5r - 7)/2 > 2r - 2$, which is a contradiction. □

Theorem 5. If $r \leq 5$ and $\Delta(G) \leq 2r - 2$ then $EDK_r$ can be solved in polynomial time.

Proof. If $r \leq 2$ then $EDK_r$ is trivial. Caprara and Rizzi [4] prove the case when $r = 3$ and $\Delta(G) \leq 4$. If $r \in \{4, 5\}$ and $\Delta(G) \leq 2r - 2$ then by Lemma 6, the $K_r$-edge intersection graph $K_r^G$ is claw-free. It follows that a maximum edge-disjoint $K_r$-packing can be found in polynomial time by constructing $K_r^G$, in $O(|V|^2)$ time, and finding in it a maximum independent set, which can also be accomplished in polynomial time [24, 28]. □

As we remarked for $VDK_r$ in Section 3.1, it seems that the results shown in Theorems 4 and 5 can be generalised to a version of $EDK_r$ in which vertices or edges have weights, and the goal is to find a $K_r$-packing of maximum total weight.

4. APX-hardness

4.1. Vertex-disjoint $K_r$-packing

We now show that if $r \geq 3$ then $VDK_r$ is APX-hard even when $\Delta \geq \lceil 5r/3 \rceil - 1$. In other words, for any $r \geq 3$ there exists some fixed constant $\varepsilon > 0$ such that no polynomial-time $\varepsilon$-approximation algorithm exists for $VDK_r$ even when $\Delta \geq \lceil 5r/3 \rceil - 1$, unless $P = \text{NP}$.

We reduce from the problem of finding a Maximum Independent Set (MIS) in a graph $G = (V, E)$ that has maximum degree 3 and is triangle-free. Berman and Karpinski [3] show that this optimisation problem, which we refer to as MIS-3-TF, is APX-hard, notably providing an explicit lower bound on the approximation ratio (specifically, they showed that it is NP-hard to approximate MIS-3-TF within $140/139 - \varepsilon$, for any fixed $\varepsilon > 0$).

The reduction from MIS-3-TF is as follows. Our goal is to construct a new graph $G' = (V', E')$ where each $K_r$ in $G'$ corresponds to exactly one vertex in $V$ and each vertex in $V$ corresponds to exactly one $K_r$ in $G'$. For any two adjacent vertices in $G$, the intersection of the two corresponding $K_r$'s in $G'$ will contain exactly $(r/3)$ vertices.

To do this, first construct a set of $|V|$ disjoint $K_r$'s in $G'$, labelled $U = \{U_1, U_2, \ldots, U_{|V|}\}$ where $U_i = \{u_1^i, u_2^i, \ldots, u_{r}^i\}$ for any $i$ where $1 \leq i \leq |V|$. Next, consider each edge $\{v_i, v_j\} \in E$. Let $U_i' = \{u_1^{i_1}, u_2^{i_2}, \ldots, u_{r}^{i_r}\}$ be any set of $(r/3)$ vertices in $U_i$ with degree $r - 1$ and $U_j' = \{u_1^{j_1}, u_2^{j_2}, \ldots, u_{r}^{j_r}\}$ be any set of $(r/3)$ vertices in $U_j$ with degree $r - 1$. For each $q$ from 1 to $|E|$ inclusive, identify $u_{a}^{i}$ and $u_{b}^{j}$ to create a single vertex labelled $u_{a}^{i\oplus j}$. Label $U'_i = U_i'$ as $W_{ij}$.

Finally, for each vertex $v_i \in V$ let $X_i$ be the set of (at least $r$ mod 3) vertices in $U_i$ with degree $r - 1$. Note that any vertex in $G'$ either belongs to some set $W_{ij}$ or some set $X_i$ where $v_i \in V$.

We first show that the set of $K_r$'s in $G'$ is $\mathcal{U}$.

Lemma 7. $\mathcal{U} = K_r^{G'}$.

Proof. By definition, $\mathcal{U} \subseteq K_r^{G'}$ so it remains to show that each $K_r$ in $G'$ belongs to $\mathcal{U}$. Suppose $K$ is an arbitrary $K_r$ in $G'$. By definition, any vertex in any set $X_i$ has degree $r - 1$ in $G'$ and thus belongs to exactly one $K_r$ in $G'$, namely $U_i$, which belongs to $K_r^{G'}$. It follows that each vertex in $K$ belongs to some set $W_{ij}$ where $1 \leq i, j \leq |V|$. Since $|W_{ij}| = \lceil r/3 \rceil$ it must be that either there exist three sets $W_{i1}, W_{i2}, W_{i3}$ where $1 \leq i_1, i_2, \ldots, i_3 \leq |V|$ and $K = W_{i1} \cup W_{i2} \cup W_{i3}$, or there exist
four or more sets $W_{ij_1}, W_{ij_2}, W_{ij_3}, W_{ij_4}, \ldots$ where $1 \leq i_1, i_2, \ldots \leq |V|$ and $K$ contains at least one vertex in each set. In the latter case, we may assume without loss of generality that $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$. By the construction of $G'$ it follows that no edge exists between any vertex in $W_{ij_1}$ and any vertex in $W_{ij_2}$, which contradicts the supposition that $K$ is a $K_r$ in $G'$. It remains that there exist three sets $W_{ij_1}, W_{ij_2}, W_{ij_3}$ where $K = W_{ij_1} \cup W_{ij_2} \cup W_{ij_3}$ and $1 \leq i_1, i_2, \ldots, j_2 \leq |V|$.

By construction, the closed neighbourhood of any vertex in $W_{ij_1}$ is $U_{i_1} \cup U_{j_1}$ so since $K$ is a $K_r$, without loss of generality assume that $i_1 \in \{i_2, j_2\}$ and $j_1 \in \{i_3, j_3\}$. A symmetric argument shows that $i_2 \in \{i_1, j_1\}$ and $j_2 \in \{i_3, j_3\}$, and $i_3 \in \{i_1, j_1\}$ and $j_3 \in \{i_2, j_2\}$. By symmetry, we only need consider the two cases, in which $i_1 = i_2 = i_3$ and in which $K = W_{i_1, i_2} \cup W_{i_2, i_3} \cup W_{i_3, i_1}$. In the former case, $K$ must be labelled $U_{i_1}$ and thus belongs to $K_r^C$. In the latter case, by the construction of $G'$ the three vertices $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ in $G$ form a triangle, which is a contradiction. □

Lemma 8. $\Delta(G') = \lceil 5r/3 \rceil - 1$.

Proof. By definition, any vertex in any set $X_i$ has degree $r - 1$. Any vertex in any set $W_i$ has degree $|U_i| + |U_i| - |W_i| - 1 = 2r - |r/3| - 1 = \lceil 5r/3 \rceil - 1$, since $r$ is an integer. □

Theorem 6. If $r \geq 3$ and $\Delta \geq \lceil 5r/3 \rceil - 1$ then $VDK_r$ is APX-hard.

Proof. We first show that a vertex-disjoint $K_r$-packing of size $B$ exists in $G'$ if and only if an independent set of size $B$ exists in $G$. By the design of the reduction, and Lemma 7, any two $K_r$'s in $K_r^C$ that are not vertex disjoint in $G'$ correspond to two vertices in $G$ that are adjacent. Conversely, for any two vertices in $G$ that are adjacent, by the design of the reduction it must be the case that the two corresponding $K_r$'s in $K_r^C$ are not vertex disjoint in $G'$. It follows that, for any graph $G$, $opt_{M33-TF}(G) = opt_{V33}(G')$. Moreover, for any vertex-disjoint $K_r$-packing $P$ in $G'$, there exists a corresponding independent set $S$ in $G$ where $|S| = |P|$, and thus that $m_{M33-TF}(G, S) = m_{V33}(G', P)$.

To show that $VDK_r$ is APX-hard in this case, we use an $L$-reduction [6]. An $L$-reduction from optimisation problem $Q$ to an optimisation problem $P$ shows that if there exists a $(1 + \delta)$-approximation algorithm for $P$ then there exists a $(1 + \alpha + \delta)$-approximation algorithm for $Q$. The reduction that we have described from MIS-3-TF to VDK_r is an $L$-reduction with $\alpha = \beta = 1$ (also called a strict reduction [6]), which shows that $VDK_r$ is APX-hard even when $\Delta(G') = \lceil 5r/3 \rceil - 1$ (shown in Lemma 8). To show that $VDK_r$ is APX-hard even when $\Delta(G') \geq \lceil 5r/3 \rceil - 1$, one can add to $G'$ a disconnected star. □

4.2. Edge-disjoint packing

4.2.1. Edge-disjoint $K_r$-packing when $r \geq 6$

If $r \geq 6$ and $\Delta = \lceil 5r/3 \rceil - 1$ then it must be that $\Delta < 2r - 2$, so by Observation 1, any edge-disjoint $K_r$-packing is also vertex disjoint. Theorem 7 then follows, which shows that if $r \geq 6$ then $EDK_r$ is APX-hard even when $\Delta \geq \lceil 5r/3 \rceil - 1$.

Theorem 7. If $r \geq 6$ and $\Delta \geq \lceil 5r/3 \rceil - 1$ then $EDK_r$ is APX-hard.

Proof. Suppose $\Delta = \lceil 5r/3 \rceil - 1$. By definition, any vertex-disjoint $K_r$-packing is also edge disjoint. Since $r \geq 6$ it follows that $\Delta = \lceil 5r/3 \rceil - 1 < 2r - 2$ so by Observation 1, any edge-disjoint $K_r$-packing is also vertex disjoint. We have shown that an edge-disjoint $K_r$-packing of size $B$ exists in $G$ if and only if a vertex-disjoint $K_r$-packing of size $B$ exists in $G$. This fact constitutes an L-reduction with $\alpha = \beta = 1$ from the restricted case of VDK_r, in which $\Delta = \lceil 5r/3 \rceil - 1$. The lemma follows by Theorem 6. As in the proof of Theorem 6, to show that $EDK_r$ is APX-hard if $r \geq 6$ even when $\Delta(G) \geq \lceil 5r/3 \rceil - 1$, one can add to $G'$ a disconnected star. □

4.2.2. Edge-disjoint $K_4$-packing

We now show that $EDK_4$ is APX-hard even when $\Delta = 7$. We present an L-reduction from a variant of Maximum Satisfiability [1], inspired by the L-reduction of Caprara and Rizzi [4] for EDK_3 when $\Delta = 5$.

An instance of Maximum Satisfiability is a boolean formula $\phi$ in conjunctive normal form with clauses $C = \{c_1, c_2, \ldots, c_C\}$ and variables $X = \{x_1, x_2, \ldots, x_X\}$. Each clause contains a set of literals. Each literal is formed by either a variable or its negation. A truth assignment $f$ is a function $f : X \mapsto \{\text{true, false}\}$. A clause is satisfied by $f$ if any of its literals are true. The goal is to find a truth assignment that satisfies the maximum number of clauses. We denote by Max 2SAT_{\leq 3} the special case of Maximum Satisfiability in which each clause contains at most two literals and each variable occurs in at most three clauses. Let $m_i$ be the number of occurrences of variable $x_i$ in $\phi$ for each variable $x_i \in X$. We assume that $2 \leq m_i \leq 3$ for each $x_i \in X$, since if any variable occurs in exactly one clause it can be set to the value satisfying that clause. Max 2SAT_{\leq 3} is APX-hard [11].

Given an instance $\phi$ of Max 2SAT_{\leq 3}, we construct a graph $G$ such that a truth assignment for $\phi$ exists that satisfies at least $k$ clauses if and only if there exists an edge-disjoint $K_4$-packing of size at least $\sum_{i=1}^{X} 3m_i + k$. As in the case of the reduction presented for EDK_3 by Caprara and Rizzi, the reduction here is one of local replacement [11]. The reduction, shown in Fig. 1, involves the construction and connection of variable and clause gadgets, which is a common technique when reducing from a variant of Maximum Satisfiability. The reduction itself is as follows.

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For each variable $x_i$, construct a variable gadget of $10m_i$ vertices, labelled $R_i = \{a'_i, b'_i, c'_i, d'_i, e'_i, h'_i, u'_i, w'_i, y'_i\}$ for each $j$ where $1 \leq j \leq m_i$. For each $j$ where $1 \leq j \leq m_i$, add an edge (if it does not exist already) between each pair of vertices in $\{a'_i, b'_i, u'_i, v'_i\}$, $\{d'_i, e'_i, v'_i\}$, $\{c'_i, u'_i, d'_i, w'_i\}$, $\{d'_i, w'_i, e'_i, y'_i\}$, $\{d'_i, e'_i, h'_i, y'_i\}$, and finally $\{h'_i, a_i^{j+1}, y'_i, u_i^{j+1}\}$ if $j < m_i$ and otherwise $\{h'_i, a_i^1, y'_i, u_i^1\}$.

We shall refer to $\{a'_i, b'_i, u'_i, v'_i\}$, $\{c'_i, v'_i, d'_i, w'_i\}$, and $\{d'_i, e'_i, h'_i, y'_i\}$ as the even $K_4$s in $R_i$, and $\{a'_i, b'_i, c'_i, v'_i\}$, $\{d'_i, w'_i, e'_i, y'_i\}$, and $\{h'_i, a_i^{j+1}, y'_i, u_i^{j+1}\}$ (and $\{h'_i, a_i^1, y'_i, u_i^1\}$) as the odd $K_4$s in $R_i$. Note that at this point in construction, $\text{deg}_G(a'_i) = \text{deg}_G(v'_i) = \text{deg}_G(d'_i) = \text{deg}_G(h'_i) = 5$, and $\text{deg}_G(b'_i) = \text{deg}_G(e'_i) = 4$ for each $j$ where $1 \leq j \leq m_i$.

We shall now construct the clause gadgets. For each clause $c_j$, construct a clause gadget of 5 vertices labelled $S_j = \{s_1^j, t_1^j, s_2^j, t_2^j, w_j\}$. Add an edge (if it does not exist already) between each pair of vertices in $\{s_1^j, t_1^j, s_2^j, w_j\}$ and $\{s_1^j, s_2^j, t_2^j, w_j\}$. We shall refer to $\{s_1^j, t_1^j, s_2^j, w_j\}$ and $\{s_1^j, s_2^j, t_2^j, w_j\}$ as $P^j$ and $P_j$ supposing the variables of the first and second literals in $c_j$ are $x_i$ and $x_j$. Note that at this point in construction, $\text{deg}_G(s_1^j) = \text{deg}_G(s_2^j) = \text{deg}_G(w_j) = 4$ and $\text{deg}_G(t_1^j) = \text{deg}_G(t_2^j) = 3$.

We shall now connect the variable and clause gadgets. For each clause $c_j$, suppose $x_k$ is the variable of some literal in $c_j$ where $c_j$ contains the $j$th occurrence of $x_k$ in $\phi$. If $x_k$ is the first literal in $c_j$ and occurs positively in $c_j$, then identify $b'_k$ and $s_1^j$, and $c'_j$ and $t_1^j$. We shall hereafter refer to the first identified vertex as either $b'_k$ or $s_1^j$ and the second identified vertex as either $c'_j$ or $t_1^j$. Note that now $\text{deg}_G(h'_i) = \text{deg}_G(c'_j) = 7$. Similarly, if $x_k$ is the first literal in $c_j$ and occurs negatively in $c_j$ then identify $e'_k$ and $s_1^j$, and $h'_i$ and $t_1^j$. In this case $\text{deg}_G(e'_k) = \text{deg}_G(h'_i) = 7$. If $x_k$ is the second literal in $c_j$ and occurs positively in $c_j$ then identify $b'_i$ and $s_2^j$, and $c'_i$ and $t_2^j$. Similarly, if $x_k$ is the second literal in $c_j$ and occurs negatively in $c_j$ then identify $e'_i$ and $s_2^j$, and $h'_i$ and $t_2^j$. This completes the construction of $G$. Observe that $\Delta = 7$.

It is straightforward to show that the reduction can be performed in polynomial time. We now prove that the reduction is correct in the first direction. By construction, no $K_4$ exists in $G$ that contains at least one vertex in a variable gadget and at least one vertex in a clause gadget. Thus, we shall say that some $K_4$ is in a variable or clause gadget if it is a strict subset of that gadget.

**Lemma 9.** If a truth assignment $\bar{f}$ for $\phi$ satisfies at least $k$ clauses then an edge-disjoint $K_4$-packing $T$ exists in $G$ where $|T| \geq \sum_{i=1}^{X} 3m_i + k$.

**Proof.** Suppose $\bar{f}$ is a truth assignment for $\phi$ that satisfies at least $k$ clauses. We shall construct an edge-disjoint $K_4$-packing $T$ where $|T| \geq \sum_{i=1}^{X} 3m_i + k$.

For each variable $x_i$, if $\bar{f}(x_i)$ is true then add the set of even $K_4$s in $R_i$ to $T$. Similarly, if $\bar{f}(x_i)$ is false then add the set of odd $K_4$s in $R_i$ to $T$. Now $|T| = \sum_{i=1}^{X} 3m_i$. For each clause gadget $c_j$ that is satisfied by $\bar{f}$, it must be that there exists some variable $x_i$ where either $\bar{f}(x_i)$ is true and $x_i$ occurs positively in $c_j$ or $\bar{f}(x_i)$ is false and $x_i$ occurs negatively in $c_j$. In either case, add $P_j$ to $T$. Now, $T$ contains exactly $\sum_{i=1}^{X} 3m_i$ $K_4$s in variable gadgets and at least $k$ $K_4$s in clause gadgets. It remains to show that $T$ is edge-disjoint. By the construction of $G$, any two $K_4$s in $T$ in the same variable gadget are
edge disjoint. Consider an arbitrary $P_i^j$ in some clause gadget $c_i$ that belongs to $T$. It must be that either $f(x_i) = 0$ and $x_i$ occurs negatively in $c_i$, or $f(x_i) = 1$ and $x_i$ occurs positively in $c_i$. In the former case, $T$ contains the set of even $K_4$s in $R_i$ so since $P_i^j \cap R_i = \{b_j, c_j\}$ where $1 \leq j \leq 3$ it follows that $T$ is edge disjoint. In the latter case, $T$ contains the set of odd $K_4$s in $R_i$ so since $P_i^j \cap R_i = \{e_j, h_j\}$ where $1 \leq j \leq 3$ it also follows that $T$ is edge disjoint. □

We now prove that the reduction is correct in the second direction. We say that some edge-disjoint $K_4$-packing $T$ in $G$ is canonical if for any variable gadget $R_i$, $T$ contains either the set of even $K_4$s in $R_i$ or the set of odd $K_4$s in $R_i$. By the construction of $G$, no edge-disjoint $K_4$-packing can contain all even $K_4$s and all odd $K_4$s.

We first show that for any variable gadget $R_i$ and edge-disjoint $K_4$-packing $T$, if $T$ contains neither all even $K_4$s in $R_i$ nor all odd $K_4$s in $R_i$ then the number of $K_4$s in $T$ is at most $3m_i - 1$.

**Proposition 1.** Suppose $T$ is an arbitrary edge-disjoint $K_4$-packing in $G$. For any variable gadget $R_i$, if $T$ contains neither all even $K_4$s in $R_i$ nor all odd $K_4$s in $R_i$ then the number of $K_4$s in $T$ is at most $3m_i - 1$.

**Proof.** By the construction of $G$, each even $K_4$ in $R_i$ intersects exactly two odd $K_4$s in $R_i$ by at least two vertices and each odd $K_4$ in $R_i$ intersects exactly two even $K_4$s in $R_i$ by at least two vertices. It follows that the $K_4$-edge intersection graph $K_{4\text{e}}^i$ contains a cycle of $6m_i$ vertices corresponding to the $6m_i K_4$s in $R_i$.

It then follows that any edge-disjoint $K_4$-packing that contains $3m_i K_4$s in $R_i$ corresponds to an independent set of size $3m_i$ in $K_{4\text{e}}^i$, and thus is either the set of even $K_4$s in $R_i$ or the set of odd $K_4$s in $R_i$. Since $T$ contains neither all even $K_4$s in $R_i$ nor all odd $K_4$s in $R_i$ it follows that $|T| < 3m_i$. □

We can now prove that for any edge-disjoint $K_4$-packing in $G$ that is not canonical, there exists a canonical edge-disjoint $K_4$-packing in $G$ of at least the same cardinality.

**Lemma 10.** If $T$ is an edge-disjoint $K_4$-packing then there exists a canonical edge-disjoint $K_4$-packing $T'$ where $|T'| \geq |T|$.

**Proof.** If $T$ is already canonical then let $T' = T$. Otherwise, by the definition of canonical, there must exist at least one variable gadget $i$ such that $T$ contains neither all even $K_4$s in $R_i$ nor all odd $K_4$s in $R_i$. For any such $i$ where $1 \leq i \leq |X|$, we show how to modify $T$ to ensure that it either contains the set of even $K_4$s in $R_i$ or the set of odd $K_4$s in $R_i$ and the cardinality of $T$ does not decrease. It follows that there exists a canonical edge-disjoint $K_4$-packing $T'$ where $|T'| \geq |T|$.

Note that by Proposition 1, the number of $K_4$s in $R_i$ in $T$ is at most $3m_i - 1$.

Suppose the variable $x_i$ corresponding to $R_i$ occurs in clauses $c_{r_1}, c_{r_2}, \ldots, c_{r_m}$, corresponding to the sets $P_i^{r_1}, P_i^{r_2}, \ldots, P_i^{r_m}$. It must be that either at most one $K_4$ in $\{P_i^{r_1}, P_i^{r_2}, \ldots, P_i^{r_m}\}$ exists in $T$ where the corresponding occurrence of $x_i$ is positive; or at most one $K_4$ in $\{P_i^{r_1}, P_i^{r_2}, \ldots, P_i^{r_m}\}$ exists in $T$ where the corresponding occurrence of $x_i$ is negative. Suppose the former case is true. Remove the $K_4$ in $\{P_i^{r_1}, P_i^{r_2}, \ldots, P_i^{r_m}\}$ in $T$ where the corresponding occurrence of $x_i$ is positive. Next, remove any even $K_4$s in $R_i$ in $T$ and add the set of odd $K_4$s in $R_i$ not already in $T$. The number of $K_4$s in $R_i$ in $T$ is now $3m_i$ so since at most one $K_4$ was removed, which was not in $R_i$, it follows that the cardinality of $T$ has not decreased. To see that $T$ is still edge-disjoint, observe that any $K_4$ in $\{P_i^{r_1}, P_i^{r_2}, \ldots, P_i^{r_m}\}$ in $T$ intersects any odd $K_4$ in $R_i$ by at most one vertex. The construction and proof in the latter case are symmetric. □

**Lemma 11.** If $T$ is an edge-disjoint $K_4$-packing where $|T| = \sum_{i=1}^{|X|} 3m_i + k$ for some integer $k \geq 1$ then exists a truth assignment $f$ that satisfies at least $k$ clauses.

**Proof.** Assume by Lemma 10 that $T$ is canonical. It follows that $T$ contains exactly $3m_i$ $K_4$s in variable gadgets and at least $k K_4$s in clause gadgets. For each variable $x_i$, set $f(x_i)$ to be true if $T$ contains all even $K_4$s in $R_i$ and false otherwise. Now consider each clause gadget $c_i$, where $S_i$ contains some $K_4$s in $T$, denoted $P_i^s$. Suppose $x_i$ occurs positively in $c_i$. It follows that $P_i^s$ contains $b_j, c_j$ for some $j$ where $1 \leq j \leq 3$. Since $T$ is canonical and edge-disjoint it follows that $T$ contains the set of even $K_4$s in $R_i$. By the construction of $f$ it follows that $f(x_i)$ is true and thus that $c_i$ is satisfied. The proof for when $x_i$ occurs negatively in $c_i$ is symmetric. It follows that at least $k$ clauses are satisfied by $f$. □

**Lemma 12.** If $r = 4$ and $\Delta = 7$ then $EDK_r$ is APX-hard.

**Proof.** We now show that the $L$-reduction we have described from Max 2SAT$_{\leq 3}$ (which is APX-hard [1]) to $EDK_4$ when $\Delta = 7$ is valid. For compactness, we abbreviate Max 2SAT$_{\leq 3}$ when appearing in a subscript to M2S3.

An $L$-reduction is characterised by a pair $(f, g)$ of functions that can be computed in polynomial time. Here, $f$ is the reduction described at the start of this section (Section 4.2.2) in which an instance $G$ of $EDK_4$ is constructed from an arbitrary instance $\phi$ of Max 2SAT$_{\leq 3}$. It is straightforward to show that $f$ can be computed in polynomial time.

The function $g$ is described by Lemma 11. For any instance $\phi$ of Max 2SAT$_{\leq 3}$ and edge-disjoint $K_4$-packing in $f(\phi)$, $g$ computes a truth assignment $f$ for $\phi$. It is also straightforward to show that $g$ can be computed in polynomial time.
To show that $f$ and $g$ constitute a valid $L$-reduction, we must show that there exist fixed constants $\alpha$ and $\beta$ such that for any instance $\phi$ of Max 2SAT\leq_3,

$$\text{opt}_{\text{EDK}_4}(f(\phi)) \leq \alpha \text{opt}_{\text{M2S3}}(\phi) \quad \text{and that for any instance $\phi$ and any edge-disjoint $K_4$-packing $T$ in $f(\phi)$,}$$

$$\text{opt}_{\text{M2S3}}(\phi) - \text{m}_{\text{M2S3}}(\phi, g(\phi, T)) \leq \beta (\text{opt}_{\text{EDK}_4}(f(\phi)) - \text{m}_{\text{EDK}_4}(f(\phi), T)) .$$

We shall now demonstrate the existence of some such $\alpha$ and $\beta$. Recall that in the instance of Max 2SAT\leq_3, $X$ is the set of variables, $C$ is the set of clauses, and $m_i$ is the number of occurrences of each variable $x_i$. Note that by the definition of Max 2SAT\leq_3, $\sum_{i=1}^{[K]} m_i$ is the total number of literals, which must be at most $2|C|$. Note also that for any instance $\phi$ of Max 2SAT\leq_3, it must be that $\text{opt}_{\text{M2S3}}(\phi) \geq |C|/2$. This is because a truth assignment satisfying $|C|/2$ clauses can be found using a greedy algorithm that in each step assigns a truth value to a variable occurring in the maximum number of clauses [30]. We can now show that

$$\text{opt}_{\text{EDK}_4}(f(\phi)) \leq \sum_{i=1}^{[K]} 3m_i + \text{opt}_{\text{M2S3}}(\phi) \leq 3 \sum_{i=1}^{[K]} m_i + \text{opt}_{\text{M2S3}}(\phi) \leq 6|C| + \text{opt}_{\text{M2S3}}(\phi) \leq 13\text{opt}_{\text{M2S3}}(\phi)$$

by Lemma 11, since $2|C| \geq \sum_{i=1}^{[K]} m_i$ and $\text{opt}_{\text{M2S3}}(\phi) \geq |C|/2$

so $\alpha = 13$. We can also show that for any instance $\phi$ and any edge-disjoint $K_4$-packing $T$ in $f(\phi)$,

$$\text{opt}_{\text{M2S3}}(\phi) - \text{m}_{\text{M2S3}}(\phi, g(\phi, T)) \leq \text{opt}_{\text{M2S3}}(\phi) - \left| T - \sum_{i=1}^{[K]} 3m_i \right| \leq \text{opt}_{\text{EDK}_4}(f(\phi)) - |T| \leq \text{opt}_{\text{EDK}_4}(f(\phi)) - |T| - \text{m}_{\text{EDK}_4}(f(\phi), T)$$

by Lemma 11, since $|T| \geq \text{opt}_{\text{EDK}_4}(f(\phi), T) = |T|$

which shows that $\beta = 1$. □

4.2.3. Edge-disjoint $K_5$-packing

We now show that EDK$_5$ is APX-hard even when $\Delta = 9$. We present an $L$-reduction that follows the same pattern as the one shown in Section 4.2.2. This reduction, shown in Fig. 2, is as follows. For each variable $x_i$, construct a variable gadget of $8m_i$ vertices, labelled $R_i = \{a_i^1, b_i^1, c_i^1, d_i^1, e_i^1, h_i^1, u_i^1, v_i^1\}$ for each $j$ where $1 \leq j \leq m_i$. For each $j$ where $1 \leq j \leq m_i$, add an edge (if it does not exist already) between each pair of vertices in $\{a_i^j, b_i^j, e_i^j, u_i^j, v_i^j\}$, $\{b_i^j, c_i^j, e_i^j, h_i^j, v_i^j\}$, and finally $\{c_i^j, d_i^j, h_i^j, v_i^j, u_i^j\}$ and $\{d_i^j, d_i^{j+1}, h_i^j, e_i^{j+1}, u_i^{j+1}\}$ if $j < m_i$, otherwise $\{c_i^j, d_i^j, h_i^j, v_i^j, u_i^j\}$ and $\{d_i^j, d_i^{j+1}, h_i^j, e_i^{j+1}, u_i^{j+1}\}$. We shall refer to $\{a_i^j, b_i^j, e_i^j, u_i^j, v_i^j\}$ and $\{c_i^j, d_i^j, h_i^j, v_i^j, u_i^j\}$ as odd $K_5$s in $R_i$, and $\{b_i^j, c_i^j, e_i^j, h_i^j, v_i^j\}$ and $\{d_i^j, d_i^{j+1}, h_i^j, e_i^{j+1}, u_i^{j+1}\}$ (and $\{d_i^j, a_i^j, h_i^j, e_i^j, u_i^j\}$) as even $K_5$s in $R_i$. At this point $\text{deg}_{C}(a_i^j) = \text{deg}_{C}(b_i^j) = \text{deg}_{C}(c_i^j) = \text{deg}_{C}(d_i^j) = 6$ and $\text{deg}_{C}(a_i^j) = \text{deg}_{C}(h_i^j) = \text{deg}_{C}(u_i^j) = \text{deg}_{C}(v_i^j) = 8$ for any $j$ where $1 \leq j \leq m_i$.

We shall now construct the clause gadgets. For each clause $c_i$, construct a clause gadget of 7 vertices labelled $S_i = \{s_i^1, t_i^1, s_i^2, t_i^2, w_i^1, w_i^2, w_i^3\}$. Add an edge (if it does not exist already) between each pair of vertices in $\{s_i^1, t_i^1, w_i^1, w_i^2, w_i^3\}$ and $\{s_i^2, t_i^2, w_i^2, w_i^3\}$. Label $\{s_i^1, t_i^1, w_i^1, w_i^2, w_i^3\}$ and $\{s_i^2, t_i^2, w_i^2, w_i^3\}$ as $P_i^1$ and $P_i^2$, where the variables of the literals in $c_i$ are $x_i$ and $\bar{x}_i$.

The connection of variable and clause gadgets follows the same pattern as for EDK$_4$. For each clause $c_i$, suppose $x_i$ is the variable of some literal in $c_i$ where $c_i$ contains the $j$th occurrence of $x_i$ in $\phi$. If $x_i$ is the first literal in $c_i$ and occurs positively in $c_i$, then identify $a_i^1$ and $s_i^1$, and $b_i^1$ and $t_i^1$. Now $\text{deg}_{C}(a_i^1) = \text{deg}_{C}(b_i^1) = 9$. Similarly, if $x_i$ is the first literal in $c_i$ and occurs negatively in $c_i$, then identify $b_i^1$ and $s_i^1$, and $c_i^1$ and $t_i^1$. If $x_i$ is the second literal in $c_i$ and occurs positively in $c_i$, then identify $a_i^2$ and $s_i^2$, and $b_i^1$ and $t_i^2$. If $x_i$ is the second literal in $c_i$ and occurs negatively in $c_i$, then identify $b_i^1$ and $s_i^2$, and $c_i^1$ and $t_i^2$. Now $\Delta = 9$.

As before, the reduction can be performed in polynomial time. We now prove correctness in the first direction.

**Lemma 13.** If a truth assignment $\bar{f}$ for $\phi$ satisfies at least $k$ clauses then an edge-disjoint $K_5$-packing $T$ exists in $G$ where $|T| \geq \sum_{i=1}^{[K]} 2m_i + k$.  

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Fig. 2. The reduction from Max 2SAT$\leq_3$ to EDK$_5$.

Proof. Suppose $\mathcal{f}$ is a truth assignment for $\varphi$ that satisfies at least $k$ clauses. We shall construct an edge-disjoint $K_5$-packing $T$ where $|T| \geq \sum_{i=1}^{X} 2m_i + k$. For each variable $x_i$, add to $T$ the set of even $K_5$s in $R_i$ if $f(x_i)$ is true and otherwise the set of odd $K_5$s in $R_i$. Now $|T| = \sum_{i=1}^{X} 3m_i$. For each clause $c_r$ satisfied by $f$, it must be that there exists some variable $x_i$ where $f(x_i)$ is true and $x_i$ occurs positively in $c_r$, or there exists some variable $x_i$ where $f(x_i)$ is false and $x_i$ occurs negatively in $c_r$. As before, in either case add $P_{r}^{i}$ to $T$. Now $|T| = \sum_{i=1}^{X} 2m_i + k$. The proof that $T$ is edge-disjoint is analogous to the proof in Lemma 9. □

We now prove the second direction. Like before, we say that some edge-disjoint $K_5$-packing $T$ in $G$ is canonical if for any $R_i$, $T$ contains either the set of even $K_5$s in $R_i$ or the set of odd $K_5$s in $R_i$.

Lemma 14. If $T$ is an edge-disjoint $K_5$-packing then there exists a canonical edge-disjoint $K_5$-packing $T'$ where $|T'| \geq |T|$.

Proof. The proof is analogous to the proof of Lemma 10. Here we describe the modification of a single variable gadget $R_i$ where $T$ contains neither all even $K_5$s nor all odd $K_5$s in $R_i$. It must be that the number of $K_5$s in $R_i$ in $T$ is at most $2m_i$.

Suppose $x_i$ occurs in clauses $c_{r_1}, c_{r_2}, \ldots, c_{r_{m_i}}$, corresponding to the sets $P_{r_1}^{i_1}, P_{r_2}^{i_1}, \ldots, P_{r_{m_i}}^{i_1}$. It must be that either at most one $K_5$ in $\{P_{r_1}^{i_1}, P_{r_2}^{i_1}, \ldots, P_{r_{m_i}}^{i_1}\}$ exists in $T$ where the corresponding occurrence of $x_i$ is positive, or at most one $K_5$ in $\{P_{r_1}^{i_2}, P_{r_2}^{i_2}, \ldots, P_{r_{m_i}}^{i_2}\}$ exists in $T$ where the corresponding occurrence of $x_i$ is negative. In the former case, remove the $K_5$ in $\{P_{r_1}^{i_1}, P_{r_2}^{i_1}, \ldots, P_{r_{m_i}}^{i_1}\}$ where the corresponding occurrence of $x_i$ is positive as well as any even $K_5$s in $R_i$ in $T$, then add the set of odd $K_5$s not already in $T$. The number of $K_5$s in $R_i$ is now $2m_i$ so since at most one $K_5$ was removed, which was not in $R_i$, it follows that the cardinality of $T$ has not decreased. To see that $T$ is still edge-disjoint, observe that any $K_5$ in $\{P_{r_1}^{i_3}, P_{r_2}^{i_3}, \ldots, P_{r_{m_i}}^{i_3}\}$ in $T$ intersects any odd $K_5$ by at most one vertex. The construction and proof in the latter case is symmetric. □

Lemma 15. If $T$ is an edge-disjoint $K_5$-packing where $|T| = \sum_{i=1}^{X} 2m_i + k$ for some integer $k \geq 1$ then exists a truth assignment $\mathcal{f}$ that satisfies at least $k$ clauses.
Proof. Assume by Lemma 14 that $T$ is canonical. It follows that $T$ contains exactly $\sum_{i=1}^{|X|} 2m_i K_5$s in variable gadgets and at least $k$ $K_5$s in clause gadgets. For each variable $x_i$, set $f(x_i)$ to be true if $T$ contains all even $K_5$s in $R_i$ and false otherwise. Now consider each clause gadget $c_r$ where $S_r$ contains some $K_5$ in $T$, which we label $P'_i$. Suppose $x_i$ occurs positively in $c_r$. It follows that $P'_i$ contains $d_j, b'_j$ for some $j$ where $1 \leq j \leq 3$. Since $T$ is edge disjoint it follows that $T$ contains the even $K_5$s in $R_i$. By the construction of $f$ it follows that $f(x_i)$ is true and thus $c_r$ is satisfied. The proof when $x_i$ occurs negatively in $c_r$ is symmetric. It follows thus that at least $k$ clauses are satisfied by $f$. □

Lemma 16. If $r = 5$ and $\Delta = 9$ then $EDK_r$ is APX-hard.

Proof. The reduction described runs in polynomial time, and Lemma 15 shows how to construct a truth assignment $f$ that satisfies $k$ clauses given an edge-disjoint $K_5$-packing of cardinality $\sum_{i=1}^{\Delta} 3m_i + k$ where $k \geq 1$. By Lemmas 13 and 15, in the reduction a truth assignment $f$ for $\phi$ exists that satisfies at least $k$ clauses if and only if there exists an edge-disjoint $K_5$-packing of size at least $\sum_{i=1}^{\Delta} 3m_i + k$. This reduction is thus an L-reduction with $\alpha = 9$ and $\beta = 1$. □

We now combine Lemmas 12 and 16 with the existing result of Caprara and Rizzi [4] in Theorem 8.

Theorem 8. If $3 \leq r \leq 5$ and $\Delta > 2r - 2$ then $EDK_r$ is APX-hard.

Proof. Caprara and Rizzi [4] prove the case when $r = 3$ and $\Delta = 5$. In Lemma 12 we prove the case when $r = 4$ and $\Delta = 7$. In Lemma 16 we prove the case when $r = 5$ and $\Delta = 9$. □

5. Conclusion and future work

To recap, we considered two problems that involve finding a maximum-cardinality $K_r$-packing in an undirected graph of fixed maximum degree $\Delta$. In the first problem (VDK$_r$), the $K_r$-packing must be vertex disjoint. In the second problem (EDK$_r$), it must be edge disjoint. It is known that VDK$_3$ is solvable in linear time if $\Delta = 3$ but APX-hard if $\Delta \geq 4$, and EDK$_3$ is solvable in linear time if $\Delta = 4$ but APX-hard if $\Delta \geq 5$ [4]. We generalised these results and presented a full complexity classification for both VDK$_r$ and EDK$_r$.

Specifically, we first showed that both VDK$_r$ and EDK$_r$ are solvable in linear time if $\Delta < 3r/2 - 1$ (Theorem 2 and Corollary 1). We then showed that both VDK$_r$ and EDK$_r$ are solvable in polynomial time if $\Delta < 5r/3 - 1$ (Theorems 3 and 4). We then showed that if $r \leq 5$ then EDK$_r$ is actually solvable in polynomial time in the slightly more general case in which $\Delta \leq 2r - 2$ (Theorem 5). We then showed that VDK$_r$ is APX-hard if $r \geq 3$ and $\Delta \geq [5r/3] - 1$ (Theorem 6) and EDK$_r$ is APX-hard if either $r \geq 6$ and $\Delta \geq [5r/3] - 1$ (Theorem 7), or $3 \leq r \leq 5$ and $\Delta > 2r - 2$ (Theorem 8).

Some of our polynomial-time algorithms involved finding a maximum independent set in a corresponding intersection graph. In each case, we showed that this intersection graph was claw-free, from which it follows that a maximum independent set in the intersection graph can be found in polynomial time [24,28]. As we noted in Section 3, in a more general setting in which the vertices of a claw-free graph have real weights, it is possible to find an independent set of maximum weight in polynomial time [24,25]. We remarked that our polynomial-time solvability results involving claw-free graphs can therefore be generalised to versions of VDK$_r$ and EDK$_r$ in which vertices or edges have weights and the goal is to find a $K_r$-packing of maximum total weight. An interesting direction for future work could be to consider other weighted versions of VDK$_r$ and EDK$_r$. More generally, it might be interesting to explore whether there are other types of packing problem in which a “natural” condition implies that the intersection graph is claw-free.

Another direction for future work involves approximation algorithms. For example, Manić and Wakabayashi [23] showed that the known approximation ratio of $(3/2+\varepsilon)$ for VDK$_3$ and EDK$_3$ which can be improved upon in the restricted settings where $\Delta = 4$ and $\Delta = 5$, respectively. It might be possible to show a similar improvement of the corresponding approximation ratio for VDK$_r$ and EDK$_r$ in the setting of an arbitrary fixed maximum degree.

Another possible direction is parameterised complexity. If $r$ is not a fixed constant then our linear-time algorithms could be seen as FPT algorithms and our other polynomial-time algorithms could be seen as XP algorithms, in both cases relative to parameter $r$. It might be interesting to explore parameterised hardness with respect to parameter $r$, for values of $r$ where only XP algorithms are currently known.

Data availability

No data was used for the research described in the article.

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