I. ADDITIONAL DETAILS ON THE CLASSICAL AND QUANTUM VARIANTS OF THE EXPERIMENTAL SETUP

Figure 2 of the manuscript shows a simplified drawing of the experimental setup. A ‘classical’ variant of this setup was used to obtain the results in Figure 3 and 4 of the manuscript, while a ‘quantum’ variant was used to obtain the results in Figure 5 of the manuscript. Figure S1 shows detailed and complete drawings of these two variants.

A. Classical variant

The experimental setup is depicted in Figure S1a. The pump is a collimated continuous-wave laser at 405nm (Coherent OBIS-LX) with an output power of 200mW and a beam diameter of 0.8±0.1mm. The BBO crystal has dimensions $0.5 \times 5 \times 5$mm and is cut for Type I SPDC at 405nm with a half opening angle of 3 degrees (Newlight Photonics). The crystal is slightly rotated around the horizontal axis to ensure near-collinear phase matching of photons at the output (i.e. the ring is collapsed to a disk to produce approximately uniform illumination of the object). A 650nm cut-off long pass filter is used to remove the pump photons after the crystal, along with a band-pass filter at 810±10nm. The correcting SLM is a PLUTO-NIR-15 from Holoeye. It is a liquid-crystal-on-silicon device with a resolution of $1920 \times 1080$ pixels, and a pixel pitch of 8µm. The beam radius on the SLM is approximately 2.4mm, corresponding to 300 pixels. The EMCCD is the model HNü 128 from Nüü Cameras, which has a resolution of 128x128 pixels with a pixel pitch of 16µm. The camera is operated with a region of interest (ROI) of 100 × 100 pixels, at an effective frame-rate of approximately 650 frames per second (fps) and with a gain of 1000. In Figure S1a, for clarity, lenses $f_1$ and $f_5$ are depicted as single lenses; however, in reality, they correspond to multiple lenses. $f_1$ represents three lenses arranged in the confocal configuration 50mm - 100mm - 50mm. The first lens is positioned 50mm after the NLC, and the last lens is situated 100mm before the sample plane. $f_5$ represents five lenses arranged in the confocal configuration 100mm - 100mm - 50mm - 50mm - 100mm, with the first positioned 100mm after the correction SLM, and the last positioned 100mm before the camera. Lenses $f_2-f_4$ all have a focal length of 100mm,

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arranged in a confocal configuration, with \( f_2 \) positioned 100mm after the sample and \( f_5 \) positioned 100mm before the SLM. Magnification from the sample plane to the camera is 1.

**B. Quantum variant**

To obtain the results shown in Figure 5, a ‘quantum’ variant of the experimental setup was used. This experimental setup is depicted in Figure S1b. It has the same arrangement as the setup shown in Figure 2, except that the sample is positioned on only one half of the sample plane (Figs.5ab), and the focal lengths of the lenses and the camera model are different. In this variant, the EMCCD is the model Ixon Ultra 897 from Andor, which has a resolution of 512x512 pixels with a pixel pitch of 16µm. It is operated with a region of interest (ROI) of 150 \( \times \) 150 pixels, at an effective frame-rate of approximately 130 frames per second (fps) and with a gain of 1000. This variant has the following lenses arrangement: \( f_1 \) represents three lenses arranged in the confocal configuration 50mm - 150mm - 100mm, with the first lens positioned 50mm after the NLC and the last lens positioned 100mm before the sample plane. Then, \( f_2, f_3, f_4 \) and \( f_5 \) have focal length of 200mm - 150mm - 50mm - 75mm, respectively, arranged in a confocal configuration, with \( f_2 \) positioned 200mm after the sample and \( f_5 \) positioned 75mm before the camera. Magnification from the sample plane to the camera is 1.125. In this variant, the correction SLM is positioned in plane A2 and the PDMS layer used to induce aberration is positioned in a conjugated plane of A1. More details can be found in section A of the supplementary document.

**II. MEASUREMENT \( G^{(2)}, C^+ \) AND \( R \)**

In each experiment of our work, we measure the spatially-resolved second-order correlation function \( G^{(2)} \), and then use it to compute the sum-coordinate projection. \( G^{(2)} \) takes the form of a 4-dimensional matrix containing \((N_x \times N_y)^2\) pixels, where \( N_x \times N_y \) corresponds to the region of the sensor used to capture data. An element of the matrix is written \( G_{ijkl}^{(2)} \), where \((i,j)\) and \((k,l)\) are pixel labels corresponding to spatial positions \((x_i, y_j)\) and \((x_k, y_l)\). It is measured by acquiring a set of \( M + 1 \) frames, denoted \( \{ I^{(l)} \}_{l \in [1,M+1]} \), using a fixed exposure time of 0.002s and then processing them using the formula provided in Ref. [1]:

\[
\Gamma_{ijkl} = \frac{1}{M} \sum_{l=1}^{M} \left[ I_{ij}^{(l)} I_{kl}^{(l)} - I_{ij}^{(l)} I_{kl}^{(l+1)} \right].
\]  

(B1)

Since the EMCCD camera cannot resolve the number of photons incident on a single pixel, the photon coincidences at the same pixel cannot be measured, and so the corresponding values \( G_{iijj}^{(2)} \) are set to 0. \( G_{ijkl}^{(2)} \) is a discrete version of the continuous second-order intensity correlation function \( G^{(2)}(r_1, r_2) = |\phi(r_1, r_2)|^2 \) where \( \phi \) is the spatial two-photon wave-function associated with the photon pairs. This formalism has been employed in various studies describing the propagation of entangled photon pairs [2, 3]. Then, \( C^+ (\delta r^+) \) can be computed from \( G^{(2)} \) by performing a
sum-coordinate projection. Using the continuous-variable formalism, $C^+$ is defined as:

$$C^+(\delta r^+) = \int G^{(2)}(r, \delta r^+ - r) dr$$  \hspace{1cm} (B2)

with $\delta r^+ = r_1 + r_2$, $r_1, r_2$ being the spatial coordinates of each photon in a pair, respectively. In practice, $C^+$ is calculated using the discrete-variable formula:

$$C^+_{i+j} = \sum_{i=1}^{N_R} \sum_{j=1}^{N_R} G^{(2)}(i, j, i-j).$$  \hspace{1cm} (B3)

In addition to $C^+$, we can also retrieve the anti-correlation image, denoted $R$, from $G^{(2)}$. Figures. 5.c.d.f show examples of such anti-correlation images. $R$ is by definition the anti-diagonal of $G^{(2)}$ i.e. $R(r) = G^{(2)}(r, -r)$, or $R_{ij} = G^{(2)}_{ij}(-i-j)$ when using the discrete formalism. In practice, however, due to small asymmetries in the photon correlations structure, it is preferable to average several anti-diagonals to obtain a more homogeneous image. In our case, averaging is performed over an small area of $5 \times 5$ pixels:

$$R_{ij} = \sum_{k,l \in [-2,2]} G^{(2)}_{ij}(-i-k)(-j+l).$$  \hspace{1cm} (B4)

**a. Classical variant**

**b. Quantum variant**

FIG. S1. Details on the classical and quantum variant of the experimental setup.  
a, Classical variant of the setup shown in Figure 2 of the manuscript. Lenses: $f_{11} = 50$ mm, $f_{12} = 100$ mm, $f_{13} = 50$ mm, $f_2 = 100$ mm, $f_3 = 100$ mm, $f_4 = 100$ mm, $f_{51} = 100$, $f_{52} = 100$ mm, $f_{53} = 50$ mm, $f_{54} = 50$ mm and $f_{55} = 50$ mm. All lens arrangements are confocal: the distance between two lenses is equal to the sum of their focal lengths. The distance between the crystal and $f_{11}$ is 50 mm. The aberration SLM is positioned at a distance of 100 mm from each $f_2$ and $f_3$ lens. The correction SLM is positioned at a distance of 100 mm from each $f_4$ and $f_{51}$ lens. The EMCCD camera is the model HNuv from Nuvi Cameras, and is positioned at a distance of 50 mm from the lens $f_{52}$. The distance between the crystal and $f_{11}$ is 50 mm. The aberration SLM is positioned at a distance of 100 mm from each $f_2$ and $f_3$ lens. The correction SLM is positioned at a distance of 100 mm from each $f_4$ and $f_{51}$ lens. The EMCCD camera is the model HNuv 128 from Nuvi Cameras, and is positioned at a distance of 50 mm from the lens $f_{52}$.  

b, Quantum variant of the setup shown in Figure 2 of the manuscript. Lenses: $f_{11} = 50$ mm, $f_{12} = 150$ mm, $f_{13} = 100$ mm, $f_2 = 200$ mm, $f_3 = 150$ mm, $f_4 = 50$ mm and $f_5 = 75$ mm. All lens arrangements are confocal: the distance between two lenses is equal to the sum of their focal lengths. The distance between the crystal and $f_{11}$ is 50 mm. The correction SLM is positioned at a distance of 200 mm from $f_2$ and 150 mm from $f_3$. Optical aberrations are induced by inserting a 1-cm-thick piece of polydimethylsiloxane (PDMS) in plane A1 of the setup, that is a plane located at a short distance from the sample’s conjugate plane. A1 is deliberately not placed in the conjugate plane to introduce sufficient aberrations. The EMCCD camera is the model Ixon Ultra 897 from Andor, and is positioned at a distance of 75 mm from the lens $f_5$. Optical paths associated with each photon are drawn separately to emphasize the fact that one illuminates the object, while the other propagates in free space and serves as a reference. LP: Low-pass filter; BBO: Beta-Barium Borate non-linear crystal.
Additional details are provided in section F of the supplementary document.

III. ADDITIONAL DETAILS ON THE OPTIMIZATION PROCESS

Our QAO approach is operated by maximizing the central value $C_0^+$ of the sum-coordinate projection $C^+$. In this section, we provide theoretical details, simulations and experimental results supporting this approach.

A. Demonstration of equation 2 of the manuscript

In the following, we describe the different steps used to derive Equation 2 of the manuscript, which is central to understand the QAO process.

The spatial two-photon wavefunction $\phi$, characterizing a pure two-photon state, propagates according to the rules of Fourier optics [4] as:

$$\phi(r'_1, r'_2) = \int \int \phi(r_1, r_2) t(r_1) t(r_2) h(r'_1, r_1) h(r'_2, r_2) dr_1 dr_2.$$

where $r_1$ and $r_2$ denote spatial coordinates in the input optical plane, $r'_1$ and $r'_2$ denote spatial coordinates in the output optical plane, $t$ is an object positioned in the input plane and $h$ is the point spread function (PSF) linking the input and output planes. Then, by definition, $G^{(2)} = |\phi|^2$. In our work, we also use the following assumptions:

1. We assume that the PSF is shift-invariant. In fact, Equation 1 of the manuscript is obtained from Equation B5 after making this assumption. Although it will simplify our calculations, this assumption is not strictly necessary.

2. We assume that the input plane (i.e. sample plane) is positioned in a Fourier plane of the crystal, and that in this plane photons are almost perfectly anti-correlated:

$$\phi(r_1, r_2) \approx \delta(r_1 + r_2).$$

The assumption that the pairs are perfectly anti-correlated in the sample plane is justified under our experimental conditions, since the crystal is illuminated by a collimated beam (0.8mm diameter), and its thickness (0.5mm) is much smaller than the corresponding Rayleigh length ($\sim 1m$) [3]. Note that, in Equation B6 the photon pair illumination beam is assumed to be infinitely large. To account for the fact that, in reality, the beam has a finite diameter, we can simply modify the definition of object $t$ and include edges $t(r) \rightarrow t(r) \text{rect}(r/a)$, where $a^2$ is the illumination area.

Under these assumptions, Equation B5 becomes:

$$\phi(r'_1, r'_2) = \int \int \delta(r_1 + r_2) t(r_1) t(r_2) h(r'_1 - r_1) h(r'_2 - r_2) dr_1 dr_2.$$
Then, using the following change of variables:

\[ r_+ = \frac{r_1 + r_2}{2}, \]
\[ r_- = \frac{r_1 - r_2}{2}, \]  

one can express the two-photon wavefunction as:

\[
\phi(r'_+, r'_-) = \int \int \delta(r_+) t(r_+ + r_-) t(r_+ - r_-) h(r'_+ + r'_- - (r_+ + r_-)) h(r'_+ - r'_- - (r_+ - r_-)) dr_+ dr_-
= \int t(r_-) t(-r_-) h(r'_+ + r'_- - r_-) h(r'_+ - r'_- + r_-) dr_-.
\]  

From there, we use another change of variables: \( r = r'_+ + r'_- - r_- \), to obtain:

\[
\phi(r'_+, r'_-) = \int t(r'_+ + r'_- - r) t(-r'_+ - r'_- + r) h(r) h(2r'_+ - r) dr.
\]  

Then, by definition: \( C^+(\delta r^+) = \int G^{(2)}(r, \delta r^+ - r) dr \). Using again the change of variable i.e. \( (r_1, r_2) \rightarrow (r_+, r_-) \), one can express \( C^+ \) as:

\[
C^+(\delta r^+) = \int \phi(r_1, \delta r^+ - r_1) dr_1
= \int \int \phi(r_1, r_2)^2 \delta(r_2 - (\delta r^+ - r_1)) dr_1
= \int \left| \phi \left( \frac{\delta r^+}{2}, r_- \right) \right|^2 dr_-.
\]

Equation (B12) can then be expanded as:

\[
C^+(\delta r^+) = \int \int \left| t \left( \frac{\delta r^+}{2} + r_- - r \right) t \left( -\frac{\delta r^+}{2} - r_+ + r \right) h(r) h(\delta r^+ - r) dr \right|^2 dr_-
= \int \int t \left( \frac{\delta r^+}{2} + r_- - r_+ \right) t \left( -\frac{\delta r^+}{2} - r_- + r_+ \right) h(r) h(\delta r^+ - r_+) t^* \left( \frac{\delta r^+}{2} - r_- - r_+ \right)
\]

\[
t^* \left( -\frac{\delta r^+}{2} - r_- + r_+ \right) h^*(r_B) h^*(\delta r^+ - r_B) dr_A dr_B dr_-
= \int h(r) h(\delta r^+ - r_+) h^*(r_B) h^*(\delta r^+ - r_B) dr_A dr_B
dr_-
= \int t \left( \frac{\delta r^+}{2} + r_- - r_+ \right) t \left( -\frac{\delta r^+}{2} - r_- + r_+ \right) t^* \left( \frac{\delta r^+}{2} + r_- - r_+ \right) t^* \left( -\frac{\delta r^+}{2} - r_- + r_+ \right) dr_-
= \int K(r_+, r_+; r) h(r) h(\delta r^+ - r_+) h^*(r_B) h^*(\delta r^+ - r_B) dr_A dr_B.
\]
Then, using the change of variable \( r = \frac{\delta r^+}{2} + r_+ - r_A \), \( K(r_A, r_B) \) can be simplified as:

\[
K(r_A, r_B) = \int t\left(\frac{\delta r^+}{2} + r_+ - r_A\right) t\left(-\frac{\delta r^+}{2} - r_+ + r_A\right) t^*\left(\frac{\delta r^+}{2} + r_+ - r_B\right) t^*\left(-\frac{\delta r^+}{2} - r_+ + r_B\right) dr_-
= \int t(r) t(-r) t(r + (r_A - r_B)) t(-r - (r_A - r_B)) dr.
\] (B14)

As a result, we have shown that \( C^+(\delta r^+) \) can be written:

\[
C^+(\delta r^+) = \iint K(r_A, r_B) h(r_A) h(\delta r^+ - r_A) h^*(r_B) h^*(\delta r^+ - r_B) dr_A dr_B.
\] (B15)

where

\[
K(r_A, r_B) = \int t(r) t(-r) t^*(r + (r_A - r_B)) t^*(-r - (r_A - r_B)) dr.
\] (B16)

Equation [B15] is the most general form of \( C^+ \). It is at the basis of all the numerical simulations shown in the supplementary document. In addition, assuming that the optical aberrations present in the system are sufficiently weak, Equation [B15] can be further simplified to obtain Equation 2 of the manuscript. Indeed, if the aberrations are weak, the PSF \( h \) is quite narrow. This implies that the term \( h(r) h(\delta r^+ - r_A) h^*(r_B) h^*(\delta r^+ - r_B) \) is non-negligible only for small values of \( |r_A| \) and \( |r_B| \) i.e. close to 0. In this case, the values \( |r_A - r_B| \) are also small, and Equation [B16] simplifies into:

\[
K(r_A, r_B) \approx \int |t(r) t(-r)|^2 dr = K.
\] (B17)

Under this assumption, \( K \) is now a constant that can be pulled out of the integral in Equation [B15] to obtain Equation 2 of the manuscript. The validity of the weak aberration hypothesis is confirmed by the simulations and further experimental results shown in Figures S5 and S6. Note that \( K \) represents the rate of photon pairs transmitted through the object. In particular, it should be noted that there are very special cases for which \( K = 0 \), such as when the object blocks light in a very specific anti-symmetric manner, i.e., \( \forall r, t(r) \neq 0 \Rightarrow t(-r) = 0 \). This is one of the pathological cases for which QAO will not work.

**B. An simple analytical example**

The central value of the sum-coordinate projection, \( C_0^+ = C^+(\delta r^+ = 0) \), can be expressed as:

\[
C_0^+ \propto |[h \ast h](0)|^2.
\] (B18)

From Equation [B18] one can conclude that the narrower is the PSF, the larger is \( C_0^+ \). This is demonstrated experimentally in Figure 3 and by numerical simulations in the next sections. To provide an intuitive understanding
of this result, let’s consider the case of the PSF approximated by a ‘Jinc’ function. This PSF corresponds to an imaging system whose numerical aperture is limited by a circular pupil. It can be written

\[ h(r) = A/\sigma \text{jinc}(|r|/\sigma) = 2AJ_1(|r|/\sigma)/|r|, \]

where \( J_1 \) is the first order Bessel function, \( \sigma \) is a length coefficient proportional to the central peak width and depends on both the wavelength and the pupil diameter, and \( A \) is a unitless constant ensuring that \( \int |h(r)|^2 \, dr = 1 \). For simplicity, we consider optical aberrations only in the form of a defocus. In this case, the PSF remains Gaussian and its width \( \sigma \) varies according to the severity of the defocus. The integral in Equation B18 can then be written as

\[
C^+_0 \propto |\text{jinc} \ast \text{jinc}|(0) = \frac{A^2}{\sigma^2} |\text{jinc}(0)|^2 
\]

\[
\propto \frac{1}{\sigma^2} \tag{B19}
\]

showing that \( C^+_0 \) increases as \( \sigma \) decreases i.e. as the defocus gets corrected.

C. Numerical simulations of Equation 2

For clarity, and without loss of generality, all simulations are performed in 1 spatial dimension i.e. \( r \to x \). These simulations are performed with Matlab. Their main components are described in the following:

- **Two-photon input field**: A two-photon wave-function \( \phi(x_1, x_2) \) is produced. Assuming perfect anti-correlations, such a function in Matlab takes the form of a 1001 × 1001 pixel matrix with zeros everywhere and ones on its anti-diagonal. It is denoted \( \Phi_{\text{en}} \). 1001 is the number of pixels considered in our simulations (spatial discretization). This number can be chosen arbitrarily, but must be sufficient to prevent resulting images from being too ‘pixelated’.

- **Object**: In Matlab, an object \( t \) takes the form of a vector with 1001 pixels.

- **Point spread function**: In Matlab, the point spread function \( h(x' - x) \) takes the form of a 1001 × 1001 pixel matrix, denoted \( T \). To simulate aberrations, \( T \) is generated by multiplying three matrices: \( T = FDF \), where \( F \) is the transfer matrix associated to light propagation through a convergent lens from the object plane to the focal plane (i.e. it is the matrix of a discrete Fourier transform), and \( D \) is a complex diagonal matrix. All terms in \( D \) are of absolute value 1. Optical aberrations in the system are simulated by modulating the phase terms on the diagonal of \( D \). For example, if they are all equal, the phase is flat, there is no aberration, and the imaging system is diffraction-limited. If, on the other hand, the phase terms are chosen randomly, then we create strong optical aberrations. In particular, the severity of aberrations can be controlled by modifying the degree of randomness of the phase terms, i.e. their degree of correlation. We then define the ‘aberration strength’. For that, we calculate the field-field correlation of the central column of \( T \), which corresponds to \( h(r) \). This function reveals a central peak, corresponding to the spatial correlation width of the field. The ‘aberration strength’,
FIG. S2. Controlling the strength of the optical aberrations in the simulations a, Flat phase in the diagonal of \( D \). b, Resulting diffraction-limited PSF. c, Phase profile in the diagonal of \( D \) with \( f_{ab} = 1 \). d, Resulting aberrated PSF. e, Phase profile in the diagonal of \( D \) with \( f_{ab} = 4 \). f, Resulting aberrated PSF.

denoted as \( f_{ab} \), is defined as the inverse of the full width at half maximum of this central peak. Figure S2 shows two examples of simulated aberrations. In figure S2.c, the phase fluctuates very little over the entire width of the diagonal. Consequently, there are few aberrations in the resulting PSF (fig. S2.d). In this case, the correlation width is equal to the total width of the space, corresponding to an aberration strength equal to 1. In Figure S2.e, on the other hand, the phase fluctuates much more, resulting in significant aberrations in the PSF (fig. S2.d). In this example, the correlation width has been chosen to be equal to a quarter of the total width of the space, corresponding to an aberration strength of 4.

• Propagation: To simulate the propagation of a two-photon wavefunction \( \Phi_{in} \) through an object \( t \) (positioned in the input plane) and an optical system \( T \), we perform the following matrix multiplication:

\[
\Phi_{out} = T \text{diag}(t) \Phi_{in} \text{diag}(t) \text{trans} \text{pose}(T),
\]  

(B20)

where \( \Phi_{out} \) is the two-photon wavefunction at the output i.e. in the camera plane. Then, by definition,
\[ G^{(2)} = |\Phi_{\text{out}}|^2. \] To calculate \( C^+ \), we sum \( G^{(2)} \) along its sum-coordinate axis using the formula:

\[
C^+_i = \sum_{i=1}^{N_x} G^{(2)}(i+i_i), \tag{B21}
\]

In addition, to calculate the output intensity image \( I \), we perform the following matrix-vector multiplication:

\[
I = T^2 t^2. \]

It simulates the propagation of an object \( t \) illuminated by incoherent light through the system \( T \).

Figure S3 shows examples of 1D simulations. Figure S3a shows the \( G^{(2)} \) matrix after propagation of \( \Phi_{\text{in}} \) through the optical system \( T \) with no aberrations and no object. The output intensity (Fig. S3d) is constant and the sum-coordinate projection (red curve) reveals the diffraction-limited PSF. Figure S3b shows the output \( G^{(2)} \) after introducing an object. In this example, we chose an object that has a simple shape:

\[
t(350 \leq x < 450) = t(550 \leq x < 650) = 1, t(150 \leq x < 250) = t(750 \leq x < 850) = 0, \text{ and } 0 \text{ elsewhere.}
\]

We observe that: (i) the anti-diagonal of \( G^{(2)} \) becomes modulated, (ii) the intensity \( I \) (Fig. S3e) shows an image corresponding to the convolution of the object with the PSF-squared, and (iii) the sum-coordinate projection still look very similar to the PSF, with a narrow and intense peak at its center. Finally, Figure S3c shows the output \( G^{(2)} \) after introducing an object and also in the presence of aberrations. In this example, we generated smooth aberrations to just slightly deform the PSF (\( f_{ab} = 1 \)). We observe that the anti-diagonal of \( G^{(2)} \) is still modulated by the object, but in the same time its width is modified by the presence of the aberrations. In Figure S3f, the intensity \( I \) shows a distorted image of the object, and when summing along the sum-coordinate axis, we retrieve an image very similar to the system PSF (red curve).

D. Comparison of the sum-coordinate projections \( C^+ \) with and without an object.

To justify the approximation shown in equation B17, we perform here simulations of the sum-coordinate projections \( C^+ \) obtained in the presence of optical aberration, with and without an object. Figure S4a show the sum-coordinate projections obtained without object (blue curve), with a relatively simple object shown in Figure S4c (red curve) and with a more complex object shown in Figure S4d (green curve). Figure S4b show the same curves as in Figure S4a after normalization to their maximum value. The aberration strength was set to \( f_{ab} = 0.9 \). On these examples, we observe two important behaviors:

1. The presence of an object does not change the spatial shape of the sum-coordinate projection, but only decreases its amplitude. This observation is consistent with the approximation of Equation B17.

2. Furthermore, we can see that the spatial shape of the sum-coordinate projection is very similar to that of the true PSF (dashed-black curves).

Figure S5 provides a more quantitative and robust analysis of the behaviors observed in the examples in Figure S4. The blue curve shows the similarity of the sum-coordinate projection obtained in the presence of an object in function of the aberration strength. One similarity value is obtained by calculating the correlation coefficient (using the ‘corrcoef’ function in Matlab) between the sum-coordinate projection generated without an object (used as a reference) and the
**FIG. S3.** 1D simulation of photon-pairs propagating through an imaging system in the presence of an object and optical aberrations. 

- **a,** $G^{(2)}$ matrix obtained by propagating entangled photon-pairs through an imaging system with no object and no optical aberrations, together with the sum-coordinate projection $C^+$ (red curve). 
- **b,** $G^{(2)}$ and $C^+$ (red curve) in the presence of an object and with no aberrations. 
- **c,** $G^{(2)}$ and $C^+$ (red curve) in the presence of an object and optical aberrations. 
- **d,** Intensity image $I$ obtained with no object and no aberrations. 
- **e,** $I$ obtained with an object and no aberrations. 
- **f,** $I$ obtained with an object and aberrations.

**FIG. S4.** Sum-coordinate projections $C^+$ with and without an object. 

- **a,** Sum-coordinate projections $C^+$ obtained in the presence of aberrations with strength $f_{ab} = 0.9$ without object (blue curve), with a simple object (red curve) and with a more complex object (green curve). The absolute-squared of the PSF is also represented by the dashed-black line. 
- **b,** Same curves as in (a) after normalization to their maximum value. 
- **c and d,** Simple and more complex object used in the simulations, respectively.

Sum-coordinate projection generated with an object. For each aberration strength $f_{ab}$, we calculated 400 correlation values between the reference sum-coordinate projection and 400 sum-coordinate projections generated for 20 different randomly generated aberration patterns and 20 different randomly generated objects (similar to the one shown in Figure S4d). After averaging, we obtained the points in blue curve. As we can see, the sum-coordinate projection
FIG. S5. **Similarity of the sum-coordinate projection with and without an object in function of the aberration strength.** (Blue curve) Similarity values obtained by comparing the sum-coordinate projection simulated with an object and this without object. To obtain a similarity value, 400 correlation values were calculated using the Matlab function ‘corrcoef’ between the reference sum-coordinate projection (no object) and 400 sum-coordinate projections generated for 20 different randomly generated aberration patterns and 20 different randomly generated objects. (Red curve) Similarity values obtained with the same process but using the absolute value squared-PSF as the reference.

with an object remains very similar (> 0.97 similarity) to the sum-coordinate projection obtained without an object. These simulations confirm the approximation shown in Equation B17 and thus justify Equation 2 of the manuscript.

In addition, using the same approach, we also compared sum-coordinate projections obtained with an object with the absolute value squared-PSF. Similarity values form the red curve in Figure S5. As we can see, the sum-coordinate projection is also very similar to the absolute value squared-PSF for weak aberrations i.e. the similarity only goes below 0.8 at \( f_{ab} = 1 \). As long as the aberrations are not too severe - which is the case in all our experiments - it is thus valid to consider the sum-coordinate projection as a approximation of the imaging system’s PSF.

In addition to these simulations, Figure S6 shows experimental results of the sum-coordinate projections measured with and without an object in the presence of optical aberrations.

E. **Graphical interpretation of the sum-coordinate projections obtained with and without an object**

As detailed previously, we found that the sum-coordinate depends very little on the object, especially if the aberrations are small. This is well understood mathematically with the theory described in section III. To understand these calculations in a more intuitive way, we consider some simple simulations shown in Figure S7. Figure S7 a,b,c show \( G^{(2)} \) and \( C^+ \) functions obtained with (a) no object and no aberrations, (b) with no object and in the presence of aberrations, and (c) with object and aberrations. The aberrations simulated here were weak i.e. \( f_{ab} = 1 \).
FIG. S6. **Comparison of sum-coordinate projection with and without the presence of an object.** First row (a,b,c) acquired with no induced aberrations. Second row (d,e,f) acquired with moderate aberrations. Third row (g,h,i) acquired with strong aberrations. First column (a,d,g) shows direct intensity images with the object, a 100µm resolution target grid. Second column (b,e,h) shows sum-coordinate projections acquired with the object present. Third column (c,f,i) shows the sum coordinate projection without the object present. Each intensity and sum-coordinate image acquired from $10^7$ frames, taking $\sim 5$ hours. Aberrations induced with a second SLM placed at plane A2. a.u. - arbitrary units

As expected, the $G^{(2)}$ are different in the case with and without the object. The resulting sum-coordinate projections (red curves) are nevertheless very similar, as predicted in theory. Looking more closely at the images, we understand that each value in the sum-coordinate projection is obtained by summing all the values of a given anti-diagonal of $G^{(2)}$. Each sum-coordinate value can therefore be interpreted as an ‘average’ value of the correlation values on each $G^{(2)}$ anti-diagonal. It is precisely this averaging effect that makes the sum-coordinate projection always ‘similar’ with or without an object present in the system.
F. Simulations of maximizing $C_0^+$ to correct optical aberrations

To demonstrate the suitability of $C_0^+$ as an optimisation target, we show numerical simulations of the effects of aberration strength on the sum-coordinate projection and specifically $C_0^+$. As above, the aberration strength is parameterised by $f_{ab}$, the inverse of the inverse of the width of the field-field correlation. Figure S8 shows the results of these simulations. Figure S8a demonstrates that $C_0^+$ decreases with increasing aberrations. Data shown in the plot is an average over 20 different base aberrations. Figure S8b shows sum-coordinate projections for select values of $f_{ab}$ to show the effect of aberration strength on the shape of the projection. To smoothly vary the aberration strength, a random base vector of 1001 elements between 0 and 1 is generated using Matlab’s rand function. This is used as the phase of our complex aberration vector, which is subsequently filtered in the Fourier domain such that its correlation length corresponds to the desired value.

These results clearly demonstrate that $C_0^+$ is dependent on aberration strength, i.e. it is inversely proportional. In
FIG. S8. **Effect of aberration strength on sum-coordinate projection.** 

**a,** Value of $C_0^+$ as a function of aberration strength. Aberration strength corresponds to $f_{ab}$, as introduced above. **b,** Five examples of sum-coordinate projections for a range of aberration strengths with $f_{ab}$ values shown in inset legend.

In other words, the stronger the aberrations, the more spread the sum-coordinate projection, and so $C_0^+$ is reduced. This implies that, if an optimisation process maximised $C_0^+$, it is indirectly minimising the optical aberrations present. This is the demonstration for why $C_0^+$ is an appropriate optimisation parameter to perform adaptive optics.

### IV. ADDITIONAL DETAILS ON THE MODAL-BASED OPTIMIZATION

In the following, we provide more details about the modal-based optimization process.
In our work, the Zernike polynomials are used as the modal basis for correction, since they form an orthonormal basis on the unit disk and result in a smooth phase mask \[5\]. The phase masks \( \theta(\psi, \rho) \) are generated using

\[
\theta(\psi, \rho) = \sum_{n=n_{\text{min}}}^{n_{\text{max}}} \sum_{m=-n}^{n} \alpha_{mn} Z_{n}^{m}(\psi, \rho),
\]

where \( \alpha_{mn} \) is the coefficient for the \( m,n \)th Zernike polynomial \( Z_{n}^{m} \), and \( \psi = \arctan2(y, x) \), \( \rho = \sqrt{x^2 + y^2} \) are the 2-dimensional radial coordinates. Note that \( Z_{n}^{m} = 0 \) when \( m + n \) is odd, so the total number of modes is given by \( N = \frac{1}{2}[(n_{\text{max}} + 1)(n_{\text{max}} + 2) - n_{\text{min}}(n_{\text{min}} + 1)] \) with the condition that \( n_{\text{max}} \geq n_{\text{min}} \geq 0 \). The correction is done modally, meaning the best correction for each Zernike polynomial is found sequentially. To reduce the errors due to any crosstalk between modes, the correction is repeated starting from the phase mask found during the previous iteration. For each mode \( mn \), a set of sum-coordinate projections is acquired for a range of correction coefficients \( \alpha_{mn} \). The maximum value for each projection is fitted as a function of \( \alpha_{mn} \), and the value of \( \alpha_{mn}^{\text{corr}} \) that maximises the projection peak is calculated. A Gaussian model is used for the fitting process. In the manuscript, Figure 3.h shows an example of acquisition and fitting for two modes.

Figure S9 shows in more details another example where the aberrations consist of three modes: \( Z_{2}^{0}, Z_{2}^{-2}, Z_{2}^{2} \) (defocus, vertical astigmatism, and oblique astigmatism). The coincidence image peak was calculated for a range of 10 different values of \( \alpha_{mn}^{i} \) for each mode, plotted in Figure S9a. In principle, the fitting can be done with as few as two points per mode plus one value for zero correction, allowing for correction with only \( 2N + 1 \) points \[6\]. However, to reduce the fitting error due to noise, more points are typically used.

From experimental observations, we assume that the correlation peak \( C_0 \) can be written as a function of correction coefficient \( \alpha_{mn} \) with the form

\[
C_0(\alpha_{mn}) = \beta + Ae^{-\frac{1}{2\sigma^2}(\alpha_{mn} - \alpha_{mn}^{\text{corr}})^2}.
\]

That is, \( C_0^+(\alpha_{mn}) \) is a Gaussian centered at the best correction \( \alpha_{mn}^{\text{corr}} \). \( \beta \) represents the noise floor of the measurements, since \( C_0^+ \) typically never goes to 0, as can be seen in Figure S9a. \( A \) is the amplitude of the Gaussian function, and \( \sigma \) is the width, both of which are dependent on acquisition time for \( C_0^+(\tau) \). MATLAB’s curve fitting functionality is used to perform the fits, and the value of \( \alpha_{mn}^{\text{corr}} \) is extracted.

To achieve a suitable SNR, \( \sim10^5 \) frames are acquired for each point. At 650Hz, to acquire \( 10^5 \) frames takes 150s, giving an approximate acquisition time of \( \sim3 \) minutes per point. To further reduce the effects of noise, the optimal correction is added to SLM B at each step so that \( C_0^+ \) becomes more visible with each corrected mode. Additionally, the projection images can be binned into macropixels to improve the SNR of \( C_0^+ \). This can be seen in Figure S9b,c,d which shows the coincidence images after correcting for each mode. The optimisation process was done for 2 iterations.
FIG. S9. Modal optimisation of $C_0^+$ For each Zernike function, we find the coefficient $\alpha_{corr}^{mn}$ that maximises $C_0^+$, then add this to our correction mask. a, Value of the peak of the sum-coordinate projection plotted as a function of correction strength for each of the n=2 Zernike functions. Solid line shows the fitted Gaussian function for each mode. b,c,d, $C^+(r)$ after optimising $C_0^+$ for $Z_2^0$, $Z_2^{-2}$, $Z_2^2$, respectively.

V. GENERATION OF ABERRATIONS

To test our QAO approach, we induce aberrations in the imaging system. These aberrations are produced by introducing aberrant elements into the A1 and A2 optical planes shown in Figure 2. These artificial aberrations replicate two prevalent types of aberration in microscopy - system-induced aberration (e.g. due to microscope objectives, lenses, or misalignment), and weak sample-induced aberrations (e.g. due to translucent tissue surrounding the sample, immersion liquid, or sample support). These systems induce 'weak optical aberrations' in the imaging system, meaning aberrations of (i) low frequency and that (ii) preserve the shift invariance of the PSF across the entire field of view i.e. $h(r, r') = h(r - r')$. This is the regime in which we demonstrate QAO. Nevertheless, in section XII Figures S14, S15 and S16 we have obtained additional simulation and preliminary experimental results in the regime of more complex aberrations.

In our work, we used a second SLM in plane A2 to obtain the results shown in Figures 3 and 4, and a piece of polydimethylsiloxane (PDMS) in plane A1 to obtain the results shown in Figures 5. In particular, to generate optical aberrations with the second SLM, we program a random phase mask on it. This phase mask ($\theta^{aber}$) is generated by randomly combining Zernike polynomials i.e. $\theta^{aber} = \sum_{n,m} \alpha_{n,m}^{aber} Z_n^m$. Bias coefficients $\alpha_{n,m}^{aber}$ are randomly
generated using MATLAB’s `rand()` function for modes with $n \leq 5$ and $|m| \leq n$.

VI. STRUCTURAL SIMILARITY (SSIM) BETWEEN IMAGES AND SIGNAL-TO-NOISE RATIO (SNR)

The structural similarity (SSIM) quantifies the similarity between two images, taking into account properties of the human visual system [7]. The SSIM between images $A$ and $B$ is calculated using the formula [7]

$$SSIM = \frac{(2\mu_A\mu_B + C_1)(2\sigma_{AB} + C_2)}{(\mu_A^2 + \mu_B^2 + C_1)(\sigma_A^2 + \sigma_B^2 + C_2)}, \quad (B24)$$

where $\mu_A, \mu_B$ are the means of images $A$ and $B$, $\sigma_A, \sigma_B$ are the standard deviations of $A$ and $B$, and $\sigma_{AB}$ is the cross-covariance between $A$ and $B$. $C_1 = 0.01L$ and $C_2 = 0.03L$ are regularisation constants where $L$ is the dynamic range of image $A$. SSIM values can be between 0 and 1, where 1 means $A$ and $B$ are identical, and 0 means that $A$ has no similarity to $B$.

The signal-to-noise ratio (SNR) is obtained by calculating an average value in a transparent region of the retrieved image, and then dividing it by the standard deviation of the noise in the same region. To have a common reference, SNR values are calculated using the same areas in all anti-correlation images shown in Figure 5. For a fixed exposure time and pump power, the SNR varies as $\sqrt{M}$, where $M$ is the total number of frames acquired to reconstruct $G^{(2)}$ [8].

VII. CLASSICAL IMAGE METRICS

Power in Bucket (PIB): This metric is simply the sum of each pixel within a circular region of the image. The diameter of this region was chosen to be 100 pixels, i.e. the full width of the image.

$$M_{PIB} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} I_{ij} \tau_{ij} \quad (B25)$$

where

$$\tau_{ij} = \begin{cases} 
1, & \text{if } (i - N_x/2)^2 + (j - N_y/2)^2 \leq r^2 \\
0, & \text{otherwise} 
\end{cases} \quad (B26)$$

with $r = 50$.

Contrast: This metric is the difference between the maximum and minimum values of the image:

$$M_{\text{contrast}} = \max(I) - \min(I) \quad (B27)$$
Since the illumination does not cover the full sensor region of interest, only the values within the same circular region \( \tau \) as defined for the PIB are considered.

**Low Frequencies:** This metric is introduced in [9], where they show that optimising the low spatial frequency content of an image will optimise the image sharpness. The Fourier transform of the image is computed, and the sum of the low-frequency values are taken as the metric:

\[
M_{LF} = \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} a_{ij} F[I]_{ij},
\]

where \( F[I] \) is the 2-dimensional fast Fourier transform of the image \( I \), and \( a \) is an annulus where

\[
a_{ij} = \begin{cases} 
1, & \text{if } r_0^2 \leq (i - N_x/2)^2 + (j - N_y/2)^2 \leq r_1^2 \\
0, & \text{otherwise},
\end{cases}
\]

with \( r_0 = 4 \) and \( r_1 = 10 \).

**VIII. RESULTS OF QAO WITH VARIOUS TYPES OF SAMPLES**

As mentioned in the manuscript, one advantage of QAO is that it does not depend on the sample structure. In the results shown in the manuscript, we used it for imaging biological samples (bee head and leg) and thin copper wires. Figure S10 shows other types of objects we have imaged in the presence of aberrations using QAO.

**IX. ADDITIONAL DATA TO FIGURE 3 OF THE MANUSCRIPT**

Figures S11 show experimental results of QAO in a bright-field imaging configuration, similar to the one used to obtain the result in Figure 3, but with PDMS-induced aberrations.

**X. ADDITIONAL DATA TO FIGURE 5 OF THE MANUSCRIPT**

In this section, we provide additional data to complement the images shown in Figure 5 of the manuscript.

As explained in the manuscript, an anti-correlation image \( R \) can be extracted from \( G^{(2)} \) (in addition to \( C^+ \), which is used for QAO). Figures 5.e-g of the manuscript show anti-correlation images acquired in all cases i.e. without aberrations, with aberrations and after correction, respectively. Figure S12 shows data that complement the results shown in Figure 5 of the manuscript:

- **No aberrations:** Figures S12a,b and d are the same images as Figures 5.a,b and e of the manuscript, respectively, acquired without aberrations. In addition, Figure S12c shows a so-called conditional image, defined as \( G^{(2)}(r|\mathbf{A}) \), where \( \mathbf{A} = (-60, 73) \) is the position of a pixel in the reference side. This image shows the probability
FIG. S10. **Examples of AO for different samples.** a,b,c, Intensity images with aberrations induced with SLM at plane A2. d,e,f, Intensity images after correction. Central region of sum-coordinate projection images corresponding to each acquisition are inset. Samples are: 100 µm resolution grid target, including label text (a,d); 500 µm resolution grid target (b,e); slice of a honeybee mouthpart (c,d). Each acquisition is $10^5$ frames, taking $\sim$ 2 mins.

of detecting one photon from a pair conditioned on the detection of its twin at pixel A. It has a narrow peak centred around $-\text{A}$.  

- **With aberrations:** Figures S12e and f shows the intensity images (reference and sample) acquired in the presence of aberrations (PDMS). As expected, the intensity image of the sample is blurred. Figure S12g shows the conditional image, $G^{(2)}(r|\text{A})$, in which the anti-correlation peak has almost completely disappeared due to aberrations. Figures S12h shows the poor-quality anti-correlation image, the same as the one shown in Figure 5.f of the manuscript.

- **With aberrations and after correction:** Figures S12i and j shows the intensity images (reference and sample) acquired after QAO correction. The image of the object has become much sharper. Figure S12k shows the conditional image, $G^{(2)}(r|\text{A})$, in which the anti-correlation peak has reappeared. Figures S12l shows the anti-correlation image and is the same as Figure 5.g of the manuscript.
FIG. S11. Results of QAO in a classical bright-field imaging configuration with PDMS-induced aberrations. These results were obtained using the experimental setup in Figure S1 but in a classical imaging configuration. Aberrations were induced by inserting the PDMS in plane A1. a, b and c, Direct intensity images acquired without aberrations, with aberrations and after QAO correction, respectively. d, e and f, Sum-coordinate projections acquired without aberrations, with aberrations and after QAO correction, respectively. The white scale bar represents 400 µm.

XI. IMPACT OF LOSSES

Figure S13 shows the variation of the signal-to-noise ratio (SNR) in the sum-coordinate projection $C^+$ in function of the transmission losses. The SNR is defined as the ratio between the central peak values ($C_0^+$) and the standard deviation of the noise surrounding it. In this experiment, transmission losses were introduced in a controlled way by rotating a linear polarizer positioned on the photon pairs optical path (i.e. photon pairs produced by type-I SPDC are horizontally polarized). As predicted in Ref. [8], the SNR decreases linearly with the losses.

XII. SIMULATIONS AND EXPERIMENTAL RESULTS IN THE CASE OF MORE COMPLEX ABERRATIONS

The experimental results presented in the manuscript were limited to the case of weak optical aberrations. We define ‘weak aberrations’ as aberrations that (i) have a low frequency and (ii) preserve the shift invariance of the PSF. The experimental results obtained confirm that we are in this regime: the phase masks obtained are composed of Zernike polynomials with a radial parameter lower than 5, and the image correction is always valid across the entire
field of view. On the contrary, ‘more complex aberrations’ are defined as aberrations (i) of higher frequency and (ii) that break the shift invariance of the PSF. It is important to note that this case does not include the scattering regime, which is beyond the scope of our study.

The reason we do not explore the regime of more complex aberrations, and also the scattering regime, in our study is technical: in the presence of aberrations that are too complex, the PSF is strongly distorted and spread. As a result, EMCCD cameras are unable to reconstruct sum-coordinate projections with a high enough SNR to run the QAO algorithm for a reasonable length of time.

However, QAO could also in principle also be extended to the case of complex aberrations. In this section, we present simulation results and preliminary experimental results demonstrating this.

A. Simulations

Figure S14 shows results from simulations demonstrating that QAO could in principle function with more complex aberrations. To generate a PSF with spatially-variant and higher-frequency aberrations \( h(x, x') \neq h(x - x') \), we generate its associated matrix \( T \) using the same approach as in section III, except that this time the matrix \( D \) is no longer perfectly diagonal. As seen in Figures S14.a,b, the matrix \( D \) is thus a quasi-diagonal matrix with a certain thickness and comprises a random phase component that varies along both spatial variables. Figure S14.c shows the amplitude of the resulting PSF of the system i.e. the amplitude of \( T \). It is spatially variant.

In this simulation, the object is a simple amplitude grating (Fig. S14.d). In the presence of complex aberrations, it is no longer visible at the output (Fig. S14.e). Similarly, in Figure S14.h, it is evident that the sum-coordinate projection has degraded and spread out. After correction, Figure S14.i shows that it is possible to find a correction phase pattern that refocuses the correlations at the center of the sum-coordinate projection. In this case, the object becomes visible again (Fig. S14.f) but only in a reduced part of the field of view (FOV) known as the isoplanatic patch. This reduction effect in the FOV is well known in classical adaptive optics [10]. It is not linked to the technique used to find the correction pattern (e.g. QAO or classical AO) but rather to the design of the AO system, specifically the placement of the SLM. In our simulations, the SLM is placed in the Fourier plane of the object, this form of AO is known as pupil AO. However, there are other possible configurations such as conjugate AO [11] or multi-conjugate AO [12], which could be adapted with QAO to increase the corrected FOV.

B. Experimental results

We have also obtained experimental results that corroborate the simulations. To perform the experiment, we replace the EMCCD camera by a single-photon avalanche diode (SPAD) array, as shown in Figure S15.a. With this device, a sum-coordinate projections can be measured using the method described in Ref. [13]. Then, to produce higher-frequency and spatially-variant aberrations, we insert a thin scattering medium i.e. a layer of parafilm (Fig. S15.b) at a controlled distance \( d \) from a conjugate plane of the SLM. The greater the distance \( d \), the more the PSF loses its
shift invariance over a wide FOV i.e. \( h(\mathbf{r}, \mathbf{r}') \neq h(\mathbf{r} - \mathbf{r}') \) \[14\].

Using such a medium, however, the SNR of the sum-coordinate projection measured with the SPAD array is still not sufficient to perform the QAO. To overcome this problem, we use a hybrid digital-optical optimization approach. As described in Ref. \[15\], such an hybrid approach performs the optimization numerically after measuring the system’s transmission matrix with classical light.

Figures S15c.d.e. show three sum-coordinate projections \( C^+ \) measured in the presence of the scattering medium positioned at three different distances \( d = 0 \text{cm}, 1 \text{cm} \) and \( 2 \text{cm} \) from the conjugate SLM plane, respectively. These images clearly show that the PSF is very distorted and spread out. Using the hybrid approach, it is possible to find phase patterns that refocus correlations at the center for every distance. Figures S15f.g.h. shows the sum-coordinate projections \( C^+ \) measured by programming these phase patterns (insets). These results confirm that correlations can be refocused in the image plane in the presence of more complex aberrations. It therefore suggest that QAO can also operate in this regime, provided that sufficiently efficient cameras are used.

As observed in the simulations, given that the PSF is spatially variant, we anticipate that the correction will only occur over a reduced portion of the FOV.

In our experiment, the further the medium is from the Fourier plane (i.e. the more the PSF shift invariance is broken), the smaller the size of the isoplanatic patch. This is exactly what we observe in Figures S16a.b.c. These intensity images were obtained by scanning a (classical) point source in the object plane after correcting the aberrations with a medium placed at a distance of \( d = 2.5 \text{cm} \). It shows clearly that when the point source is at the center of the image (Fig. S16a), it appears as a diffraction-limited spot, whereas when it approaches the edge, i.e. when it moves out of the isoplanatic patch (Fig. S16b.c), it get distorted and becomes a speckle.

Moreover, using the method described in Ref. \[16\], it is possible to measure the width of the isoplanatic patches with entangled photon pairs by measuring the anti-diagonal of the second-order correlation function \( G^{(2)}(\mathbf{r}, -\mathbf{r}) \). Figures S16d.e.f show these anti-diagonals, i.e. the isoplanatic patches, for medium placed at different distances \( d \). The widths of the isoplanatic patches are depicted in Figure S16g as a function of the distance \( d \). We observe that the further the medium is from the conjugate SLM plane (i.e. the more the PSF shift invariance is broken), the smaller the size of the isoplanatic patch. As detailed above, this reduction effect in the FOV is well known in classical adaptive optics \[10\], and is not linked to the technique used to find the correction pattern (e.g. QAO or classical AO) but rather to the design of the AO system, specifically the placement of the SLM.

It is important to note that these simulation and experimental results are preliminary: they do not formally demonstrate that QAO functions in this regime of more complex aberrations, but they strongly suggest that it does. These results require a more in-depth study to be fully validated.

XIII. SIMULATION RESULTS OF QAO IN A REFLECTION IMAGING CONFIGURATION

In this section, we present simulation results showing that the use of QAO could potentially be extended to reflection imaging configurations. Figure S17a shows the experimental configuration that we consider here. Note that even
though the diagram is depicted in transmission, it also accounts for the case of imaging in reflection. Indeed, we can consider that the aberrations on the illumination path $h_{ill}$ correspond to those between the source and the object, and those on the imaging path $h_{im}$ correspond to those between the object and the camera, after reflection by the object.

In this simple case, aberrations in the illumination and the imaging path are generated in Fourier planes of the object, and the two SLMs are positioned in conjugate planes. The object is an amplitude grating (Fig. S17d). In addition, we note that this configuration is slightly different than the one use in our experiment. In this case, the crystal surface is imaged onto the object, which means that photon pairs are strongly correlated when they pass through the object (without aberrations), and not anti-correlated. This arrangement is theoretically detailed in section XV and can also be found in the experimental results shown in Figures S18 and S20. In practice, this means that the optimisation process must be performed using the minus-coordinate projection of $G^{(2)}$, instead of the sum-coordinate projection.

The black curve in Figure S17g shows such a minus-coordinate projection measured in the presence of aberrations, that is spread and distorted. As expected, the object is also not visible in the direct intensity image (Figure S17e). When performing the optimisation using both SLMs, the central correlation peak of the minus-coordinate projection increases, to finally reach the red curve shown in Figure S17g. Figures S17b,c show the two optimal SLM correction phase patterns. As shown in Figure S17f, the object reappears at the output after correction, showing that QAO works. These simulations were repeated several times with different aberrations and object to confirm their relevance.

This result is promising yet quite surprising. Indeed, it is clear that this problem has multiple solutions: there exist different phase masks that optimize the overall PSF of the system, without necessarily optimizing the PSF on the imaging arm (which is our goal). To explain why QAO converges to the optimal solution, we must consider another effect that also influences the correlation peak value in the minus-coordinate projection, that is the total number of pairs passing through the object. This number is maximum when the pairs are perfectly correlated. Indeed, the farther apart they are, the more likely it is that only one of the two photons will be transmitted, leading to a loss of their correlation. When optimizing the peak at the output by modulating the illumination path SLM, we are then also optimizing the total number of pairs that pass through the object. Consequently, this SLM converges toward the solution that maximizes the transmission of the pairs, thereby correcting the illumination PSF $h_{ill}$. As a result, the imaging path SLM corrects only the imaging PSF $h_{im}$, making the image visible.

It’s essential to note that these simulation results are preliminary and require a more in-depth study to be fully validated.

\textbf{XIV. COMPARISON BETWEEN ENTANGLEMENT AND CLASSICAL CORRELATIONS TO ACHIEVE QAO}

This section derives $C_0^+$ for classically correlated photons with no entanglement and compare the result to the case with entanglement.
Let’s go through the complete reasoning with classical correlations. We assume the existence of an object point in the sample plane emitting perfectly anti-correlated photons, modelled by the following joint probability density function:

\[ p_0(r_1, r_2) = \delta(r_1 + r_1) \] (B30)

When propagating photons through the imaging system, the blurring process caused by the PSF can be described by introducing a random variable \( N \) to their initial positions in the object plane:

\[ r_k \to r_k + N_k, \quad k \in 1, 2 \] (B31)

Since both photons are influenced by the PSF independently, it is necessary to introduce a distinct random spread term for each of them. Their probability density function is determined by the PSF of the imaging system:

\[ P_{N_k}(n) = |h(n)|^2 \]. Applying basis statistics, we obtain the probability density in the image plane:

\[ p_i(r_1', r_2') = \iint p_0(r_1, r_2) |t(r_1)t(r_2)h(r_1' - r_1)|^2 |h(r_2' - r_2)|^2 dr_2 dr_1, \] (B32)

where \( t \) is an object. Following the same mathematical reasoning as in Appendix III, we obtain the following result:

\[ p_i(r_1', r_2') = \int |t(r_1' + r_2' - r)t(-r_1' - r_2' + r)h(r)h(2r_1' - r)|^2 dr. \] (B33)

Finally, we define \( C^+_cl(\delta r^+) = \iint p_i(r_1, \delta r^+ - r_1) dr_1 = \int p_i(\delta r^+ + \frac{1}{2}, r_1) dr_1 \) and obtain:

\[ C^+_cl(\delta r^+) = K_{cl} \int |h(r)h(\delta r^+ - r)|^2 dr \]

\[ = K_{cl} |h|^2 |h|^2(\delta r^+) \] (B34)

where

\[ K_{cl} = \int |t(r) t(-r)|^2 dr. \] (B36)

As one can see, classically correlated photons would lead to a result that is formally different from that obtained with entanglement. However, the QAO algorithm could still work, but in a less efficient manner. Indeed, as shown in Figure S18, \( |h* h|^2 \) is more sensitive to the deformations of the PSF than \( |h|^2 |h|^2 \) is. As a results, the same holds for their central values i.e. \( C^+(0) \) and \( C^+_cl(0) \). We observe that the central value decreases more rapidly when increasing the aberration strength with entangled photons than with classical correlations. This increased sensitivity is a genuine
advantage provided by entanglement. In addition, the red curve shows the variation $|h|^2(0)$, corresponding to the case of classical AO, which is even less sensitive.

Qualitatively, this increased sensitivity is easy to understand when making an analogy with classical imaging: imaging with coherent light $|t \ast h|^2$ is known to produce lower-quality images (e.g., presence of speckle-artifacts) because it is more sensitive to system aberrations, than with incoherent imaging $|t|^2|h|^2$.

XV. QAO WITH PHOTONS CORRELATED IN POSITIONS

A. Theory with entangled photons

In this section, we derive the Equations to use QOA but in the case where the object is placed in a plane conjugated to the surface of the crystal. In this case, we can then assume that the photons are perfectly correlated in positions:

$$\phi(r_1, r_2) \approx \delta(r_1 - r_2). \quad (B37)$$

Starting from Equation B35 and performing the following variable change described in B39 we obtain:

$$\phi(r'_+, r'_-) = \int t(r_+)^2 h(r'_+ + r'_- - r_+) h(r'_+ - r'_- - r_+)^d r_+. \quad (B38)$$

From there, we use another change of variables: $r = r'_+ + r'_- - r_+$, to obtain:

$$\phi(r'_+, r'_-) = \int t(r'_+ + r'_- - r)^2 h(r) h(r - 2r'_+)^d r. \quad (B39)$$

Then, one can define the minus-coordinate projection of $G^{(2)}$ as: $C^-(\delta r^-) = \int G^{(2)}(r_1, -\delta r^- + r_1) d r_1$. Using again the change of variable i.e. $(r_1, r_2) \rightarrow (r_+, r_-)$, one can express $C^-$ as:

$$C^-(\delta r^-) = \int \left| \phi \left( r_+, \frac{\delta r^-}{2} \right) \right|^2 d r_+. \quad (B40)$$

Then, we can expand and simplify the previous Equation as follow:

$$C^-(\delta r^-) = \iiint K_-(r_A, r_B) h(r) h(r_A - \delta r^-) h^*(r_B) h^*(r_B - \delta r^-) d r_A d r_B. \quad (B41)$$

where:

$$K_-(r_A, r_B) = \int t(r)^2 t^*(r + (r_A - r_B))^2 d r. \quad (B42)$$
Equation [B41] is the most general form of $C^-$. In addition, assuming that the optical aberrations present in the system are sufficiently weak, Equation [B41] can be further simplified. Indeed, if the aberrations are weak, the PSF $h$ quite narrow. This implies that the term $h(r)h(r_A - \delta r^+)h^*(r_B)h^*(r_B - \delta r^+)$ is non-negligible only for small values of $|r_A|$ and $|r_B|$ i.e. close to 0. In this case, the values $|r_A - r_B|$ are also small, and Equation [B42] simplifies into:

$$K_-(r_A, r_B) \approx \int |t(r)|^4 dr = K_. \tag{B43}$$

Under this assumption, $K$ is now a constant that can be pull out of the integral in Equation [B41]. Finally, the minus-coordinate projection can be expressed as:

$$C^-(\delta r^-) \approx K_-|h \star h(\delta r^-)|^2, \tag{B44}$$

where $\star$ is the correlation product.

**B. Experimental results with entangled photons**

Figures [S19] and [S20] show experimental results of the effect of aberrations when the sample is illuminated by correlated photons i.e. the crystal surface is imaged onto the sample:

- Figure [S19] shows the presence of PDMS-induced aberrations in a bright-field imaging configuration involving two high numerical aperture (NA) microscope objectives i.e. 20X 0.5NA.

- Figure [S20] shows the presence of PDMS-induced aberrations in a quantum-enhanced phase imaging configuration, similar to the one used in Refs. [17, 18].

**C. Theory with classically correlated photons**

We assume perfectly classically correlated photons modelled by the following joint probability density function:

$$p_0(r_1, r_2) = \delta(r_1 - r_2). \tag{B45}$$

Then, following the same reasoning than for classically anti-correlated photons (section X), we find:

$$C^- (\delta r^-) = K_-[|h|^2 \star |h|^2(\delta r^-)], \tag{B46}$$
where:

\[ K_- = \int |t(r)|^4 \, dr. \]  

(B47)

FIG. S12. Additional images to Figure 5 of the manuscript. (a,b) Intensity images, (c) conditional image $C^{(2)}(r|A)$ (where $A = (-60, 73)$) and (d) anti-correlation image $R$ without aberrations. (e,f) Intensity images, (g) conditional image $C^{(2)}(r|A)$ and (h) anti-correlation image $R$ in the presence of aberrations. (i,j) Intensity images, (k) conditional image $C^{(2)}(r|A)$ and (l) anti-correlation image $R$ in the presence of aberrations after correction.
FIG. S14. Simulations of QAO in the case of more complex aberrations. a and b, Amplitude and phase of the matrix $D$ used to generate the higher frequency and spatially-variant PSF, respectively. c, PSF amplitude of an imaging system with complex aberrations. d - e, Intensity image of the object (d) without aberrations, (e) with with complex aberrations and no correction, (c) with correction. g - i, Sum-coordinate projection (g) without aberrations, (h) with complex aberrations and no correction, (i) with correction.
FIG. S15. **Refocusing of photon pairs correlations through more complex aberrations.** (a) Experimental setup. Spatially-entangled photon pairs are produced by type-I SPDC in a BBO crystal using a blue pump laser. The surface of the crystal is imaged on the SLM. The object plane is conjugated with the Fourier plane of the crystal and the camera plane. The camera is a single-photon avalanche diode (SPAD) array. The scattering medium (SM) is positioned at a distance $d$ from the conjugate plane of the SLM. Low pass (BPF) and band pass filters (BPF) block the pump photons. All lenses are positioned in an afocal configuration. $f_1-f_2$ is represented by two lenses for clarity but is in reality composed of a series of four lenses with focal lengths 50 mm, 150 mm, 100 mm, and 200 mm, respectively. $f_3-f_4$ is composed of a series of four consecutive lenses with focal lengths 200 mm, 100 mm, 75 mm, and 50 mm, respectively. $f_5 = 150$ mm. (c) Sum-coordinate projection measured without correction and $d = 0$ cm. (d) Sum-coordinate projection measured without correction and $d = 1$ cm. (e) Sum-coordinate projection measured without correction and $d = 2$ cm. (f) Sum-coordinate projection measured after correction and $d = 0$ cm. (g) Sum-coordinate projection measured after correction and $d = 1$ cm. (h) Sum-coordinate projection measured after correction and $d = 2$ cm. The corresponding phase pattern are in insets.
**FIG. S16. Isoplanatic patches.** (a) Classical intensity image of a point source positioned at the center of the object plane, after correction and with a medium positioned at \( d = 2.5 \text{cm} \). (b) Classical intensity image of a point source slightly off-center. (c) Classical intensity image of a point source positioned on the edge of the object plane. These three intensity images were acquired by replacing the SPAD camera by a conventional scientific CCD camera to have a better pixel resolution. The green part appearing on the right side of image (g) is just some camera noise. As detailed in Ref. [16], anti-diagonal component of \( G^{(2)}(r, -r) \) can be displayed to reveal the isoplanatic patch of the imaging system. (d) Isoplanatic patch i.e. \( G^{(2)}(r, -r) \) after correction at \( d = 0 \text{cm} \). (e) Isoplanatic patch i.e. \( G^{(2)}(r, -r) \) after correction at \( d = 1 \text{cm} \). (f) Isoplanatic patch i.e. \( G^{(2)}(r, -r) \) after correction at \( d = 2 \text{cm} \). (g) Isoplanatic patch width are estimated using a Gaussian fit and represented in function of the distance \( d \).
FIG. S17. Simulations of QAO in a reflection imaging configuration. a, Diagram of the experimental setup considered in this simulation. Aberrations are generated in Fourier planes in both the illumination and imaging paths. The SLMs are positioned in conjugate planes of the aberrations planes. $h_{\text{ill}}$ and $h_{\text{im}}$ are the illumination and imaging PSFs, respectively. The scheme is depicted in transmission for clarity, but in reality represents a reflection configuration (the illumination path aberrations $h_{\text{ill}}$ corresponds to light passing from the source to the object, and those in the imaging path $h_{\text{im}}$ refer to the return after object reflection). b and c, Phase profiles on the illumination and imaging SLM obtained after QAO optimisation.

d, Intensity profile of the object measured without aberrations. e, Intensity profile of the object measured in the presence of aberrations.

f, Intensity profile of the object measured in the presence of aberrations after correction. g, Minus-coordinate projection profiles in the presence of aberrations without correction (black curve) and after correction (red curve).
FIG. S18. Comparison of the sensitivity of QAO of entangled photons and classical correlated photons. Variation of the central values (blue) $C^+(0)$ and (black) $C_{cl}(0)^+$ in function of the aberrations strength. $C^+(0) \propto |h^+ h(0)|^2$ in the case of entangled photons, and $C_{cl}^+(0) \propto |h|^2|h|^2(0)$ in the case of classically correlated photons. As a comparison, the red curve shows also the variations of $|h|^2(0)$, which corresponds to the case of classical AO.
FIG. S19. Aberrations in a bright-field imaging configuration composed on high-NA microscope objectives. 

**a**, Experimental setup. Laser: diode at 405nm; BBO: Beta Baryum Borate; LPF: low pass filter; Microscope objectives: Nikon Plan Fluo 20X 0.5; BPF: Band-pass filter; All lenses are positioned in an afocal configuration. $f_1-f_2$ is represented by two lenses for clarity but is in reality composed of a series of four lenses with focal lengths 40 mm, 100 mm, 150 mm, and 100 mm, respectively. $f_3$ is represented by one lens, but in reality it is composed of a series of three consecutive lenses with focal lengths 50 mm, 75 mm, and 100 mm, respectively. $f_4 = 100$ mm. 

**b**, Photon of the microscope objectives. 

**c**, Photon of the sample (Fiber wool). 

**d and e**, Intensity and minus-coordinate projection without aberrations, respectively. **d and e**, Intensity and minus-coordinate projection in the presence of aberrations, respectively. The peak in the minus-coordinate gets distorted and spread in the presence of aberrations. The white scale bar represents 50µm.
Aberrations in a quantum-enhanced phase imaging configuration. a, Spatially and polarization entangled photon pairs are produced by SPDC in a pair of BBO crystals (0.5 mm thickness each). The polarisation of the pump is 45 degrees. The crystal output surface is imaged onto an SLM. In this demonstration, the SLM is used both to generate the birefringent phase object (a smiley face) and to perform phase-shifting holography. Aberrations are created by inserting a PDMS layer in the imaging path. A polariser at 45 degrees is positioned in front of the EMCCD camera, used to detect photon correlations. All lenses are positioned in an afocal configuration. \( f_1-f_2 \) is represented by two lenses for clarity but is in reality composed of a series of four lenses with focal lengths 35 mm, 60 mm, 50 mm, and 125 mm, respectively. \( f_3 = 200 \text{ mm}, f_4 = 75 \text{ mm}, f_5 = 125 \text{ mm} \) and \( f_6 = 150 \text{ mm} \). Another lens of 25 mm is also placed between the lens \( f_6 \) and the camera in a single-lens imaging configuration with magnification of 1 (not shown for clarity). Laser: diode at 405 nm; BBO: Beta Baryum Borate; BPF: Band-pass filter.

c, Phase image reconstructed via correlation measurements and phase shifting-holography, without aberrations. d, Minus-coordinate projection measured without aberrations. e and f, Phase image reconstructed and minus-coordinate projection measured in the presence of aberrations in the imaging path, respectively. The quantum advantage of this type of setup is linked to the fact that the measured phase (c) is doubled when using photon pairs compared to the one that would be measured classically (b), providing better sensitivity. In the presence of aberrations, this advantage is lost. More details about this experiment can be found in Refs. [17] and [18]. White scale bar is 1 mm.