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Flat coordinates of algebraic Frobenius manifolds in small dimensions

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ABSTRACT

Orbit spaces of the reflection representation of finite irreducible Coxeter groups provide polynomial Frobenius manifolds. Flat coordinates of the Frobenius manifold metric η are Saito polynomials which are distinguished basic invariants of the Coxeter group.

Algebraic Frobenius manifolds are typically related to quasi-Coxeter conjugacy classes in finite Coxeter groups. We find explicit relations between flat coordinates of the metric η and flat coordinates of the intersection form g for most known examples of algebraic Frobenius manifolds up to dimension 4. In all the cases, flat coordinates of the metric η appear to be algebraic functions on the orbit space of the Coxeter group.

1. Introduction

A Frobenius manifold with a polynomial prepotential is known as a polynomial Frobenius manifold. Coxeter orbit spaces, constructed from finite irreducible Coxeter groups, can be given the structure of a semisimple polynomial Frobenius manifold [10]. Dubrovin conjectured that these classify all irreducible semisimple polynomial Frobenius manifolds, which Hertling proved with the added assumption that the Euler vector field has positive degrees [16]. A Frobenius manifold with an algebraic prepotential is known as an algebraic Frobenius manifold. This is a natural case to consider after classifying the polynomial Frobenius manifolds.

The first non-rational algebraic Frobenius manifolds were found by Dubrovin and Mazzocco in 2000, which they derived from the Coxeter group H_3 in relation to Painlevé VI equation [14]. Explicit prepotentials of these Frobenius manifolds were given more recently by Kato, Mano and Sekiguchi [19] (see also Remark 6.1 in that paper).

The local monodromy group of a semisimple Frobenius manifold is generated by finitely many reflections [12]. It comes together with a particular set of generating reflections $R_1, \ldots, R_n$, where n is the dimension of the Frobenius manifold. In the case of an algebraic Frobenius manifold there is a finite orbit of the braid group $\mathfrak{B}_n$ acting on n-tuples of reflections in the monodromy group, this action is known as the Hurwitz action [14]. The local monodromy group is then necessarily a finite group [20], and the product of reflections R_i gives a quasi-Coxeter element w in this group. An equivalent property of w is that it does not belong to any proper reflection subgroup of the Coxeter group (see [9]).

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It is expected that irreducible semisimple algebraic Frobenius manifolds are closely related to the quasi-Coxeter conjugacy classes of finite irreducible Coxeter groups, where polynomial Frobenius manifolds correspond to the conjugacy class of a Coxeter element [4]. We recall some findings of algebraic Frobenius manifolds below together with their links to quasi-Coxeter elements. It seems not clear though whether these constructions give the same quasi-Coxeter conjugacy class as described above following [9].

Pavlyk constructed bi-Hamiltonian structures of hydrodynamic type by considering dispersionless limit of generalised Drinfeld–Sokolov hierarchies associated to a regular element of a Heisenberg subalgebra $\mathcal{H}_w$ of an affine Lie algebra $\widehat{\mathfrak{g}}$ [22]. To make dispersionless limit finite one has to restrict analysis to a suitable submanifold of the phase space. In this construction the Heisenberg subalgebra $\mathcal{H}_w$ is associated with a regular quasi-Coxeter element w of the Weyl group of the finite-dimensional Lie algebra $\mathfrak{g}$ (in general, non-equivalent Heisenberg subalgebras are in one-to-one correspondence with conjugacy classes of the Weyl group [18]). Dubrovin had previously shown that bi-Hamiltonian structures of hydrodynamic type have a correspondence with Frobenius manifolds [11]. Pavlyk claimed that his construction produces algebraic Frobenius manifolds and he gave an explicit expression for the prepotential in the case of the conjugacy class $D_4(a_1)$ [22] (in the notation for conjugacy classes of Weyl groups from Carter [1]).

Dinar also gave a construction of algebraic Frobenius manifolds [7]. Starting with a regular quasi-Coxeter element w in a Weyl group there is a distinguished nilpotent element e in the associated simple Lie algebra $\mathfrak{g}$ [3], [25]. Dinar constructed a bi-Hamiltonian structure of hydrodynamic type on a subvariety of the Slodowy slice $\mathcal{S}_e \subseteq \mathfrak{g}^*$ using Dirac reduction and gave an explicit expression for the prepotential in the case of nilpotent orbit $F_4(a_2)$ [4] (in the notation for nilpotent orbits from [2]). He also derived prepotentials for $D_4(a_1)$ [6] and $E_8(a_1)$ [5], the latter of which was simplified in a joint work with Sekiguchi [8]. The eigenvalues of the quasi-Coxeter element w have the form $e^{\frac{2\pi i}{|w|}\eta_j}$, where $|w|$ denotes the order of w and $0 \le \eta_j \le |w|-1$. The degrees d_j of the corresponding Frobenius manifold are $d_j = \frac{\eta_j+1}{|w|}$.

Two algebraic prepotentials related to Weyl groups E_6 and E_7, and seven algebraic prepotentials related to the Coxeter group H_4 were found by Sekiguchi [24] who used degrees of the latter Frobenius manifolds conjectured by Douvropoulos (see further details in [9]). These prepotentials are denoted by $H_4(k)$, where $k = 1, 2, 3, 4, 6, 7, 9$.

The degrees of these Frobenius manifolds are determined as follows [9]. For a regular quasi-Coxeter element w with eigenvalues $e^{\frac{2\pi i}{|w|}\eta_j}$, there exists a regular element $w_0 \in W$ such that w is conjugate to w_0^l for some $l \in \mathbb{N}$, $|w| = |w_0|$ and the eigenvalues of w_0 have the form $e^{\frac{2\pi i}{|w|}(d_j^W-1)}$, where d_j^W are the fundamental degrees of W, assuming l is the smallest such positive integer. Then, the degrees d_j of the Frobenius manifolds $H_4(k)$ are

$$d_j = \frac{(\eta_j + l)\,(\mathrm{mod}\,|w|)}{|w|}, \tag{1.1}$$

where the remainder $(\eta_j + l)\,(\mathrm{mod}\,|w|)$ is between 1 and $|w|$. The algebraic degree of these Frobenius manifolds also have a combinatorial interpretation [9].

Any Frobenius manifold has two compatible flat metrics η and g, where metric g is usually referred to as the intersection form. It is a complicated problem in general to express one flat coordinate system in terms of the other. For polynomial Frobenius manifolds expressing flat coordinates of η via that of g gives a distinguished set of basic invariants of a Coxeter group, known as Saito polynomials [23]. These polynomials play an important role in the representation theory of Cherednik algebras [15].

Let us explain the relation between the two sets of flat coordinates for two-dimensional algebraic Frobenius manifolds. Prepotentials for two-dimensional (semisimple) algebraic Frobenius manifolds have the following form [10]:

$$F = \frac{1}{2}t_1^2 t_2 + \frac{k(2k)^k}{k^2-1}t_2^{k+1}, \tag{1.2}$$

where $k \in \mathbb{Q} \setminus \{-1, 0, 1\}$. The degrees of the Frobenius manifold are $d_1 = 1$ and $d_2 = \frac{2}{k}$, and the charge is $d = \frac{k-2}{k}$. Let x_1, x_2 be flat coordinates of the intersection form. Then we have the relations

$$t_1 = (x_1 + ix_2)^k + (x_1 - ix_2)^k, \qquad t_2 = \frac{x_1^2 + x_2^2}{2k}, \tag{1.3}$$

which can be checked similarly to the polynomial case $k \in \mathbb{N}$ considered in [10].

Now, let w be a quasi-Coxeter element in the dihedral group $I_2(m)$. It must be the product of two reflections that generate $I_2(m)$. Hence $w = c^l$, where c is a Coxeter element of $I_2(m)$ and $(m, l) = 1$. The eigenvalues of w are $e^{\pm\frac{2\pi i}{m}l}$. We can assume $l \le \frac{m}{2}$ as the corresponding elements $w = c^l$ give representatives for all the quasi-Coxeter conjugacy classes. Then the smallest positive integer r such that w is conjugate to c^r is $r = l$. Thus, the degrees of the Frobenius manifold, using prescription (1.1) and following [9], are

$$d_1 = 1, \qquad d_2 = \frac{2l}{m},$$

since $\eta_1 = m - l$, $\eta_2 = l$ and $|w| = m$. From the general form (1.2) of a prepotential of an algebraic two-dimensional Frobenius manifold, we see that $k = \frac{m}{l}$ and thus

$$F=\frac{1}{2}t_1^2t_2+\frac{ml}{m^2-l^2}\left(\frac{2m}{l}\right)^{\frac{m}{l}}t_2^{\frac{m}{l}+1},$$

and this has charge $d=\frac{m-2l}{m}$. Note that when $l=1$ we get the polynomial two-dimensional Frobenius manifolds.

The Coxeter group $I_2(m)$ has basic invariants

$$y_1=(x_1+ix_2)^m+(x_1-ix_2)^m, \qquad y_2=\frac{x_1^2+x_2^2}{2m}. \tag{1.4}$$

We can express these basic invariants in terms of the flat coordinates of the metric in the following form:

$$y_1=\left(\frac{t_1+\sqrt{t_1^2-4\left(\frac{2m}{l}\right)^{\frac{m}{l}}t_2^{\frac{m}{l}}}}{2}\right)^l+\left(\frac{t_1-\sqrt{t_1^2-4\left(\frac{2m}{l}\right)^{\frac{m}{l}}t_2^{\frac{m}{l}}}}{2}\right)^l, \qquad y_2=\frac{t_2}{l}. \tag{1.5}$$

Formulas (1.5) may be thought of as inverse relations to formulas (1.3), where we replace flat coordinates x_1, x_2 with basic invariants given by (1.4).

Note that in the above analysis we relate an algebraic Frobenius manifold (1.2) with a conjugacy class of a quasi-Coxeter element in a dihedral group provided that $k\geq 2$. Two-dimensional Frobenius manifolds with $0<k<2$ have positive degrees but a relation to quasi-Coxeter elements seems unclear in this range. Note that the charge $d<0$ in this case, whereas $d\geq 0$ when $k\geq 2$. The general conjecture on the relation of algebraic Frobenius manifolds with quasi-Coxeter elements in [4] assumes that the degrees are positive. A possible way to exclude the examples (1.2) with $0<k<2$ is to impose an additional assumption to the conjecture that the charge $d\geq 0$. For $k<0$ the Frobenius manifolds with prepotential (1.2) have $d_2<0$.

In this work we establish relations between the two sets of flat coordinates for all but one of the known non-rational algebraic Frobenius manifolds of dimensions 3 and 4. Thus, we deal with the two Dubrovin–Mazzocco examples $(H_3)'$ and $(H_3)''$, Pavlyk's example for $D_4(a_1)$ and Dinar's example for $F_4(a_2)$. We also cover most of the examples from [24] related to H_4. The only known algebraic Frobenius manifolds in dimensions 3 or 4 which we do not deal with are the examples $H_4(6)$ and $H_4(9)$ from [24]. In the latter case the prepotential is a rational function, rather than a polynomial function, of the flat coordinates and an additional variable Z, which is algebraic in the flat coordinates. In the case of $H_4(6)$ the method which we use in all other examples and explain below cannot be applied as prescribed (see also Remark 2.4).

In all the cases we consider the flat coordinates of the metric η appear to be functions on a finite cover of the orbit space of the corresponding Coxeter group, while coordinates on the orbit space are basic invariants in flat coordinates of the intersection form g. Formulas (1.4) and (1.5) demonstrate this in dimension 2.

We note that the inversion symmetry [10] of a polynomial Frobenius manifold gives a Frobenius manifold with a rational prepotential, and, more generally, the inversion of an algebraic Frobenius manifold gives a Frobenius manifold with an algebraic prepotential. For the polynomial cases and the algebraic cases listed above their inversions have a negative degree. Relations between the two sets of flat coordinates of the resulting Frobenius manifolds can be deduced directly from the relations between the original two sets of the flat coordinates by considering how the inversion changes the flat coordinates of the intersection form following [21] and how the inversion changes the flat coordinates of the metric following [10].

The structure of the paper is as follows. In Section 2 we present some preliminary results on Frobenius manifolds, Coxeter invariant polynomials, the Laplace operator and we explain the method we use in order to relate the two flat coordinate systems. We assume that flat coordinates of η are algebraic in the basic invariants of the corresponding Coxeter group which turns out to be the case as we manage to find flat coordinates in such a form. The key tool to use is the Laplace operator in the flat coordinates of the intersection form g. This operator can also be rewritten in the flat coordinates of the Frobenius manifold metric η, see Proposition 2.1. This allows us to investigate harmonic functions in the flat coordinates. On the other hand we find harmonic basic invariants (see Proposition 2.3) which we equate to harmonic functions of suitable degree in the flat coordinates. We still have some coefficients to be found in these relations. In order to find them we compute the intersection form g in two different ways. After the general method is explained in subsection 2.4 we deal with all the examples in the subsequent Sections.

In all the examples we express explicitly the basic invariants of the flat coordinates of the intersection form g in terms of the flat coordinates of the Frobenius manifold metric η, generalising formulas (1.4) and (1.5) from the two-dimensional case. The explicit formulas for the flat coordinates of the metric η in terms of the basic invariants of the flat coordinates of the intersection form g are given for the examples related to H_3, D_4, $H_4(1)$ and $H_4(2)$. These formulas are also obtained but are too long to include for the example related to F_4 and for $H_4(3)$. Calculations are done by Mathematica, see Data availability section below.

In the final Section 7, we present some observations on the dual prepotentials of algebraic Frobenius manifolds. In subsection 7.1 we find the dual prepotentials for Frobenius manifolds with prepotentials of the form (1.2) with $k=\frac{1}{l}$. For $l\geq 2$ we express the dual prepotential using a hypergeometric function and we apply the inversion symmetry to find the dual prepotentials for $k=-\frac{1}{l}$. In subsection 7.2, for $(H_3)''$ and $D_4(a_1)$ we analyse singularities of the third order derivatives of their dual prepotentials on the Coxeter mirrors. While determining dual prepotentials for algebraic Frobenius manifolds was a motivation for the present work results in Section 7 demonstrate that these prepotentials are considerably more involved comparing to the polynomial Frobenius manifolds.

2. General approach

2.1. Notations

For any two vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{C}^n$, we define

$$(a,\, b) = \sum_{\alpha=1}^{n} a_\alpha b_\alpha \in \mathbb{C}.$$

Let M be a smooth manifold with coordinate system $z_1, \ldots, z_n$. If $f \in C^\infty(M)$ is homogeneous in the z coordinates, and has degree k, then we may write

$$\deg f(z) = k.$$

For an (r,s)-tensor field T on M expressed in the z coordinates we denote

$$T^{a_1\ldots a_r}_{b_1\ldots b_s;i}(z) := \frac{\partial T^{a_1\ldots a_r}_{b_1\ldots b_s}(z)}{\partial z_i}.$$

For a vector field $X = X^\lambda(z)\partial_{z_\lambda}$ on M, the Lie derivative $\mathcal{L}_X T$ of an (r,s)-tensor field T along X is an (r,s)-tensor field which is defined in the z coordinates by the following formula:

$$(\mathcal{L}_X T)^{a_1\ldots a_r}_{b_1\ldots b_s}(z) = X^\lambda(z) T^{a_1\ldots a_r}_{b_1\ldots b_s;\lambda}(z) - \sum_{\alpha=1}^{r} X^{a_\alpha}_{;\lambda}(z) T^{a_1\ldots a_{\alpha-1}\lambda\, a_{\alpha+1}\ldots a_r}_{b_1\ldots b_s}(z) + \sum_{\beta=1}^{s} X^{\lambda}_{;b_\beta}(z) T^{a_1\ldots a_r}_{b_1\ldots b_{\beta-1}\lambda\, b_{\beta+1}\ldots b_s}(z). \tag{2.1}$$

Note that, throughout, we are assuming summation over repeated upper and lower indices.

2.2. Laplace operator on Frobenius manifolds

Let M be an n-dimensional Frobenius manifold M with prepotential $F(t_1, \ldots, t_n)$ [10]. The third order derivatives of the prepotential are used to define the symmetric $(0,3)$-tensor c as

$$c_{ijk}(t) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}. \tag{2.2}$$

We define the metric η, which is constant in the t coordinates, as

$$\eta_{ij}(t) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_1}.$$

Let us denote $\eta^{ij} = \left(\eta^{-1}\right)^{ij}$ to be the inverse of the metric. We assume that

$$\eta_{ij}(t) = \eta^{ij}(t) = \delta_{i+j,\,n+1}.$$

The prepotential F satisfies the WDVV equations

$$c_{ij\lambda}(t)\eta^{\lambda\mu}(t)c_{\mu kl}(t) = c_{kj\lambda}(t)\eta^{\lambda\mu}(t)c_{\mu il}(t).$$

We assume that F is quasihomogeneous:

$$\mathcal{L}_E F(t) = (3-d)F(t),$$

where the Euler vector field E has the form

$$E(t) = \sum_{\alpha=1}^{n} d_\alpha t_\alpha \partial_{t_\alpha}, \tag{2.3}$$

with $d_1 = 1$ and the charge $d \neq 1$. Let us use the shorthand notations

$$c^i_{jk} = \eta^{i\lambda} c_{\lambda jk}, \qquad c^{ij}_k = \eta^{j\lambda} c^i_{\lambda k}. \tag{2.4}$$

Then c^i_{jk} are structure constants of a commutative Frobenius algebra defined on the tangent space $T_t M$ and $e = \partial_{t_1}$ is its unity. It follows that

$$(\mathcal{L}_E c)^{ij}_k = (d-1)c^{ij}_k. \tag{2.5}$$

The intersection form g is defined in the t coordinates by the formula

$$g^{ij}(t) = E^\lambda(t) c^{ij}_\lambda(t), \tag{2.6}$$

which is known to be a flat metric on a dense open subset of M. Let $x_1, \dots, x_n$ be flat coordinates for g such that

$$g^{ij}(x) = \delta^{ij}. \tag{2.7}$$

In these coordinates,

$$E(x) = \frac{1-d}{2} \sum_{\alpha=1}^n x_\alpha \partial_{x_\alpha}.$$

Let us denote $g_{ij} = \left(g^{-1}\right)_{ij}$ to be inverse of the intersection form. We know that

$$g_{ij} = c_{ij\lambda}(E^{-1})^\lambda,$$

where E^{-1} is the multiplicative inverse of the Euler vector field E. We define the tensor field

$$\overset{*}{c}{}^i_{jk} := g_{j\lambda} c^{i\lambda}_k, \tag{2.8}$$

and we see that

$$\overset{*}{c}{}^i_{jk} = g_{j\lambda} c^{i\lambda}_k = c_{j\lambda\mu}(E^{-1})^\mu c^{i\lambda}_k = c_{k\lambda\mu}(E^{-1})^\mu c^{i\lambda}_j = g_{k\lambda} c^{i\lambda}_j = \overset{*}{c}{}^i_{kj}.$$

Thus

$$g^{i\lambda} g_{k\mu} c^{j\mu}_\lambda = g^{i\lambda} \overset{*}{c}{}^j_{k\lambda} = g^{i\lambda} \overset{*}{c}{}^j_{\lambda k} = g^{i\lambda} g_{\lambda\mu} c^{j\mu}_k = c^{ji}_k. \tag{2.9}$$

Define Δ to be the Laplace operator in the x coordinates and ∇ to be the gradient operator in the x coordinates, so for a function $f \in C^\infty(M)$ we have

$$\Delta(f) = \sum_{\alpha=1}^n \frac{\partial^2 f}{\partial x_\alpha^2}, \qquad \nabla(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}\right).$$

Let $z_1, \dots, z_n$ be any coordinate system on M. Then

$$g^{ij}(z) = \big(\nabla(z_i),\, \nabla(z_j)\big) = \sum_{\alpha=1}^n \frac{\partial z_i}{\partial x_\alpha}\frac{\partial z_j}{\partial x_\alpha}. \tag{2.10}$$

We also have

$$g^{i\lambda}(z) g_{\lambda j;k}(z) = (g^{i\lambda} g_{\lambda j})_{;k}(z) - g^{i\lambda}_{;k}(z) g_{\lambda j}(z) = \delta^i_{j;k} - g^{i\lambda}_{;k}(z) g_{\lambda j}(z) = -g^{i\lambda}_{;k}(z) g_{\lambda j}(z). \tag{2.11}$$

Let ${}^g\Gamma^i_{jk}(z)$ be the Christoffel symbols for the metric g in the z coordinates. Then, in the coordinate system $x_1, \dots, x_n$, the Christoffel symbols satisfy the following coordinate transformation law:

$${}^g\Gamma^\lambda_{\mu\nu}(z) \frac{\partial z_\mu}{\partial x_j}\frac{\partial z_\nu}{\partial x_k}\frac{\partial x_i}{\partial z_\lambda} + \frac{\partial^2 z_\lambda}{\partial x_j \partial x_k}\frac{\partial x_i}{\partial z_\lambda} = {}^g\Gamma^i_{jk}(x) = 0. \tag{2.12}$$

The following proposition can be extracted from [10] (see formula (G.6) and Lemma 3.4). We include a complete proof below.

Proposition 2.1. *For a function $f \in C^\infty(M)$, we have*

$$\Delta(f) = g^{\nu\mu}(t) \frac{\partial^2 f}{\partial t_\nu \partial t_\mu} + \Delta(t_\nu)\frac{\partial f}{\partial t_\nu}. \tag{2.13}$$

Furthermore,

$$\Delta(t_i) = \left(\frac{d-1}{2} + d_i\right) c^{i\lambda}_\lambda(t),$$

where we sum over the index λ and $i = 1, \dots, n$ is fixed.

Proof. We have

$$\Delta(f)=\sum_{\alpha=1}^{n}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial f}{\partial t_\nu}\frac{\partial t_\nu}{\partial x_\alpha}\right)=\sum_{\alpha=1}^{n}\left(\frac{\partial^2 f}{\partial t_\nu\partial t_\mu}\frac{\partial t_\nu}{\partial x_\alpha}\frac{\partial t_\mu}{\partial x_\alpha}+\frac{\partial f}{\partial t_\nu}\frac{\partial^2 t_\nu}{\partial x_\alpha^2}\right),$$

which gives the equality (2.13) by formula (2.10). Now we find $\Delta(t_j)$. By relation (2.12) we have that

$$\Delta(t_i)=\sum_{\alpha=1}^{n}\frac{\partial^2 t_\lambda}{\partial x_\alpha^2}\delta^i_\lambda=\sum_{\alpha=1}^{n}\frac{\partial^2 t_\lambda}{\partial x_\alpha^2}\frac{\partial x_\mu}{\partial t_\lambda}\frac{\partial t_i}{\partial x_\mu}=\sum_{\alpha=1}^{n}-{}^g\Gamma^\nu_{\sigma\omega}(t)\frac{\partial t_\sigma}{\partial x_\alpha}\frac{\partial t_\omega}{\partial x_\alpha}\frac{\partial x_\mu}{\partial t_\nu}\frac{\partial t_i}{\partial x_\mu}=-{}^g\Gamma^i_{\sigma\omega}(t)\sum_{\alpha=1}^{n}\frac{\partial t_\sigma}{\partial x_\alpha}\frac{\partial t_\omega}{\partial x_\alpha}.$$

Using equations (2.10) and (2.11) we get that

$$\begin{aligned}\Delta(t_i)&=-g^{\sigma\omega}(t){}^g\Gamma^i_{\sigma\omega}(t)=-\frac{1}{2}g^{\sigma\omega}(t)g^{i\lambda}(t)\left(g_{\lambda\omega;\sigma}(t)+g_{\sigma\lambda;\omega}(t)-g_{\sigma\omega;\lambda}(t)\right)\\&=\frac{1}{2}\left(g^{\sigma\omega}(t)g^{i\lambda}_{;\sigma}(t)g_{\lambda\omega}(t)+g^{\sigma\omega}(t)g^{i\lambda}_{;\omega}(t)g_{\sigma\lambda}(t)-g^{\sigma\omega}_{;\lambda}(t)g^{i\lambda}(t)g_{\sigma\omega}(t)\right)\\&=g^{i\lambda}_{;\lambda}(t)-\frac{1}{2}g^{\sigma\omega}_{;\lambda}(t)g^{i\lambda}(t)g_{\sigma\omega}(t).\end{aligned}\tag{2.14}$$

By relation (2.6) we can rearrange equation (2.14) as

$$\begin{aligned}\Delta(t_i)&=(E^\mu(t)c^{i\lambda}_\mu(t))_{;\lambda}-\frac{1}{2}(E^\mu(t)c^{\sigma\omega}_\mu(t))_{;\lambda}g^{i\lambda}(t)g_{\sigma\omega}(t)\\&=E^\mu_{;\lambda}(t)c^{i\lambda}_\mu(t)+E^\mu(t)c^{i\lambda}_{\mu;\lambda}(t)-\frac{1}{2}\left(E^\mu_{;\lambda}(t)c^{\sigma\omega}_\mu(t)+E^\mu(t)c^{\sigma\omega}_{\mu;\lambda}(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t).\end{aligned}$$

From relations (2.4) we see that $c^{ij}_{k;l}(t)=c^{ij}_{l;k}(t)$, since η is constant in the t coordinates, hence

$$\Delta(t_i)=E^\mu_{;\lambda}(t)c^{i\lambda}_\mu(t)+E^\mu(t)c^{i\lambda}_{\lambda;\mu}(t)-\frac{1}{2}\left(E^\mu_{;\lambda}(t)c^{\sigma\omega}_\mu(t)+E^\mu(t)c^{\sigma\omega}_{\lambda;\mu}(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t).$$

By relation (2.1) the Lie derivative of the tensor field c^{ij}_k has the form

$$(\mathcal{L}_E c)^{ij}_k(t)=E^\lambda(t)c^{ij}_{k;\lambda}(t)-E^i_{;\lambda}(t)c^{\lambda j}_k(t)-E^j_{;\lambda}(t)c^{i\lambda}_k(t)+E^\lambda_{;k}(t)c^{ij}_\lambda(t).$$

Therefore

$$\begin{aligned}\Delta(t_i)=&(\mathcal{L}_E c)^{i\lambda}_\lambda(t)+E^i_{;\mu}(t)c^{\mu\lambda}_\lambda(t)+E^\lambda_{;\mu}(t)c^{i\mu}_\lambda(t)\\&-\frac{1}{2}\left((\mathcal{L}_E c)^{\sigma\omega}_\lambda(t)+E^\sigma_{;\mu}(t)c^{\mu\omega}_\lambda(t)+E^\omega_{;\mu}(t)c^{\sigma\mu}_\lambda(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t).\end{aligned}$$

By relations (2.5) and (2.9) we have that

$$\begin{aligned}\Delta(t_i)=&(d-1)c^{i\lambda}_\lambda(t)+E^i_{;\mu}(t)c^{\mu\lambda}_\lambda(t)+E^\lambda_{;\mu}(t)c^{i\mu}_\lambda(t)\\&-\frac{1}{2}\left((d-1)c^{\sigma\omega}_\lambda(t)+E^\sigma_{;\mu}(t)c^{\mu\omega}_\lambda(t)+E^\omega_{;\mu}(t)c^{\sigma\mu}_\lambda(t)\right)g^{i\lambda}(t)g_{\sigma\omega}(t)\\=&(d-1)c^{i\lambda}_\lambda(t)+E^i_{;\mu}(t)c^{\mu\lambda}_\lambda(t)+E^\lambda_{;\mu}(t)c^{i\mu}_\lambda(t)\\&-\frac{d-1}{2}c^{i\lambda}_\lambda(t)-\frac{1}{2}E^\sigma_{;\mu}(t)c^{\mu i}_\sigma(t)-\frac{1}{2}E^\omega_{;\mu}(t)c^{i\mu}_\omega(t)\\=&\frac{d-1}{2}c^{i\lambda}_\lambda(t)+E^i_{;\mu}(t)c^{\mu\lambda}_\lambda(t).\end{aligned}$$

The statement follows by formula (2.3). □

2.3. Coxeter-invariant coordinates

Let W be a finite irreducible Coxeter group of rank n acting on its complexified reflection representation $V\cong\mathbb{C}^n$ by orthogonal transformations with respect to $(\cdot,\cdot)$. Consider an orthonormal basis $e_1,\dots,e_n$, and the coordinates $x_1,\dots,x_n$ defined as

$$x_i(v)=(v,e_i),$$

for all $v \in V$ and all $i = 1, \dots, n$. Let $y_1, \dots, y_n$ be a set of homogeneous generators of the algebra $\mathbb{C}[x_1, \dots, x_n]^W$. It is well-known that such a set always exists [17] and we have an algebra isomorphism

$$\mathbb{C}[y_1, \dots, y_n] \cong \mathbb{C}[x_1, \dots, x_n]^W.$$

These generators are called basic invariants. The degrees d_i^W of basic invariants y_i do not depend on the choice of basic invariants [17]. We assume that $d_1^W \geq d_2^W \geq \dots \geq d_{n-1}^W > d_n^W = 2$.

Lemma 2.2. *Let $p, q \in \mathbb{C}[x_1, \dots, x_n]^W$. Then $\Delta(p), \Delta(q) \in \mathbb{C}[x_1, \dots, x_n]^W$ and $(\nabla(p), \nabla(q)) \in \mathbb{C}[x_1, \dots, x_n]^W$.*

Proof. The first claim follows from the invariance of Δ under orthogonal transformations. We have

$$(\nabla(p), \nabla(q)) = \frac{1}{2}\Big(\Delta(pq) - \Delta(p)q - p\Delta(q)\Big),$$

which implies the second statement. $\square$

We will use the following statement.

Proposition 2.3. *There exists a set of basic invariants $Y_1, \dots, Y_n \in \mathbb{C}[x_1, \dots, x_n]^W$ such that*

$$\Delta(Y_n) = 1, \qquad \Delta(Y_j) = 0,$$

for $j = 1, \dots, n-1$.

Proof. Let $y_1, \dots, y_n \in \mathbb{C}[x_1, \dots, x_n]^W$ be a set of basic invariants. Define Y_n as

$$Y_n := \frac{1}{2n} \sum_{i=1}^{n} x_i^2,$$

so that $\Delta(Y_n) = 1$. Now, it is well-known that

$$\mathbb{C}[x_1, \dots, x_n] = Y_n \mathbb{C}[x_1, \dots, x_n] \oplus H, \tag{2.15}$$

where $H = \operatorname{Ker}(\Delta)$ is the vector space of harmonic polynomials. Consider the vector spaces V_k of homogeneous W-invariant polynomials of degree k and the linear maps

$$\Delta : \operatorname{Span}\{y_j, Y_n V_{\deg y_j(x)-2}\} \to V_{\deg y_j(x)-2},$$

for $j = 1, \dots, n-1$. Since the dimension of the domain is larger than the dimension of the range, there must be a nontrivial kernel that is not contained in $Y_n V_{\deg y_j(x)-2}$ by the direct sum decomposition (2.15). Let Y_j be a nonzero element of this kernel. The polynomials Y_j, $1 \leq j \leq n$, are homogeneous and each basic invariant y_i can be expressed as a polynomial in Y_j, thus Y_j generate $\mathbb{C}[x_1, \dots, x_n]^W$ and we have that $\Delta(Y_j) = 0$ for all $j \leq n-1$. $\square$

Suppose we can write the prepotential F of a Frobenius manifold M as a polynomial $F(t, Z)$ in the t coordinates and Z, where Z satisfies an equation of the form

$$P(t; Z) = \sum_{k=0}^{N} a_k(t) Z^k = 0, \tag{2.16}$$

where $a_k \in \mathbb{C}[t_2, \dots, t_n]$, such Frobenius manifolds are called *algebraic*. We say that an algebraic Frobenius manifold M is *associated to the Coxeter group W* if there exist basic invariants $y_1, \dots, y_n$ in the flat coordinates $x_1, \dots, x_n$ of the intersection form g which are simultaneously polynomial in $t_1, \dots, t_n$ and Z. All of the examples of Frobenius manifolds which we consider below are associated to Coxeter groups.

We will sometimes need to consider the t coordinates and Z as independent variables (see e.g. Proposition 2.5 below). In such cases, for a rational function f of $n+1$ variables, we will write $f^F(t, Z)$ instead of $f(t, Z)$.

2.4. Relating flat coordinates with basic invariants

In this subsection we will explain how to relate flat coordinates t_i with flat coordinates x_j of the intersection form g, or rather with basic invariants y_j of a Coxeter group. It is known [10] that for $d \neq 1$ we have

$$t_n = \frac{1-d}{4}\sum_{i=1}^{n} x_i^2.$$

Since E is diagonal, we have $\deg t_i(x) = \frac{2d_i}{d_n}$ for all i. The general method for finding basic invariants as polynomials $y_i(t, Z)$ below will go through the following steps:

1) Set $y_n = \sum_{i=1}^n x_i^2 = \frac{4}{1-d} t_n$. Choose $y_1, \dots, y_{n-1}$ so that $y_1, \dots, y_n$ form a set of basic invariants for a finite irreducible Coxeter group W.

2) Let $Y_1, \dots, Y_n$ be a set of basic invariants such that $\Delta(Y_n) = 1$ and $\Delta(Y_j) = 0$ for $j = 1, \dots, n-1$, which exist by Proposition 2.3. Each Y_i can be expressed as a polynomial in $y_1, \dots, y_n$. In particular, $Y_n = \frac{1}{2n} y_n$.

3) Let V_j be the vector space of polynomials in $t_1, \dots, t_n$ and Z which are homogeneous in the x coordinates of degree d_j^W. Find the harmonic elements of V_j using Proposition 2.1, for $j = 1, \dots, n-1$.

4) Equate $Y_j = Y_j(y)$ with a general harmonic element of V_j. Rearrange these equations to find each y_j as a polynomial in $t_1, \dots, t_n$ and Z up to some coefficients to be found. This can be done successively for $j = n-1, n-2, \dots, 1$.

5) Find the intersection form g^{ij} in the y coordinates by the formula $g^{ij}(y) = \big(\nabla(y_i), \nabla(y_j)\big)$, and express the entries as polynomials in the y coordinates, which can be done by Lemma 2.2. Substitute the expressions for $y_i(t, Z)$ into these entries, so we have $g^{ij}(y(t))$.

6) Calculate the components $g^{ij}(y(t))$ of the intersection form g in the y coordinates by performing a change of coordinates $y = y(t)$ on the intersection form $g^{\lambda\mu}(t)$ given by formula (2.6):

$$g^{ij}(y(t)) = g^{\lambda\mu}(t)\frac{\partial y_i}{\partial t_\lambda}\frac{\partial y_j}{\partial t_\mu}.$$

Here, the derivatives $\frac{\partial y_i}{\partial t_\lambda}$ are found via their expressions in the t coordinates and Z which still contain some coefficients to be found.

7) We equate the two expressions for $g^{ij}(y(t))$ from steps 5) and 6), and find the values for the remaining coefficients, which is possible in all the examples we consider. Thus we get basic invariants y_j expressed as polynomials in t_i and Z. Note that the polynomials we find may not be unique if the Coxeter graph of W has non-trivial symmetries.

One may alternatively try to skip steps 2) and 4), but this increases the difficulty of the calculations needed to equate the two expressions for $g^{ij}(y(t))$.

Remark 2.4. In the case of algebraic Frobenius manifold $H_4(6)$ there do not exist any non-zero harmonic polynomials in the space V_j at step 3) of degrees 20 and 30. This prevents us from equating basic invariants of H_4 with polynomials in t_i and Z, and thus we cannot apply the above method as prescribed.

Proposition 2.5. *Let $e = e^i(y)\partial_{y_i}$ be the unity vector field of an algebraic Frobenius manifold associated to W with prepotential $F(t, Z)$. Then $e^i(y) \in \mathbb{C}[t, Z]$ for each $i = 1, \dots, n$.*

Proof. We know that $e = \partial_{t_1}$. Hence

$$e^i(y) = e^\alpha(t)\frac{\partial y_i}{\partial t_\alpha} = \frac{\partial y_i}{\partial t_1} = \frac{\partial y_i^F}{\partial t_1} + \frac{\partial y_i^F}{\partial Z}\frac{\partial Z}{\partial t_1} \in \mathbb{C}[t, Z]$$

since $\frac{\partial Z}{\partial t_1} = 0$ by relation (2.16). $\square$

Proposition 2.6. *Let g be the intersection form of an algebraic Frobenius manifold associated to W with root system R_W. Then*

$$\det(g^{ij}(t)) = \frac{c \prod\limits_{\alpha \in R_W} (\alpha, x)}{(\det J)^2},$$

where $J = \left(\frac{\partial y_i}{\partial t_j}\right)_{i,j=1}^n$ is the Jacobi matrix and $c \in \mathbb{C}$.

Proof. It follows from [17] that

$$\det(g^{\lambda\mu}(y)) = c \prod_{\alpha\in R_W} (\alpha, x)$$

for some $c \in \mathbb{C}$. From relation (2.10), we see that

$$\det(g^{ij}(t)) = \det\left(g^{\lambda\mu}(y)\frac{\partial t_i}{\partial y_\lambda}\frac{\partial t_j}{\partial y_\mu}\right) = \det(g^{\lambda\mu}(y))\det\left(J^{-1}\right)^2 = \frac{c \prod_{\alpha\in R_W}(\alpha, x)}{(\det J)^2}. \quad \square$$

3. Algebraic Frobenius manifolds related to H_3

There are two non-polynomial algebraic Frobenius manifolds which we can be associated to H_3, both found by Dubrovin and Mazzocco [14]. Prepotentials of these three dimensional Frobenius manifolds were given explicitly by Kato, Mano and Sekiguchi [19] (see also Remark 6.1 in [19]). Let R_{H_3} be the following root system for H_3:

$$R_{H_3} = \{\pm e_i \mid 1 \le i \le 3\} \cup \left\{ \frac{1}{2}\left(\pm e_{\sigma(1)} \pm \varphi e_{\sigma(2)} \pm \overline{\varphi} e_{\sigma(3)}\right) \middle| \sigma \in \mathfrak{A}_3 \right\},$$

where

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\varphi} = \frac{1-\sqrt{5}}{2},$$

and $\mathfrak{A}_3$ is the alternating group on 3 elements. Let us introduce the following basic invariants for H_3 (cf. [23]):

$$y_1 = 95\epsilon_2\epsilon_3 - 32\epsilon_1^2\epsilon_3 - 5\epsilon_1\epsilon_2^2 + 2\epsilon_1^3\epsilon_2 + 3\sqrt{5}\delta\epsilon_2, \tag{3.1}$$
$$y_2 = \sqrt{5}\delta + \epsilon_1\epsilon_2 - 11\epsilon_3, \tag{3.2}$$
$$y_3 = \epsilon_1, \tag{3.3}$$

where

$$\epsilon_1 = x_1^2 + x_2^2 + x_3^2, \tag{3.4}$$
$$\epsilon_2 = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, \tag{3.5}$$
$$\epsilon_3 = x_1^2x_2^2x_3^2, \tag{3.6}$$
$$\delta = (x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2). \tag{3.7}$$

The basic invariants y_1, y_2, y_3 have degrees 10, 6, 2, respectively.

Lemma 3.1. *(cf. [23]) The intersection form $g^{ij}(y)$ takes the form*

$$g^{ij}(y) = \begin{pmatrix} 30y_2^3 + 36y_2^2y_3^3 + 8y_1y_3^4 & 28y_2^2y_3 + 8y_2y_3^4 & 20y_1 \\ 28y_2^2y_3 + 8y_2y_3^4 & 8y_1 + 8y_2y_3^2 & 12y_2 \\ 20y_1 & 12y_2 & 4y_3 \end{pmatrix}.$$

Consider another set of basic invariants for H_3 given by

$$Y_1 = y_1 - \frac{9}{17}y_2y_3^2 - \frac{10}{187}y_3^5, \tag{3.8}$$
$$Y_2 = y_2 - \frac{2}{21}y_3^3, \tag{3.9}$$
$$Y_3 = \frac{1}{6}y_3. \tag{3.10}$$

The following statement can be checked directly.

Lemma 3.2. *We have $\Delta(Y_3) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = 0$.*

3.1. $(H_3)'$ example

The prepotential for $(H_3)'$ is

$$F(t)=\frac{1}{2}\left(t_1t_2^2+t_1^2t_3\right)-\frac{1}{18}t_3^4Z-\frac{7}{72}t_3^3Z^4-\frac{17}{105}t_3^2Z^7-\frac{2}{9}t_3Z^{10}-\frac{64}{585}Z^{13},$$

where

$$P(t_2,t_3,Z):=Z^4+t_3Z+t_2=0. \tag{3.11}$$

The Euler vector field is

$$E(t)=t_1\partial_{t_1}+\frac{4}{5}t_2\partial_{t_2}+\frac{3}{5}t_3\partial_{t_3},$$

the unity vector field is $e(t)=\partial_{t_1}$, and the charge is $d=\frac{2}{5}$. The intersection form (2.6) is then given by

$$g^{ij}(t)=\begin{pmatrix}\frac{1}{60}(16t_2Z^3+19t_2t_3-9t_3^2Z) & \frac{1}{5}(2t_2Z^2+t_3Z^3+t_3^2) & t_1\\ \frac{1}{5}(2t_2Z^2+t_3^2+t_3Z^3) & t_1+\frac{Z}{10}(8t_2+3t_3Z) & \frac{4}{5}t_2\\ t_1 & \frac{4}{5}t_2 & \frac{3}{5}t_3\end{pmatrix}. \tag{3.12}$$

We have that $\deg t_1(x)=\frac{10}{3}$, $\deg t_2(x)=\frac{8}{3}$, $\deg t_3(x)=2$ and $\deg Z(x)=\frac{2}{3}$.

Proposition 3.3. *Let $V_1=\{p\in\mathbb{C}[t_1,t_2,t_3,Z]\mid \deg p(x)=10\}$ and let $V_2=\{p\in\mathbb{C}[t_1,t_2,t_3,Z]\mid \deg p(x)=6\}$. The harmonic elements of V_1 are proportional to*

$$\begin{aligned}&2244000t_1^3-628320t_1t_2^2Z^2-1168530t_1t_2t_3^2-583440t_1t_2t_3Z^3+151470t_1t_3^3Z\\&+768944t_2^3t_3+406912t_2^3Z^3-311872t_2^2t_3^2Z+43087t_2t_3^3Z^2+32000t_3^5+37103t_3^4Z^3,\end{aligned}$$

and the harmonic elements of V_2 are proportional to

$$1260t_1t_2-224t_2^2Z-154t_2t_3Z^2-80t_3^3-35t_3^2Z^3.$$

Proof. Using Proposition 2.1 we can directly calculate

$$\Delta(t_1)=\frac{7}{20}Z^2, \tag{3.13}$$

$$\Delta(t_2)=-\frac{1}{2}Z, \tag{3.14}$$

$$\Delta(t_3)=\frac{9}{10}. \tag{3.15}$$

A general element of V_1 is of the form

$$\begin{aligned}&a_1t_1^3+a_2t_1^2t_2Z+a_3t_1^2t_3Z^2+a_4t_1t_2^2Z^2+a_5t_1t_2t_3^2+a_6t_1t_2t_3Z^3+a_7t_1t_3^3Z\\&\quad+a_8t_2^3t_3+a_9t_2^3Z^3+a_{10}t_2^2t_3^2Z+a_{11}t_2t_3^3Z^2+a_{12}t_3^5+a_{13}t_3^4Z^3,\end{aligned} \tag{3.16}$$

where $a_i\in\mathbb{C}$. By calculating the Laplacian of this general element (3.16) using Proposition 2.1 and formulas (3.13)–(3.15) we find that the only harmonic elements of V_1 are as claimed. A general element of V_2 has the form

$$b_1t_1t_2+b_2t_1t_3Z+b_3t_2^2Z+b_4t_2t_3Z^2+b_5t_3^3+b_6t_3^2Z^3, \tag{3.17}$$

where $b_i\in\mathbb{C}$. By calculating the Laplacian of this general element (3.17) using Proposition 2.1 and formulas (3.13)–(3.15) we find that the only harmonic elements of V_2 are as claimed. □

Theorem 3.4. *We have the following relations*

$$\begin{aligned}y_1=&\frac{128000}{19683}\Big(12000t_1^3-3360t_1t_2^2Z^2-3390t_1t_2t_3^2-3120t_1t_2t_3Z^3+810t_1t_3^3Z\\&+4112t_2^3t_3+2176t_2^3Z^3-2176t_2^2t_3^2Z-119t_2t_3^3Z^2+200t_3^5+119t_3^4Z^3\Big),\end{aligned} \tag{3.18}$$

$$y_2=\frac{3200}{729}\left(180t_1t_2-32t_2^2Z-22t_2t_3Z^2-5t_3^3-5t_3^2Z^3\right), \tag{3.19}$$

$$y_3=\frac{20}{3}t_3. \tag{3.20}$$

Proof. Note that $Y_3 = \frac{1}{6}y_3 = \frac{10}{9}t_3$. We now equate Y_1 and Y_2 given by relations (3.8)–(3.10) with general harmonic elements of V_1 and V_2, respectively, given by Proposition 3.3. We then rearrange these equations to find y_i in terms of t_j and Z. We find

$$\begin{aligned} y_1 = {} & \frac{25600000}{18711}t_3^5 + \frac{a}{583440}(2244000t_1^3 - 628320t_1t_2^2Z^2 - 1168530t_1t_2t_3^2 \\ & - 583440t_1t_2t_3Z^3 + 151470t_1t_3^3Z + 768944t_2^3t_3 + 406912t_2^3Z^3 \\ & - 311872t_2^2t_3^2Z + 43087t_2t_3^3Z^2 + 32000t_3^5 + 37103t_3^4Z^3) \\ & - \frac{80b}{119}(1260t_1t_2 - 224t_2^2Z - 154t_2t_3Z^2 - 80t_3^3 - 35t_3^2Z^3), \end{aligned} \tag{3.21}$$

$$y_2 = \frac{16000}{567}t_3^3 - \frac{b}{35}(1260t_1t_2 - 224t_2^2Z - 154t_2t_3Z^2 - 80t_3^3 - 35t_3^2Z^3), \tag{3.22}$$

$$y_3 = \frac{20}{3}t_3, \tag{3.23}$$

where $a, b \in \mathbb{C}$. In order to find a and b we perform steps 5–7 from Section 2.4. That is, we transform the intersection form (3.12) into y coordinates by applying formulas (3.21)–(3.23) and compare it with the expression given by Lemma 3.1. We find that $a = \frac{133120000}{6561}$ and $b = -\frac{16000}{729}$, which implies the statement. □

Proposition 3.5. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, Z]$.

Proof. We have $P(t, Z) = 0$ by relation (3.11). Hence

$$0 = \frac{\partial P}{\partial t_j} = \frac{\partial P^F}{\partial t_j} + \frac{\partial P^F}{\partial Z}\frac{\partial Z}{\partial t_j}.$$

Therefore

$$\frac{\partial Z}{\partial t_j} = -\frac{\frac{\partial P^F}{\partial t_j}}{\frac{\partial P^F}{\partial Z}}.$$

We thus have that

$$\frac{\partial y_i}{\partial t_j} = \frac{\partial y_i^F}{\partial t_j} - \frac{\partial y_i^F}{\partial Z}\frac{\frac{\partial P^F}{\partial t_j}}{\frac{\partial P^F}{\partial Z}}. \tag{3.24}$$

The first term is polynomial in t_1, t_2, t_3 and Z. The polynomial P^F is irreducible over $\mathbb{C}[t_1, t_2, t_3]$ and thus $\frac{\partial P^F}{\partial Z}$ is invertible in the field $\mathbb{C}(t_1, t_2, t_3)[Z]/(P^F)$, where $\mathbb{C}(t_1, t_2, t_3)$ is the field of rational functions in t_1, t_2 and t_3. Hence the second term in equality (3.24) can be represented as an element of the ring $\mathbb{C}(t_1, t_2, t_3)[Z]$, when we reduce it modulo P^F as a polynomial in Z. It can be checked that it is a polynomial in t_1, t_2 and t_3. □

Proposition 3.6. *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod\limits_{\alpha \in R_{H_3}} (\alpha, x)}{Q(t, Z)^2},$$

where $c = -3^{26} \cdot 5$ *and*

$$\begin{aligned} Q(t, Z) = 2^3 \cdot 5^7 \Big(& 24000t_1^3 - 10208t_2^3t_3 + 540t_1t_2t_3^2 + (2484t_2^2t_3^2 + 540t_1t_3^3 - 9600t_1^2t_2)Z \\ & + (3360t_1t_2^2 - 3600t_1^2t_3 - 189t_2t_3^3)Z^2 + (720t_1t_2t_3 - 4544t_2^3 - 81t_3^4)Z^3 \Big). \end{aligned}$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 3.4, which leads to Proposition 3.6.

In the next statement we express flat coordinates t_i via basic invariants y_j and Z, which is an inversion of the formulas from Theorem 3.4.

Theorem 3.7. *We have the following relations:*

$$t_1 = -\frac{1}{28800Z(20Z^3+3y_3)}\left(102400Z^9 + 20160y_3Z^6 + 1080y_3^2Z^3 + 729y_2 + 54y_3^3\right), \tag{3.25}$$

$$t_2 = -\frac{Z}{20}\left(20Z^3 + 3y_3\right), \tag{3.26}$$

$$t_3 = \frac{3}{20}y_3, \tag{3.27}$$

where Z satisfies the equation

$$\begin{aligned} &2^{29}5^{11}Z^{27} + 2^{27}3^3 5^{10} y_3 Z^{24} + 2^{22}3^4 5^8 151\, y_3^2 Z^{21} + 2^{18}3^4 5^7\left(2^2 7^2 19\, y_3^3 - 3^3 y_2\right) Z^{18} \\ &+ 2^{13}3^6 5^5\left(60089\, y_3^4 - 2^2 3^3 11\, y_2 y_3\right) Z^{15} + 2^{10}3^7 5^3\left(5^2 2\cdot 11\cdot 19\cdot 41\, y_3^5 - 3^3 263\, y_2 y_3^2\right. \\ &\left.+\, 2^2 3^7 y_1\right) Z^{12} + 2^9 3^7 5^2\left(2^3 3^6 y_2^2 + 3^9 2\, y_1 y_3 + 3^3 19\cdot 41\, y_2 y_3^3 + 5^2 2\cdot 4987\, y_3^6\right) Z^9 \\ &+ 2^6 3^9 5\left(3^6 7\, y_2^2 y_3 + 2^2 3^8 y_1 y_3^2 + 2^5 3^3 23\, y_2 y_3^4 + 2^2 5^3 53\, y_3^7\right) Z^6 \\ &+ 2^3 3^{10}\left(3^6 5\, y_2^2 y_3^2 + 2^3 3^7 y_1 y_3^3 + 2^2 3^3 131\, y_2 y_3^5 - 2^2 5^2 7\, y_3^8\right) Z^3 \\ &+ 3^9\left(3^9 y_2^3 + 3^7 2\, y_2^2 y_3^3 + 2^2 3^4 y_2 y_3^6 + 2^3 y_3^9\right) = 0, \end{aligned} \tag{3.28}$$

and y_i are given by relations (3.1)–(3.7).

Proof. Formula (3.27) follows immediately from Theorem 3.4, and formula (3.26) follows from relation (3.11). Substituting the relations (3.26) and (3.27) into formula (3.19) we get the expression (3.25). Finally, substituting relations (3.25)–(3.27) into formula (3.18) we get the formula (3.28). $\square$

Proposition 3.8. *The unity vector field $e = \partial_{t_1}$ in the y coordinates has the form*

$$e(y) = \frac{64000}{81}t_2\partial_{y_2} + \frac{1280000}{6561}\left(1200t_1^2 - 112t_2^2Z^2 - 113t_2t_3^2 - 104t_2t_3Z^3 + 27t_3^3\right)\partial_{y_1}.$$

Proof. We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 3.4. $\square$

3.2. $(H_3)''$ example

The prepotential for $(H_3)''$ is

$$F(t) = \frac{1}{2}\left(t_2^2 t_1 + t_3 t_1^2\right) + \frac{4063}{1701}t_3^7 + \frac{19}{135}t_3^5Z^2 - \frac{73}{27}t_3^3Z^4 + \frac{11}{9}t_3Z^6 - \frac{16}{35}Z^7,$$

where

$$P(t_2, t_3, Z) := Z^2 + t_2 - t_3^2 = 0. \tag{3.29}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{2}{3}t_2\partial_{t_2} + \frac{1}{3}t_3\partial_{t_3},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{2}{3}$. The intersection form (2.6) is then given by

$$g^{ij}(t) = \begin{pmatrix} \frac{4}{243}(585t_2^2t_3 + 3240t_2t_3^3 + 4456t_3^5 - 324Z(t_2^2 - 7t_2t_3^2 + 6t_3^4)) & -\frac{4}{27}(33t_2^2 + 4t_2t_3(18Z - 13t_3) - 72t_3^3(Z+t_3)) & t_1 \\ -\frac{4}{27}(33t_2^2 + 4t_2t_3(18Z - 13t_3) - 72t_3^3(Z+t_3) & t_1 - \frac{22}{3}t_2t_3 + \frac{52}{27}t_3^3 + 4Z(t_2 - t_3^2) & \frac{2}{3}t_2 \\ t_1 & \frac{2}{3}t_2 & \frac{1}{3}t_3 \end{pmatrix}. \tag{3.30}$$

We have that $\deg t_1(x) = 6$, $\deg t_2(x) = 4$, $\deg t_3(x) = 2$ and $\deg Z(x) = 2$.

Proposition 3.9. *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 10\}$ and let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$. The harmonic elements of V_1 are proportional to*

$$25245t_1t_2 + 22275t_1t_3^2 - 16830t_2^2t_3 - 20196t_2^2Z + 21890t_2t_3^3 + 40392t_2t_3^2Z - 104196t_3^5 - 20196t_3^4Z,$$

and the harmonic elements of V_2 are proportional to

$$189t_1 + 630t_2t_3 + 400t_3^3.$$

Proof. Using Proposition 2.1 we can directly calculate

$$\Delta(t_1) = -\frac{5}{27}(33t_2 - 26t_3^2 + 54t_3Z), \tag{3.31}$$

$$\Delta(t_2) = \frac{1}{3}(9Z - 11t_3), \tag{3.32}$$

$$\Delta(t_3) = \frac{1}{2}. \tag{3.33}$$

A general element of V_1 is of the form

$$a_1t_1t_2 + a_2t_1t_3^2 + a_3t_1t_3Z + a_4t_2^2t_3 + a_5t_2^2Z + a_6t_2t_3^3 + a_7t_2t_3^2Z + a_8t_3^5 + a_9t_3^4Z, \tag{3.34}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (3.34) using Proposition 2.1 and formulas (3.31)–(3.33) we find that the only harmonic elements of V_1 are as claimed. A general element of V_2 has the form

$$b_1t_1 + b_2t_2t_3 + b_3t_2Z + b_4t_3^3 + b_5t_2^2Z, \tag{3.35}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (3.35) using Proposition 2.1 and formulas (3.31)–(3.33) we find that the only harmonic elements of V_2 are as claimed. □

Theorem 3.10. *We have the following relations*

$$y_1 = \frac{288}{25}\Big(135t_1t_2 + 405t_1t_3^2 - 90t_2^2t_3 - 108t_2^2Z + 1070t_2t_3^3 + 216t_2t_3^2Z + 2292t_3^5 - 108t_3^4Z\Big) \tag{3.36}$$

$$y_2 = \frac{8}{5}\left(27t_1 + 90t_2t_3 + 160t_3^3\right), \tag{3.37}$$

$$y_3 = 12t_3. \tag{3.38}$$

Proof. Note that $Y_3 = \frac{1}{6}y_3 = 2t_3$. We now equate Y_1 and Y_2 given by relations (3.8)–(3.10) with general harmonic elements of V_1 and V_2, respectively, given by Proposition 3.9. We then rearrange these equations to find y_i in terms of t_j and Z. We find

$$y_1 = \frac{1990656}{77}t_3^5 + \frac{a}{40392}(25245t_1t_2 + 22275t_1t_3^2 - 16830t_2^2t_3 - 20196t_2^2Z + 21890t_2t_3^3 + 40392t_2t_3^2Z - 104196t_3^5 - 20196t_3^4Z) + \frac{81b}{425}t_3^2(189t_1 + 630t_2t_3 + 400t_3^3), \tag{3.39}$$

$$y_2 = \frac{1152}{7}t_3^3 + \frac{b}{400}(189t_1 + 630t_2t_3 + 400t_3^3), \tag{3.40}$$

$$y_3 = 12t_3, \tag{3.41}$$

where $a, b \in \mathbb{C}$. In order to find a and b we perform steps 5–7 from Section 2.4. That is, we transform the intersection form (3.30) into y coordinates by applying formulas (3.39)–(3.41) and compare it with the expression given by Lemma 3.1. We find that $a = \frac{62208}{25}$ and $b = \frac{640}{7}$, which implies the statement. □

Proposition 3.11. *The derivatives $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, Z]$.*

Proof is similar to the one for Proposition 3.5.

Proposition 3.12. *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha\in R_{H_3}} (\alpha, x)}{Q(t,Z)^2},$$

where $c = -2^{14}\cdot 5^5$ *and*

$$Q(t,Z) = 3^6\left(56t_3^3 + 126t_2t_3 - 27t_1 + 54(t_2 - t_3^2)Z\right).$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated Theorem 3.10, which leads to Proposition 3.12.

In the next statement we express flat coordinates t_i via basic invariants y_j and Z, which is an inversion of the formulas from Theorem 3.10.

Theorem 3.13. *We have the following relations:*

$$t_1 = \frac{5}{23328}\left(108y_2 - 25y_3^3 + 1296y_3Z^2\right), \tag{3.42}$$

$$t_2 = \frac{1}{144}y_3^2 - Z^2, \tag{3.43}$$

$$t_3 = \frac{1}{12}y_3, \tag{3.44}$$

where Z *satisfies the equation*

$$31104Z^5 + 12960Z^4y_3 + (900y_2 - 360y_3^3)Z^2 + (25y_1 - 25y_2y_3^2 + 2y_3^5) = 0, \tag{3.45}$$

and y_i *are given by relations (3.1)–(3.7).*

Proof. Formula (3.44) follows immediately from Theorem 3.10, and formula (3.43) follows from relation (3.29). Substituting the relations (3.43) and (3.44) into formula (3.37) we get the expression (3.42). Finally, substituting relations (3.42)–(3.44) into formula (3.36) we get the formula (3.45). □

Proposition 3.14. *The unity vector field* $e = \partial_{t_1}$ *in the* y *coordinates has the form*

$$e(y) = \frac{216}{5}\partial_{y_2} + \frac{7776}{5}\left(t_2 + 3t_3^2\right)\partial_{y_1}.$$

Proof. We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 3.10. □

4. Algebraic Frobenius manifold related to D_4

The $D_4(a_1)$ Frobenius manifold has been described by Pavlyk [22] and Dinar [4], with a prepotential given explicitly by Pavlyk. It is a four dimensional Frobenius manifold which can be associated to the Coxeter group D_4, it is denoted with the conjugacy class a_1 in the Coxeter group D_4 [1]. The prepotential for $D_4(a_1)$ is

$$F(t) = \frac{19t_4^5}{2^6 3^4 5} + \frac{7t_4^3t_3^2}{2^5 3^3} - \frac{t_4^3t_2}{2\cdot 3^3} + \frac{t_4t_3^4}{2^6 3} + \frac{t_4t_3^2t_2}{6} + \frac{t_4t_2^2}{6} + \frac{t_4t_1^2}{2} + t_2t_3t_1 - \frac{Z^5}{2^3 3^4 5},$$

where

$$P(t_2,t_3,t_4,Z) := Z^2 - (t_4^2 + 3t_3^2 + 24t_2) = 0. \tag{4.1}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{2}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{1}{2}$. We note that slightly different prepotentials and coordinates are used in Pavlyk [22] and Dinar [4]. The intersection form (2.6) is then given by

$$g^{11}(t) = \frac{1}{864}\left(t_4(19t_4^2 + 63t_3^2 - 144t_2) - 2Z(4t_4^2 + 3t_3^2 + 24t_2)\right), \tag{4.2}$$

$$g^{12}(t) = \frac{1}{96}t_3\left(t_4(7t_4 - 2Z) + 3t_3^2 + 48t_2\right), \tag{4.3}$$

$$g^{22}(t) = \frac{1}{288}\left(t_4(7t_4^2 + 27t_3^2 + 144t_2) - 2Z(t_4^2 + 12t_3^2 + 24t_2)\right), \tag{4.4}$$

$$g^{13}(t) = \frac{1}{18}\left(6t_2 + 3t_3^2 - t_4(t_4 + Z)\right), \tag{4.5}$$

$$g^{23}(t) = t_1 + \frac{1}{6}t_3(2t_4 - Z),\ g^{33}(t) = \frac{1}{6}(t_4 - 2Z), \tag{4.6}$$

$$g^{14}(t) = t_1,\ g^{24}(t) = t_2,\ g^{34}(t) = \frac{1}{2}t_3,\ g^{44}(t) = \frac{1}{2}t_4. \tag{4.7}$$

Let R_{D_4} be the following root system for D_4:

$$R_{D_4} = \left\{\pm e_i \pm e_j \mid 1 \le i < j \le 4\right\}.$$

Let us introduce the following basic invariants for D_4 (cf. [23]):

$$y_1 = x_1^6 + x_2^6 + x_3^6 + x_4^6, \tag{4.8}$$

$$y_2 = x_1 x_2 x_3 x_4, \tag{4.9}$$

$$y_3 = x_1^4 + x_2^4 + x_3^4 + x_4^4, \tag{4.10}$$

$$y_4 = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{4.11}$$

The basic invariants y_1, y_2, y_3, y_4 have degrees 6, 4, 4, 2, respectively.

Lemma 4.1. *(cf. [23]) The intersection form $g^{ij}(y)$ takes the form*

$$g^{ij}(y) = \begin{pmatrix} 30y_1y_3 - 180y_2^2y_4 + 30y_1y_4^2 - 30y_3y_4^3 + 6y_4^5 & 6y_2y_3 & 32y_1y_4 - 96y_2^2 + 12y_3^2 - 24y_3y_4^2 + 4y_4^4 & 12y_1 \\ 6y_2y_3 & \frac{1}{6}(2y_1 - 3y_3y_4 + y_4^3) & 4y_2y_4 & 8y_2 \\ 32y_1y_4 - 96y_2^2 + 12y_3^2 - 24y_3y_4^2 + 4y_4^4 & 4y_2y_4 & 16y_1 & 8y_3 \\ 12y_1 & 8y_2 & y_3 & y_4 \end{pmatrix}.$$

Consider another set of basic invariants for D_4 given by

$$Y_1 = y_1 - \frac{5}{4}y_3y_4 + \frac{5}{16}y_4^3, \tag{4.12}$$

$$Y_2 = y_2, \tag{4.13}$$

$$Y_3 = y_3 - \frac{1}{2}y_4^2, \tag{4.14}$$

$$Y_4 = \frac{1}{8}y_4. \tag{4.15}$$

The following statement can be checked directly.

Lemma 4.2. *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

We have that $\deg t_1(x) = 4$, $\deg t_2(x) = 4$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

Proposition 4.3. *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 6\}$ and let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, Z] \mid \deg p(x) = 4\}$. The harmonic elements of V_1 are proportional to*

$$216t_1t_3 + 72t_2t_4 + 24t_2Z - 9t_3^2t_4 + 3t_3^2Z + t_4^3 + t_4^2Z,$$

and the harmonic elements of V_2 are of the form

$$a(4t_1 - t_3t_4) + b(t_4^2 + 3t_3^2 - 8t_2),$$

where $a, b \in \mathbb{C}$ are constants.

Proof. Using Proposition 2.1 we can directly calculate

$$\Delta(t_1) = \frac{t_3}{Z}(2Z - t_4), \tag{4.16}$$

$$\Delta(t_2) = \frac{1}{4}\left(2t_4 - Z - \frac{3t_3^2}{Z}\right), \tag{4.17}$$

$$\Delta(t_3) = -\frac{t_3}{Z}, \tag{4.18}$$

$$\Delta(t_4) = 1. \tag{4.19}$$

A general element of V_1 is of the form

$$a_1t_4t_2 + a_2t_4t_1 + a_3t_4Z + a_4t_3t_2 + a_5t_3t_1 + a_6t_3Z + a_7t_2^3 \\ + a_8t_2^2t_1 + a_9t_2^2Z + a_{10}t_2t_1^2 + a_{11}t_2t_1Z + a_{12}t_1^3 + a_{13}t_1^2Z, \tag{4.20}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.20) using Proposition 2.1 and formulas (4.16)–(4.19) we find that the only harmonic elements of V_1 are as claimed. A general element of V_2 has the form

$$b_1t_4 + b_2t_3 + b_3t_2^2 + b_4t_2t_1 + b_5t_2Z + b_6t_1^2 + b_7t_1Z, \tag{4.21}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (4.21) using Proposition 2.1 and formulas (4.16)–(4.19) we find that the only harmonic elements of V_2 are as claimed. □

Theorem 4.4. *Define*

$$y_1 = -\frac{16}{9}\left(216t_1t_3 - 288t_2t_4 + 24t_2Z + 126t_3^2t_4 + 3t_3^2Z - 44t_4^3 + t_4^2Z\right), \tag{4.22}$$

$$y_2 = 4\left(4t_1 - t_3t_4\right), \tag{4.23}$$

$$y_3 = 8\left(3t_4^2 - 3t_3^2 + 8t_2\right), \tag{4.24}$$

$$y_4 = 8t_4. \tag{4.25}$$

Under the corresponding tensorial transformation the intersection form given by formulas (4.2)–(4.7) takes the form given in Lemma 4.1.

Proof. Note that $Y_4 = \frac{1}{8}y_4 = t_4$. We now equate Y_1 with a general harmonic element of V_1, and we equate Y_2 and Y_3 with general harmonic elements of V_2, where Y_1, Y_2 and Y_3 are given by formulas (4.12)–(4.14) and the harmonic elements of V_1 and V_2 are given by Proposition 4.3. We then rearrange these equations to find y_i in terms of t_j and Z. We find

$$y_1 = 160t_4^3 + \frac{a_1}{24}(216t_1t_3 + 72t_2t_4 + 24t_2Z - 9t_3^2t_4 + 3t_3^2Z + t_4^3 + t_4^2Z) \\ + 10t_4\left(\frac{a_3}{4}(4t_1 - t_3t_4) + b_3(t_4^2 + 3t_3^2 - 8t_2)\right), \tag{4.26}$$

$$y_2 = \frac{a_2}{4}(4t_1 - t_3t_4) + b_2(t_4^2 + 3t_3^2 - 8t_2^2), \tag{4.27}$$

$$y_3 = 32t_4^2 + \frac{a_3}{4}(4t_1 - t_3t_4) + b_3(t_4^2 + 3t_3^2 - 8t_2), \tag{4.28}$$

$$y_4 = 8t_4, \tag{4.29}$$

where $a_i, b_j \in \mathbb{C}$. In order to find a_i and b_j we perform steps 5–7 from Section 2.4. That is, we transform the intersection form (4.2)–(4.7) into y coordinates by applying formulas (4.26)–(4.29) and compare it with the expression given by Lemma 4.1. A particular solution is given by

$$a_1 = -\frac{128}{3}, \qquad a_2 = 16, \qquad a_3 = 0, \qquad b_2 = 0, \qquad b_3 = -8,$$

which implies the statement. □

Remark 4.5. There are in fact five other ways to choose y_i in Theorem 4.4 as polynomials of t_j and Z. This non-uniqueness is due to the S_3 symmetry of the Coxeter graph of D_4.

Proposition 4.6. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

Proposition 4.7. *We have that*

$$\det(g^{ij}(t)) = \frac{c \prod_{\alpha\in R_{D_4}} (\alpha, x)}{Q(t,Z)^2},$$

where $c = 9$ and

$$Q(t,Z) = 2^{14}(12t_1 + 5t_3t_4 + 2t_3Z).$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 4.4, which leads to Proposition 4.7.

In the next statement we express flat coordinates t_i via basic invariants y_j and Z, which is an inversion of formulas from Theorem 4.4.

Theorem 4.8. *We have the following relations:*

$$t_1 = -\frac{1}{13824y_2}\left(32y_4Z^3 + 24y_4^2Z^2 + 18y_1y_4 - 864y_2^2 - 27y_3y_4^2 + 7y_4^4\right), \tag{4.30}$$

$$t_2 = \frac{1}{512}\left(16Z^2 + 2y_3 - y_4^2\right), \tag{4.31}$$

$$t_3 = -\frac{1}{432y_2}\left(32Z^3 + 24y_4Z^2 + 18y_1 - 27y_3y_4 + 7y_4^3\right), \tag{4.32}$$

$$t_4 = \frac{1}{8}y_4, \tag{4.33}$$

where Z satisfies the equation

$$\begin{aligned} &2^{10}Z^6 + 2^9 3\, y_4Z^5 + 2^6 3^2 y_4^2Z^4 + 2^6\left(7\, y_4^3 - 3^3 y_3y_4 + 3^2 2\, y_1\right)Z^3 + 2^4 3\left(7\, y_4^4\right.\\ &\left.- 3^3 y_3y_4^2 + 3^2 2\, y_1y_4 - 2^2 3^4 y_2^2\right)Z^2 + 7^2 y_4^6 - 3^3 2\cdot 7\, y_3y_4^4 + 2^2 3^2 7\, y_1y_4^3\\ &+ 3^6 y_3^2y_4^2 - 2^3 3^5 y_2^2y_4^2 - 2^2 3^5 y_1y_3y_4 + 2^3 3^6 y_2^2y_3 + 2^2 3^4 y_1^2 = 0, \end{aligned} \tag{4.34}$$

and y_i are given by relations (4.8)–(4.11).

Proof. Formula (4.33) follows immediately from Theorem 4.4. Using relations (4.1) and (4.23) we see that

$$t_1 = \frac{1}{32}(2y_2 + t_3y_4), \tag{4.35}$$

$$t_2 = \frac{1}{24}\left(Z^2 - 3t_3^2 - \frac{1}{64}y_4^2\right). \tag{4.36}$$

Substituting the relations (4.33) and (4.36) into formula (4.24) and rearranging, we get

$$t_3^2 = \frac{1}{75}\left(Z^2 + \frac{71}{64}y_4^2 - 3y_3\right). \tag{4.37}$$

We can then substitute relations (4.33), (4.35) and (4.36) into formula (4.22) and reduce modulo t_3^2 using relation (4.37) to find a linear equation in t_3 which we can rearrange to find relation (4.32). Substituting relations (4.32) and (4.33) into formulas (4.35) and (4.36) gives us relation (4.30) and the following:

$$\begin{aligned} t_2 = -\frac{1}{1492992y_2^2}&(1024Z^6 + 1152y_1Z^3 + 324y_1^2 - 62208y_2^2Z^2 + 1536y_4Z^5 + 864y_1y_4Z^2\\ &- 1728y_3y_4Z^3 - 972y_1y_3y_4 + 576y_4^2Z^4 + 972y_2^2y_4^2 - 1296y_3y_4^2Z^2\\ &+ 729y_3^2y_4^2 + 448y_4^3Z^3 + 252y_1y_4^3 + 336y_4^4Z^2 - 378y_3y_4^4 + 49y_4^6). \end{aligned} \tag{4.38}$$

We then substitute relations (4.30), (4.32), (4.33) and (4.38) into formula (4.22) and we get the formula (4.34). Finally, reducing relation (4.38) modulo the polynomial (4.34) in Z gives us relation (4.31). □

Proposition 4.9. *The unity vector field $e = \partial_{t_1}$ in the y coordinates has the form*

$$e(y) = 16\left(\partial_{y_2} - 24 t_3 \partial_{y_1}\right).$$

Proof. We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 4.4. $\square$

5. Algebraic Frobenius manifold related to F_4

The $F_4(a_2)$ Frobenius manifold was described by Dinar with a prepotential given explicitly [4] (there seem to be some typos for the prepotential in [4], we include a corrected version below which was communicated to us by Dinar). It is a four dimensional Frobenius manifold which can be associated to the Coxeter group F_4, and is denoted by the conjugacy class a_2 in the Coxeter group F_4 [1]. The prepotential for $F_4(a_2)$ is

$$\begin{aligned}
F(t) = {} & \frac{t_4 t_1^2}{2} + t_3 t_2 t_1 + \frac{2^4}{13^2} t_4 t_2^2 - \frac{7}{13} t_3 t_2^2 + \frac{2^4 3^4 71 \cdot 4259}{5^5 7 \cdot 13} t_4^4 t_2 - \frac{2^3 3^4 23 \cdot 47}{5^5 7} t_4^3 t_3 t_2 - \frac{3^5 13 \cdot 103 \cdot 293}{2^3 5^5 7} t_4^2 t_3^2 t_2 \\
& - \frac{3^4 13^2 79 \cdot 467}{2^6 5^5 7} t_4 t_3^3 t_2 - \frac{3^4 13^3 157 \cdot 383}{2^{12} 5^5 7} t_3^4 t_2 + \frac{2^5 3^7 67 \cdot 521749}{5^9 7} t_4^7 + \frac{2^4 3^{10} 13 \cdot 693097}{5^9 7} t_4^6 t_3 \\
& + \frac{2^2 3^8 13^2 23^2 7 \cdot 97}{5^9} t_4^5 t_3^2 + \frac{3^7 13^3 18224639}{2^6 5^8 7} t_4^4 t_3^3 + \frac{3^8 13^4 7243667}{2^{11} 5^8 7} t_4^3 t_3^4 + \frac{3^8 13^5 8754721}{2^{14} 5^9 7} t_4^2 t_3^5 \\
& + \frac{3^7 13^6 19 \cdot 435503}{2^{18} 5^9 7} t_4 t_3^6 + \frac{3^8 13^7 41 \cdot 7129}{2^{22} 5^9 7} t_3^7 + \left(\frac{3^3 5^2}{13^2 7} t_2^2 - \frac{2^2 3^6 139}{5^2 7 \cdot 13} t_4^3 t_2 - \frac{3^8 23}{5^2 2 \cdot 7} t_4^2 t_3 t_2 \right. \\
& - \frac{3^7 13 \cdot 73}{2^5 5^2 7} t_4 t_3^2 t_2 - \frac{3^6 13^2 41}{2^9 5^2 7} t_3^3 t_2 + \frac{3^9 13^6 29 \cdot 43}{2^{20} 5^7} t_3^6 + \frac{3^{10} 13^5 15937}{2^{15} 5^7 7} t_4 t_3^5 + \frac{3^{12} 13^4 2729}{2^{12} 5^6 7} t_4^2 t_3^4 \\
& \left. + \frac{3^9 13^3 131357}{2^8 5^6 7} t_4^3 t_3^3 + \frac{3^{10} 13^3 1949}{2^3 5^6 7} t_4^4 t_3^2 + \frac{3^{11} 13 \cdot 68473}{5^7 7} t_4^5 t_3 + \frac{2^2 3^9 89 \cdot 11701}{5^7 7} t_4^6 \right) Z \\
& + \left(\frac{3^{14} 139}{5^5 2 \cdot 7} t_4^5 + \frac{3^{13} 13 \cdot 19}{5^4 2 \cdot 7} t_4^4 t_3 + \frac{3^{15} 13^3}{2^{11} 5^4} t_4^2 t_3^3 + \frac{3^{12} 13^3 101}{2^9 5^4 7} t_4^3 t_3^2 + \frac{3^{13} 13^4 31}{2^{16} 5^4 7} t_4 t_3^4 \right. \\
& \left. + \frac{3^{12} 13^5 41}{2^{20} 5^5 7} t_3^5 - \frac{3^{11}}{2^2 5 \cdot 7 \cdot 13} t_4^2 t_2 - \frac{3^{10}}{2^5 5 \cdot 7} t_4 t_3 t_2 - \frac{3^9 13}{2^{10} 5 \cdot 7} t_3^2 t_2 \right) Z^2,
\end{aligned}$$

where

$$\begin{aligned}
P(t_2, t_3, t_4, Z) := {} & Z^3 - \frac{2^3 3^4 13}{5^4} \left(\frac{2^3 3}{13} t_4^2 + t_4 t_3 + \frac{13}{2^5 3} t_3^2 \right) Z + \frac{2^2 3 \cdot 13}{5^6} \left(\frac{2^6 5^4}{3^3 13^2} t_2 \right. \\
& \left. - \frac{2^7 139}{13} t_4^3 - 2^4 3^2 23\, t_4^2 t_3 - 3 \cdot 13 \cdot 73\, t_4 t_3^2 - \frac{13^2 41}{2^4} t_3^3 \right) = 0.
\end{aligned}$$

The Euler vector field is

$$E(t) = t_1 \partial_{t_1} + t_2 \partial_{t_2} + \frac{1}{3} t_3 \partial_{t_3} + \frac{1}{3} t_4 \partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{2}{3}$. The intersection form (2.6) is then given by

$$\begin{aligned}
g^{11}(t) = {} & -\frac{3^4}{2^{13} 5^8 13} \Big(259200000000 Z^2 t_2 + 861120000000 Z t_2 t_3 + 1833744640000 t_2 t_3^2 \\
& - 94787461372500 Z^2 t_3^3 - 229118338413900 Z t_3^4 - 184708373655429 t_3^5 \\
& + 1067520000000 Z t_2 t_4 + 6818938880000 t_2 t_3 t_4 - 593226777780000 Z^2 t_3^2 t_4 \\
& - 1466753056797600 Z t_3^3 t_4 - 1556308486273320 t_3^4 t_4 - 1870069760000 t_2 t_4^2 \\
& - 996804506880000 Z^2 t_3 t_4^2 - 4153975366694400 Z t_3^2 t_4^2 - 6641896760778240 t_3^3 t_4^2 \\
& - 546311900160000 Z^2 t_4^3 - 6910723746201600 Z t_3 t_4^3 - 16691391832227840 t_3^2 t_4^3
\end{aligned}$$

$$- 5389922879078400Zt_4^4 - 17222218351902720t_3t_4^4 - 6432998677807104t_4^5\Big), \tag{5.1}$$

$$\begin{aligned}
g^{12}(t) = &- \frac{3^4}{2^{17}5^8}\Big(86400000000t_2Z^2 + 911040000000t_2t_3Z + 1440474880000t_2t_3^2 \\
&- 44608926082500t_3^3Z^2 - 143790305946300t_3^4Z - 157672431777393t_3^5 \\
&+ 1059840000000t_2t_4Z + 4513832960000t_2t_3t_4 - 330765116760000t_3^2t_4Z^2 \\
&- 1081698384499200t_3^3t_4Z - 1043122465279440t_3^4t_4 + 4196270080000t_2t_4^2 \\
&- 700228488960000t_3t_4^2Z^2 - 2682937817164800t_3^2t_4^2Z - 3140123415886080t_3^3t_4^2 \\
&- 382258206720000t_4^3Z^2 - 3411310364467200t_3t_4^3Z - 5919243052769280t_3^2t_4^3 \\
&-2126376537292800t_4^4Z - 7197815592714240t_3t_4^4 - 3176813316538368t_4^5\Big),
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
g^{22}(t) = &- \frac{3^4 13}{2^{21}5^8}\Big(28800000000t_2Z^2 + 511680000000t_2t_3Z + 1958344960000t_2t_3^2 \\
&- 19207343902500t_3^3Z^2 - 78756703307100t_3^4Z - 114276677239881t_3^5 \\
&+ 1121280000000t_2t_4Z + 3545784320000t_2t_3t_4 - 158303428920000t_3^2t_4Z^2 \\
&- 707890736966400t_3^3t_4Z - 936110404686480t_3^4t_4 + 2777743360000t_2t_4^2 \\
&- 383440936320000t_3t_4^2Z^2 - 2025454752921600t_3^2t_4^2Z - 2337422808015360t_3^3t_4^2 \\
&- 244681482240000t_4^3Z^2 - 2221827115622400t_3t_4^3Z - 3112967576309760t_3^2t_4^3 \\
&-1046938278297600t_4^4Z - 2618007725998080t_3t_4^4 - 1417602668691456t_4^5\Big),
\end{aligned} \tag{5.3}$$

$$\begin{aligned}
g^{13}(t) = &\frac{1}{2^6 5^4 13^2}\Big(1280000t_2 - 924007500t_3Z^2 - 3897258300t_3^2Z - 2181574863t_3^3 \\
&- 3411720000t_4Z^2 - 9067593600t_3t_4Z - 4518684144t_3^2t_4 - 5620492800t_4^2Z \\
&+9861336576t_3t_4^2 + 50200031232t_4^3\Big),
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
g^{23}(t) = &\frac{1}{2^{10}5^4 13}\Big(8320000t_1 - 8960000t_2 - 308002500t_3Z^2 - 2188871100t_3^2Z \\
&- 3888894321t_3^3 - 1137240000t_4Z^2 - 9593251200t_3t_4Z - 8055045648t_3^2t_4 \\
&-5580057600t_4^2Z - 5561457408t_3t_4^2 + 4045676544t_4^3\Big),
\end{aligned} \tag{5.5}$$

$$g^{33}(t) = \frac{2}{13^2 3}(150Z - 91t_3 + 16t_4), \tag{5.6}$$

$$g^{14}(t) = t_1, \qquad g^{24}(t) = t_2, \qquad g^{34}(t) = \frac{1}{3}t_3, \qquad g^{44}(t) = \frac{1}{3}t_4. \tag{5.7}$$

Let R_{F_4} be the following root system for F_4:

$$R_{F_4} = \{\pm e_i \mid 1 \le i \le 4\} \cup \left\{\pm e_i \pm e_j \mid 1 \le i < j \le 4\right\} \cup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

Let us introduce the following basic invariants for F_4 (cf. [23]):

$$\begin{aligned}
y_1 &= 288\epsilon_2\epsilon_4 - 108\epsilon_1^2\epsilon_4 - 8\epsilon_2^3 + 3\epsilon_1^2\epsilon_2^2, \\
y_2 &= 12\epsilon_4 - 3\epsilon_1\epsilon_3 + \epsilon_2^2, \\
y_3 &= 6\epsilon_3 - \epsilon_1\epsilon_2, \\
y_4 &= \epsilon_1,
\end{aligned}$$

where

$$\begin{aligned}
\epsilon_1 &= x_1^2 + x_2^2 + x_3^2 + x_4^2, \\
\epsilon_2 &= x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2, \\
\epsilon_3 &= x_1^2x_2^2x_3^2 + x_1^2x_2^2x_4^2 + x_1^2x_3^2x_4^2 + x_2^2x_3^2x_4^2,
\end{aligned}$$

$$\epsilon_4 = x_1^2 x_2^2 x_3^2 x_4^2.$$

The basic invariants y_1, y_2, y_3, y_4 have degrees $12, 8, 6, 2$, respectively.

Lemma 5.1. *(cf. [23]) The entries of the intersection form $g^{ij}(y)$ are*

$$\begin{aligned}
&g^{11}(y) = 1152 y_2^2 y_3 - 144 y_1 y_2 y_4 + 1152 y_2 y_3^2 y_4 - 144 y_1 y_3 y_4^2 + 288 y_3^3 y_4^2,\\
&g^{12}(y) = -96 y_2^2 y_4 - 48 y_2 y_3 y_4^2, \qquad g^{22}(y) = -8 y_2 y_3 - y_1 y_4 + 3 y_3^2 y_4,\\
&g^{13}(y) = 192 y_2^2 + 120 y_2 y_3 y_4 - 12 y_1 y_4^2 + 12 y_3^2 y_4^2,\\
&g^{23}(y) = 2 y_1 - 6 y_3^2 - 8 y_2 y_4^2, \qquad g^{33}(y) = 20 y_2 y_4 - 4 y_3 y_4^2,\\
&g^{14}(y) = 24 y_1, \qquad g^{24}(y) = 16 y_2, \qquad g^{34}(y) = 12 y_3, \qquad g^{44}(y) = 4 y_4.
\end{aligned}$$

Consider another set of basic invariants for F_4 given by

$$\begin{aligned}
Y_1 &= y_1 + \frac{1}{3080} y_4 \left(2520 y_2 y_4 + 1708 y_3 y_4^2 + 61 y_4^5\right),\\
Y_2 &= y_2 + \frac{1}{160} y_4 \left(40 y_3 + 3 y_4^3\right),\\
Y_3 &= y_3 - \frac{1}{8} y_4^3,\\
Y_4 &= \frac{1}{8} y_4.
\end{aligned}$$

The following statement can be checked directly.

Lemma 5.2. *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

We have that $\deg t_1(x) = 6$, $\deg t_2(x) = 6$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

Theorem 5.3. *Define*

$$\begin{aligned}
y_1 = \frac{1}{2^{15} 3^9 5^6 13^2} \Big(& 2^{17} 5^8 13^2 t_1^2 - 2^{22} 5^8 13\, t_1 t_2 + 2^{22} 5^8 17\, t_2^2 + 2^{14} 3^6 5^8 13^2 t_2 t_3 Z^2\\
& - 2^{12} 3^6 5^6 13^3 41\, t_2 t_3^2 Z + 2^{13} 3^3 5^4 13^5 7 \cdot 29\, t_1 t_3^3 - 2^{12} 3^3 5^5 13^4 491\, t_2 t_3^3\\
& - 2^2 3^9 5^4 13^6 41\, t_3^4 Z^2 + 2^2 3^9 5^2 11^2 13^8 t_3^5 Z + 3^6 13^8 1202837\, t_3^6 + 2^{18} 3^7 5^8 13\, t_1 t_3^3\\
& - 2^{17} 3^6 5^6 13^2 23\, t_2 t_3 t_4 Z + 2^{17} 3^4 5^4 13^4 59\, t_1 t_3^2 t_4 + 2^{16} 3^4 5^5 13^3 67\, t_2 t_3^2 t_4\\
& - 2^8 3^{10} 5^4 13^5 41\, t_3^3 t_4 Z^2 + 2^6 3^9 5^3 13^6 43^2 t_3^4 t_4 Z + 2^5 3^7 13^8 111347\, t_3^5 t_4\\
& + 2^{20} 3^6 5^6 13 \cdot 31\, t_2 t_4^2 Z - 2^{19} 3^4 5^4 13^3 367\, t_1 t_3 t_4^2 + 2^{20} 3^4 5^5 13^4 11\, t_2 t_3 t_4^2\\
& - 2^{11} 3^{10} 5^4 13^4 269\, t_3^2 t_4^2 Z^2 + 2^{11} 3^9 5^3 13^5 757\, t_3^3 t_4^2 Z + 2^8 11^2 5 \cdot 23633 \cdot 6892993\, t_3^4 t_4^2\\
& + 2^{22} 3^3 5^4 13^2 43 \cdot 61\, t_1 t_4^3 - 2^{25} 3^3 5^5 11 \cdot 13 \cdot 701\, t_2 t_4^3 - 2^{15} 3^9 5^4 13^3 1039\, t_3 t_4^3 Z^2\\
& - 2^{13} 3^9 5^3 13^4 9431\, t_3^2 t_4^3 Z + 2^{13} 3^6 13^5 5 \cdot 1939033\, t_3^3 t_4^3 - 2^{18} 3^{10} 5^4 13^2 557\, t_4^4 Z^2\\
& - 2^{22} 3^9 5^3 13^3 587\, t_3 t_4^4 Z - 2^{16} 3^7 13^4 5 \cdot 23 \cdot 206351\, t_3^2 t_4^4 - 2^{21} 3^9 5^2 13^3 17 \cdot 257\, t_4^5 Z\\
& - 2^{22} 3^7 13^3 19 \cdot 71 \cdot 2383\, t_3 t_4^5 + 2^5 3 \cdot 43 \cdot 103 \cdot 149 \cdot 2791 \cdot 1285517\, t_4^6 \Big),\\
y_2 = \frac{1}{2^{12} 3^5 5^4 13} \Big(& -2^{12} 5^6 3\, t_2 Z + 2^{13} 5^4 13^2 t_1 t_3 - 2^{13} 5^4 7 \cdot 13\, t_2 t_3 + 2^2 3^5 5^4 13^3 t_3^2 Z^2\\
& + 2^2 3^4 5^2 13^4 41\, t_3^3 Z + 3^3 13^5 11 \cdot 1171\, t_3^4 + 2^{17} 5^4 13\, t_1 t_4 - 2^{17} 5^6 t_2 t_4\\
& + 2^7 3^6 5^4 13^2 t_3 t_4 Z^2 + 2^6 3^5 5^2 13^3 73\, t_3^2 t_4 Z + 2^6 3^3 13^4 17 \cdot 79\, t_3^3 t_4 + 2^{10} 3^7 5^4 13\, t_4^2 Z^2\\
& + 2^{10} 3^6 5^2 13^2 23\, t_3 t_4^2 Z - 2^9 3^4 13^3 47 \cdot 593\, t_3^2 t_4^2 + 2^{13} 3^4 5^2 13 \cdot 139\, t_4^3 Z\\
& - 2^{15} 3^3 13^2 23 \cdot 1303\, t_3 t_4^3 - 2^{16} 3^3 13 \cdot 62539\, t_4^4 \Big),\\
y_3 = \frac{1}{2^4 3^4 5 \cdot 13} \Big(& -2^5 5 \cdot 13\, t_1 + 2^9 3 \cdot 5\, t_2 + 3^3 13^4 t_3^3 + 2^4 3^5 13^3 t_3^2 t_4 + 2^6 3^4 13^2 11\, t_3 t_4^2
\end{aligned}$$

$$
\begin{aligned}
&\quad -2^9 3^3 13 \cdot 79 t_4^3\Big),\\
y_4 &= 12 t_4.
\end{aligned}
$$

Under the corresponding tensorial transformation the intersection form given by formulas (5.1)–(5.7) takes the form given in Lemma 5.1.

Remark 5.4. There is in fact one other way to choose y_i in Theorem 5.3 as polynomials of t_j and Z. This non-uniqueness is due to the $\mathbb{Z}_2$ symmetry of the Coxeter graph of F_4.

Proposition 5.5. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

Proposition 5.6. *We have that*

$$
\det(g^{ij}(t)) = \frac{c \prod_{\alpha \in R_{F_4}} (\alpha, x)}{Q(t, Z)^2},
$$

where $c = 2^{36} 3^{42} 5^8 13^4$ *and*

$$
\begin{aligned}
Q(t, Z) = &- 2^{16} 5^6 13^2 t_1^2 + 2^{17} 5^6 7 \cdot 13\, t_1 t_2 + 2^{21} 5^6 t_2^2 - 2^8 3^6 5^6 13^3 Z^2 t_1 t_3 + 2^{12} 3^7 5^6 13^2 Z^2 t_2 t_3\\
&- 2^8 3^5 5^4 13^4 41\, Z t_1 t_3^2 - 2^{14} 3^5 5^4 13^4 Z t_2 t_3^2 + 2^6 3^3 5^2 13^5 17 \cdot 797\, t_1 t_3^3\\
&- 2^{10} 3^3 5^2 13^5 919\, t_2 t_3^3 - 2^2 3^9 5^4 13^6 47\, Z^2 t_3^4 + 2^2 3^8 5^2 13^7 17 \cdot 41\, Z t_3^5 - 3^6 13^8 5 \cdot 89 \cdot 97\, t_3^6\\
&- 2^{12} 3^7 5^6 13^2 Z^2 t_1 t_4 + 2^{14} 3^6 5^6 11 \cdot 13\, Z^2 t_2 t_4 - 2^{13} 3^5 5^4 13^3 73\, Z t_1 t_3 t_4\\
&+ 2^{13} 3^5 5^4 13^2 229\, Z t_2 t_3 t_4 + 2^{10} 3^4 5^2 13^5 919\, t_1 t_3^2 t_4 - 2^{12} 3^4 5^2 13^3 72889\, t_2 t_3^2 t_4\\
&- 2^3 3^9 5^4 13^5 19 \cdot 149\, Z^2 t_3^3 t_4 + 2^3 3^8 5^2 13^6 41^2 17\, Z t_3^4 t_4 + 3^7 13^7 2 \cdot 17 \cdot 23 \cdot 37 \cdot 59\, t_3^5 t_4\\
&- 2^{16} 3^6 5^4 13^2 23\, Z t_1 t_4^2 + 2^{17} 3^6 5^4 13 \cdot 227\, Z t_2 t_4^2 + 2^{14} 3^4 5^2 13^3 19 \cdot 1039\, t_1 t_3 t_4^2\\
&- 2^{17} 3^4 5^2 13^2 64871\, t_2 t_3 t_4^2 - 2^7 3^{10} 5^6 13^4 83\, Z^2 t_3^2 t_4^2 + 2^9 3^9 5^2 13^5 7 \cdot 11 \cdot 41\, Z t_3^3 t_4^2\\
&+ 2^5 3^7 13^6 5 \cdot 41 \cdot 37649\, t_3^4 t_4^2 + 2^{20} 3^3 5^2 13^3 17 \cdot 47\, t_1 t_4^3 - 2^{20} 3^3 5^2 13 \cdot 188701\, t_2 t_4^3\\
&- 2^{11} 3^9 5^4 13^3 3571\, Z^2 t_3 t_4^3 + 2^{13} 3^8 5^2 13^4 11 \cdot 97\, Z t_3^2 t_4^3 + 2^{17} 3^6 13^5 5 \cdot 52057\, t_3^3 t_4^3\\
&- 2^{17} 3^9 5^4 13^2 11\, Z^2 t_4^4 - 2^{14} 3^8 5^2 13^3 11 \cdot 37 \cdot 139\, Z t_3 t_4^4 + 2^{13} 3^7 13^4 5 \cdot 17 \cdot 499 \cdot 659\, t_3^2 t_4^4\\
&- 2^{18} 3^{11} 5^3 13^2 197\, Z t_4^5 + 2^{17} 3^7 13^3 5 \cdot 11 \cdot 247439\, t_3 t_4^5 + 2^{22} 3^6 7^2 13^2 19 \cdot 41 \cdot 61\, t_4^6.
\end{aligned}
$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 5.3, which leads to Proposition 5.6.

Note that we do not include relations for t_i, Z and e as functions of the basic invariants y_j for this example as they are too long to present here.

6. Algebraic Frobenius manifolds related to H_4

There are 7 known non-polynomial algebraic Frobenius manifolds which can be associated to H_4, they are each four-dimensional and their prepotentials have been listed by Sekiguchi [24]. Let R_{H_4} be the following root system for H_4:

$$
R_{H_4} = \{\pm e_i \mid 1 \le i \le 4\} \cup \left\{ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\} \cup \left\{ \frac{1}{2}\left(\pm e_{\sigma(2)} \pm \varphi e_{\sigma(3)} \pm \overline{\varphi} e_{\sigma(4)}\right) \middle| \sigma \in \mathfrak{A}_4 \right\},
$$

where

$$
\varphi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\varphi} = \frac{1-\sqrt{5}}{2},
$$

and $\mathfrak{A}_4$ is the alternating group on 4 elements. Let us introduce the following basic invariants for H_4 (cf. [23]):

$$
\begin{aligned}
y_1 = &\frac{32}{3}x_1^{24}h_2^3 - 40x_1^{22}\left(2h_2^4 + 3h_2h_6\right) + x_1^{20}\left(360h_{10} + \frac{1344}{5}h_2^5 + 672h_2^2h_6\right)\\
&+ x_1^{18}\left(1080h_6^2 - 1608h_2^3h_6 - \frac{1328}{3}h_2^6 - 2880h_{10}h_2\right)\\
&+ x_1^{16}\left(10024h_{10}h_2^2 + 272h_2^7 + 1248h_2^4h_6 - 5628h_2h_6^2\right)\\
&+ x_1^{14}\left(18588h_2^2h_6^2 + 272h_2^8 - 7620h_{10}h_6 - 16856h_{10}h_2^3\right)\\
&+ x_1^{12}\left(14216h_{10}h_2^4 + 23508h_{10}h_6h_2 - \frac{1328}{3}h_2^9 - 1248h_6h_2^6 - 27396h_6^2h_2^3\right.\\
&\left.-5796h_6^3\right) + x_1^{10}\left(3240h_{10}^2 - 7160h_{10}h_2^5 - 25332h_{10}h_6h_2^2 + \frac{1344}{5}h_2^{10}\right.\\
&\left.+1608h_2^7h_6 + 19968h_2^4h_6^2 + 7350h_2h_6^3\right) + x_1^8\left(2144h_{10}h_2^6 - 3232h_{10}^2h_2\right.\\
&\left.+10908h_{10}h_2^3h_6 - 906h_{10}h_6^2 - 80h_2^{11} - 672h_2^8h_6 - 6924h_2^5h_6^2 - 1956h_2^2h_6^3\right)\\
&+ x_1^6\left(1168h_{10}^2h_2^2 - 344h_{10}h_2^7 - 2172h_{10}h_2^4h_6 - 1908h_{10}h_2h_6^2 + \frac{32}{3}h_2^{12}\right.\\
&\left.+120h_2^9h_6 + 1332h_2^6h_6^2 + 288h_2^3h_6^3 + 2394h_6^4\right) + x_1^4\left(348h_{10}^2h_6 - 152h_{10}^2h_2^3\right.\\
&\left.+16h_{10}h_2^8 + 60h_{10}h_6h_2^5 + 408h_{10}h_6^2h_2^2 - 84h_2^7h_6^2 + 84h_2^4h_6^3 - 909h_2h_6^4\right)\\
&+ x_1^2\left(8h_{10}^2h_2^4 - 42h_{10}h_6^2h_2^3 - 87h_{10}h_6^3 - 6h_2^5h_6^3 + 135h_2^2h_6^4\right)\\
&+ \frac{4}{3}h_{10}^3 - 3h_{10}h_2h_6^3 + \frac{9}{5}h_6^5,
\end{aligned} \tag{6.1}
$$

$$
\begin{aligned}
y_2 = &4x_1^{16}h_2^2 - 10x_1^{14}\left(2h_2^3 + 3h_6\right) + x_1^{12}\left(44h_2^4 + 138h_2h_6\right)\\
&+ x_1^{10}\left(180h_{10} - 44h_2^5 - 402h_2^2h_6\right) + x_1^8\left(44h_2^6 - 464h_{10}h_2 + 402h_2^3h_6 + 294h_6^2\right)\\
&+ x_1^6\left(296h_{10}h_2^2 - 20h_2^7 - 138h_2^4h_6 - 306h_2h_6^2\right) + x_1^4\left(4h_2^8 - 76h_{10}h_2^3 - 114h_{10}h_6\right.\\
&\left.+30h_2^5h_6 + 168h_2^2h_6^2\right) + x_1^2\left(4h_{10}h_2^4 - 21h_2^3h_6^2 + \frac{57}{2}h_6^3\right) + h_{10}^2 - \frac{3}{2}h_2h_6^3,
\end{aligned} \tag{6.2}
$$

$$
\begin{aligned}
y_3 = &-2x_1^{10}h_2 + 6x_1^8h_2^2 + x_1^6\left(33h_6 - 14h_2^3\right) - x_1^4\left(33h_2h_6 - 6h_2^4\right)\\
&+ x_1^2\left(11h_{10} - 2h_2^5\right) - h_{10}h_2 + \frac{3}{2}h_6^2,
\end{aligned} \tag{6.3}
$$

$$
y_4 = x_1^2 + h_2, \tag{6.4}
$$

where

$$
h_2 = \epsilon_1, \tag{6.5}
$$

$$
h_6 = \sqrt{5}\delta + \epsilon_1\epsilon_2 - 11\epsilon_3, \tag{6.6}
$$

$$
h_{10} = 95\epsilon_2\epsilon_3 - 32\epsilon_1^2\epsilon_3 - 5\epsilon_1\epsilon_2^2 + 2\epsilon_1^3\epsilon_2 + 3\sqrt{5}\delta\epsilon_2, \tag{6.7}
$$

and

$$
\epsilon_1 = x_2^2 + x_3^2 + x_4^2, \tag{6.8}
$$

$$
\epsilon_2 = x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2, \tag{6.9}
$$

$$
\epsilon_3 = x_2^2x_3^2x_4^2, \tag{6.10}
$$

$$
\delta = (x_2^2 - x_3^2)(x_2^2 - x_4^2)(x_3^2 - x_4^2). \tag{6.11}
$$

The basic invariants y_1, y_2, y_3, y_4 have degrees $30, 20, 12, 2$, respectively.

Lemma 6.1. *(cf. [23]) The entries of the intersection form $g^{ij}(y)$ are*

$$g^{11}(y) = \frac{928}{3}y_2y_3^3y_4 + 240y_1y_3^2y_4^2 + 96y_2^2y_3y_4^3 + 160y_1y_2y_4^4,$$
$$g^{12}(y) = -32y_3^4 - 112y_2y_3^2y_4^2 - 120y_1y_3y_4^3 + 48y_2^2y_4^4,$$
$$g^{22}(y) = \frac{152}{3}y_3^3y_4 - 56y_2y_3y_4^3 + 20y_1y_4^4,$$
$$g^{13}(y) = -80y_2^2 - \frac{16}{3}y_3^3y_4^2 - 16y_2y_3y_4^4 - 40y_1y_4^5,$$
$$g^{23}(y) = -30y_1 + 8y_3^2y_4^3 - 24y_2y_4^5, \qquad g^{33}(y) = 44y_2y_4 - 8y_3y_4^5,$$
$$g^{14}(y) = 60y_1, \qquad g^{24}(y) = 40y_2, \qquad g^{34}(y) = 24y_3, \qquad g^{44}(y) = 5y_4.$$

Consider another set of basic invariants for H_4 given by

$$Y_1 = y_1 - \frac{y_4^3}{30030}\left(4y_4^{12} + 320y_3y_4^6 + 7051y_2y_4^2 - 715y_3^2\right), \tag{6.12}$$
$$Y_2 = y_2 + \frac{y_4^4}{748}\left(3y_4^6 + 110y_3\right), \tag{6.13}$$
$$Y_3 = y_3 + \frac{y_4^6}{14}, \tag{6.14}$$
$$Y_4 = \frac{1}{8}y_4. \tag{6.15}$$

The following statement can be checked directly.

Lemma 6.2. *We have $\Delta(Y_4) = 1$ and $\Delta(Y_1) = \Delta(Y_2) = \Delta(Y_3) = 0$.*

Note that for examples $H_4(3)$, $H_4(4)$ and $H_4(7)$ we will omit the relations for t_i and Z as functions of the basic invariants y_j, as they become too long. Likewise, we omit analogues of Propositions 4.3, 4.7 and 4.9.

6.1. $H_4(1)$ example

The prepotential for $H_4(1)$ is

$$F(t) = t_1t_2t_3 + \frac{1}{2}t_1^2t_4 + \frac{3356}{665}t_4^{21} + \frac{64}{5}t_3t_4^{16} + \frac{472}{11}t_3^2t_4^{11} + \frac{16}{3}t_3^3t_4^6 + 28t_3^4t_4 - \frac{16}{15}t_2t_4^{15}$$
$$+ 8t_2t_3t_4^{10} + 32t_2t_3^2t_4^5 - \frac{8}{3}t_2t_3^3 + \frac{19}{18}t_2^2t_4^9 - t_2^2t_3t_4^4 + \frac{1}{6}t_2^3t_4^3 + \frac{1}{105}Z^7,$$

where

$$P(t_2, t_3, t_4, Z) := Z^2 - 4t_4(t_4^5 - 3t_3) - t_2 = 0. \tag{6.16}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{3}{5}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{10}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{9}{10}$. The intersection form (2.6) is then given by

$$g^{11}(t) = \frac{1}{10}\Big(228t_2t_3^2Z + 19t_2^3t_4 - 2736t_3^3t_4Z - 228t_2^2t_3t_4^2 + 12160t_2t_3^2t_4^3 + 76t_2^2t_4^4Z$$
$$+ 3040t_3^3t_4^4 - 2736t_2t_3t_4^5Z + 22800t_3^2t_4^6Z + 1444t_2^2t_4^7 + 13680t_2t_3t_4^8 + 89680t_3^2t_4^9$$
$$+1520t_2t_4^{10}Z - 21888t_3t_4^{11}Z - 4256t_2t_4^{13} + 58368t_3t_4^{14} + 4864t_4^{16}Z + 40272t_4^{19}\Big), \tag{6.17}$$

$$g^{12}(t) = -\frac{3}{5}\Big(t_2^2Z - 280t_3^3 - 54t_2t_3t_4Z + 504t_3^2t_4^2Z + 10t_2^2t_4^3 - 800t_2t_3t_4^4 - 240t_3^2t_4^5$$
$$+68t_2t_4^6Z - 936t_3t_4^7Z - 200t_2t_4^9 - 2360t_3t_4^{10} + 256t_4^{12}Z - 512t_4^{15}\Big), \tag{6.18}$$

$$g^{22}(t) = -\frac{2}{5}\Big(44t_2t_3 - 924t_3^2t_4 - 33t_2t_4^2Z + 396t_3t_4^3Z - 176t_2t_4^5 - 88t_3t_4^6 - 132t_4^8Z \\ -236t_4^{11}\Big), \tag{6.19}$$

$$g^{13}(t) = \frac{7}{10}\Big(-2t_2t_3Z + 24t_3^2t_4Z + 3t_2^2t_4^2 - 16t_2t_3t_4^3 + 320t_3^2t_4^4 + 4t_2t_4^5Z - 56t_3t_4^6Z \\ +38t_2t_4^8 + 160t_3t_4^9 + 16t_4^{11}Z - 32t_4^{14}\Big), \tag{6.20}$$

$$g^{23}(t) = t_1 - 8t_3^2 - t_2t_4Z + 12t_3t_4^2Z - 2t_2t_4^4 + 64t_3t_4^5 - 4t_4^7Z + 8t_4^{10}, \tag{6.21}$$

$$g^{33}(t) = \frac{1}{40}\Big(3t_2Z - 36t_3t_4Z + 36t_2t_4^3 - 72t_3t_4^4 + 12t_4^6Z + 76t_4^9\Big), \tag{6.22}$$

$$g^{14}(t) = t_1, \qquad g^{24}(t) = \frac{3}{5}t_2, \qquad g^{34}(t) = \frac{1}{2}t_3, \qquad g^{44}(t) = \frac{1}{10}t_4. \tag{6.23}$$

We have that $\deg t_1(x) = 20$, $\deg t_2(x) = 12$, $\deg t_3(x) = 10$, $\deg t_4(x) = 2$ and $\deg Z(x) = 6$.

Proposition 6.3. *Let $V_1 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 30\}$, let $V_2 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 20\}$ and let $V_3 = \{p \in \mathbb{C}[t_1, t_2, t_3, t_4, Z] \mid \deg p(x) = 12\}$. The harmonic elements of V_1 are proportional to*

$$27027t_2^2Z - 1801800t_1t_3 - 6806800t_3^3 - 648648t_2t_3t_4Z + 3891888t_3^2t_4^2Z + 32175t_2^2t_4^3 \\ + 13556400t_2t_3t_4^4 - 2335080t_1t_4^5 + 90256320t_3^2t_4^5 + 216216t_2t_4^6Z \\ - 25594592t_3t_4^7Z - 4591440t_2t_4^9 + 35834480t_3t_4^{10} + 432432t_4^{12}Z - 864864t_4^{15},$$

the harmonic elements of V_2 are proportional to

$$561t_1 + 7106t_3^2 - 627t_2t_4^4 - 37620t_3t_4^5 - 12350t_4^{10},$$

and the harmonic elements of V_3 are proportional to

$$21t_2 + 308t_3t_4 - 220t_4^6.$$

Proof. Using Proposition 2.1 we can directly calculate

$$\Delta(t_1) = -\frac{19}{10}\Big(t_2Z - 30t_3t_4Z + 8t_2t_4^3 - 320t_3t_4^4 + 40t_4^6Z - 80t_4^9\Big), \tag{6.24}$$

$$\Delta(t_2) = -\frac{11}{5}\Big(8t_3 - 9t_4^2Z - 32t_4^5\Big), \tag{6.25}$$

$$\Delta(t_3) = -\frac{9}{20}t_4\Big(3Z + 4t_4^3\Big), \tag{6.26}$$

$$\Delta(t_4) = \frac{1}{5}. \tag{6.27}$$

A general element of V_1 is of the form

$$a_1t_1t_3 + a_2t_1t_4^5 + a_3t_1t_4^2Z + a_4t_2^2t_4^3 + a_5t_2^2Z + a_6t_2t_3t_4^4 + a_7t_2t_3t_4Z + a_8t_2t_4^9 + a_9t_2t_4^6Z \\ + a_{10}t_3^3 + a_{11}t_3^2t_4^5 + a_{12}t_3^2t_4^2Z + a_{13}t_3t_4^{10} + a_{14}t_3t_4^7Z + a_{15}t_4^{15} + a_{16}t_4^{12}Z, \tag{6.28}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.28) using Proposition 2.1 and formulas (6.24)–(6.27) we find that the only harmonic elements of V_1 are as claimed. A general element of V_2 has the form

$$b_1t_1 + b_2t_2t_4^4 + b_3t_2t_4Z + b_4t_3^2 + b_5t_3t_4^5 + b_6t_3t_4^2Z + b_7t_4^{10} + b_8t_4^7Z, \tag{6.29}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.29) using Proposition 2.1 and formulas (6.24)–(6.27) we find that the only harmonic elements of V_2 are as claimed. A general element of V_3 has the form

$$c_1t_2 + c_2t_3t_4 + c_3t_4^6 + c_4t_4^3Z, \tag{6.30}$$

where $c_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.30) using Proposition 2.1 and formulas (6.24)–(6.27) we find that the only harmonic elements of V_3 are as claimed. $\square$

Theorem 6.4. *We have the following relations*

$$y_1 = \frac{2^{30}5^9}{3}\Big(27t_2^2 Z - 1800t_1t_3 - 6800t_3^3 - 648t_2t_3t_4Z + 3888t_3^2t_4^2 + 12600t_2t_3t_4^4 + 6120t_1t_4^5 + 190320t_3^2t_4^5 + 216t_2t_4^6Z - 2592t_3t_4^7Z - 42840t_2t_4^9 - 953520t_3t_4^{10} + 432t_4^{12}Z + 1309136t_4^{15}\Big), \tag{6.31}$$

$$y_2 = 2^{20}5^7\left(3t_1 + 38t_3^2 - 21t_2t_4^4 - 460t_3t_4^5 + 950t_4^{10}\right), \tag{6.32}$$

$$y_3 = 2^{11}5^4\left(3t_2 + 44t_3t_4 - 260t_4^6\right), \tag{6.33}$$

$$y_4 = 40t_4. \tag{6.34}$$

Proof. Note that $Y_4 = \frac{1}{8}y_4 = 5t_4$. We now equate Y_1, Y_2 and Y_3 given by relations (6.12)–(6.14) with general harmonic elements of V_1, V_2 and V_3, respectively, given by Proposition 6.3. We then rearrange these equations to find y_i in terms of t_j and Z. We find

$$y_1 = \frac{2^{42}3^25^{14}13}{7^311}t_4^{15} + \frac{a}{2^33^37\cdot 11\cdot 13}\Big(27027t_2^2Z - 1801800t_1t_3 - 6806800t_3^3 - 648648t_2t_3t_4Z + 3891888t_3^2t_4^2Z + 32175t_2^2t_4^3 + 13556400t_2t_3t_4^4 - 2335080t_1t_4^5 + 90256320t_3^2t_4^5 + 216216t_2t_4^6Z - 2594592t_3t_4^7Z - 4591440t_2t_4^9 + 35834480t_3t_4^{10} + 432432t_4^{12}Z - 864864t_4^{15}\Big) - \frac{131276800b}{67431}t_4^5\Big(561t_1 + 7106t_3^2 - 627t_2t_4^4 - 37620t_3t_4^5 - 12350t_4^{10}\Big) + \frac{2^{23}5^82251c}{302379}t_4^9\left(21t_2 + 308t_3t_4 - 220t_4^6\right) - \frac{80c^2}{2541}t_4^3\left(21t_2 + 308t_3t_4 - 220t_4^6\right)^2, \tag{6.35}$$

$$y_2 = \frac{2^{29}5^{10}}{77}t_4^{10} + \frac{b}{12350}\Big(-561t_1 - 7106t_3^2 + 627t_2t_4^4 + 37620t_3t_4^5 + 12350t_4^{10}\Big) + \frac{320000c}{187}t_4^4\left(21t_2 + 308t_3t_4 - 220t_4^6\right), \tag{6.36}$$

$$y_3 = -\frac{2^{17}5^6}{7}t_4^6 + \frac{c}{220}\left(-21t_2 - 308t_3t_4 + 220t_4^6\right), \tag{6.37}$$

$$y_4 = 40t_4, \tag{6.38}$$

where $a, b, c \in \mathbb{C}$. In order to find a, b and c we perform steps 5–7 from Section 2.4. That is, we transform the intersection form (6.17)–(6.23) into y coordinates by applying formulas (6.35)–(6.38) and compare it with the expression given by Lemma 6.1. We find that

$$a = 2^{33}3^25^9, \qquad b = -\frac{2^{21}5^913\cdot 19}{11\cdot 17}, \qquad c = -\frac{2^{13}5^511}{7},$$

which implies the statement. $\square$

Proposition 6.5. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

Proposition 6.6. *We have that*

$$\det(g^{ij}(t)) = \frac{c\prod_{\alpha\in R_{H_4}}(\alpha, x)}{Q(t, Z)^2},$$

where $c = 5$ *and*

$$Q(t, Z) = 5^{20}\left(5t_1 - 70t_3^2 + 5t_2t_4Z - 60t_3t_4^2Z - 35t_2t_4^4 + 140t_3t_4^5 + 20t_4^7Z + 42t_4^{10}\right).$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 6.4, which leads to Proposition 6.6.

In the next statement we express flat coordinates t_i via basic invariants y_j and Z, which is an inversion of the formulas from Theorem 6.4.

Theorem 6.7. *We have the following relations:*

$$t_1 = \frac{1}{2^{27}5^{12}3y_4^2}\left(-2^{26}3^2 5^{12}19\, Z^4 + 2^{16}5^8 3\cdot 19\, y_3 Z^2 - 2^4 5^4 19\, y_3^2 \right.$$
$$\left. +2^7 5^5\, y_2 y_4^2 + 2^{13}5^6 3\cdot 7\cdot 19\, y_4^6 Z^2 + 2^2 3^4 5^2 7\, y_3 y_4^6 + 3^2 7\cdot 47\, y_4^{12}\right), \tag{6.39}$$

$$t_2 = \frac{1}{2^{15}5^7}\left(2^{13}5^6 11\, Z^2 + 2^2 5^2 3\, y_3 + 23 y_4^6\right), \tag{6.40}$$

$$t_3 = \frac{1}{2^{15}5^6 y_4}\left(-2^{14}5^6 3\, Z^2 + 2^3 5^2 y_3 + 17 y_4^6\right), \tag{6.41}$$

$$t_4 = \frac{1}{40} y_4, \tag{6.42}$$

where Z satisfies the equation

$$2^{34}3^3 5^{12} Z^6 - 2^{30}3^3 5^9 y_4^3 Z^5 - 2^{23}3^3 5^8 y_3 Z^4 + 2^{12}3^2 5^4 y_3^2 Z^2$$
$$- 2^{12}3^2 5^4 y_2 y_4^2 Z^2 - 2y_3^3 + 6 y_2 y_3 y_4^2 + 3 y_1 y_4^3 = 0, \tag{6.43}$$

and y_i are given by relations (6.1)–(6.11).

Proof. Formula (6.42) follows immediately from Theorem 6.4. Using relations (6.16) and (6.33) we see that

$$t_2 = Z^2 + \frac{3}{10} t_3 y_4 - \frac{y_4^6}{2^{16}5^6},$$
$$t_3 = \frac{1}{2^{15}5^4 11\, y_4}\left(-2^{16}5^5 3\, t_2 + 160 y_3 + 13 y_4^6\right).$$

We can solve this system of equations to find t_2 and t_3 which gives us formulas (6.40) and (6.41). Substituting formulas (6.40)–(6.42) into relation (6.32) and solving for t_1 we get formula (6.39). Finally, substituting relations (6.39)–(6.42) into formula (6.31) we get the formula (6.43). □

Proposition 6.8. *The unity vector field $e = \partial_{t_1}$ in the y coordinates has the form*

$$e(y) = 2^{20}5^7 3\, \partial_{y_2} - 2^{33}5^{10}3\left(5t_3 - 17t_4^5\right)\partial_{y_1}.$$

Proof. We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 6.4. □

6.2. $H_4(2)$ example

The prepotential for $H_4(2)$ is

$$F(t) = -\frac{66084040}{73920} t_4^{16} + \frac{143564400}{73920} t_3^2 t_4^{11} - \frac{40727610}{73920} t_3^4 t_4^6 - \frac{392931}{73920} t_3^6 t_4$$
$$+ t_1 t_2 t_3 + \frac{1}{2} t_1^2 t_4 - \frac{3}{4} t_4^4\left(2288 t_4^{10} - 1620 t_3^2 t_4^5 - 27 t_3^4\right) Z - 760 t_4^{12} Z^2$$
$$+ \left(\frac{1744}{48} t_4^{10} - \frac{4860}{48} t_3^2 t_4^5 - \frac{81}{48} t_3^4\right) Z^3 + 140 t_4^8 Z^4 + 24 t_4^6 Z^5$$
$$- \frac{53}{6} t_4^4 Z^6 - \frac{10}{7} t_4^2 Z^7 + \frac{Z^8}{4},$$

where

$$P(t_2,t_3,t_4,Z) := Z^3 - 12t_4^4 Z - 11t_4^6 + \frac{27}{4}t_3^2 t_4 - t_2 = 0. \tag{6.44}$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{4}{5}t_2\partial_{t_2} + \frac{1}{3}t_3\partial_{t_3} + \frac{2}{15}t_4\partial_{t_4},$$

the unity vector field is $e(t)=\partial_{t_1}$, and the charge is $d=\frac{13}{15}$. The intersection form (2.6) is then given by

$$\begin{aligned} g^{11}(t) =& \frac{1}{6}\Big(-32Zt_2^2 + 567Z^2t_3^4 + 1440Zt_2t_3^2t_4 - 112t_2^2t_4^2 - 10530Zt_3^4t_4^2 - 20160t_2t_3^2t_4^3 \\ &- 2784Z^2t_2t_4^4 + 20979t_3^4t_4^4 + 27864Z^2t_3^2t_4^5 + 2208Zt_2t_4^6 - 23976Zt_3^2t_4^7 \\ &+40416t_2t_4^8 + 648000t_3^2t_4^9 - 7776Z^2t_4^{10} - 186624Zt_4^{12} + 182736t_4^{14}\Big), \end{aligned} \tag{6.45}$$

$$\begin{aligned} g^{12}(t) =& \frac{3}{4}t_3\Big(-20Z^2t_2 + 81t_3^4 + 360Z^2t_3^2t_4 + 300Zt_2t_4^2 - 2925Zt_3^2t_4^3 - 1840t_2t_4^4 \\ &+1170t_3^2t_4^5 + 1980Z^2t_4^6 - 1980Zt_4^8 + 18720t_4^{10}\Big), \end{aligned} \tag{6.46}$$

$$\begin{aligned} g^{22}(t) =& -\frac{3}{10}\Big(99t_2t_3^2 + 44Z^2t_2t_4 - 891t_3^4t_4 - 1287Z^2t_3^2t_4^2 - 220Zt_2t_4^3 + 5445Zt_3^2t_4^4 \\ &+528t_2t_4^5 + 5841t_3^2t_4^6 - 396Z^2t_4^7 + 396Zt_4^9 - 3744t_4^{11}\Big), \end{aligned} \tag{6.47}$$

$$g^{13}(t) = -9Z^2t_3^2 - 8Zt_2t_4 + 90Zt_3^2t_4^2 - 8t_2t_4^3 - 468t_3^2t_4^4 - 72Z^2t_4^5 + 72Zt_4^7 + 792t_4^9, \tag{6.48}$$

$$g^{23}(t) = \frac{1}{4}\Big(4t_1 - 27t_3^3 - 60Z^2t_3t_4 + 240Zt_3t_4^3 - 240t_3t_4^5\Big), \tag{6.49}$$

$$g^{33}(t) = \frac{8}{27}\Big(2Z^2 - 8t_4^2Z - 19t_4^4\Big), \tag{6.50}$$

$$g^{14}(t) = t_1, \qquad g^{24}(t) = \frac{4}{5}t_2, \qquad g^{34}(t) = \frac{1}{3}t_3, \qquad g^{44}(t) = \frac{2}{15}t_4. \tag{6.51}$$

We have that $\deg t_1(x)=15$, $\deg t_2(x)=12$, $\deg t_3(x)=5$, $\deg t_4(x)=2$ and $\deg Z(x)=4$.

Proposition 6.9. *Let $V_1=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z] \mid \deg p(x)=30\}$, let $V_2=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z] \mid \deg p(x)=20\}$ and let $V_3=\{p\in\mathbb{C}[t_1,t_2,t_3,t_4,Z] \mid \deg p(x)=12\}$. The harmonic elements of V_1 are proportional to*

$$\begin{aligned} &5765760t_1^2 - 51891840Z^2t_2t_3^2 + 188107920t_1t_3^3 + 302837535t_3^6 + 30750720Zt_2^2t_4 \\ &\quad + 350269920Z^2t_3^4t_4 - 155675520Zt_2t_3^2t_4^2 + 274743040t_2^2t_4^3 - 350269920Zt_3^4t_4^3 \\ &\quad + 3090653280t_2t_3^2t_4^4 + 337438464Z^2t_2t_4^5 + 558122400t_1t_3t_4^5 + 33046299000t_3^4t_4^5 \\ &\quad - 1810683072Z^2t_3^2t_4^6 + 96349440Zt_2t_4^7 - 1117385280Zt_3^2t_4^8 + 1662275328t_2t_4^9 \\ &\quad + 185379977376t_3^2t_4^{10} + 1391157504Z^2t_4^{11} + 11062884096Zt_4^{13} - 21529753344t_4^{15}, \end{aligned}$$

the harmonic elements of V_2 are proportional to

$$\begin{aligned} &47872Z^2t_2 - 269280t_1t_3 - 984555t_3^4 - 323136Z^2t_3^2t_4 - 239360Zt_2t_4^2 + 1615680Zt_3^2t_4^3 \\ &\quad - 3512256t_2t_4^4 - 25208172t_3^2t_4^5 - 430848Z^2t_4^6 + 430848Zt_4^8 - 35274672t_4^{10}, \end{aligned}$$

and the harmonic elements of V_3 are proportional to

$$112t_2 + 2079t_3^2t_4 + 5940t_4^6.$$

Proof. Using Proposition 2.1 we can directly calculate

$$\begin{aligned} \Delta(t_1) =& \frac{28t_3}{4t_2 - 27t_3^2t_4 - 20t_4^6}\Big(-4Z^2t_2 + 45Z^2t_3^2t_4 + 48Zt_2t_4^2 - 360Zt_3^2t_4^3 \\ &-208t_2t_4^4 + 1260t_3^2t_4^5 + 100Z^2t_4^6 - 400Zt_4^8 + 400t_4^{10}\Big), \end{aligned} \tag{6.52}$$

$$\Delta(t_2) = -\frac{11}{10\left(4t_2 - 27t_3^2t_4 - 20t_4^6\right)}\Big(108t_2t_3^2 + 80Z^2t_2t_4 - 729t_3^4t_4$$

$$- 1260Z^2t_3^2t_4^2 - 320Zt_2t_4^3 + 3600Zt_3^2t_4^4 + 320t_2t_4^5 + 3060t_3^2t_4^6 \\ -400Z^2t_4^7 + 1600Zt_4^9 - 1600t_4^{11}\Big), \tag{6.53}$$

$$\Delta(t_3) = \frac{64t_3t_4\left(2t_4^2+Z\right)\left(4t_4^2-Z\right)}{3\left(4t_2-27t_3^2t_4-20t_4^6\right)}, \tag{6.54}$$

$$\Delta(t_4) = \frac{4}{15}. \tag{6.55}$$

A general element of V_1 is of the form

$$a_1t_1^2 + a_2t_1t_3^3 + a_3t_1t_3t_4^5 + a_4t_1t_3t_4^3Z + a_5t_1t_3t_4Z^2 + a_6t_2^2t_4^3 + a_7t_2^2t_4Z \\ + a_8t_2t_3^2t_4^4 + a_9t_2t_3^2t_4^2Z + a_{10}t_2t_3^2Z^2 + a_{11}t_2t_4^9 + a_{12}t_2t_4^7Z + a_{13}t_2t_4^5Z^2 \\ + a_{14}t_3^6 + a_{15}t_3^4t_4^5 + a_{16}t_3^4t_4^3Z + a_{17}t_3^4t_4Z^2 + a_{18}t_3^2t_4^{10} + a_{19}t_3^2t_4^8Z \\ + a_{20}t_3^2t_4^6Z^2 + a_{21}t_4^{15} + a_{22}t_4^{13}Z + a_{23}t_4^{11}Z^2, \tag{6.56}$$

where $a_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.56) using Proposition 2.1 and formulas (6.52)–(6.55) we find that the only harmonic elements of V_1 are as claimed. A general element of V_2 has the form

$$b_1t_1t_3 + b_2t_2t_4^4 + b_3t_2t_4^2Z + b_4t_2Z^2 + b_5t_3^4 + b_6t_3^2t_4^5 \\ + b_7t_3^2t_4^3Z + b_8t_3^2t_4Z^2 + b_9t_4^{10} + b_{10}t_4^8Z + b_{11}t_4^6Z^2, \tag{6.57}$$

where $b_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.57) using Proposition 2.1 and formulas (6.52)–(6.55) we find that the only harmonic elements of V_2 are as claimed. A general element of V_3 has the form

$$c_1t_2 + c_2t_3^2t_4 + c_3t_4^6 + c_4t_4^4Z + c_5t_4^2Z^2, \tag{6.58}$$

where $c_i \in \mathbb{C}$. By calculating the Laplacian of this general element (6.58) using Proposition 2.1 and formulas (6.52)–(6.55) we find that the only harmonic elements of V_3 are as claimed. □

Theorem 6.10. *We have the following relations*

$$y_1 = 2^83^25^9\Big(5760t_1^2 - 51840Z^2t_2t_3^2 + 187920t_1t_3^3 + 302535t_3^6 + 30720Zt_2^2t_4 \\ + 349920Z^2t_3^4t_4 - 155520Zt_2t_3^2t_4^2 + 266240t_2^2t_4^3 - 349920Zt_3^4t_4^3 + 2782080t_2t_3^2t_4^4 \\ - 393216Z^2t_2t_4^5 + 4665600t_1t_3t_4^5 + 45198000t_3^4t_4^5 + 3120768Z^2t_3^2t_4^6 \\ + 3747840Zt_2t_4^7 - 25764480Zt_3^2t_4^8 + 75859968t_2t_4^9 + 952477056t_3^2t_4^{10} \\ +7962624Z^2t_4^{11} + 4478976Zt_4^{13} + 4105230336t_4^{15}\Big), \tag{6.59}$$

$$y_2 = -2^43^25^6\Big(256Z^2t_2 - 1440t_1t_3 - 5265t_3^4 - 1728Z^2t_3^2t_4 - 1280Zt_2t_4^2 \\ +8640Zt_3^2t_4^3 - 31488t_2t_4^4 - 370656t_3^2t_4^5 - 2304Z^2t_4^6 + 2304Zt_4^8 - 2566656t_4^{10}\Big), \tag{6.60}$$

$$y_3 = -2^35^43\Big(16t_2 + 297t_3^2t_4 + 4320t_4^6\Big), \tag{6.61}$$

$$y_4 = 30t_4. \tag{6.62}$$

Proof. Note that $Y_4 = \frac{1}{8}y_4 = \frac{15}{4}t_4$. We now equate Y_1, Y_2 and Y_3 given by relations (6.12)–(6.14) with general harmonic elements of V_1, V_2 and V_3, respectively, given by Proposition 6.9. We then rearrange these equations to find y_i in terms of t_j and Z. We find

$$y_1 = \frac{2^{12}3^{17}5^{14}13}{7^311}t_4^{15} + \frac{a}{2^83\cdot 11\cdot 59\cdot 677}\Big(5765760t_1^2 - 51891840Z^2t_2t_3^2 \\ + 188107920t_1t_3^3 + 302837535t_3^6 + 30750720Zt_2^2t_4 + 350269920Z^2t_3^4t_4 \\ - 155675520Zt_2t_3^2t_4^2 + 274743040t_2^2t_4^3 - 350269920Zt_3^4t_4^3 + 3090653280t_2t_3^2t_4^4 \\ + 337438464Z^2t_2t_4^5 + 558122400t_1t_3t_4^5 + 33046299000t_3^4t_4^5 - 1810683072Z^2t_3^2t_4^6 \\ + 96349440Zt_2t_4^7 - 1117385280Zt_3^2t_4^8 + 1662275328t_2t_4^9 + 185379977376t_3^2t_4^{10}$$

$$+1391157504Z^2t_4^{11}+11062884096Zt_4^{13}-21529753344t_4^{15}\Big)$$
$$-\frac{3^2 5^4 641 b}{7\cdot 13\cdot 29\cdot 8447}t_4^5\Big(47872Z^2t_2-269280t_1t_3-984555t_3^4-323136Z^2t_3^2t_4$$
$$-239360Zt_2t_4^2+1615680Zt_3^2t_4^3-3512256t_2t_4^4-25208172t_3^2t_4^5-430848Z^2t_4^6$$
$$+430848Zt_4^8-35274672t_4^{10}\Big)-\frac{2^5 3^5 5^8 2251 c}{7^2 11^2 17}t_4^9\left(112t_2+2079t_3^2t_4+5940t_4^6\right)$$
$$-\frac{5c^2}{2^2 3^4 11^2 7}t_4^3\left(112t_2+2079t_3^2t_4+5940t_4^6\right)^2, \tag{6.63}$$
$$y_2=\frac{2^9 3^{10} 5^{10}}{77}t_4^{10}+\frac{b}{2^4 3^2 29\cdot 8447}\Big(-47872Z^2t_2+269280t_1t_3+984555t_3^4$$
$$+323136Z^2t_3^2t_4+239360Zt_2t_4^2-1615680Zt_3^2t_4^3+3512256t_2t_4^4+25208172t_3^2t_4^5$$
$$+430848Z^2t_4^6-430848Zt_4^8+35274672t_4^{10}\Big)-\frac{3750c}{187}t_4^4\left(112t_2+2079t_3^2t_4+5940t_4^6\right), \tag{6.64}$$
$$y_3=-\frac{2^5 3^6 5^6}{7}t_4^6+\frac{c}{5940}\left(112t_2+2079t_3^2t_4+5940t_4^6\right), \tag{6.65}$$
$$y_4=30t_4, \tag{6.66}$$

where $a,b,c\in\mathbb{C}$. In order to find a,b and c we perform steps 5–7 from Section 2.4. That is, we transform the intersection form (6.45)–(6.51) into y coordinates by applying formulas (6.63)–(6.66) and compare it with the expression given by Lemma 6.1. We find that

$$a=\frac{2^{16}3^3 5^9 59\cdot 677}{91},\qquad b=\frac{2^8 3^4 5^6 29\cdot 8447}{11\cdot 17},\qquad c=-\frac{2^5 3^4 5^5 11}{7},$$

which implies the statement. $\square$

Proposition 6.11. *The derivatives* $\frac{\partial y_i}{\partial t_j}\in\mathbb{C}[t_1,t_2,t_3,t_4,Z]$.

Proof is similar to the one for Proposition 3.5.

Proposition 6.12. *We have that*

$$\det(g^{ij}(t))=\frac{c\prod_{\alpha\in R_{H_4}}(\alpha,x)}{Q(t,Z)^2},$$

where $c=2^{72}5$ *and*

$$Q(t,Z)=3^7 5^{20}\Big(4t_1^2+36Z^2t_2t_3^2-72t_1t_3^3+324t_3^6+60Z^2t_1t_3t_4-783Z^2t_3^4t_4$$
$$+1080Zt_2t_3^2t_4^2-240Zt_1t_3t_4^3-5130Zt_3^4t_4^3+4932t_2t_3^2t_4^4-1380t_1t_3t_4^5$$
$$-20871t_3^4t_4^5+11016Z^2t_3^2t_4^6-11016Zt_3^2t_4^8-194076t_3^2t_4^{10}\Big).$$

By Proposition 2.6, we need only find $\det\left(\frac{\partial y_i}{\partial t_j}\right)$. It can be calculated by Theorem 6.10, which leads to Proposition 6.12.

In the next statement we express flat coordinates t_i via basic invariants y_j and Z, which is an inversion of the formulas from Theorem 6.10.

Theorem 6.13. *We have the following relations:*

$$t_1=\frac{1}{2^8 3^{11} 5^9\sqrt{3}\,y_4^{\frac{3}{2}}\sqrt{-2^7 3^6 5^6 Z^3-3^5 5^2 y_3+2^5 3^3 5^2 y_4^4 Z-518y_4^6}}$$
$$\Big(-2^{14}3^{12}5^{12}13\,Z^6-2^8 3^{11}5^8 13\,Z^3y_3-3^{10}5^4 13\,y_3^2+2^{12}3^{14}5^{11}Z^5y_4^2$$
$$+3^{12}5^5\,y_2y_4^2-2^{10}3^9 5^8 571\,Z^4y_4^4+2^6 3^8 5^4 13\,Zy_3y_4^4$$
$$-2^8 3^6 5^6 11\cdot 2269\,Z^3y_4^6+2^3 3^5 5^2 17\cdot 19\cdot 23\,y_3y_4^6+2^9 3^6 5^4 199\,Z^2y_4^8$$

$$+2^6 3^3 5^2 22259\, Z y_4^{10} + 2^4 7 \cdot 180569\, y_4^{12}\Big), \tag{6.67}$$

$$t_2 = \frac{1}{2^6 3^7 5^7}\left(2^6 3^6 5^6 11\, Z^3 - 3^5 5^2 2\, y_3 - 2^4 3^3 5^2 11\, y_4^4 Z - 1201 y_4^6\right), \tag{6.68}$$

$$t_3 = \frac{1}{3^4 5^3 2\sqrt{3}}\sqrt{\frac{-2^7 3^6 5^6 Z^3 - 3^5 5^2 y_3 + 2^5 3^3 5^2 y_4^4 Z - 518 y_4^6}{y_4}}, \tag{6.69}$$

$$t_4 = \frac{1}{30} y_4, \tag{6.70}$$

where Z satisfies the equation

$$\begin{aligned}
&2^{28}3^{24}5^{24} Z^{12} + 2^{23}3^{23}5^{20} Z^9 y_3 + 2^{15}3^{23}5^{16} Z^6 y_3^2 + 2^9 3^{21} 5^{12} Z^3 y_3^3 \\
&+ 3^{20}5^8 y_3^4 + 2^{27}3^{25}5^{22} Z^{11} y_4^2 - 2^{15}3^{23}5^{16} Z^6 y_2 y_4^2 + 2^{21}3^{24}5^{18} Z^8 y_3 y_4^2 \\
&- 2^9 3^{22} 5^{12} Z^3 y_2 y_3 y_4^2 + 2^{13}3^{23}5^{14} Z^5 y_3^2 y_4^2 - 3^{21}5^8 2\, y_2 y_3^2 y_4^2 \\
&- 2^8 3^{22} 5^{12} Z^3 y_1 y_4^3 - 3^{21}5^8 2\, y_1 y_3 y_4^3 + 2^{24}3^{21}5^{20} 353\, Z^{10} y_4^4 \\
&- 2^{13}3^{23}5^{14} Z^5 y_2 y_4^4 - 3^{21}5^8 y_2^2 y_4^4 + 2^{20}3^{21}5^{16} 13\, Z^7 y_3 y_4^4 \\
&- 2^{11}3^{20}5^{12} 23\, Z^4 y_3^2 y_4^4 - 2^7 3^{18} 5^8 Z y_3^3 y_4^4 + 2^{25}3^{18}5^{19} 223\, Z^9 y_4^6 \\
&+ 2^{11}3^{20}5^{12} 23\, Z^4 y_2 y_4^6 + 2^{18}3^{18}5^{14} 53\, Z^6 y_3 y_4^6 + 2^7 3^{19} 5^8 Z y_2 y_3 y_4^6 \\
&- 2^9 3^{17} 5^{10} 17 \cdot 41\, Z^3 y_3^2 y_4^6 + 2^3 3^{15} 5^6 7 \cdot 37\, y_3^3 y_4^6 + 2^6 3^{19} 5^8 Z y_1 y_4^7 \\
&+ 2^{20}3^{19}5^{17} 7 \cdot 19\, Z^8 y_4^8 + 2^9 3^{17} 5^{10} 17 \cdot 41\, Z^3 y_2 y_4^8 - 2^{15}3^{19}5^{12} 59\, Z^5 y_3 y_4^8 \\
&- 2^3 3^{16} 5^6 7 \cdot 37\, y_2 y_3 y_4^8 + 2^{10}3^{17}5^8 7\, Z^2 y_3^2 y_4^8 - 2^2 3^{16} 5^6 7 \cdot 37\, y_1 y_4^9 \\
&- 2^{19}3^{16}5^{15} 2029\, Z^7 y_4^{10} - 2^{10}3^{17}5^8 7\, Z^2 y_2 y_4^{10} - 2^{13}3^{15}5^{10} 7 \cdot 191\, Z^4 y_3 y_4^{10} \\
&+ 2^7 3^{14} 5^6 7^2 13\, Z y_3^2 y_4^{10} - 2^{16}3^{13}5^{15} 2213\, Z^6 y_4^{12} - 2^7 3^{14} 5^6 7^2 13\, Z y_2 y_4^{12} \\
&+ 2^{16}3^{12}5^8 7^2 11\, Z^3 y_3 y_4^{12} - 2^8 3^{11} 5^4 7^3 y_3^2 y_4^{12} + 2^{16}3^{13}5^{11} 7 \cdot 11 \cdot 43\, Z^5 y_4^{14} \\
&+ 2^8 3^{11} 5^4 7^3 y_2 y_4^{14} + 2^{12}3^{12}5^6 7^3 Z^2 y_3 y_4^{14} + 2^{14}3^{10}5^9 7^2 1171\, Z^4 y_4^{16} \\
&- 2^{11}3^9 5^4 7^4 Z y_3 y_4^{16} - 2^{16}3^6 5^7 7^3 163\, Z^3 y_4^{18} + 2^9 3^5 5^2 7^5 y_3 y_4^{18} \\
&- 2^{12}3^7 5^4 7^4 47\, Z^2 y_4^{20} + 2^{12}3^3 5^2 7^5 41\, Z y_4^{22} - 2^{10}7^6 11\, y_4^{24} = 0,
\end{aligned} \tag{6.71}$$

and y_i are given by relations (6.1)–(6.11).

Proof. Formula (6.70) follows immediately from Theorem 6.10. Using relations (6.44) and (6.60) we see that

$$\begin{aligned}
t_1 = &-\frac{1}{2^9 3^{11} 5^{11} t_3}\Big(-2^{12}3^9 5^{10} Z^5 + 2^4 3^{13} 5^{11} 13 t_3^4 - 3^7 5^4 y_2 + 2^{10}3^7 5^9 y_4^2 Z^4 + 2^8 3^8 5^7 y_4^4 Z^3 \\
&+ 2^4 3^{10} 5^7 y_4^5 t_3^2 - 2^9 3^3 5^5 y_4^6 Z^2 - 2^6 5^3 3 \cdot 7 \cdot 11 y_4^8 Z + 2^2 7^2 59 y_4^{10}\Big),
\end{aligned} \tag{6.72}$$

$$t_2 = \frac{1}{2^6 3^6 5^6}\left(2^6 3^6 5^6 Z^3 + 2^3 3^8 5^5 t_3^2 y_4 - 2^4 3^3 5^2 y_4^4 Z - 11 y_4^6\right). \tag{6.73}$$

We also have relation (6.61), which together with relations (6.72) and (6.73) gives formulas (6.67)–(6.69). Finally, by substituting relations (6.67)–(6.70) into formula (6.59) we get the formula (6.71). □

Proposition 6.14. *The unity vector field $e = \partial_{t_1}$ in the y coordinates has the form*

$$e(y) = 2^9 3^4 5^7 t_3 \partial_{y_2} - 2^{12} 3^4 5^{10}\left(16 t_1 + 261 t_3^3 + 6480 t_3 t_4^5\right)\partial_{y_1}.$$

Proof. We have that

$$e = \partial_{t_1} = \frac{\partial y_\alpha}{\partial t_1}\partial_{y_\alpha},$$

which gives the statement by applying the relations from Theorem 6.10. □

6.3. $H_4(3)$ example

The prepotential for $H_4(3)$ is

$$\begin{aligned}F(t) = &\frac{2016569088}{43793750}t_4^{13} + \frac{7929073152}{43793750}t_4^{10}t_3 + \frac{11291664384}{43793750}t_4^7t_3^2 - \frac{6228045824}{43793750}t_4^4t_3^3\\ &- \frac{1582124544}{43793750}t_4t_3^4 + t_1t_2t_3 + \frac{1}{2}t_1^2t_4 - \frac{256}{9375}t_4^3\left(t_4^3 - t_3\right)\left(779t_4^6 + 20532t_4^3t_3 - 18480t_3^2\right)\\ &+ \frac{32256}{3125}t_4^2\left(17t_4^3 - 10t_3\right)\left(t_4^3 - t_3\right)^2 - \frac{7168}{125}t_4\left(t_4^3 - t_3\right)^3 Z^3 + \frac{96}{5}t_4\left(t_4^3 - t_3\right)^2 Z^6\\ &- \frac{8}{2625}\left(1573t_4^9 - 27588t_4^6t_3 + 25536t_4^3t_3^2 - 2352t_3^3\right)Z^4 + \frac{544}{175}\left(t_4^3 - t_3\right)^2 Z^7\\ &- \frac{288}{125}t_4^2\left(17t_4^3 - 10t_3\right)\left(t_4^3 - t_3\right)Z^5 + \frac{9}{70}t_4^2\left(17t_4^3 - 10t_3\right)Z^8 + \frac{50}{1911}Z^{13}\\ &- \frac{15}{7}t_4\left(t_4^3 - t_3\right)Z^9 - \frac{10}{21}\left(t_4^3 - t_3\right)Z^{10} + \frac{125}{1568}t_4Z^{12},\end{aligned}$$

where

$$P(t_2, t_3, t_4, Z) := Z^4 - \frac{224}{25}\left(t_4^3 - t_3\right)Z + \frac{48}{25}t_4^4 + \frac{224}{25}t_4t_3 - t_2 = 0.$$

The Euler vector field is

$$E(t) = t_1\partial_{t_1} + \frac{2}{3}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{1}{6}t_4\partial_{t_4},$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{5}{6}$. The intersection form (2.6) is then given by

$$\begin{aligned}g^{11}(t) = &-\frac{2}{765625}\Big(4812500t_2^2t_3 - 10780000Zt_2t_3^2 + 36220800Z^2t_3^3 - 1203125Z^2t_2^2t_4\\ &+ 10010000Z^3t_2t_3t_4 + 36005200t_2t_3^2t_4 + 305220608Zt_3^3t_4 + 1375000Zt_2^2t_4^2\\ &+ 40040000Z^2t_2t_3t_4^2 - 193177600Z^3t_3^2t_4^2 - 1491331072t_3^3t_4^2 - 24234375t_2^2t_4^3\\ &- 43120000Zt_2t_3t_4^3 - 236297600Z^2t_3^2t_4^3 - 33110000Z^3t_2t_4^4 - 607006400t_2t_3t_4^4\\ &- 1372388864Zt_3^2t_4^4 + 36190000Z^2t_2t_4^5 + 260198400Z^3t_3t_4^5 + 1541281280t_3^2t_4^5\\ &- 13200000Zt_2t_4^6 - 395225600Z^2t_3t_4^6 + 119891200t_2t_4^7 + 2865967104Zt_3t_4^7\\ &+ 184307200Z^3t_4^8 - 5516836864t_3t_4^8 - 91660800Z^2t_4^9 - 1231515648Zt_4^{10}\\ &-5094508544t_4^{11}\Big),\end{aligned}$$

$$\begin{aligned}g^{12}(t) = &-\frac{8}{21875}\Big(3125Zt_2^2 - 45500Z^2t_2t_3 + 156800Z^3t_3^2 + 592704t_3^3 + 28125t_2^2t_4\\ &- 119000Zt_2t_3t_4 + 619360Z^2t_3^2t_4 + 45000Z^3t_2t_4^2 + 119700t_2t_3t_4^2\\ &+ 2847488Zt_3^2t_4^2 + 75500Z^2t_2t_4^3 - 851200Z^3t_3t_4^3 - 7902720t_3^2t_4^3\\ &- 66000Zt_2t_4^4 - 619360Z^2t_3t_4^4 - 673200t_2t_4^5 - 3351936Zt_3t_4^5\\ &+204800Z^3t_4^6 + 4163712t_3t_4^6 - 326400Z^2t_4^7 + 2147328Zt_4^8 - 4627456t_4^9\Big),\end{aligned}$$

$$\begin{aligned}g^{22}(t) = &-\frac{32}{9375}\Big(-1250Z^3t_2 + 6300t_2t_3 - 56448Zt_3^2 - 4375Z^2t_2t_4 + 30800Z^3t_3t_4\\ &+ 91728t_3^2t_4 - 5250Zt_2t_4^2 + 56840Z^2t_3t_4^2 + 3325t_2t_4^3 + 159936Zt_3t_4^3\\ &-17200Z^3t_4^4 - 369600t_3t_4^4 - 9240Z^2t_4^5 - 46368Zt_4^6 + 80672t_4^7\Big),\end{aligned}$$

$$\begin{aligned}g^{13}(t) = &\frac{1}{9800}\Big(3125t_2^2 - 22400Zt_2t_3 + 75264Z^2t_3^2 - 179200t_2t_3t_4 + 200704Zt_3^2t_4\\ &+ 28000Z^2t_2t_4^2 - 125440Z^3t_3t_4^2 - 551936t_3^2t_4^2 - 19200Zt_2t_4^3 - 261632Z^2t_3t_4^3\\ &+ 225600t_2t_4^4 + 215040Zt_3t_4^4 + 125440Z^3t_4^5 + 2867200t_3t_4^5 - 118272Z^2t_4^6\\ &+36864Zt_4^7 - 604160t_4^8\Big),\end{aligned}$$

$$g^{23}(t) = \frac{1}{175}\Big(175t_1 - 125Z^2t_2 + 560Z^3t_3 - 300Zt_2t_4 + 2128Z^2t_3t_4 - 1200t_2t_4^2$$
$$+2688Zt_3t_4^2 - 560Z^3t_4^3 - 4928t_3t_4^3 - 768Z^2t_4^4 + 576Zt_4^5 + 6400t_4^6\Big),$$
$$g^{33}(t) = \frac{1}{1568}\Big(250Zt_2 - 840Z^2t_3 + 625t_2t_4 - 2240Zt_3t_4 - 8960t_3t_4^2 + 840Z^2t_4^3$$
$$-480Zt_4^4 + 4512t_4^5\Big),$$
$$g^{14}(t) = t_1, \qquad g^{24}(t) = \frac{2}{3}t_2, \qquad g^{34}(t) = \frac{1}{2}t_3, \qquad g^{44}(t) = \frac{1}{6}t_4.$$

We have that $\deg t_1(x) = 12$, $\deg t_2(x) = 8$, $\deg t_3(x) = 6$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

Theorem 6.15. *We have the following relations*

$$y_1 = \frac{32768}{45}\Big(273437500Zt_1t_2^2 + 166015625Z^3t_2^3 - 7717500000t_1^2t_3 - 1684375000Z^2t_1t_2t_3$$
$$+ 2810937500t_2^3t_3 + 3430000000Z^3t_1t_3^2 + 10143000000Zt_2^2t_3^2 + 44562560000t_1t_3^3$$
$$+ 13088880000Z^2t_2t_3^3 - 98467891200Z^3t_3^4 - 1066905133056t_3^5 - 3691406250t_1t_2^2t_4$$
$$- 615234375Z^2t_2^3t_4 - 4900000000Zt_1t_2t_3t_4 + 656250000Z^3t_2^2t_3t_4$$
$$+ 15092000000Z^2t_1t_3^2t_4 + 17272500000t_2^2t_3^2t_4 - 75075840000Zt_2t_3^3t_4$$
$$- 117276364800Z^2t_3^4t_4 + 1289062500Zt_2^3t_4^2 - 85995000000t_1t_2t_3t_4^2$$
$$- 4633125000Z^2t_2^2t_3t_4^2 + 21952000000Zt_1t_3^2t_4^2 + 74499600000Z^3t_2t_3^2t_4^2$$
$$+ 1219784832000t_2t_3^3t_4^2 + 2152574484480Zt_3^4t_4^2 + 7717500000t_1^2t_4^3$$
$$+ 1684375000Z^2t_1t_2t_4^3 - 7557031250t_2^3t_4^3 - 6860000000Z^3t_1t_3t_4^3$$
$$- 17206000000Zt_2^2t_3t_4^3 - 1196603520000t_1t_3^2t_4^3 - 40481840000Z^2t_2t_3^2t_4^3$$
$$+ 27536588800Z^3t_3^3t_4^3 - 2854462464000t_3^4t_4^3 - 1050000000Zt_1t_2t_4^4$$
$$- 6075000000Z^3t_2^2t_4^4 - 11858000000Z^2t_1t_3t_4^4 - 237331500000t_2^2t_3t_4^4$$
$$- 361141760000Zt_2t_3^2t_4^4 + 1152093644800Z^2t_3^3t_4^4 + 166320000000t_1t_2t_4^5$$
$$+ 24714375000Z^2t_2^2t_4^5 + 9408000000Zt_1t_3t_4^5 - 163279200000Z^3t_2t_3t_4^5$$
$$- 4338329856000t_2t_3^2t_4^5 - 5691700510720Zt_3^3t_4^5 + 3430000000Z^3t_1t_4^6$$
$$- 59062000000Zt_2^2t_4^6 + 4552719360000t_1t_3t_4^6 + 290661840000Z^2t_2t_3t_4^6$$
$$- 604041267200Z^3t_3^2t_4^6 - 1144791531520t_3^3t_4^6 - 3234000000Z^2t_1t_4^7$$
$$+ 699646500000t_2^2t_4^7 + 1311466240000Zt_2t_3t_4^7 - 4066807449600Z^2t_3^2t_4^7$$
$$+ 1008000000Zt_1t_4^8 + 147735600000Z^3t_2t_4^8 + 15995339136000t_2t_3t_4^8$$
$$+ 7407908372480Zt_3^2t_4^8 - 3837646400000t_1t_4^9 - 481752880000Z^2t_2t_4^9$$
$$+ 1597027532800Z^3t_3t_4^9 + 130060834283520t_3^2t_4^9 + 105855360000Zt_2t_4^{10}$$
$$+ 2986145792000Z^2t_3t_4^{10} - 21627170112000t_2t_4^{11} - 10675438878720Zt_3t_4^{11}$$
$$- 1135868723200Z^3t_4^{12} - 398693684838400t_3t_4^{12} + 838213017600Z^2t_4^{13}$$
$$+2299551252480Zt_4^{14} + 318244776083456t_4^{15}\Big),$$

$$y_2 = \frac{256}{3}\Big(656250t_1t_2 + 109375Z^2t_2^2 - 910000Z^3t_2t_3 - 16503200t_2t_3^2$$
$$- 18966528Zt_3^3 - 500000Zt_2^2t_4 + 22344000t_1t_3t_4 + 1120000Z^2t_2t_3t_4$$
$$+ 1881600Z^3t_3^2t_4 + 66382848t_3^3t_4 + 3093750t_2^2t_4^2 + 8960000Zt_2t_3t_4^2$$
$$- 18816000Z^2t_3^2t_4^2 + 910000Z^3t_2t_4^3 + 56022400t_2t_3t_4^3 + 16758784Zt_3^2t_4^3$$
$$- 29484000t_1t_4^4 - 3500000Z^2t_2t_4^4 + 6137600Z^3t_3t_4^4 + 298923520t_3^2t_4^4$$

$$
\begin{aligned}
&+1920000Zt_2t_4^5+25446400Z^2t_3t_4^5-164559200t_2t_4^6-74102784Zt_3t_4^6\\
&-8019200Z^3t_4^7-3012660224t_3t_4^7+6316800Z^2t_4^8+17123328Zt_4^9\\
&+3641568256t_4^{10}\Big),\\
y_3 &= 64\left(875t_1+8624t_3^2+4125t_2t_4^2+66528t_3t_4^3-196992t_4^6\right),\\
y_4 &= 24t_4.
\end{aligned}
$$

Proposition 6.16. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

6.4. $H_4(4)$ example

The prepotential for $H_4(4)$ is

$$
\begin{aligned}
F(t) = &-\frac{25}{33}Z^{11}+\frac{1}{2}Z^{10}(-4t_3-5t_4)-\frac{125}{9}Z^9(t_3+t_4)(t_3+3t_4)-5Z^8\Big(8t_3^3+57t_3^2t_4\\
&+124t_3t_4^2+95t_4^3\Big)-\frac{35}{3}Z^6(t_3+t_4)(t_3+3t_4)\left(12t_3^3+123t_3^2t_4+336t_3t_4^2+305t_4^3\right)\\
&-10Z^7\left(5t_3^4+64t_3^3t_4+236t_3^2t_4^2+344t_3t_4^3+175t_4^4\right)+Z^5\Big(135t_3^6-840t_3^5t_4\\
&-17130t_3^4t_4^2-80920t_3^3t_4^3-174765t_3^2t_4^4-181952t_3t_4^5-74420t_4^6\Big)\\
&+5Zt_4(t_3+t_4)^2(t_3+3t_4)^2\Big(180t_3^5+2115t_3^4t_4+11760t_3^3t_4^2+30270t_3^2t_4^3\\
&+32316t_3t_4^4+9035t_4^5\Big)+\frac{5}{3}Z^3(t_3+t_4)(t_3+3t_4)\Big(-45t_3^6+1260t_3^5t_4+13185t_3^4t_4^2\\
&+42360t_3^3t_4^3+58125t_3^2t_4^4+37788t_3t_4^5+15815t_4^6\Big)-100Z^4\Big(-6t_3^7-87t_3^6t_4\\
&-450t_3^5t_4^2-1015t_3^4t_4^3-650t_3^3t_4^4+1283t_3^2t_4^5+2450t_3t_4^6+1195t_4^7\Big)\\
&-\frac{5}{6}Z^2\Big(540t_3^9+9315t_3^8t_4+73440t_3^7t_4^2+356940t_3^6t_4^3+1252440t_3^5t_4^4\\
&+3321450t_3^4t_4^5+6239424t_3^3t_4^6+7356636t_3^2t_4^7+4563564t_3t_4^8+994315t_4^9\Big)\\
&+\frac{1}{198}\Big(198t_1t_2t_3+99t_1^2t_4-4455t_3^{10}t_4-178200t_3^9t_4^2-2569050t_3^8t_4^3\\
&-21740400t_3^7t_4^4-120561210t_3^6t_4^5-458678880t_3^5t_4^6-1191552120t_3^4t_4^7\\
&-2004227280t_3^3t_4^8-1955070535t_3^2t_4^9-858257224t_3t_4^{10}-45669270t_4^{11}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
P(t_2,t_3,t_4,Z) := {} & Z^5-t_2+15t_3^4t_4+120t_3^3t_4^2+530t_3^2t_4^3+1160t_3t_4^4+843t_4^5\\
&+10Z^3(t_3+t_4)(t_3+3t_4)-15Z(t_3+t_4)^2(t_3+3t_4)^2\\
&+20Z^2t_4(3t_3^2+12t_3t_4+13t_4^2)=0.
\end{aligned}
$$

The Euler vector field is

$$
E(t)=t_1\partial_{t_1}+t_2\partial_{t_2}+\frac{1}{5}t_3\partial_{t_3}+\frac{1}{5}t_4\partial_{t_4},
$$

the unity vector field is $e(t)=\partial_{t_1}$, and the charge is $d=\frac{4}{5}$. The intersection form (2.6) is then given by

$$
\begin{aligned}
g^{11}(t) = &-15\Big(-2Z^4t_2-24Z^3t_2t_3+116Z^2t_2t_3^2+192Zt_2t_3^3-956t_2t_3^4-3360Z^4t_3^5\\
&+12560Z^3t_3^6+2880Z^2t_3^7-14826Zt_3^8+108t_3^9-78Z^3t_2t_4+368Z^2t_2t_3t_4\\
&+1584Zt_2t_3^2t_4-4096t_2t_3^3t_4-48360Z^4t_3^4t_4+106920Z^3t_3^5t_4+104472Z^2t_3^6t_4
\end{aligned}
$$

$$
\begin{aligned}
&- 194592Zt_3^7t_4 + 17013t_3^8t_4 + 276Z^2t_2t_4^2 + 2976Zt_2t_3t_4^2 + 2824t_2t_3^2t_4^2\\
&- 238560Z^4t_3^3t_4^2 + 221106Z^3t_3^4t_4^2 + 806976Z^2t_3^5t_4^2 - 962208Zt_3^6t_4^2\\
&+ 199488t_3^7t_4^2 + 1048Zt_2t_4^3 + 27264t_2t_3t_4^3 - 526312Z^4t_3^2t_4^3 - 417856Z^3t_3^3t_4^3\\
&+ 2293152Z^2t_3^4t_4^3 - 1907424Zt_3^5t_4^3 + 1003828t_3^6t_4^3 + 24108t_2t_4^4\\
&- 527680Z^4t_3t_4^4 - 2204820Z^3t_3^2t_4^4 + 1151040Z^2t_3^3t_4^4 + 655860Zt_3^4t_4^4\\
&+ 2033688t_3^5t_4^4 - 193800Z^4t_4^5 - 2871240Z^3t_3t_4^5 - 6809880Z^2t_3^2t_4^5\\
&+ 10362720Zt_3^3t_4^5 - 3485610t_3^4t_4^5 - 1183430Z^3t_4^6 - 13507200Z^2t_3t_4^6\\
&+ 20695840Zt_3^2t_4^6 - 29726240t_3^3t_4^6 - 7758720Z^2t_4^7 + 19063840Zt_3t_4^7\\
&- 68995740t_3^2t_4^7 + 7219830Zt_4^8 - 73570020t_3t_4^8 - 30991507t_4^9\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{12}(t) = &- 5\Big(-4Z^4t_2 - 18Z^3t_2t_3 + 184Z^2t_2t_3^2 + 144Zt_2t_3^3 - 1792t_2t_3^4\\
&- 2520Z^4t_3^5 + 21760Z^3t_3^6 + 2160Z^2t_3^7 - 26880Zt_3^8 + 81t_3^9\\
&- 72Z^3t_2t_4 + 664Z^2t_2t_3t_4 + 1968Zt_2t_3^2t_4 - 11672t_2t_3^3t_4\\
&- 49920Z^4t_3^4t_4 + 228270Z^3t_3^5t_4 + 148320Z^2t_3^6t_4 - 398112Zt_3^7t_4\\
&+ 29796t_3^8t_4 + 552Z^2t_2t_4^2 + 5352Zt_2t_3t_4^2 - 18832t_2t_3^2t_4^2 - 288120Z^4t_3^3t_4^2\\
&+ 813000Z^3t_3^4t_4^2 + 1444824Z^2t_3^5t_4^2 - 2417184Zt_3^6t_4^2 + 422196t_3^7t_4^2\\
&+ 3536Zt_2t_4^3 + 5448t_2t_3t_4^3 - 708320Z^4t_3^2t_4^3 + 915172Z^3t_3^3t_4^3\\
&+ 5680992Z^2t_3^4t_4^3 - 7655496Zt_3^5t_4^3 + 2785376t_3^6t_4^3 + 20448t_2t_4^4\\
&- 776504Z^4t_3t_4^4 - 881952Z^3t_3^2t_4^4 + 10163184Z^2t_3^3t_4^4 - 13244448Zt_3^4t_4^4\\
&+ 10650846t_3^5t_4^4 - 307680Z^4t_4^5 - 2577210Z^3t_3t_4^5 + 6008160Z^2t_3^2t_4^5\\
&- 11434800Zt_3^3t_4^5 + 23025000t_3^4t_4^5 - 1401160Z^3t_4^6 - 4244760Z^2t_3t_4^6\\
&- 1991200Zt_3^2t_4^6 + 20880740t_3^3t_4^6 - 5101920Z^2t_4^7 + 4450040Zt_3t_4^7\\
&- 12170880t_3^2t_4^7 + 2911200Zt_4^8 - 39000375t_3t_4^8 - 21754220t_4^9\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{22}(t) = 5\Big(&2Z^4t_2 - 92Z^2t_2t_3^2 + 896t_2t_3^4 - 10880Z^3t_3^6 + 13440Zt_3^8 + 18Z^3t_2t_4\\
&- 368Z^2t_2t_3t_4 - 552Zt_2t_3^2t_4 + 7168t_2t_3^3t_4 + 12360Z^4t_3^4t_4 - 130560Z^3t_3^5t_4\\
&- 59040Z^2t_3^6t_4 + 215040Zt_3^7t_4 - 14169t_3^8t_4 - 348Z^2t_2t_4^2 - 2208Zt_2t_3t_4^2\\
&+ 17408t_2t_3^2t_4^2 + 98880Z^4t_3^3t_4^2 - 570750Z^3t_3^4t_4^2 - 708480Z^2t_3^5t_4^2\\
&+ 1432368Zt_3^6t_4^2 - 226704t_3^7t_4^2 - 1984Zt_2t_4^3 + 12288t_2t_3t_4^3 + 284680Z^4t_3^2t_4^3\\
&- 1084400Z^3t_3^3t_4^3 - 3305976Z^2t_3^4t_4^3 + 5146176Zt_3^5t_4^3 - 1665604t_3^6t_4^3\\
&- 1512t_2t_4^4 + 347680Z^4t_3t_4^4 - 691908Z^3t_3^2t_4^4 - 7555008Z^2t_3^3t_4^4\\
&+ 10855464Zt_3^4t_4^4 - 7291824t_3^5t_4^4 + 150456Z^4t_4^5 + 337008Z^3t_3t_4^5\\
&- 8461536Z^2t_3^2t_4^5 + 13837632Zt_3^3t_4^5 - 19824414t_3^4t_4^5 + 416810Z^3t_4^6\\
&- 3634560Z^2t_3t_4^6 + 10628480Zt_3^2t_4^6 - 31855600t_3^3t_4^6 + 143640Z^2t_4^7\\
&+ 4582400Zt_3t_4^7 - 26339220t_3^2t_4^7 + 771720Zt_4^8 - 6866640t_3t_4^8 + 1889215t_4^9\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{13}(t) = &- 5\Big(t_2 - 4Z^4t_3 - 8Z^3t_3^2 + 24Z^2t_3^3 - 6Zt_3^4 - 6Z^4t_4 - 64Z^3t_3t_4 + 132Z^2t_3^2t_4\\
&- 96Zt_3^3t_4 + 60t_3^4t_4 - 104Z^3t_4^2 + 216Z^2t_3t_4^2 - 636Zt_3^2t_4^2 + 720t_3^3t_4^2 + 108Z^2t_4^3\\
&- 1856Zt_3t_4^3 + 2240t_3^2t_4^3 - 1686Zt_4^4 + 1600t_3t_4^4 - 1444t_4^5\Big),
\end{aligned}
$$

$$
\begin{aligned}
g^{23}(t) = &\,t_1 - 4t_2 + 10Z^4t_3 - 60Z^2t_3^3 + 20Z^4t_4 + 80Z^3t_3t_4 - 360Z^2t_3^2t_4 + 120Zt_3^3t_4\\
&+ 160Z^3t_4^2 - 660Z^2t_3t_4^2 + 720Zt_3^2t_4^2 - 600t_3^3t_4^2 - 360Z^2t_4^3 + 2120Zt_3t_4^3
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 3600t_3^2t_4^3 + 2320Zt_4^4 - 5600t_3t_4^4 - 1600t_4^5,\\
&g^{33}(t) = -\frac{1}{5}\left(2Z + 4t_3 + 5t_4\right),\\
&g^{14}(t) = t_1, \qquad g^{24}(t) = t_2, \qquad g^{34}(t) = \frac{1}{5}t_3, \qquad g^{44}(t) = \frac{1}{5}t_4.
\end{aligned}
$$

We have that $\deg t_1(x) = 10$, $\deg t_2(x) = 10$, $\deg t_3(x) = 2$, $\deg t_4(x) = 2$ and $\deg Z(x) = 2$.

Theorem 6.17. *We have the following relations*

$$
\begin{aligned}
y_1 = -\frac{2^{19}5^9}{3^{13}7}\Big(&42t_1^2t_2 - 168t_1t_2^2 + 168t_2^3 - 210Z^4t_1t_2t_3 + 420Z^4t_2^2t_3 + 1050Z^3t_2^2t_3^2\\
&+ 2100Z^2t_1t_2t_3^3 - 4200Z^2t_2^2t_3^3 + 27300Zt_2^2t_3^4 + 89250t_1t_2t_3^5 - 178500t_2^2t_3^5\\
&+ 228620Z^4t_2t_3^6 + 114240Z^3t_1t_3^7 - 228480Z^3t_2t_3^7 + 2465680Z^2t_2t_3^8\\
&- 141120Zt_1t_3^9 + 282240Zt_2t_3^9 - 5667634t_2t_3^{10} - 13482560Z^3t_3^{12}\\
&+ 42611520Zt_3^{14} - 420Z^4t_1t_2t_4 + 1050Z^4t_2^2t_4 - 1890Z^3t_1t_2t_3t_4 + 7980Z^3t_2^2t_3t_4\\
&+ 12600Z^2t_1t_2t_3^2t_4 - 20790Z^2t_2^2t_3^2t_4 - 104580Zt_1t_2t_3^3t_4 + 427560Zt_2^2t_3^3t_4\\
&- 164430t_1^2t_3^4t_4 + 1550220t_1t_2t_3^4t_4 - 2334570t_2^2t_3^4t_4 - 46620Z^4t_1t_3^5t_4\\
&+ 2836680Z^4t_2t_3^5t_4 + 1599360Z^3t_1t_3^6t_4 - 2240700Z^3t_2t_3^6t_4 - 15120Z^2t_1t_3^7t_4\\
&+ 39481120Z^2t_2t_3^7t_4 - 2540160Zt_1t_3^8t_4 + 38725680Zt_2t_3^8t_4 + 28513800t_1t_3^9t_4\\
&- 170380280t_2t_3^9t_4 + 16561860Z^4t_3^{10}t_4 - 323581440Z^3t_3^{11}t_4 - 64723680Z^2t_3^{12}t_4\\
&+ 1193122560Zt_3^{13}t_4 - 29292690t_3^{14}t_4 - 3780Z^3t_1t_2t_4^2 + 8820Z^3t_2^2t_4^2\\
&+ 23100Z^2t_1t_2t_3t_4^2 - 28560Z^2t_2^2t_3t_4^2 - 627480Zt_1t_2t_3^2t_4^2 + 1892100Zt_2^2t_3^2t_4^2\\
&- 1315440t_1^2t_3^3t_4^2 + 8585220t_1t_2t_3^3t_4^2 - 11043480t_2^2t_3^3t_4^2 - 466200Z^4t_1t_3^4t_4^2\\
&+ 15817200Z^4t_2t_3^4t_4^2 + 9705150Z^3t_1t_3^5t_4^2 - 7914060Z^3t_2t_3^5t_4^2 - 211680Z^2t_1t_3^6t_4^2\\
&+ 289891980Z^2t_2t_3^6t_4^2 - 17575740Zt_1t_3^7t_4^2 + 573477240Zt_2t_3^7t_4^2 + 513248400t_1t_3^8t_4^2\\
&- 1955803360t_2t_3^8t_4^2 + 331237200Z^4t_3^9t_4^2 - 3429078870Z^3t_3^{10}t_4^2 - 1553368320Z^2t_3^{11}t_4^2\\
&+ 14721448140Zt_3^{12}t_4^2 - 820195320t_3^{13}t_4^2 + 12600Z^2t_1t_2t_4^3 - 3290Z^2t_2^2t_4^3\\
&- 1244460Zt_1t_2t_3t_4^3 + 3290280Zt_2^2t_3t_4^3 - 3844260t_1^2t_3^2t_4^3 - 450Z^5t_2t_3^2t_4^3\\
&+ 21037800t_1t_2t_3^2t_4^3 - 24410930t_2^2t_3^2t_4^3 - 1773240Z^4t_1t_3^3t_4^3 + 49466480Z^4t_2t_3^3t_4^3\\
&+ 225Z^8t_3^4t_4^3 + 33077100Z^3t_1t_3^4t_4^3 - 11938395Z^3t_2t_3^4t_4^3 - 3845520Z^2t_1t_3^5t_4^3\\
&+ 1272065200Z^2t_2t_3^5t_4^3 - 2700Z^6t_3^6t_4^3 - 56395080Zt_1t_3^6t_4^3 + 3895595560Zt_2t_3^6t_4^3\\
&+ 4000050600t_1t_3^7t_4^3 - 11987148880t_2t_3^7t_4^3 + 2939165645Z^4t_3^8t_4^3\\
&- 21122966200Z^3t_3^9t_4^3 - 17379662880Z^2t_3^{10}t_4^3 + 105145262880Zt_3^{11}t_4^3\\
&- 6999178480t_3^{12}t_4^3 - 815640Zt_1t_2t_4^4 + 2080680Zt_2^2t_4^4 - 4853520t_1^2t_3t_4^4\\
&- 1800Z^5t_2t_3t_4^4 + 23828490t_1t_2t_3t_4^4 - 26013980t_2^2t_3t_4^4 - 3180240Z^4t_1t_3^2t_4^4\\
&+ 91783860Z^4t_2t_3^2t_4^4 + 1800Z^8t_3^3t_4^4 + 69522180Z^3t_1t_3^3t_4^4 - 11884320Z^3t_2t_3^3t_4^4\\
&+ 5400Z^7t_3^4t_4^4 - 29988000Z^2t_1t_3^4t_4^4 + 3617141020Z^2t_2t_3^4t_4^4 - 32400Z^6t_3^5t_4^4\\
&- 38118780Zt_1t_3^5t_4^4 + 15323659800Zt_2t_3^5t_4^4 - 25650Z^5t_3^6t_4^4 + 17678161200t_1t_3^6t_4^4\\
&- 44692143980t_2t_3^6t_4^4 + 15227879120Z^4t_3^7t_4^4 - 83502915015Z^3t_3^8t_4^4\\
&- 119765904000Z^2t_3^9t_4^4 + 469572887280Zt_3^{10}t_4^4 + 2620343040t_3^{11}t_4^4 - 2164806t_1^2t_4^5\\
&- 1350Z^5t_2t_4^5 + 10008684t_1t_2t_4^5 - 11210172t_2^2t_4^5 - 2653140Z^4t_1t_3t_4^5\\
&+ 96693240Z^4t_2t_3t_4^5 + 4950Z^8t_3^2t_4^5 + 92617560Z^3t_1t_3^2t_4^5 - 43361430Z^3t_2t_3^2t_4^5\\
&+ 43200Z^7t_3^3t_4^5 - 113047200Z^2t_1t_3^3t_4^5 + 6749433120Z^2t_2t_3^3t_4^5 - 121500Z^6t_3^4t_4^5
\end{aligned}
$$

$$
\begin{aligned}
&+357293160Zt_1t_3^4t_4^5+38092662600Zt_2t_3^4t_4^5-307800Z^5t_3^5t_4^5+49036806000t_1t_3^5t_4^5\\
&-108705602416t_2t_3^5t_4^5+50844400500Z^4t_3^6t_4^5-219447220080Z^3t_3^7t_4^5\\
&-570111869640Z^2t_3^8t_4^5+1249790969280Zt_3^9t_4^5+479633884512t_3^{10}t_4^5-803880Z^4t_1t_4^6\\
&+45585960Z^4t_2t_4^6+5400Z^8t_3t_4^6+74024790Z^3t_1t_3t_4^6-107328900Z^3t_2t_3t_4^6\\
&+126000Z^7t_3^2t_4^6-225570240Z^2t_1t_3^2t_4^6+7964410020Z^2t_2t_3^2t_4^6-108000Z^6t_3^3t_4^6\\
&+1399923420Zt_1t_3^3t_4^6+61840178440Zt_2t_3^3t_4^6-1415250Z^5t_3^4t_4^6+89821989600t_1t_3^4t_4^6\\
&-183359111760t_2t_3^4t_4^6+113517812720Z^4t_3^5t_4^6-379424004920Z^3t_3^6t_4^6\\
&-1998548319360Z^2t_3^7t_4^6+1114813850860Zt_3^8t_4^6+4204075665280t_3^9t_4^6+2025Z^8t_4^7\\
&+27351660Z^3t_1t_4^7-87125115Z^3t_2t_4^7+158400Z^7t_3t_4^7-231759360Z^2t_1t_3t_4^7\\
&+5344717200Z^2t_2t_3t_4^7+337500Z^6t_3^2t_4^7+2400609960Zt_1t_3^2t_4^7+65461451880Zt_2t_3^2t_4^7\\
&-3114000Z^5t_3^3t_4^7+113150026920t_1t_3^3t_4^7-234714455760t_2t_3^3t_4^7+169129871150Z^4t_3^4t_4^7\\
&-396171632160Z^3t_3^5t_4^7-5361270379440Z^2t_3^6t_4^7-5723934259520Zt_3^7t_4^7\\
&+20067909135750t_3^8t_4^7+70200Z^7t_4^8-97251840Z^2t_1t_4^8+1534267620Z^2t_2t_4^8\\
&+831600Z^6t_3t_4^8+2057206620Zt_1t_3t_4^8+42875701800Zt_2t_3t_4^8-3307950Z^5t_3^2t_4^8\\
&+103042663920t_1t_3^2t_4^8-249632019090t_2t_3^2t_4^8+160462051440Z^4t_3^3t_4^8\\
&-159576392790Z^3t_3^4t_4^8-11200275624000Z^2t_3^5t_4^8-29740325063280Zt_3^6t_4^8\\
&+62134558132320t_3^7t_4^8+535500Z^6t_4^9+712487160Zt_1t_4^9+13938177080Zt_2t_4^9\\
&-1452600Z^5t_3t_4^9+68111856120t_1t_3t_4^9-203506674360t_2t_3t_4^9+84058972000Z^4t_3^2t_4^9\\
&+151452210000Z^3t_3^3t_4^9-18131069931360Z^2t_3^4t_4^9-75535900424000Zt_3^5t_4^9\\
&+130861824336350t_3^6t_4^9-125550Z^5t_4^{10}+25090865520t_1t_4^{10}-86556812352t_2t_4^{10}\\
&+12789015040Z^4t_3t_4^{10}+241285828350Z^3t_3^2t_4^{10}-22060526165760Z^2t_3^3t_4^{10}\\
&-125256834868940Zt_3^4t_4^{10}+186992066838888t_3^5t_4^{10}-5145601875Z^4t_4^{11}\\
&+114867728760Z^3t_3t_4^{11}-18961639423440Z^2t_3^2t_4^{11}-142926704208480Zt_3^3t_4^{11}\\
&+171096028587180t_3^4t_4^{11}+11533193965Z^3t_4^{12}-10237991411520Z^2t_3t_4^{12}\\
&-110581088197120Zt_3^2t_4^{12}+78681803681760t_3^3t_4^{12}-2609003447640Z^2t_4^{13}\\
&-53312566785280Zt_3t_4^{13}-16250437730220t_3^2t_4^{13}-12255008872620Zt_4^{14}\\
&-46426601177520t_3t_4^{14}-22667569904962t_4^{15}\Big),
\end{aligned}
$$

$$
\begin{aligned}
y_2=-\frac{2^{12}5^6}{3^8}\Big(&t_1^2-4t_1t_2+3t_2^2-45Z^3t_2t_3^2-810Zt_2t_3^4-1026t_1t_3^5+2052t_2t_3^5\\
&-405Z^4t_3^6-4050Z^2t_3^8+10044t_3^{10}+10Z^4t_2t_4-180Z^3t_2t_3t_4-100Z^2t_2t_3^2t_4\\
&-6480Zt_2t_3^3t_4-10260t_1t_3^4t_4+12220t_2t_3^4t_4-4860Z^4t_3^5t_4-9085Z^3t_3^6t_4\\
&-64800Z^2t_3^7t_4+33450Zt_3^8t_4+200880t_3^9t_4-225Z^3t_2t_4^2-400Z^2t_2t_3t_4^2\\
&-16080Zt_2t_3^2t_4^2-30780t_1t_3^3t_4^2-4840t_2t_3^3t_4^2-22485Z^4t_3^4t_4^2-109020Z^3t_3^5t_4^2\\
&-511200Z^2t_3^6t_4^2+535200Zt_3^7t_4^2+1253640t_3^8t_4^2-300Z^2t_2t_4^3-12480Zt_2t_3t_4^3\\
&-20520t_1t_3^2t_4^3-163520t_2t_3^2t_4^3-50280Z^4t_3^3t_4^3-550695Z^3t_3^4t_4^3-2505600Z^2t_3^5t_4^3\\
&+3797100Zt_3^6t_4^3+773760t_3^7t_4^3+2410Zt_2t_4^4+46710t_1t_3t_4^4-380460t_2t_3t_4^4\\
&-55975Z^4t_3^2t_4^4-1498360Z^3t_3^3t_4^4-8058900Z^2t_3^4t_4^4+15594000Zt_3^5t_4^4\\
&-27933080t_3^6t_4^4+60588t_1t_4^5-287748t_2t_4^5-29020Z^4t_3t_4^5-2341695Z^3t_3^2t_4^5\\
&-17008800Z^2t_3^3t_4^5+39864360Zt_3^4t_4^5-162541344t_3^5t_4^5-6375Z^4t_4^6\\
&-2031420Z^3t_3t_4^6-22768320Z^2t_3^2t_4^6+62921280Zt_3^3t_4^6-465541320t_3^4t_4^6
\end{aligned}
$$

$$
\begin{aligned}
&- 778805Z^3t_4^7 - 17671680Z^2t_3t_4^7 + 56624500Zt_3^2t_4^7 - 804407360t_3^3t_4^7\\
&- 6115770Z^2t_4^8 + 23108560Zt_3t_4^8 - 886614660t_3^2t_4^8 + 1355790Zt_4^9\\
&-625055760t_3t_4^9 - 240893344t_4^{10}\Big),
\end{aligned}
$$

$$
\begin{aligned}
y_3 = &-\frac{2^7 5^4}{3^5}\Big(-Zt_2 - 6t_1t_3 + 12t_2t_3 + Z^4t_3^2 - 6Z^2t_3^4 - 297t_3^6 - 12t_1t_4 + 30t_2t_4\\
&+ 4Z^4t_3t_4 + 12Z^3t_3^2t_4 - 48Z^2t_3^3t_4 + 15Zt_3^4t_4 - 3564t_3^5t_4 + 3Z^4t_4^2 + 48Z^3t_3t_4^2\\
&- 132Z^2t_3^2t_4^2 + 120Zt_3^3t_4^2 - 22365t_3^4t_4^2 + 52Z^3t_4^3 - 144Z^2t_3t_4^3 + 530Zt_3^2t_4^3\\
&- 83880t_3^3t_4^3 - 54Z^2t_4^4 + 1160Zt_3t_4^4 - 166515t_3^2t_4^4 + 843Zt_4^5 - 147084t_3t_4^5\\
&-20311t_4^6\Big),\\
y_4 =& 20t_4.
\end{aligned}
$$

Proposition 6.18. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

6.5. $H_4(7)$ example

The prepotential for $H_4(7)$ is

$$
\begin{aligned}
F(t) = &t_1t_2t_3 + \frac{1}{2}t_1^2t_4 - \frac{4096}{135}t_3t_4\left(315t_3^3 + 32t_4^5\right) + \frac{32768}{1125}t_4^2\left(75t_3^3 + 2t_4^5\right)Z^2\\
&- \frac{32768}{225}t_3\left(75t_3^3 + 2t_4^5\right)Z^3 - \frac{16384}{5625}t_4\left(375t_3^3 + 14t_4^5\right)Z^5 + \frac{34816}{225}t_3t_4^4Z^6\\
&- \frac{116736}{175}t_3^2t_4^2Z^7 + \frac{256}{75}\left(220t_3^3 + 3t_4^5\right)Z^8 - \frac{118784}{945}t_3t_4^3Z^9 + \frac{44544}{175}t_3^2t_4Z^{10}\\
&- \frac{5632}{1575}t_4^4Z^{11} + \frac{832}{225}t_3t_4^2Z^{12} + \frac{17664}{455}t_3^2Z^{13} - \frac{352}{315}t_4^3Z^{14} + \frac{1568}{225}t_3t_4Z^{15}\\
&+ \frac{496}{2975}t_4^2Z^{17} + \frac{496}{945}t_3Z^{18} + \frac{71}{1575}t_4Z^{20} + \frac{16}{7245}Z^{23},
\end{aligned}
$$

where

$$
P(t_2, t_3, t_4, Z) := Z^8 + \frac{32}{5}t_4Z^5 + 64t_3Z^3 - \frac{64}{5}t_4^2Z^2 + 128t_3t_4 - t_2 = 0.
$$

The Euler vector field is

$$
E(t) = t_1\partial_{t_1} + \frac{4}{5}t_2\partial_{t_2} + \frac{1}{2}t_3\partial_{t_3} + \frac{3}{10}t_4\partial_{t_4},
$$

the unity vector field is $e(t) = \partial_{t_1}$, and the charge is $d = \frac{7}{10}$. The intersection form (2.6) is then given by

$$
\begin{aligned}
g^{11}(t) = &\frac{128}{703125}\Big(-1250Zt_2^2 + 11250Z^4t_2t_3 + 1850000Z^7t_3^2 + 153000000Z^2t_3^3\\
&+ 5375Z^6t_2t_4 + 758000Zt_2t_3t_4 + 13878000Z^4t_3^2t_4 - 32200Z^3t_2t_4^2\\
&+ 745600Z^6t_3t_4^2 - 117344000Zt_3^2t_4^2 - 754420t_2t_4^3 - 1829120Z^3t_3t_4^3\\
&- 819072Z^5t_4^4 - 121034240t_3t_4^4 + 1223424Z^2t_4^5\Big),\\
g^{12}(t) = &\frac{128}{9375}\Big(-125Z^7t_2 - 11500Z^2t_2t_3 + 16000Z^5t_3^2 + 5400000t_3^3\\
&- 450Z^4t_2t_4 + 28400Z^7t_3t_4 + 3272000Z^2t_3^2t_4 + 5280Zt_2t_4^2\\
&+ 202480Z^4t_3t_4^2 + 4448Z^6t_4^3 - 1155840Zt_3t_4^3 + 3584Z^3t_4^4 - 256000t_4^5\Big),\\
g^{22}(t) = &\frac{512}{1875}\Big(-25Z^5t_2 - 4250t_2t_3 - 40000Z^3t_3^2 - 295Z^2t_2t_4 + 2480Z^5t_3t_4
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 622000t_3^2t_4 + 788Z^7t_4^2 + 123760Z^2t_3t_4^2 + 5324Z^4t_4^3 - 20800Zt_4^4\Big),\\
g^{13}(t) &= \frac{1}{1875}\Big(25Z^4t_2 - 4000Z^7t_3 - 288000Z^2t_3^2 - 560Zt_2t_4 - 20160Z^4t_3t_4\\
&\quad + 544Z^6t_4^2 + 148480Zt_3t_4^2 - 2048Z^3t_4^3 - 51200t_4^4\Big),\\
g^{23}(t) &= \frac{1}{75}\Big(75t_1 + 20Z^2t_2 - 320Z^5t_3 - 38400t_3^2 - 128Z^7t_4 - 12160Z^2t_3t_4\\
&\quad - 704Z^4t_4^2 + 2560Zt_4^3\Big),\\
g^{33}(t) &= \frac{Z}{600}\Big(5Z^6 + 420Zt_3 + 35Z^3t_4 - 112t_4^2\Big),\\
g^{14}(t) &= t_1, \qquad g^{24}(t) = \frac{4}{5}t_2, \qquad g^{34}(t) = \frac{1}{2}t_3, \qquad g^{44}(t) = \frac{3}{10}t_4.
\end{aligned}
$$

We have that $\deg t_1(x) = \frac{20}{3}$, $\deg t_2(x) = \frac{16}{3}$, $\deg t_3(x) = \frac{10}{3}$, $\deg t_4(x) = 2$ and $\deg Z(x) = \frac{2}{3}$.

Theorem 6.19. *We have the following relations*

$$
\begin{aligned}
y_1 = -\frac{5^4}{3^{28}7}\Big(& 2^{3}3^{4}5^{18}7\, Z^7t_1^3t_2 - 2^{3}3^{7}5^{15}7\, Zt_1^2t_2^3 + 3^{3}5^{14}7\cdot 11\cdot 31\, Z^3t_1t_2^4\\
& - 5^{13}7\cdot 1831\, Z^5t_2^5 - 2^{4}3^{6}5^{20}7\, t_1^4t_3 + 2^{4}3^{4}5^{18}7^{3}\, Z^2t_1^3t_2t_3 + 3^{5}5^{15}2\cdot 7\cdot 967\, Z^4t_1^2t_2^2t_3\\
& - 3^{4}5^{14}2\cdot 7\cdot 2383\, Z^6t_1t_2^3t_3 + 2^{2}5^{13}7\cdot 19121\, t_2^5t_3 - 2^{8}3^{4}5^{18}7^{2}\, Z^5t_1^3t_3^2\\
& - 2^{6}3^{6}5^{15}7\cdot 283\, Z^7t_1^2t_2t_3^2 + 2^{5}3^{4}5^{16}7\cdot 11\cdot 17\, Zt_1t_2^3t_3^2 - 2^{6}5^{13}3\cdot 7\cdot 31547\, Z^3t_2^4t_3^2\\
& - 2^{12}3^{5}5^{18}7\cdot 61\, t_1^3t_3^3 - 2^{12}3^{5}5^{15}7\cdot 257\, Z^2t_1^2t_2t_3^3 - 2^{8}3^{3}5^{14}7\cdot 47\cdot 421\, Z^4t_1t_2^2t_3^3\\
& - 2^{7}5^{13}7\cdot 131\cdot 2647\, Z^6t_2^3t_3^3 - 2^{14}3^{5}5^{15}7\cdot 1153\, Z^5t_1^2t_3^4 - 2^{21}3^{3}5^{14}7\cdot 239\, Z^7t_1t_2t_3^4\\
& + 2^{11}5^{13}3\cdot 7\cdot 181\cdot 6269\, Zt_2^3t_3^4 - 2^{17}3^{6}5^{15}7\cdot 13\cdot 1291\, t_1^2t_3^5 + 2^{16}3^{5}5^{14}7\cdot 16447\, Z^2t_1t_2t_3^5\\
& - 2^{13}5^{14}3\cdot 7\cdot 3183137\, Z^4t_2^2t_3^5 - 2^{20}3^{5}5^{15}7\cdot 17\cdot 643\, Z^5t_1t_3^6 + 2^{18}5^{14}7\cdot 1747\cdot 1789\, Z^7t_2t_3^6\\
& - 2^{24}3^{5}5^{15}7\cdot 34649\, t_1t_3^7 + 2^{23}5^{14}7\cdot 2511151\, Z^2t_2t_3^7 - 2^{26}5^{15}7^{2}112687\, Z^5t_3^8\\
& - 2^{28}3^{3}5^{15}7\cdot 530807\, t_3^9 + 3^{5}5^{17}7^{3}2\, Z^4t_1^3t_2t_4 + 2^{2}3^{7}5^{14}7\cdot 67\, Z^6t_1^2t_2^2t_4\\
& - 3^{4}5^{13}7\cdot 41\, t_1t_2^4t_4 - 5^{13}2\cdot 7\cdot 4871\, Z^2t_2^5t_4 - 2^{6}3^{4}5^{17}7^{2}13\, Z^7t_1^3t_3t_4\\
& + 2^{5}3^{5}5^{14}7\cdot 23\cdot 199\, Zt_1^2t_2^2t_3t_4 - 2^{4}3^{3}5^{14}7\cdot 1091\, Z^3t_1t_2^3t_3t_4 - 2^{5}5^{13}7\cdot 11\cdot 3119\, Z^5t_2^4t_3t_4\\
& - 2^{11}3^{4}5^{18}7^{3}\, Z^2t_1^3t_3^2t_4 + 2^{7}3^{6}5^{14}7\cdot 10169\, Z^4t_1^2t_2t_3^2t_4 - 2^{7}3^{4}5^{13}7\cdot 53\cdot 401\, Z^6t_1t_2^2t_3^2t_4\\
& + 2^{6}5^{12}7\cdot 4272533\, t_2^4t_3^2t_4 - 2^{12}3^{5}5^{14}7\cdot 37\cdot 367\, Z^7t_1^2t_3^3t_4 - 2^{12}3^{3}5^{13}7\cdot 127\cdot 251\, Zt_1t_2^2t_3^3t_4\\
& - 2^{10}5^{12}7^{2}3\cdot 19\cdot 48157\, Z^3t_2^3t_3^3t_4 + 2^{19}3^{5}5^{15}7\cdot 257\, Z^2t_1^2t_3^4t_4 - 2^{13}3^{3}5^{13}7\cdot 1883993\, Z^4t_1t_2t_3^4t_4\\
& + 2^{13}5^{13}7^{2}3\cdot 397\cdot 557\, Z^6t_2^2t_3^4t_4 - 2^{18}3^{3}5^{14}7\cdot 67\cdot 293\, Z^7t_1t_3^5t_4 - 2^{17}5^{13}13^{2}3\cdot 7\cdot 22063\, Zt_2^2t_3^5t_4\\
& - 2^{23}3^{5}5^{14}7\cdot 16447\, Z^2t_1t_3^6t_4 + 2^{19}5^{13}73^{2}7\cdot 12347\, Z^4t_2t_3^6t_4 - 2^{24}5^{14}7^{2}3\cdot 31\cdot 6673\, Z^7t_3^7t_4\\
& - 2^{30}5^{14}7\cdot 2511151\, Z^2t_3^8t_4 - 2^{5}3^{7}5^{16}7^{2}\, Zt_1^3t_2t_4^2 + 2^{5}3^{7}5^{13}7\cdot 367\, Z^3t_1^2t_2^2t_4^2\\
& + 2^{5}3^{4}5^{13}7\cdot 2099\, Z^5t_1t_2^3t_4^2 - 2^{4}5^{12}7\cdot 11\cdot 3517\, Z^7t_2^4t_4^2 - 2^{7}3^{4}5^{16}7^{2}173\, Z^4t_1^3t_3t_4^2\\
& - 2^{6}3^{5}5^{13}7\cdot 19\cdot 29\cdot 59\, Z^6t_1^2t_2t_3t_4^2 + 2^{5}3^{4}5^{12}7\cdot 407717\, t_1t_2^3t_3t_4^2 - 2^{5}3^{3}5^{13}7\cdot 149\cdot 271\, Z^2t_2^4t_3t_4^2\\
& - 2^{11}3^{5}5^{13}7\cdot 19\cdot 3119\, Zt_1^2t_2t_3^2t_4^2 + 2^{9}3^{4}5^{12}7\cdot 61\cdot 48311\, Z^3t_1t_2^2t_3^2t_4^2 + 2^{9}5^{13}7\cdot 11\cdot 19\cdot 479\, Z^5t_2^3t_3^2t_4^2\\
& - 2^{13}3^{5}5^{13}7\cdot 311\cdot 439\, Z^4t_1^2t_3^3t_4^2 - 2^{14}3^{4}5^{13}7\cdot 61\cdot 4663\, Z^6t_1t_2t_3^3t_4^2 + 2^{11}5^{12}7\cdot 13\cdot 3501503\, t_2^3t_3^3t_4^2\\
& + 2^{17}3^{4}5^{13}7\cdot 14029\, Zt_1t_2t_3^4t_4^2 - 2^{15}5^{12}3\cdot 7\cdot 11\cdot 13\cdot 104297\, Z^3t_2^2t_3^4t_4^2 - 2^{19}3^{3}5^{13}7\cdot 4814471\, Z^4t_1t_3^5t_4^2\\
& - 2^{18}3^{2}5^{13}7\cdot 3231829\, Z^6t_2t_3^5t_4^2 + 2^{23}5^{13}3\cdot 7\cdot 47\cdot 263303\, Zt_2t_3^6t_4^2 - 2^{25}3^{2}5^{13}7\cdot 1951\cdot 29207\, Z^4t_3^7t_4^2\\
& + 2^{6}3^{4}5^{15}7^{2}17\, Z^6t_1^3t_4^3 - 2^{4}3^{7}5^{12}7\cdot 11\cdot 79\, t_1^2t_2^2t_4^3 + 2^{4}3^{3}5^{12}7\cdot 89\cdot 6449\, Z^2t_1t_2^3t_4^3\\
& + 2^{4}5^{11}3\cdot 7\cdot 211\cdot 6449\, Z^4t_2^4t_4^3 + 2^{12}3^{7}5^{16}7^{2}\, Zt_1^3t_3t_4^3 - 2^{10}3^{5}5^{12}7\cdot 208253\, Z^3t_1^2t_2t_3t_4^3\\
& - 2^{15}3^{3}5^{12}7\cdot 1429\, Z^5t_1t_2^2t_3t_4^3 - 2^{8}5^{11}7\cdot 31\cdot 181\cdot 3659\, Z^7t_2^3t_3t_4^3 + 2^{12}3^{5}5^{14}7\cdot 13\cdot 37\cdot 59\, Z^6t_1^2t_3^2t_4^3
\end{aligned}
$$

$$+ 2^{16}3^{2}5^{13}23^{2} Z^{10}t_2^2t_3^2t_4^3 + 2^{12}3^{3}5^{12}7^{2}1178167\, t_1t_2^2t_3^2t_4^3 - 2^{10}5^{11}2107634939\, Z^2t_2^3t_3^2t_4^3$$
$$+ 2^{18}3^{8}5^{13}7\cdot 1033\, Zt_1^2t_3^3t_4^3 - 2^{17}3^{3}5^{12}7\cdot 13\cdot 23\cdot 4099\, Z^3t_1t_2t_3^3t_4^3 + 2^{18}5^{11}107\cdot 330509\, Z^5t_2^2t_3^3t_4^3$$
$$- 2^{18}3^{4}5^{13}7\cdot 377563\, Z^6t_1t_3^4t_4^3 + 2^{24}3^{2}5^{16}23\, Z^8t_2t_3^4t_4^3 + 2^{16}5^{13}491035057\, t_2^2t_3^4t_4^3$$
$$- 2^{24}3^{3}5^{13}7\cdot 373\cdot 523\, Zt_1t_3^5t_4^3 - 2^{22}5^{12}3\cdot 658847099\, Z^3t_2t_3^5t_4^3 + 2^{24}5^{12}7\cdot 25229\cdot 55439\, Z^6t_3^6t_4^3$$
$$- 2^{30}5^{13}3\cdot 7\cdot 23\cdot 71\cdot 5791\, Zt_3^7t_4^3 - 2^{12}3^{4}5^{15}7^{2} Z^3t_1^3t_4^4 + 2^{9}3^{12}5^{12}7\, Z^5t_1^2t_2t_4^4$$
$$+ 2^{7}3^{3}5^{11}7\cdot 13\cdot 17\cdot 5981\, Z^7t_1t_2^2t_4^4 + 2^{12}3^{2}5^{12}13\, Z^9t_2^3t_4^4 + 2^{8}5^{10}1553\cdot 74317\, Zt_2^4t_4^4$$
$$- 2^{10}3^{5}5^{12}7\cdot 23\cdot 97849\, t_1^2t_2t_3t_4^4 - 2^{16}3^{2}5^{12}23\, Z^{12}t_2^2t_3t_4^4 + 2^{9}3^{4}5^{11}7^{2}644789\, Z^2t_1t_2^2t_3t_4^4$$
$$- 2^{10}5^{10}684923563\, Z^4t_2^3t_3t_4^4 + 2^{16}3^{5}5^{13}7\cdot 11\cdot 23\cdot 157\, Z^3t_1^2t_3^2t_4^4 + 2^{14}3^{5}5^{11}7\cdot 239\cdot 1361\, Z^5t_1t_2t_3^2t_4^4$$
$$- 2^{15}5^{10}3\cdot 7\cdot 105745993\, Z^7t_2^2t_3^2t_4^4 + 2^{24}3^{4}5^{14} Z^{10}t_2t_3^3t_4^4 + 2^{17}3^{5}5^{13}7^{2}144073\, t_1t_2t_3^3t_4^4$$
$$- 2^{15}5^{10}31\cdot 2275515659\, Z^2t_2^2t_3^3t_4^4 - 2^{25}3^{3}5^{12}7^{2}19\cdot 1987\, Z^3t_1t_3^4t_4^4$$
$$- 2^{20}5^{10}7\cdot 17\cdot 683\cdot 247601\, Z^5t_2t_3^4t_4^4 + 2^{30}3^{2}5^{17} Z^8t_3^5t_4^4 - 2^{22}5^{11}56634970591\, t_2t_3^5t_4^4$$
$$+ 2^{28}3^{2}5^{11}59\cdot 56802241\, Z^3t_3^6t_4^4 - 2^{16}3^{4}5^{15}7\cdot 97\, t_1^3t_4^5 + 2^{10}3^{12}5^{10}7\cdot 103\, Z^2t_1^2t_2t_4^5$$
$$+ 2^{14}3^{2}5^{11} Z^{14}t_2^2t_4^5 + 2^{8}3^{3}5^{9}7\cdot 17\cdot 6436589\, Z^4t_1t_2^2t_4^5 + 2^{8}5^{8}3\cdot 83\cdot 113\cdot 283487\, Z^6t_2^3t_4^5$$
$$- 2^{15}3^{10}5^{10}7^{2}367\, Z^5t_1^2t_3t_4^5 + 2^{12}3^{3}5^{10}7\cdot 47\cdot 409\cdot 1801\, Z^7t_1t_2t_3t_4^5 - 2^{15}3^{3}5^{11}89\cdot 241\, Z^9t_2^2t_3t_4^5$$
$$+ 2^{15}5^{8}19^{2}379\cdot 72043\, Zt_2^3t_3t_4^5 - 2^{17}3^{5}5^{12}7\cdot 67\cdot 151\cdot 443\, t_1^2t_3^2t_4^5 + 2^{20}3^{2}5^{12}29\cdot 173\, Z^{12}t_2t_3^2t_4^5$$
$$+ 2^{17}3^{3}5^{9}7\cdot 242346031\, Z^2t_1t_2t_3^2t_4^5 - 2^{14}5^{8}3\cdot 17\cdot 29\cdot 307\cdot 2364179\, Z^4t_2^2t_3^2t_4^5$$
$$+ 2^{21}3^{3}5^{9}7\cdot 151247651\, Z^5t_1t_3^3t_4^5 + 2^{18}5^{9}21157\cdot 4697569\, Z^7t_2t_3^3t_4^5 + 2^{27}3^{2}5^{15}13\cdot 17\, Z^{10}t_3^4t_4^5$$
$$+ 2^{24}3^{3}5^{13}7\cdot 2004073\, t_1t_3^4t_4^5 - 2^{22}5^{8}3\cdot 883\cdot 159406111\, Z^2t_2t_3^4t_4^5 - 2^{29}5^{8}3343\cdot 11789651\, Z^5t_3^5t_4^5$$
$$- 2^{29}5^{11}85909378939\, t_3^6t_4^5 + 2^{14}3^{14}5^{9}7\, Z^7t_1^2t_4^6 + 2^{14}3^{2}5^{10}1523\, Z^{11}t_2^2t_4^6$$
$$- 2^{11}3^{3}5^{8}7\cdot 13\cdot 73\cdot 627131\, Zt_1t_2^2t_4^6 + 2^{11}5^{7}6691\cdot 10945717\, Z^3t_2^3t_4^6 - 2^{16}3^{10}5^{10}7\cdot 739\, Z^2t_1^2t_3t_4^6$$
$$- 2^{19}3^{3}5^{11}73\, Z^{14}t_2t_3t_4^6 + 2^{13}3^{3}5^{8}7\cdot 6560672719\, Z^4t_1t_2t_3t_4^6 - 2^{13}5^{8}3\cdot 19\cdot 107\cdot 271\cdot 4603\, Z^6t_2^2t_3t_4^6$$
$$- 2^{21}3^{4}5^{8}7\cdot 195868609\, Z^7t_1t_3^2t_4^6 + 2^{21}3^{2}5^{10}173\cdot 2039\, Z^9t_2t_3^2t_4^6 - 2^{17}5^{7}3\cdot 175949\cdot 534167\, Zt_2^2t_3^2t_4^6$$
$$+ 2^{26}3^{3}5^{13}73\, Z^{12}t_3^3t_4^6 - 2^{25}3^{3}5^{9}7\cdot 307854203\, Z^2t_1t_3^3t_4^6 + 2^{19}5^{7}17\cdot 1171\cdot 755025827\, Z^4t_2t_3^3t_4^6$$
$$+ 2^{26}5^{7}3\cdot 11\cdot 29\cdot 1522702439\, Z^7t_3^4t_4^6 + 2^{28}5^{8}17\cdot 23\cdot 99971\cdot 187073\, Z^2t_3^5t_4^6 + 2^{17}3^{14}5^{9}7\, Z^4t_1^2t_4^7$$
$$+ 2^{12}3^{3}5^{7}7\cdot 17\cdot 61\cdot 545533\, Z^6t_1t_2t_4^7 + 2^{12}3^{2}5^{9}1747\cdot 2179\, Z^8t_2^2t_4^7 + 2^{12}5^{6}3\cdot 7\cdot 25411781761\, t_2^3t_4^7$$
$$- 2^{18}3^{2}5^{9}7\cdot 31\cdot 101\cdot 109\, Z^{11}t_2t_3t_4^7 + 2^{18}3^{4}5^{7}7\cdot 3491\cdot 87421\, Zt_1t_2t_3t_4^7$$
$$+ 2^{16}5^{6}3\cdot 11\cdot 41\cdot 625672687\, Z^3t_2^2t_3t_4^7 + 2^{22}3^{4}5^{11}73^{2} Z^{14}t_3^2t_4^7$$
$$- 2^{20}3^{3}5^{7}7^{2}13^{2}607\cdot 45503\, Z^4t_1t_3^2t_4^7 - 2^{18}3^{3}5^{6}128237\cdot 6437531\, Z^6t_2t_3^2t_4^7$$
$$- 2^{28}3^{3}5^{11}6197\, Z^9t_3^3t_4^7 - 2^{24}5^{6}3\cdot 443\cdot 8098119613\, Zt_2t_3^3t_4^7$$
$$+ 2^{26}5^{6}3\cdot 7\cdot 715783571707\, Z^4t_3^4t_4^7 + 2^{19}3^{2}5^{8}7541\, Z^{13}t_2t_4^8$$
$$- 2^{19}3^{3}5^{6}7^{2}17^{2}19\cdot 31\cdot 199\, Z^3t_1t_2t_4^8 + 2^{15}5^{6}3\cdot 23\cdot 29\cdot 219793529\, Z^5t_2^2t_4^8$$
$$+ 2^{20}3^{3}5^{6}7\cdot 23\cdot 67\cdot 1297\cdot 3301\, Z^6t_1t_3t_4^8 - 2^{20}3^{4}5^{9}28807\, Z^8t_2t_3t_4^8$$
$$+ 2^{17}5^{5}3\cdot 7\cdot 2544911109121\, t_2^2t_3t_4^8 + 2^{25}3^{2}5^{9}19\cdot 43\cdot 829\, Z^{11}t_3^2t_4^8$$
$$+ 2^{26}3^{3}5^{7}7\cdot 53\cdot 167\cdot 116381\, Zt_1t_3^2t_4^8 - 2^{26}3^{2}5^{5}167\cdot 5651\cdot 157637\, Z^3t_2t_3^2t_4^8$$
$$+ 2^{26}3^{2}5^{5}19\cdot 47\cdot 359\cdot 6805069\, Z^6t_3^3t_4^8 + 2^{32}5^{6}3\cdot 31\cdot 1249\cdot 8516173\, Zt_3^4t_4^8$$
$$+ 2^{18}3^{3}5^{7}11\cdot 151\cdot 2281\, Z^{10}t_2t_4^9 - 2^{18}3^{3}5^{5}7^{2}1453\cdot 9140533\, t_1t_2t_4^9$$
$$+ 2^{16}5^{5}8861954242873\, Z^2t_2^2t_4^9 - 2^{23}3^{3}5^{8}73\cdot 7541\, Z^{13}t_3t_4^9$$
$$- 2^{25}3^{3}5^{5}7\cdot 4139\cdot 129119\, Z^3t_1t_3t_4^9 - 2^{23}5^{5}151\cdot 5680110559\, Z^5t_2t_3t_4^9$$
$$- 2^{26}3^{2}5^{7}40185941\, Z^8t_3^2t_4^9 + 2^{24}5^{4}17\cdot 23621998776373\, t_2t_3^2t_4^9$$
$$+ 2^{30}5^{4}11\cdot 764381\cdot 5742169\, Z^3t_3^3t_4^9 - 2^{24}3^{14}5^{4}7\cdot 11\cdot 863\, Z^5t_1t_4^{10}$$

$$
\begin{aligned}
&+2^{20}5^3 17\cdot 233941\cdot 6976421\, Z^7 t_2 t_4^{10} - 2^{25}3^2 5^6 7\cdot 1381231\, Z^{10} t_3 t_4^{10}\\
&-2^{25}3^3 5^5 7\cdot 7457\cdot 33020201\, t_1 t_3 t_4^{10} - 2^{25}5^3 58812201936403\, Z^2 t_2 t_3 t_4^{10}\\
&+2^{29}5^3 39878573708081\, Z^5 t_3^2 t_4^{10} + 2^{32}5^4 3909211\cdot 4328957\, t_3^3 t_4^{10}\\
&+2^{22}3^2 5^5 7541^2\, Z^{12} t_4^{11} + 2^{24}3^{13}5^4 7^2 5399\, Z^2 t_1 t_4^{11} - 2^{28}3^2 5^5 7\cdot 53\cdot 7541\, Z^9 t_4^{12}\\
&+2^{22}5^2 225349\cdot 1564621501\, Z^4 t_2 t_4^{11} - 2^{27}5^2 3\cdot 107\cdot 127\cdot 4920163967\, Z^7 t_3 t_4^{11}\\
&-2^{31}5^3 13\cdot 1583\cdot 4421\cdot 158759\, Z^2 t_3^2 t_4^{11} - 2^{26}5\cdot 296367312358063\, Z t_2 t_4^{12}\\
&-2^{32}5\cdot 1383659\cdot 123837583\, Z^4 t_3 t_4^{12} + 2^{30}7\cdot 249677\cdot 73045429\, Z^6 t_4^{13}\\
&+2^{33}5\cdot 296367312358063\, Z t_3 t_4^{13} - 2^{32}11\cdot 43\cdot 104455205831\, Z^3 t_4^{14}\\
&-2^{50}5^{10}7\cdot 349\, t_4^{15}\Big),
\end{aligned}
$$

$$
\begin{aligned}
y_2 = \frac{5^3}{2^3 3^{18}} \Big(&2^2 3^3 5^{13} t_1^3 - 3^2 5^{10} 17\, Z^4 t_1 t_2^2 - 5^8 2\cdot 3\cdot 7\, Z^6 t_2^3 - 2^5 3^3 5^{10} 11\, Z^7 t_1 t_2 t_3\\
&+2^5 3^3 5^8 31\, Z t_2^3 t_3 + 2^8 3^4 5^{12} 19\, t_1^2 t_3^2 - 2^9 3^2 5^{10} 353\, Z^2 t_1 t_2 t_3^2 - 2^6 3^2 5^8 367\, Z^4 t_2^2 t_3^2\\
&+2^{15}3^2 5^{10} 29\, Z^5 t_1 t_3^3 - 2^{11}5^8 3\cdot 4327\, Z^7 t_2 t_3^3 + 2^{14}3^3 5^{10} 1319\, t_1 t_3^4\\
&-2^{15}5^8 3\cdot 109\cdot 137\, Z^2 t_2 t_3^4 - 2^{21}5^{11} 3\cdot 11\, Z^5 t_3^5 + 2^{20}3^2 5^{10} 97\, t_3^6 - 2^4 5^7 5023\, Z^3 t_2^3 t_4\\
&+2^4 3^2 5^9 11\cdot 13\, Z t_1 t_2^2 t_4 - 2^7 3^3 5^{10} 7^2\, Z^4 t_1 t_2 t_3 t_4 + 2^7 5^7 3\cdot 4817\, Z^6 t_2^2 t_3 t_4\\
&+2^{15}3^2 5^9 53\, Z^7 t_1 t_3^2 t_4 - 2^{10}5^7 3\cdot 13\cdot 397\, Z t_2^2 t_3^2 t_4 + 2^{16}3^2 5^{10} 353\, Z^2 t_1 t_3^3 t_4\\
&-2^{13}5^7 19\cdot 41\cdot 317\, Z^4 t_2 t_3^3 t_4 + 2^{19}5^8 11\cdot 1723\, Z^7 t_3^4 t_4 + 2^{22}5^8 3\cdot 109\cdot 137\, Z^2 t_3^5 t_4\\
&-2^5 3^2 5^8 547\, Z^6 t_1 t_2 t_4^2 - 2^4 3^2 5^6 7\cdot 13\cdot 167\, t_2^3 t_4^2 - 2^{10}3^2 5^8 11\cdot 103\, Z t_1 t_2 t_3 t_4^2\\
&-2^9 5^6 3\cdot 7\cdot 3907\, Z^3 t_2^2 t_3 t_4^2 + 2^{15}3^2 5^8 881\, Z^4 t_1 t_3^2 t_4^2 + 2^{11}3^2 5^7 31883\, Z^6 t_2 t_3^2 t_4^2\\
&-2^{16}5^8 43\cdot 461\, Z t_2 t_3^3 t_4^2 + 2^{22}5^7 127\cdot 1301\, Z^4 t_3^4 t_4^2 + 2^9 3^2 5^7 823\, Z^3 t_1 t_2 t_4^3\\
&-2^8 5^6 3\cdot 7\cdot 37\cdot 53\, Z^5 t_2^2 t_4^3 - 2^{13}3^2 5^8 7\cdot 53\, Z^6 t_1 t_3 t_4^3 - 2^{12}5^6 3\cdot 73\cdot 1877\, t_2^2 t_3 t_4^3\\
&-2^{17}3^5 5^8 11\, Z t_1 t_3^2 t_4^3 + 2^{17}3^2 5^6 13\cdot 137\, Z^3 t_2 t_3^2 t_4^3 - 2^{19}3^4 5^7 641\, Z^6 t_3^3 t_4^3\\
&+2^{23}5^7 96601\, Z t_3^4 t_4^3 + 2^8 3^2 5^6 7\cdot 13\cdot 67\cdot 79\, t_1 t_2 t_4^4 - 2^{10}5^5 7^2 10343\, Z^2 t_2^2 t_4^4\\
&+2^{14}3^2 5^6 19\cdot 443\, Z^3 t_1 t_3 t_4^4 + 2^{13}5^5 7^2 29\cdot 409\, Z^5 t_2 t_3 t_4^4 - 2^{14}5^5 17\cdot 379\cdot 17209\, t_2 t_3^2 t_4^4\\
&-2^{20}5^5 17\cdot 19\cdot 13931\, Z^3 t_3^3 t_4^4 + 2^{13}3^7 5^5 79\, Z^5 t_1 t_4^5 - 2^{12}5^3 13\cdot 59\cdot 5623\, Z^7 t_2 t_4^5\\
&+2^{15}3^2 5^6 13\cdot 103\cdot 883\, t_1 t_3 t_4^5 + 2^{15}5^3 11\cdot 5478043\, Z^2 t_2 t_3 t_4^5\\
&-2^{19}5^3 11\cdot 83\cdot 32839\, Z^5 t_3^2 t_4^5 + 2^{21}5^5 5189507\, t_3^3 t_4^5 - 2^{14}3^7 5^5 59\, Z^2 t_1 t_4^6\\
&-2^{12}5^2 327966773\, Z^4 t_2 t_4^6 + 2^{18}5^2 3\cdot 83\cdot 593\cdot 613\, Z^7 t_3 t_4^6\\
&+2^{20}5^3 1249\cdot 3677\, Z^2 t_3^2 t_4^6 + 2^{17}5\cdot 139571863\, Z t_2 t_4^7 + 2^{19}5\cdot 1099915517\, Z^4 t_3 t_4^7\\
&-2^{18}13\cdot 49685821\, Z^6 t_4^8 - 2^{24}5\cdot 139571863\, Z t_3 t_4^8 + 2^{23}19\cdot 1661677\, Z^3 t_4^9\\
&+2^{35}5^7 17\, t_4^{10}\Big),
\end{aligned}
$$

$$
\begin{aligned}
y_3 = -\frac{5^3}{2^2 3^{11}} \Big(&3^3 5^6 t_1 t_2 - 2^2 5^4 13\, Z^2 t_2^2 + 2^6 5^4 23\, Z^5 t_2 t_3 + 2^6 5^4 587\, t_2 t_3^2\\
&+2^{13}5^7 Z^3 t_3^3 - 2^5 5^3 Z^7 t_2 t_4 + 2^7 3^3 5^5 11\, t_1 t_3 t_4 + 2^9 5^3 53\, Z^2 t_2 t_3 t_4\\
&+2^{12}5^5 Z^5 t_3^2 t_4 + 2^{13}5^4 23\cdot 131\, t_3^3 t_4 - 2^4 5^2 1523\, Z^4 t_2 t_4^2 + 2^9 5^3 3\cdot 73\, Z^7 t_3 t_4^2\\
&-2^{13}5^3 29\, Z^2 t_3^2 t_4^2 + 2^8 5\cdot 31\cdot 41\, Z t_2 t_4^3 + 2^{11}5\cdot 7\cdot 491\, Z^4 t_3 t_4^3 - 2^9 7541\, Z^6 t_4^4\\
&-2^{15}5\cdot 31\cdot 41\, Z t_3 t_4^4 + 2^{14}7\cdot 53\, Z^3 t_4^5 + 2^{21}5^3 11\, t_4^6\Big),
\end{aligned}
$$

$$
y_4 = \frac{40}{3} t_4.
$$

Proposition 6.20. *The derivatives* $\frac{\partial y_i}{\partial t_j} \in \mathbb{C}[t_1, t_2, t_3, t_4, Z]$.

Proof is similar to the one for Proposition 3.5.

7. Remarks on almost duality

Let us define on a Frobenius manifold M the following tensor fields:

$$\overset{*}{c}_{ijk} := g_{i\lambda}\overset{*}{c}{}^{\lambda}_{jk}, \qquad \overset{*}{c}{}^{ijk} := g^{i\lambda}c^{jk}_{\lambda}, \tag{7.1}$$

where $\overset{*}{c}{}^{\lambda}_{jk}$ and c^{jk}_{λ} are given by formulas (2.8) and (2.4), respectively. It can be shown that $\overset{*}{c}_{ijk}(x) = \frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}$ for a function $F_*(x)$, which is the dual prepotential of M [13].

For irreducible polynomial Frobenius manifolds, it was shown in [13] that their dual prepotentials (up to rescaling) have the following simple form:

$$F_*(x) = \sum_{\alpha \in R_+} \frac{(\alpha, x)^2}{(\alpha, \alpha)} \log(\alpha, x), \tag{7.2}$$

where R_+ is a positive root system for the associated Coxeter group W. Below we give some partial results about dual prepotentials for some algebraic Frobenius manifolds.

7.1. Two-dimensional examples

A two-dimensional (semisimple) algebraic Frobenius manifold has a prepotential of the form

$$F(t) = \frac{1}{2}t_1^2 t_2 + \frac{k(2k)^k}{k^2-1} t_2^{k+1}, \tag{7.3}$$

with $k \in \mathbb{Q} \setminus \{-1, 0, 1\}$ (see [10]). This has degrees $d_1 = 1$ and $d_2 = \frac{2}{k}$, and charge $d = \frac{k-2}{k}$. The choice of the coefficient of t_2^{k+1} in formula (7.3) is convenient for having a simple relation between coordinates t_1, t_2 and the flat coordinates of the intersection form x_1, x_2. Using formulas (2.2), (2.3) and (2.6), we find the intersection form:

$$g^{ij}(t) = \begin{pmatrix} (2k)^{k+1} t_2^{k-1} & t_1 \\ t_1 & \frac{2}{k} t_2 \end{pmatrix}.$$

Using formulas (2.4), (2.8) and (7.1), we get

$$\overset{*}{c}_{111}(t) = -4k^{-1} t_1 t_2 D, \qquad \overset{*}{c}_{112}(t) = \left(4(2k)^k t_2^k + t_1^2\right) D, \tag{7.4}$$

$$\overset{*}{c}_{122}(t) = -2(2k)^{k+1} t_1 t_2^{k-1} D, \qquad \overset{*}{c}_{222}(t) = (2k)^k k^2 t_2^{k-2}\left(4(2k)^k t_2^k + t_1^2\right) D, \tag{7.5}$$

where $D = \det\left(g^{ij}(t)\right)^{-2} = \left(4(2k)^k t_2^k - t_1^2\right)^{-2}$. Similar to the polynomial case $k \in \mathbb{Z}_{\geq 2}$ considered in [10], the flat coordinates of the metric t_1, t_2 are related to the flat coordinates of the intersection form x_1, x_2 by the following formulas:

$$t_1 = z^k + \overline{z}^k, \qquad t_2 = \frac{z\overline{z}}{2k}, \tag{7.6}$$

where $z := x_1 + ix_2$ and $\overline{z} := x_1 - ix_2$.

Here and in the next two theorems, we assume that when taking powers of k we are working in an open set $U \subseteq \mathbb{C}$ which contains points z, $\overline{z}$, $2k$, $\frac{z\overline{z}}{2k}$ and $2ix_2$. In the open set U we choose a single branch of the function $f(w) = w^k$ so that we have the relation $f(z)f(\overline{z}) = f(2k)f(\frac{z\overline{z}}{2k})$. For example, we can assume that U does not contain the non-positive imaginary axis which can be achieved for $k > 0$ by taking $|\mathrm{Re}(x_1)|, |\mathrm{Im}(x_1)| < 1$ and $\mathrm{Re}(x_2), \mathrm{Im}(x_2) > 1$. Similarly, for $k < 0$ we can assume that U does not contain the non-negative imaginary axis and take the same conditions for x_1 and x_2.

Performing a tensorial transformation of (7.4) and (7.5) with the relations (7.6), we get the following third order derivatives of the dual prepotential:

$$\overset{*}{c}_{111}(x) = \frac{kx_1(x_1^2+3x_2^2)}{(z\overline{z})^2} + \frac{2kix_2^3}{(z\overline{z})^2}\frac{\overline{z}^k + z^k}{\overline{z}^k - z^k}, \qquad \overset{*}{c}_{112}(x) = \frac{kx_2(x_2^2-x_1^2)}{(z\overline{z})^2} - \frac{2kix_1x_2^2}{(z\overline{z})^2}\frac{\overline{z}^k + z^k}{\overline{z}^k - z^k}, \tag{7.7}$$

$$\overset{*}{c}_{122}(x) = \frac{kx_1(x_1^2-x_2^2)}{(z\overline{z})^2} + \frac{2kix_1^2x_2}{(z\overline{z})^2}\frac{\overline{z}^k + z^k}{\overline{z}^k - z^k}, \qquad \overset{*}{c}_{222}(x) = \frac{kx_2(x_2^2+3x_1^2)}{(z\overline{z})^2} - \frac{2kix_1^3}{(z\overline{z})^2}\frac{\overline{z}^k + z^k}{\overline{z}^k - z^k}. \tag{7.8}$$

For the next result we assume that $k = l^{-1}$ with $l \in \mathbb{Z}_{\geq 2}$. Also, we will make use of the hypergeometric function ${}_2F_1(a, b; c; w)$ which is single-valued for the argument $|w| < 1$. This condition holds for $w = \frac{ix_1+x_2}{2x_2}$ when we use the constraints specified above.

Theorem 7.1. *Let M be a two-dimensional Frobenius manifold with prepotential (7.3) with $k = l^{-1}$, where $l \in \mathbb{Z}_{\geq 2}$. Then the dual prepotential of M has the form*

$$\begin{aligned} F_*(x) &= \frac{x_2^2}{l}\log x_2 + \frac{\overline{z}^2}{4l}\log\overline{z} + \frac{z^2}{4l}\log z \\ &+ \sum_{j=1}^{l-1}\frac{\overline{z}^{\frac{j}{l}}}{4j}\left(\frac{lx_1 + (l-2j)ix_2}{(j-l)z^{\frac{j}{l}-1}} + (2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right)\right), \end{aligned} \tag{7.9}$$

where ${}_2F_1(a, b; c; w)$ is the hypergeometric function.

Proof. For $k = l^{-1}$, the third order derivatives of the dual prepotential given by formulas (7.7)–(7.8) may be simplified as

$$\overset{*}{c}_{111}(x) = \frac{x_1}{lz\overline{z}} - \frac{2x_2^2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \tag{7.10}$$

$$\overset{*}{c}_{112}(x) = \frac{x_2}{lz\overline{z}} + \frac{2x_1x_2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \tag{7.11}$$

$$\overset{*}{c}_{122}(x) = -\frac{x_1}{lz\overline{z}} - \frac{2x_1^2}{l(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \tag{7.12}$$

$$\overset{*}{c}_{222}(x) = \frac{1}{l}\left(\frac{2}{x_2} - \frac{x_2}{z\overline{z}}\right) + \frac{2x_1^3}{lx_2(z\overline{z})^2}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}, \tag{7.13}$$

where we use the identity

$$\frac{\overline{z}^{\frac{1}{l}} + z^{\frac{1}{l}}}{\overline{z}^{\frac{1}{l}} - z^{\frac{1}{l}}} = \frac{\overline{z}+z}{\overline{z}-z} + \frac{2}{\overline{z}-z}\sum_{j=1}^{l-1}\overline{z}^{\frac{j}{l}}z^{\frac{l-j}{l}}.$$

Let us define the following functions:

$$\begin{aligned} A(x) &:= \frac{x_2^2}{l}\log x_2 + \frac{\overline{z}^2}{4l}\log\overline{z} + \frac{z^2}{4l}\log z, \\ B_j(x) &:= \frac{\overline{z}^{\frac{j}{l}}\,(lx_1 + (l-2j)ix_2)}{4j(j-l)z^{\frac{j}{l}-1}}, \\ C_j(x) &:= \frac{\overline{z}^{\frac{j}{l}}}{4j}(2ix_2)^{2-\frac{j}{l}}\,{}_2F_1\left(\frac{j}{l},\frac{j}{l};\frac{j}{l}+1;\frac{ix_1+x_2}{2x_2}\right), \end{aligned}$$

for $j = 1, \ldots, l-1$. Then, we want to show that

$$\frac{\partial^3}{\partial x_a\partial x_b\partial x_c}\left(A(x) + \sum_{j=1}^{l-1}\left(B_j(x) + C_j(x)\right)\right) = \overset{*}{c}_{abc}(x), \tag{7.14}$$

where $\overset{*}{c}_{abc}(x)$ are given by formulas (7.10)–(7.13). The third order derivatives of $A(x)$ are

$$\frac{\partial^3 A}{\partial x_1^3} = \frac{x_1}{lz\overline{z}}, \qquad \frac{\partial^3 A}{\partial x_1^2\partial x_2} = \frac{x_2}{lz\overline{z}}, \qquad \frac{\partial^3 A}{\partial x_1\partial x_2^2} = -\frac{x_1}{lz\overline{z}}, \qquad \frac{\partial^3 A}{\partial x_2^3} = \frac{1}{l}\left(\frac{2}{x_2} - \frac{x_2}{z\overline{z}}\right). \tag{7.15}$$

Next, we calculate the third order derivatives of $B_j(x)$ for $j = 1, \ldots, l-1$ to be

$$\frac{\partial^3 B_j}{\partial x_1^3} = \frac{4ix_2^3\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 B_j}{\partial x_1^2\partial x_2} = -\frac{4ix_1x_2^2\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3z^{\frac{j}{l}+2}}, \tag{7.16}$$

$$\frac{\partial^3 B_j}{\partial x_1\partial x_2^2} = \frac{4ix_1^2x_2\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 B_j}{\partial x_2^3} = -\frac{4ix_1^3\,\overline{z}^{\frac{j}{l}-3}b_j}{l^3z^{\frac{j}{l}+2}}, \tag{7.17}$$

where $b_j = l(l-2j)x_1 + i(j^2 - jl + l^2)x_2$. Now, let us consider the first order derivatives of $C_j(x)$ for $j = 1, \dots, l-1$. We get

$$\begin{aligned}
\frac{\partial C_j}{\partial x_1} &= \frac{\bar{z}^{\frac{j}{l}-1}}{4l}(2ix_2)^{2-\frac{j}{l}}\, {}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right) \\
&\quad - \frac{\bar{z}^{\frac{j}{l}}}{4j}(2ix_2)^{1-\frac{j}{l}}\, {}_2F_1'\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right), \\
\frac{\partial C_j}{\partial x_2} &= \left(\frac{ix_1+x_2}{j} - \frac{ix_1}{2l}\right)\bar{z}^{\frac{j}{l}-1}(2ix_2)^{1-\frac{j}{l}}\, {}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right) \\
&\quad + \frac{ix_1}{2j}\bar{z}^{\frac{j}{l}}(2ix_2)^{-\frac{j}{l}}\, {}_2F_1'\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right).
\end{aligned}$$

The hypergeometric function has the properties

$$\begin{aligned}
{}_2F_1'(a, b; c; w) &= \frac{c-1}{w}({}_2F_1(a, b; c-1; w) - {}_2F_1(a, b; c; w)), \\
{}_2F_1(a, b; b; w) &= (1-w)^{-a},
\end{aligned}$$

for all $a, b, c \in \mathbb{C}$. Here the branch of $(1-w)^{-a}$ is the one which equals 1 at $w = 0$. When $a = \frac{j}{l}$ and $w = \frac{ix_1+x_2}{2x_2}$ we have the relation

$$ {}_2F_1\left(\frac{j}{l}, b; b; \frac{ix_1+x_2}{2x_2}\right) = \left(1 - \frac{ix_1+x_2}{2x_2}\right)^{-\frac{j}{l}} = \frac{(2ix_2)^{\frac{j}{l}}}{z^{\frac{j}{l}}}, \tag{7.18}$$

for any $b \in \mathbb{C}$, where the functions $f(t) = t^{\frac{1}{l}}$ on the right-hand side of (7.18) are taken on the same branch, which is possible since the open set U contains both z and $2ix_2$. Hence we have

$$ {}_2F_1'\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right) = \frac{2jx_2}{l(ix_1+x_2)}\left(\frac{(2ix_2)^{\frac{j}{l}}}{z^{\frac{j}{l}}} - {}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right)\right). \tag{7.19}$$

Substitution of relation (7.19) into the above formulas for the derivatives $\frac{\partial C_j}{\partial x_i}$ gives

$$\begin{aligned}
\frac{\partial C_j}{\partial x_1} &= -\frac{x_2^2\bar{z}^{\frac{j}{l}-1}}{lz^{\frac{j}{l}}}, \\
\frac{\partial C_j}{\partial x_2} &= \frac{i}{j}\bar{z}^{\frac{j}{l}}(2ix_2)^{1-\frac{j}{l}}\, {}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right) + \frac{x_1x_2\bar{z}^{\frac{j}{l}-1}}{lz^{\frac{j}{l}}}.
\end{aligned}$$

Since $\frac{\partial C_j}{\partial x_1}$ contains no hypergeometric functions, its derivatives are more easily attainable, and we see that

$$\frac{\partial^3 C_j}{\partial x_1^3} = -\frac{2x_2^2\bar{z}^{\frac{j}{l}-3}c_j}{l^3z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 C_j}{\partial x_1^2\partial x_2} = \frac{2x_1x_2\bar{z}^{\frac{j}{l}-3}c_j}{l^3z^{\frac{j}{l}+2}}, \qquad \frac{\partial^3 C_j}{\partial x_1\partial x_2^2} = -\frac{2x_1^2\bar{z}^{\frac{j}{l}-3}c_j}{l^3z^{\frac{j}{l}+2}}, \tag{7.20}$$

where $c_j = l^2x_1^2 + 2il(l-2j)x_1x_2 - (l^2 - 2jl + 2j^2)x_2^2$. On the other hand, $\frac{\partial C_j}{\partial x_2}$ still contains hypergeometric functions. Looking at the second order derivative, and using relation (7.19), we see that

$$\frac{\partial^2 C_j}{\partial x_2^2} = -\frac{2}{j}\bar{z}^{\frac{j}{l}}(2ix_2)^{-\frac{j}{l}}\, {}_2F_1\left(\frac{j}{l}, \frac{j}{l}; \frac{j}{l}+1; \frac{ix_1+x_2}{2x_2}\right) + \frac{x_1\bar{z}^{\frac{j}{l}-2}}{l^2z^{\frac{j}{l}+1}}(3lx_1^2 + i(l-2j)x_1x_2 + 2lx_2^2).$$

Differentiating $C_j(x)$ with respect to x_2 for the third time and substituting relation (7.19) into this expression, we get

$$\frac{\partial^3 C_j}{\partial x_2^3} = \frac{2x_1^3\bar{z}^{\frac{j}{l}-3}}{l^3x_2z^{\frac{j}{l}+2}}(l^2x_1^2 + 2il(2j-l)x_1x_2 - (l^2 - 2jl + 2j^2)x_2^2). \tag{7.21}$$

From relations (7.15)–(7.17) and (7.20)–(7.21), one can check directly that formula (7.14) holds. □

Theorem 7.2. *Let $\widetilde{M}$ be a two-dimensional Frobenius manifold with prepotential (7.3) with $k = -l^{-1}$, where $l \in \mathbb{Z}_{\geq 2}$. Then the dual prepotential of $\widetilde{M}$ has the form*

$$\widetilde{F_*}(x) = F_*(x) - \frac{x_1^2 + x_2^2}{2l} \log(x_1^2 + x_2^2),$$

where $F_*(x)$ is the function given by formula (7.9).

Proof. Given a two-dimensional Frobenius manifold M, with charge $d \neq 1$ and $\eta_{11} = 0$ one can construct a two-dimensional Frobenius manifold $\widetilde{M}$ with charge $\widetilde{d} = 2 - d$ using a symmetry of the WDVV equations known as an inversion [10]. The flat coordinates x of the intersection form of M may be expressed in terms of the flat coordinates $\widetilde{x}$ of the intersection form of $\widetilde{M}$ via the following relation:

$$x_i = \frac{2\widetilde{x}_i}{(1-\widetilde{d})\left(\widetilde{x}_1^2 + \widetilde{x}_2^2\right)},$$

for $i = 1, 2$. Moreover, the dual prepotential $\widetilde{F_*}$ of $\widetilde{M}$ may be expressed as

$$\widetilde{F_*}(\widetilde{x}) = \frac{4F_*(x(\widetilde{x}))}{(1-d)^2\left(x_1(\widetilde{x})^2 + x_2(\widetilde{x})^2\right)^2}, \tag{7.22}$$

where F_* is the dual prepotential of M [21]. In two dimensions, semisimple Frobenius manifolds with $d \neq 1$ and $\eta_{11} = 0$ are uniquely parametrized, up to isomorphism, by their charge [10]. A Frobenius manifold with prepotential (7.3) has charge $d = \frac{k-2}{k}$. Let M be the Frobenius manifold with prepotential (7.3) with $k = l^{-1}$, thus M has charge $d = 1 - 2l$. We know from Theorem 7.1 that this Frobenius manifold has a dual prepotential of the form (7.9). The inversion $\widetilde{M}$ must have charge $\widetilde{d} = 2l + 1$ and therefore its prepotential must be of the form (7.3) with $k = -l^{-1}$. The dual prepotential of $\widetilde{M}$ is given by equation (7.22) from which the statement follows. □

7.2. $(H_3)''$ and $D_4(a_1)$

Recall that for a polynomial Frobenius manifold associated to a Coxeter group W with root system $R = R_W$, the dual prepotential has the form (7.2). Let $\alpha \in R$ and define $\alpha_i = (\alpha, e_i)$, then we have the following relations (for generic points on $(\alpha, x) = 0$):

$$\left((\alpha, x)\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k}\right)\Bigg|_{(\alpha, x)=0} = \frac{2\alpha_i \alpha_j \alpha_k}{(\alpha, \alpha)},$$

for all $i, j, k = 1, \ldots, n$. Below we give related results for the algebraic Frobenius manifolds $(H_3)''$ and $D_4(a_1)$.

Proposition 7.3. *Let $P^M(x, Z)$ be the polynomial from relation (3.45) for $M = (H_3)''$ and let it be the polynomial from relation (4.34) for $M = D_4(a_1)$ expressed in the x coordinates. Then for each $\alpha \in R$, where $R = R_{H_3}$ for $M = (H_3)''$ and $R = R_{D_4}$ for $M = D_4(a_1)$, we have that*

$$P^M(x, Z)|_{(\alpha, x)=0} = K_\alpha^M (L_\alpha^M)^2,$$

where $K_\alpha^M, L_\alpha^M \in \mathbb{C}[x; Z]$ and L_α^M is linear in Z. K_α^M is cubic in Z for $M = (H_3)''$ and quartic in Z for $M = D_4(a_1)$.

To check that the polynomial $P^M(x, Z)$ factorises on the hyperplanes $(\alpha, x) = 0$ we first substitute the expressions for $y_i(x)$ from relations (3.1)–(3.7), or (4.8)–(4.11), into the left-hand side of equation (3.45), or equation (4.34), respectively. We then restrict to the hyperplane $(\alpha, x) = 0$ and see that the polynomial factorises as claimed.

Proposition 7.4. *Let $\alpha \in R$, where $R = R_{H_3}$ for $M = (H_3)''$ and $R = R_{D_4}$ for $M = D_4(a_1)$. The third order derivatives $\overset{*}{c}_{ijk}(x)$ of the dual prepotential F_* of $M = (H_3)''$ or $M = D_4(a_1)$ satisfy*

$$\left((\alpha, x)\overset{*}{c}_{ijk}(x)\right)\Big|_{(\alpha, x)=0} = 0$$

if $L_\alpha^M(x, Z) = 0$. If $K_\alpha^M(x, Z) = 0$ then we have

$$\left((\alpha, x)\overset{*}{c}_{ijk}(x)\right)\Big|_{(\alpha, x)=0} = \frac{2\alpha_i \alpha_j \alpha_k}{(\alpha, \alpha)}.$$

Proof. By formulas (2.7) and (7.1) we have

$$\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k} = \overset{*}{c}_{ijk}(x) = g_{i\lambda}(x) g_{j\mu}(x) g_{k\nu}(x) \overset{*}{c}^{\lambda\mu\nu}(x) = \overset{*}{c}^{ijk}(x).$$

Then

$$\frac{\partial^3 F_*}{\partial x_i \partial x_j \partial x_k} = \overset{*}{c}^{\alpha\beta\gamma}(t) \frac{\partial x_i}{\partial t_\alpha} \frac{\partial x_j}{\partial t_\beta} \frac{\partial x_k}{\partial t_\gamma} = g^{\alpha\delta}(t) c_\delta^{\beta\gamma}(t) \frac{\partial x_i}{\partial y_r} \frac{\partial y_r}{\partial t_\alpha} \frac{\partial x_j}{\partial y_s} \frac{\partial y_s}{\partial t_\beta} \frac{\partial x_k}{\partial y_t} \frac{\partial y_t}{\partial t_\gamma}. \tag{7.23}$$

Now we express the right-hand side of (7.23) in x coordinates and Z. For the terms $g^{\alpha\delta}(t)$ and $c_\delta^{\beta\gamma}(t)$ we apply Theorem 3.13. The derivatives $\frac{\partial x_i}{\partial y_r}$, $\frac{\partial x_j}{\partial y_s}$ and $\frac{\partial x_k}{\partial y_t}$ can be found by inverting the Jacobi matrix $J = \left(\frac{\partial y_i}{\partial x_j}\right)$. The derivatives $\frac{\partial y_r}{\partial t_\alpha}$, $\frac{\partial y_s}{\partial t_\beta}$ and $\frac{\partial y_t}{\partial t_\gamma}$ can be found by Theorem 3.10. We then reduce the resulting expression for $\overset{*}{c}_{ijk}(x)$ as a polynomial in Z modulo the relation (3.45) for $M = (H_3)''$, or modulo the relation (4.34) for $M = D_4(a_1)$.

Then, for any $\alpha \in R$ we get $(\alpha, x)\overset{*}{c}_{ijk}(x)$ which can be restricted to $(\alpha, x) = 0$. Using Proposition 7.3 we can then reduce the restricted expression as a polynomial in Z modulo K_α^M or modulo L_α^M depending on which branch of Z we consider on the hyperplane. This leads to the claim. □

Data availability

Mathematica code is available at https://notebookarchive.org/2024-03-2sleam2.

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