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Some results regarding the ideal structure of C^*-algebras of étale groupoids

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Abstract

We prove a sandwiching lemma for inner-exact locally compact Hausdorff étale groupoids. Our lemma says that every ideal of the reduced C^*-algebra of such a groupoid is sandwiched between the ideals associated to two uniquely defined open invariant subsets of the unit space. We obtain a bijection between ideals of the reduced C^*-algebra, and triples consisting of two nested open invariant sets and an ideal in the C^*-algebra of the subquotient they determine that has trivial intersection with the diagonal subalgebra and full support. We then introduce a generalisation to groupoids of Ara and Lolk's relative strong topological freeness condition for partial actions, and prove that the reduced C^*-algebras of inner-exact locally compact Hausdorff étale groupoids satisfying this condition admit an obstruction ideal in Ara and Lolk's sense.

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1 | INTRODUCTION

The purpose of this paper is to investigate the ideal structure of the reduced C^*-algebras of locally compact Hausdorff étale groupoids. This very broad class of C^*-algebras contains all reduced crossed products of commutative C^*-algebras by discrete groups. It also includes graph

C^*-algebras [18], higher rank graph C^*-algebras [17], the models described by Spielberg [27] and Katsura [14] for Kirchberg algebras, the stable and unstable Ruelle algebras of Smale spaces (up to Morita equivalence), and many self-similar action C^*-algebras [11].

Among the more natural invariants of a C^*-algebra, but also among the most difficult to compute, is its lattice of ideals. In the situation of étale groupoid C^*-algebras, definitive theorems are available for C^*-algebras of amenable groupoids that are essentially principal in the sense of Renault [22], graph C^*-algebras [12, 13], and for C^*-algebras of a single local homeomorphisms [15], but few other truly general results about ideal structure of groupoid C^*-algebras are available.

The analysis of ideals in étale groupoid C^*-algebras typically has two components. The first is concerned with what we call here dynamical ideals, and is well-understood. The continuous functions on the unit space $G^{(0)}$ of an étale groupoid G embed as a σ-unital subalgebra D of the groupoid C^*-algebra. So, each ideal I of $C^*(G)$ yields an ideal $I \cap D$ of D and hence an open subset U_I of $G^{(0)}$ on which it is supported. This U_I is invariant in the sense that if $s(\gamma) \in U_I$ then $r(\gamma) \in U_I$. If I is generated as an ideal by $I \cap D$, we call it a *dynamical ideal*. The assignment $I \mapsto U_I$ is a lattice isomorphism between dynamical ideals of $C^*(G)$ and open invariant sets of $G^{(0)}$, giving a complete description of the dynamical ideals. In particular, by identifying the *essentially principle* (now sometimes referred to instead as *strongly effective*) and amenable groupoids for which every ideal of $C_r^*(G)$ is dynamical, Renault gives a complete description of the ideal structure for this class of groupoid C^*-algebras [22].

The second component of the analysis is more complicated. It amounts to understanding the collection of all possible ideals that have fixed intersection with D. For full C^*-algebras, this is, in general, hopelessly intractable: there is a zoo of ideals contained in the kernel of the regular representation, which has trivial intersection with D, alone. So, we are led to restrict our attention to reduced C^*-algebras. Another problem arises almost immediately: given an open invariant set U, the restriction map $f \mapsto f|_{G\setminus G|_U}$ on $C_c(G)$ extends to a homomorphism $C_r^*(G) \to C_r^*(G \setminus G|_U)$ whose kernel contains the dynamical ideal I_U associated to U. In the setting of full C^*-algebras, this containment is an equality, but for reduced C^*-algebras it need not be: as Willet's example [28] shows, the quotient $C_r^*(G)/I_U$ can coincide with the full C^*-algebra $C^*(G \setminus G|_U)$, and we encounter the same zoo of ideals as before. So, we restrict attention further to groupoids that are inner-exact in the sense that $C_r^*(G)/I_U \cong C_r^*(G \setminus G|_U)$ for every open invariant set U. Lest this seem overly restrictive, note that this includes all amenable étale groupoids G, and therefore all nuclear étale groupoid C^*-algebras [4].

In this setting, existing results rely, explicitly or otherwise, on a kind of sandwiching lemma. This technique was developed by an Huef and Raeburn [13] to analyse Cuntz–Krieger algebras. Here the dynamical ideals are better known as *gauge-invariant ideals* (see Proposition 3.9). To understand the ideals of a Cuntz–Krieger algebra, an Huef and Raeburn concentrate on primitive ideals and demonstrate that for each primitive ideal I there are a unique smallest gauge-invariant ideal K containing I and largest gauge-invariant ideal J contained in I. They then analyse the quotient K/J, which is itself Morita equivalent to a graph algebra (but of a graph consisting of just one vertex and one edge). The C^*-algebra of this graph is $C(\mathbb{T})$, so its ideal structure is well-understood, and their analysis proceeds from there. A similar idea was used in [12], and again in [15] to understand ideal structure first for graph C^*-algebras and then for topological-graph algebras, viewed as C^*-algebras associated to singly generated irreversible dynamics.

Another instance of the same idea appears in Ara and Lolk's very interesting work on partial actions [5]. They identify a relative strong topological freeness condition that generalises Renault's topologically principle condition in the setting of transformation groupoids for partial actions. They show that relative strong topological freeness guarantees the existence of an *obstruction*

ideal: a smallest dynamical ideal of $C_r^*(G)$ that contains every ideal with trivial intersection with D. This can again be regarded as a kind of sandwiching result, but with the quantifiers switched: there exists a pair of dynamical ideals, namely the zero ideal and the obstruction ideal, that sandwich every ideal that has trivial intersection with D. One of our motivations in writing this paper is that, because this particular aspect of Ara and Lolk's paper appears as a technical step along the way to their main objective, it is in danger of receiving less attention than we think it deserves, and we want to advertise the idea more broadly.

In this paper, we take up the idea of the sandwiching lemma and of Ara and Lolk's relative strong topological freeness condition and obstruction ideal. We first establish a general sandwiching lemma for groupoid C^*-algebras (Lemma 3.4): given any inner-exact locally compact Hausdorff étale groupoid G, and any ideal I of $C_r^*(G)$, there are a unique smallest dynamical ideal K containing I and largest dynamical ideal contained in I. As a result the ideals of $C_r^*(G)$ are parameterised by triples (U, V, J) consisting of open invariant sets $U \subseteq V \subseteq G^{(0)}$, and an ideal J of $C_r^*(G|_V \setminus G|_U)$ that has trivial intersection with D and vanishes nowhere on $G|_V \setminus G|_U$ (Theorem 3.7).

We then adapt Ara and Lolk's notions of topological freeness and strong topological freeness at a point (see also Renault's notion of discretely trivial isotropy [22]), and of relative strong topological freeness, from their setting of partial actions of groups to the setting of étale groupoids. We identify a condition on étale groupoids, which we phrase as being *jointly effective where they are effective*, that ensures that $C_r^*(G)$ admits an obstruction ideal in the sense of Ara and Lolk (see Theorem 4.12 and Corollary 4.14). We also show that this obstruction ideal is minimal in the strong sense that there exists an ideal that has trivial intersection with D and whose support exhausts the support of the obstruction ideal. We show that any groupoid whose isotropy groups are all either trivial or infinite cyclic is jointly effective where it is effective. This includes all graph groupoids and groupoids arising from single local homeomorphisms. In our companion paper [7], we show how to use our results to give a complete description of the ideal structure of a large class of Deaconu–Renault groupoid C^*-algebras, including those considered in [12, 13, 15] and all C^*-algebras of rank-2 graphs.

The paper is arranged as follows. We introduce the background we need in Section 2. In Section 3, we prove our sandwiching lemma and explore its consequences. In Section 4, we introduce the notions of a groupoid being effective at a unit, strongly effective at a unit, and being strongly effective where it is effective. We then prove that such groupoids admit an obstruction ideal, and discuss some examples. Finally in Section 5, we present examples of groupoids that are jointly effective where they are effective, and describe the support of the obstruction ideal; in particular, we devote Subsection 5.2 to showing exactly how our work in Section 4 generalises the ideas of Ara and Lolk.

2 | PRELIMINARIES

2.1 | Hausdorff étale groupoids

We will always be working with topological groupoids that are locally compact, Hausdorff, and étale, and we shall adopt most of the notation and terminology from [25] (see also [21]).

We consider the unit space $G^{(0)}$ as a locally compact Hausdorff subspace of G, and we denote range and source maps by $r, s\colon G \to G^{(0)}$. A *bisection* is a subset B of G such that both r and s

restrict to injective maps on B. That G is Hausdorff means that the unit space $G^{(0)}$ is a closed subset of G, and that G is étale (in the sense that the range and source maps are local homeomorphisms) implies that $G^{(0)}$ is also open, that G has a basis consisting of open bisections, and the range and source fibres over a unit $x \in G^{(0)}$ given by $G^x = \{\gamma \in G : r(\gamma) = x\}$ and $G_x = \{\gamma \in G : s(\gamma) = x\}$, respectively, are discrete in the relative topology. In particular, the isotropy group over a unit $x \in G^{(0)}$ given as the intersection

$$\mathcal{I}(G)_x = G^x \cap G_x = \{\gamma \in G : r(\gamma) = x = s(\gamma)\}$$

is a discrete subgroup of G. A unit x is said to have trivial isotropy if $\mathcal{I}(G)_x = \{x\}$. The isotropy subgroupoid is then the group bundle

$$\mathcal{I}(G) = \bigsqcup_{x \in G^{(0)}} \mathcal{I}(G)_x = \{\gamma \in G : r(\gamma) = s(\gamma)\}.$$

We write $\mathcal{I}^\circ(G)$ for the topological interior of the isotropy of G. A Hausdorff groupoid G is said to be *effective* if $\mathcal{I}^\circ(G) = G^{(0)}$, that is, the interior of the isotropy subgroupoid with the subspace topology coincides with the unit space. When G is second-countable this coincides (using a Baire category argument) with the notion of G being *topologically principal* in the sense that $G^{(0)}$ has a dense set of points with trivial isotropy.

2.2 | Reduced groupoid C*-algebra

We will be working with the reduced groupoid C^*-algebras of locally compact Hausdorff étale groupoids. We follow the exposition of [25].

The convolution algebra $C_c(G)$ of a locally compact Hausdorff étale groupoid G is the set of compactly supported and complex-valued functions on G equipped with the convolution product

$$f * g(\gamma) = \sum_{\alpha \in G^{r(\gamma)}} f(\alpha) g(\alpha^{-1}\gamma)$$

for all $f, g \in C_c(G)$ and $\gamma \in G$, and the involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$ for all $f \in C_c(G)$ and $\gamma \in G$. Each unit $x \in G^{(0)}$ determines a regular representation $\pi_x : C_c(G) \to B(\ell^2(G_x))$ given by

$$\pi_x(f)\delta_\gamma = \sum_{\alpha \in G_{r(\gamma)}} f(\alpha)\delta_{\alpha\gamma},$$

for all $f \in C_c(G)$ and $\gamma \in G_x$. The *reduced groupoid C*-algebra* $C_r^*(G)$ of G is the completion of $\bigoplus_{x \in G^{(0)}} \pi_x(C_c(G))$ in $\bigoplus_{x \in G^{(0)}} B(\ell^2(G_x))$. As the unit space $G^{(0)}$ is both open and closed in G, the commutative algebra $C_0(G^{(0)})$ sits naturally as a subalgebra of $C_r^*(G)$, and we refer to $C_0(G)$ as the diagonal subalgebra. Note that $C_0(G^{(0)})$ need not be a C^*-diagonal (in the sense of Kumjian [16]) nor a Cartan subalgebra (in the sense of Renault [23]).

Renault [21, Proposition II.4.2] shows (Renault makes the standing assumption that the groupoids considered there are second-countable, but that assumption is not needed for the following) that any element in the reduced groupoid C^*-algebra may be thought of as a function on the groupoid. More precisely, there exists a linear and norm-decreasing map $j : C_r^*(G) \to C_0(G)$

given by

$$j(a)(\gamma) = \left(\pi_{s(\gamma)}(a)\delta_{s(\gamma)} \mid \delta_\gamma\right)$$

for all $a \in C_r^*(G)$ and $\gamma \in G$, and j is the identity on $C_c(G)$. The reduced groupoid C^*-algebra admits a faithful conditional expectation $E : C_r^*(G) \to C_0(G^{(0)})$ onto the diagonal given by restriction of functions in the sense that $j(E(a)) = j(a)|_{G^{(0)}}$ for all $a \in C_r^*(G)$ [25, Proposition 10.2.6]. Renault shows that for $a, b \in C_r^*(G)$, the convolution formula for $j(a) * j(b)$ is a convergent series that converges to $j(a * b)$.

A subset U of $G^{(0)}$ is G-invariant (or simply invariant) if $r(GU) \subseteq U$, and the reduction of G to U is $G|_U = \{\gamma \in G : r(\gamma), s(\gamma) \in U\}$. If U is an open and invariant subset of $G^{(0)}$, then $G|_U$ is an open subgroupoid of G (and hence locally compact, Hausdorff, and étale), and the inclusion $i_U : C_c(G|_U) \to C_c(G)$ extends to an injective *-homomorphism $i_U : C_r^*(G|_U) \to C_r^*(G)$. We let $I_U := i_U(C_r^*(G|_U))$ be the image of i_U in $C_r^*(G)$. This is an ideal with the property that $I_U \cap C_0(G^{(0)}) = C_0(U)$ and I_U is generated as an ideal by $C_0(U)$. We shall refer to such ideals as *dynamical ideals* (see Definition 3.1).

The complement $G^{(0)} \setminus U$ is a closed invariant set of units, and there is a *-homomorphism $\pi_U : C_r^*(G) \to C_r^*(G|_{G^{(0)}\setminus U})$ determined by $\pi_U(f) = f|_{G|_{G^{(0)}\setminus U}}$ for all $f \in C_c(G)$.

Given an ideal I in $C_r^*(G)$, we write $\operatorname{supp}(I) := \{\gamma \in G : j(I)(\gamma) \neq \{0\}\}$. So, $\operatorname{supp}(I_U) = G|_U$ for every open invariant $U \subseteq G^{(0)}$ (for completeness, we prove this in Proposition 3.3).

Lemma 2.1. *Let G be a locally compact Hausdorff étale groupoid. Suppose that I is an ideal of $C_r^*(G)$. Then* $\operatorname{supp}(I)$ *is invariant under multiplication and inversion in G.*

Proof. Fix $\gamma \in \operatorname{supp}(I)$ and take $\alpha \in G_{r(\gamma)}$ and $\beta \in G_{s(\gamma)}$. Fix $a \in I$ such that $j(a)\gamma \neq 0$. Take open bisections B and C containing α and β, respectively, and take $h \in C_c(B)$ and $k \in C_c(C)$ with $h(\alpha) = k(\beta) = 1$. Then $j(hak)(\alpha\gamma\beta) = j(a)(\gamma) \neq 0$, so $\alpha\gamma\beta \in \operatorname{supp}(I)$. Putting $\beta = s(\gamma)$ gives invariance under left multiplication, putting $\alpha = r(\gamma)$ gives invariance under right multiplication, and putting $\alpha = \beta = \gamma^{-1}$ gives invariance under inversion. □

The groupoid G is *inner-exact* if, for every open invariant subset $U \subseteq G^{(0)}$, the resulting sequence

$$0 \to C_r^*(G|_U) \to C_r^*(G) \to C_r^*(G|_{G^{(0)}\setminus U}) \to 0 \tag{2.1}$$

is exact, (see [2, Definition 3.7] and also [6, Definition 3.5]). Any amenable groupoid is inner exact. Combining [3, Proposition 4.23 and Theorem 7.10], we also see that the (partial) crossed product groupoid of an exact group acting (partially) on a second-countable locally compact Hausdorff space is inner-exact. Willett's example of a non-amenable groupoid whose full and reduced C^*-algebras coincide is not inner-exact [28].

Remark 2.2. The empty set satisfies the axioms defining a locally compact Hausdorff étale groupoid. By convention, we take the C^*-algebra of the empty groupoid to be the zero C^*-algebra; in particular (2.1) collapses to the exact sequence $0 \to C_r^*(G) \to C_r^*(G) \to 0$ if $U \in \{\varnothing, G^{(0)}\}$. We thank the referee for pressing us on this point.

3 | A SANDWICHING LEMMA FOR HAUSDORFF ÉTALE GROUPOIDS

The characterisations of the primitive-ideal spaces of graph C^*-algebras of [12] and [13] were founded on the 'sandwiching lemmas' [12, Lemma 2.6] and [13, Lemma 4.5] that show that every primitive ideal is sandwiched between a pair of uniquely determined gauge-invariant ideals. Here we observe that a similar sandwiching lemma holds for ideals of reduced Hausdorff étale groupoid C^*-algebras.

Definition 3.1. We say that an ideal I in a reduced groupoid C^*-algebra $C_r^*(G)$ is *dynamical* if it is generated as an ideal by its intersection with the diagonal subalgebra $C_0(G^{(0)})$. Equivalently, I is dynamical if it is of the form I_U for an open invariant subset U of $G^{(0)}$. We say that I is *purely non-dynamical* if $I \cap C_0(G^{(0)}) = \{0\}$.

Remark 3.2. According to Definition 3.1, the trivial ideal $\{0\}$ is the unique ideal of $C_r^*(G)$ that is both a dynamical ideal and a purely non-dynamical ideal. Though linguistically unsatisfactory, this convention simplifies the statements of our key results: in Proposition 3.3 treating $\{0\}$ as a dynamical ideal avoids treating the open invariant set $\emptyset$ as a special case; but later in Theorem 3.7, treating $\{0\}$ as a purely non-dynamical ideal avoids treating dynamical ideals as a special case (see Remark 3.8).

In the context of Deaconu–Renault groupoids, the dynamical ideals are precisely the usual gauge-invariant ideals (see Proposition 3.9).

Proposition 3.3. *Let G be a locally compact Hausdorff étale groupoid. The map $U \mapsto I_U$ is a lattice isomorphism from the lattice of open invariant subsets of X to the lattice of dynamical ideals of $C_r^*(G)$. For each open invariant $U \subseteq G^{(0)}$, we have $I_U \cap C_0(G^{(0)}) = C_0(U)$, and* $\mathrm{supp}(I_U) = G|_U$.

Proof. The map $U \mapsto I_U$ is always an injection [25, Theorem 10.3.3], and surjectivity follows from the definition of dynamical ideals. Proposition 10.3.2 of [25] shows that I_U is the closure of $C_c(G|_U) \subseteq C_c(G)$. In particular, $\mathrm{supp}(I_U) \subseteq G|_U$, and $I_U \cap C_0(G^{(0)}) \subseteq C_0(U)$ by continuity of $j(a)$ for each $a \in C_r^*(G)$. The reverse containments hold because if $\gamma \in G|_U$, then there is a map $f \in C_c(G|_U) \subseteq I_U$ such that $f(\gamma) \neq 0$, so $\gamma \in \mathrm{supp}(I_U)$, and $C_c(U)$ is contained in $C_c(G|_U)$, so $C_0(U)$ is contained in I_U. □

As lattice isomorphisms preserve least upper bounds and greatest lower bounds, it follows from Proposition 3.3 that, for example, $I_U \cap I_V = I_{U\cap V}$ and $I_U + I_V = I_{U\cup V}$ for all open invariant U and V.

We now state our sandwiching lemma.

Lemma 3.4 (The sandwiching lemma). *Let G be a locally compact Hausdorff étale groupoid that is inner-exact and let I be an ideal of $C_r^*(G)$. Consider the open and invariant subsets*

$$U = \{x \in G^{(0)} : f(x) \neq 0 \text{ for some } f \in I \cap C_0(G^{(0)})\}$$

and

$$V = \{x \in G^{(0)} : j(a)(x) \neq 0 \text{ for some } a \in I\}.$$

Then I_U is the largest dynamical ideal of $C_r^*(G)$ contained in I and I_V is the smallest dynamical ideal of $C_r^*(G)$ containing I.

Proof. The set U is open because every $f \in C_0(G^{(0)})$ is continuous. To see that U is invariant, take $x \in U$ and fix $\gamma \in G_x$. We will show that $r(\gamma) \in U$. As $x \in U$ there exists $f \in I \cap C_0(G^{(0)})$ such that $f(x) \neq 0$. Let B be an open bisection containing γ and fix $h \in C_c(B)$ such that $h(\gamma) = 1$. Then $hfh^* \in I \cap C_c(r(B)) \subseteq I \cap C_0(G^{(0)})$. Moreover, $hfh^*(r(\gamma)) = h(\gamma)f(x)h^*(\gamma^{-1}) = f(x) \neq 0$, so we conclude that $r(\gamma) \in U$.

For each $x \in U$, choose $f_x \in I \cap C_0(G^{(0)})$ such that $f_x(x) \neq 0$. Then $\{f_x : x \in U\}$ generates $C_0(U)$ as an ideal of $C_0(G^{(0)})$, and it is contained in I. Hence, $I_U \subseteq I$. Suppose that U' is an open subset of $G^{(0)}$ strictly containing U and fix $x \in U' \setminus U$ and $f \in C_c(U')$ with $f(x) \neq 0$. Then $f \in I_{U'}$ but $f \notin I_U$ by definition of I_U. In particular, $f \notin I$, so $I_{U'} \nsubseteq I$. This proves that I_U is the largest dynamical ideal contained in I.

The set V is open because $j(a)$ is continuous for every $a \in C_r^*(G)$. We claim that $V = s(\mathrm{supp}(I))$. That $V \subseteq s(\mathrm{supp}(I))$ is obvious. For the reverse inclusion, suppose that $a \in I$ and $j(a)(\gamma) \neq 0$. For any open bisection B containing γ^{-1} and any $f \in C_c(B)$ satisfying $f(\gamma^{-1}) = 1$, we have $j(fa)(s(\gamma)) = j(a)(\gamma) \neq 0$. As $j(a)(\gamma) = \overline{j(a^*)(\gamma^{-1})}$, we have $r(\gamma) \in V$ if and only if $s(\gamma) \in V$, so V is invariant.

We now show that $I \subseteq I_V$. Let $E : C_r^*(G) \to C_0(G^{(0)})$ be the faithful conditional expectation onto the diagonal and observe that $E(I) \subseteq C_0(V)$. As G is inner-exact, it follows from [6, Lemma 3.6] that I is contained in the ideal in $C_r^*(G)$ generated by $E(I)$, so we find that $I \subseteq I_V$ as wanted. To see that V is minimal with this property, suppose that $V' \subsetneq V$ is an open invariant set. By definition of V there exists $x \in V \setminus V'$ and $a \in I$ such that $j(a)(x) \neq 0$. Hence, $\mathrm{supp}(I) \nsubseteq \mathrm{supp}(I_{V'})$, so $I \nsubseteq I_{V'}$. □

Remark 3.5. If the ideal I in Lemma 3.4 is a purely non-dynamical ideal of $C_r^*(G)$, then U is empty, and then $I_U = \{0\}$; if I is a dynamical ideal, then $V = U$ and $I_U = I$.

Consider a pair of nested open invariant subsets $U \subseteq V \subseteq G^{(0)}$. Recall that we obtain C^*-homomorphisms $\imath_V : C_r^*(G|_V) \to C_r^*(G)$ and $\imath_{V\setminus U} : C_r^*(G|_{V\setminus U}) \to C_r^*(G|_{G^{(0)}\setminus U})$ extending the canonical inclusion of algebras of compactly supported functions. For these maps, the diagram

$$\begin{array}{ccccc} C_r^*(G|_U) & \xrightarrow{\imath_U} & C_r^*(G) & \xrightarrow{\pi_U} & C_r^*(G|_{G^{(0)}\setminus U}) \\ & & \uparrow{\scriptstyle \imath_V} & & \uparrow{\scriptstyle \imath_{V\setminus U}} \\ C_r^*(G|_U) & \xrightarrow[\imath_U^V]{} & C_r^*(G|_V) & \xrightarrow[\pi_U^V]{} & C_r^*(G|_{V\setminus U}) \end{array}$$

commutes.

Lemma 3.6. *Let G be a locally compact Hausdorff étale groupoid that is inner-exact. Let I be an ideal of $C_r^*(G)$ and let U and V be the open invariant sets of Lemma 3.4. Then $J := \pi_U^V(\imath_V^{-1}(I))$ is an ideal in $C_r^*(G|_{V\setminus U})$ that is purely non-dynamical and has full support.*

Proof. It is clear that J is an ideal of $C_r^*(G|_{V\setminus U})$. To see that J is purely non-dynamical, take $f \in J \cap C_c(V \setminus U)$. Pick $\widetilde{f} \in \pi_U^{-1}(f)$ and note that $\widetilde{f} \in C_0(V)$ extends f (because π_U^V implements restriction of functions). Then $\iota_V(\widetilde{f}) \in I$ by definition of J. If $x \in V \setminus U$, then $\iota_V(\widetilde{f})(x) = 0$ by definition of U, so $f(x) = 0$. Hence, $f = 0$. So, J is purely non-dynamical.

Next we show that J has full support. Clearly, $\mathrm{supp}(J) \subseteq G|_{V\setminus U}$ (as J is an ideal of $C_r^*(G|_{V\setminus U})$). We must prove the reverse inclusion. Fix $\gamma \in G$ with $s(\gamma) \in V \setminus U$. As $V = s(\mathrm{supp}(I))$, there exists $a \in I$ such that $j(a)(\gamma) \neq 0$. The inclusion map $C_r^*(G|_V) \to I_V \subseteq C_r^*(G)$ extends the canonical inclusion $C_c(G_V) \to C_c(G)$, so it intertwines the maps $j^V : C_r^*(G|_V) \to C_0(G|_V)$ and $j : C_r^*(G) \to C_0(G)$. Therefore, $j^V(\iota_V^{-1}(a))(\gamma) = j(a)(\gamma) \neq 0$, and we conclude that $\mathrm{supp}(J) = G|_{V\setminus U}$. □

Let $\mathcal{T}(G)$ be the collection of triples (U, V, J) where $U \subseteq V \subseteq G^{(0)}$ are nested open and invariant subsets and J is a purely non-dynamical ideal in $C_r^*(G|_{V\setminus U})$ with full support.

Theorem 3.7. *Let G be a locally compact Hausdorff étale groupoid that is inner-exact. There is a bijection Θ from $\mathcal{T}(G)$ to the collection of ideals of $C_r^*(G)$ such that*

$$\Theta(U, V, J) = \pi_U^{-1}(\iota_{V\setminus U}(J))$$

for all $(U, V, J) \in \mathcal{T}(G)$. The inverse Θ^{-1} takes $I \lhd C_r^*(G)$ to the triple $(U, V, J) \in \mathcal{T}(G)$ consisting of the sandwich sets $U \subseteq V$ and the purely non-dynamical ideal $J \lhd C_r^*(G|_{V\setminus U})$ with full support of Lemma 3.4.

Remark 3.8. It is important in the statement of Theorem 3.7 that $\emptyset$ is a groupoid, that its reduced C^*-algebra is $\{0\}$, and that $\{0\}$ is a purely non-dynamical ideal of $C_r^*(G)$: the dynamical ideals of $C_r^*(G)$ are in the range of Θ because each $I_U = \Theta(U, U, \{0\})$.

Proof of Theorem 3.7. The map Θ takes values in the ideals of $C_r^*(G)$ by definition.

To see that Θ is injective, fix $(U, V, J) \in \mathcal{T}(G)$ and let $I = \Theta(U, V, J)$. We will prove that U and V are the sandwiching sets U_I, V_I obtained from Lemma 3.4 applied to I, and that $J = \iota_{V\setminus U}^{-1}(\pi_U(I))$. This defines a left inverse to Θ, defined on the image of Θ, which implies that Θ is injective.

We have $I_U = \pi_U^{-1}(0) \subseteq \Theta(U, V, J)$ by definition of Θ. If $U' \subseteq G^{(0)}$ is an open invariant set containing U such that $I_{U'} \subseteq \Theta(U, V, J)$, then $\pi_U(I_{U'}) \subseteq \pi_U(\Theta(U, V, J)) = \iota_{V\setminus U}(J)$ and the latter has trivial intersection with $G^{(0)} \setminus U$ (as J is purely non-dynamical). As π_U implements restriction of functions, we see that $I_{U'} \cap C_0(G^{(0)}) \subseteq C_0(U)$, so $U' = U$. Let E be the faithful conditional expectation of $C_r^*(G)$ onto $C_0(G^{(0)})$. Observe that $E(\Theta(U, V, J)) \subseteq C_0(V)$. By [6, Lemma 3.6], $\Theta(U, V, J)$ is contained in the ideal generated by $E(\Theta(U, V, J))$, so we see that $\Theta(U, V, J) \subseteq I_V$. In particular, $\mathrm{supp}(\Theta(U, V, J)) \subseteq \mathrm{supp}(I_V) = G|_V$. On the other hand, as $\mathrm{supp}(J) = G|_{V\setminus U}$ we have $G|_V \subseteq \mathrm{supp}(\Theta(U, V, J))$. Now if V' is a proper open invariant subset of V such that $\Theta(U, V, J) \subseteq I_{V'}$, then $G|_V = \mathrm{supp}(\Theta(U, V, J)) = \mathrm{supp}(I_{V'}) = G|_{V'} \subsetneq G|_V$ which contradicts our observation above. Therefore, V is the smallest such open invariant subset. Finally, observe that

$$\iota_{V\setminus U}^{-1}(\pi_U(I)) = \iota_{V\setminus U}^{-1}(\pi_U(\Theta(U, V, J))) = \iota_{V\setminus U}^{-1}(\iota_{V\setminus U}(J)) = J,$$

and this completes the proof that Θ is injective.

To see that it is surjective, fix an ideal I of $C_r^*(G)$. By Lemma 3.4, there are open invariant sets $U \subseteq V \subseteq G^{(0)}$ such that $I_U \subseteq I \subseteq I_V$ and $\mathrm{supp}(\pi_U^V(I/I_U)) = G|_{V\setminus U}$. As $I \subseteq I_V = \iota_V(C^*(G|_V))$, we

obtain an ideal $\iota_V^{-1}(I)$ of $C^*(G|_V)$. Let $J := \pi_U^V(\iota_V^{-1}(I))$. We claim that $K := \Theta(U,V,J)$ is equal to I, which will establish surjectivity of Θ. By definition, both I and K are ideals of $C_r^*(G)$ that contain I_U, so it suffices to show that $I/I_U = K/I_U$. By inner-exactness, $\pi_U : C_r^*(G) \to C_r^*(G|_{G^{(0)}\setminus U})$ has kernel I_U, so it suffices to show that $\pi_U(K) = \pi_U(\Theta(U,V,J))$. By definition of Θ, we have $\pi_U(K) = \iota_{V\setminus U}(J) = \iota_{V\setminus U}(\pi_U^V(\iota_V^{-1}(I)))$. By definition of the two maps, $\iota_{V\setminus U}\circ\pi_U^V = \pi_U\circ\iota_V$, so we obtain $\pi_U(K) = \pi_U(I)$ as required. □

To link Lemma 3.4 back to the results [12, Lemma 2.6] and [13, Lemma 4.5] that inspired it, we observe that for Deaconu–Renault groupoids, the dynamical ideals employed above are precisely the gauge-invariant ideals of the C^*-algebra of a Deaconu–Renault groupoid. The result is certainly well-known, but we are not aware that it has been recorded explicitly elsewhere in this generality. For the case of finitely aligned higher rank graphs, this was observed in [20, Lemma 7.5].

Recall that if $T : \mathbb{N}^d \curvearrowright X$ is an action by d local homeomorphisms, then we let G_T denote the Deaconu–Renault groupoid of T as in, for example, [26, section 3]. An ideal I of $C^*(G_T)$ is *gauge-invariant* if the canonical gauge action γ of $\mathbb{T}^d$ on $C_r^*(G_T)$ satisfies $\gamma_z(I) \subseteq I$ for all $z \in \mathbb{T}^d$.

Proposition 3.9. *Let X be a locally compact Hausdorff space and suppose $T : \mathbb{N}^d \curvearrowright X$ is an action on X by d commuting local homeomorphisms. The map that carries an open invariant subset U of X to the ideal I_U generated by $C_0(U)$ is a lattice isomorphism from the lattice of open invariant subsets of X to the lattice of gauge-invariant ideals of $C^*(G_T)$.*

Proof. As $\gamma_z(f) = f$ for all $z \in \mathbb{T}^k$ and $f \in C_0(G^{(0)})$, the ideals of $C^*(G_T)$ generated by subsets of $C_0(G^{(0)})$ are gauge invariant. In particular, each I_U is gauge-invariant.

The map $U \mapsto I_U$ is an injection [25, Theorem 10.3.3]. For surjectivity, we follow the second paragraph of the proof of [25, Theorem 10.3.3], dropping the assumption that G is strongly effective but fixing a *gauge-invariant* ideal I, until its penultimate sentence. At that point, while GW need not be effective, we observe that GW is identical to the groupoid of the topological higher rank graph Λ defined by $\Lambda^n = X \times \{n\}$ for all n, whose range and source maps are given by $s(x,n) = (T^n(x),0)$ and $r(x,n) = (x,0)$ and with the factorisation rules $(x,m)(T^m(x),n) = (x,m+n) = (x,n)(T^n(x),m)$. We may now apply the gauge-invariant uniqueness theorem of [9, Corollary 5.21] in place of [25, Theorem 10.3.3] to see that $\widetilde{\pi}$ is injective, and the surjectivity of $U \mapsto I_U$ follows. The final statement follows from Proposition 3.3. □

4 | EFFECTIVENESS AT A UNIT AND THE OBSTRUCTION IDEAL

In this section, we introduce the notions of effectiveness at a unit and joint effectiveness at a unit for étale groupoids. The key property that emerges is that of being jointly effective where effective. This is inspired by the notions in [5, section 7] of (strong) topological freeness at a point for a partial group action. The points in the unit space of a groupoid that are not effective comprise an open invariant set and hence a dynamical ideal that we call the obstruction ideal. Our main results in this section (Theorem 4.12 and Corollary 4.14) say that if a Hausdorff étale groupoid is inner-exact and its full and reduced C^*-algebras coincide (Anantharaman–Delaroche calls this the weak containment property [2]), then the obstruction ideal contains all purely non-dynamical ideals, and is minimal with this property.

Recall that a groupoid G is *effective* if the interior $\mathcal{I}^\circ(G)$ of the isotropy is equal to the unit space $G^{(0)}$. For $x \in G^{(0)}$, we write $\mathcal{I}^\circ(G)_x$ for the intersection of G_x with $\mathcal{I}^\circ(G)$.

Definition 4.1. A locally compact Hausdorff étale groupoid G is *effective at a unit* $x \in G^{(0)}$ if $\mathcal{I}^\circ(G)_x = \{x\}$. Equivalently, G is effective at x if for any non-trivial isotropy element $\gamma \in \mathcal{I}(G)_x \setminus \{x\}$ and any open bisection B in $G \setminus G^{(0)}$ containing γ there exists $y \in s(B)$ such that $r(By) \neq y$. When the groupoid is understood, we may just say that the unit is effective. We let $G^{(0)}_{\mathrm{eff}}$ denote the collection of effective units.

Any unit with trivial isotropy is effective. An isolated unit with non-trivial isotropy is not effective.

We have the following general description of the units that are not effective. This also shows that our terminology is consistent with the literature on effective groupoids.

Lemma 4.2. *Let G be a locally compact Hausdorff étale groupoid. We have*

$$G^{(0)} \setminus G^{(0)}_{\mathrm{eff}} = s(\mathcal{I}^\circ(G) \setminus G^{(0)}), \tag{4.1}$$

and this is an open and invariant subset of $G^{(0)}$. Consequently, $G^{(0)}_{\mathrm{eff}}$ is closed and invariant. Moreover, G is effective if and only if G is effective at each of its units.

Proof. Suppose that G is not effective at $x \in G^{(0)}$. Then x has non-trivial isotropy and any $\gamma \in \mathcal{I}^\circ(G)_x \setminus \{x\}$ is contained in an open bisection B in $\mathcal{I}^\circ(G) \setminus G^{(0)}$. Therefore, $s(B)$ is an open neighbourhood of x consisting of points that are not effective, so $G^{(0)} \setminus G^{(0)}_{\mathrm{eff}}$ is open and contained in $s(\mathcal{I}^\circ(G) \setminus G^{(0)})$. On the other hand, if $\gamma \in \mathcal{I}^\circ(G) \setminus G^{(0)}$, then there is an open bisection B in $\mathcal{I}^\circ(G) \setminus G^{(0)}$ containing γ. If $x = s(\gamma)$, this means that $\mathcal{I}^\circ(G)_x \neq \{x\}$, so G is not effective at x.

To see invariance, let $x \in s(\mathcal{I}^\circ(G) \setminus G^{(0)})$ and take $\gamma \in G$ with $x = s(\gamma)$ and $r(\gamma) = z \neq x$. We will show that z is not effective. Let $\eta \in \mathcal{I}^\circ(G) \setminus G^{(0)}$ with $s(\eta) = x = r(\eta)$. Choose an open bisection B_γ in $G \setminus G^{(0)}$ containing γ and an open bisection B_η in $\mathcal{I}(G)^\circ \setminus G^{(0)}$ containing η. Then $B_\gamma B_\eta B_\gamma^{-1}$ is an open bisection containing $\gamma\eta\gamma^{-1}$ (which is isotropy over z), and it consists only of isotropy elements, because B_η consists only of isotropy elements. Therefore, $B_\gamma B_\eta B_\gamma^{-1} \subseteq \mathcal{I}(G)^\circ \setminus G^{(0)}$ and $z \in s(B_\gamma B_\eta B_\gamma^{-1})$, so z is not effective.

The final statement is a direct consequence of (4.1). □

The obstruction ideal defined below will play a central role in Theorem 4.12.

Definition 4.3. Let G be a locally compact Hausdorff étale groupoid. The set of all units that are not effective is an open and invariant subset of $G^{(0)}$, so it determines a dynamical ideal $I_{G^{(0)} \setminus G^{(0)}_{\mathrm{eff}}}$ of $C^*_r(G)$. We call this the *obstruction ideal* and denote it by J^{ob}. This terminology is explained in Remark 4.16.

We let G_{eff} denote the reduction of G to the closed invariant subset of effective points. The unit space of G_{eff} then coincides with $G^{(0)}_{\mathrm{eff}}$.

We require a groupoid analogue of the notion of strong topological freeness introduced in [5, section 7].

Definition 4.4. A locally compact Hausdorff groupoid G is *jointly effective* at a unit $x \in G^{(0)}$ if for any finite collection of non-trivial isotropy elements $\gamma_1, \dots, \gamma_n \in \mathcal{I}(G)_x \setminus \{x\}$ and any open bisections $B_1, \dots, B_n$ in $G \setminus G^{(0)}$ such that $\gamma_i \in B_i$ there exists $y \in \bigcap_{i=1}^n s(B_i)$ such that $r(B_i y) \neq y$ for all $i = 1, \dots, n$.

Remark 4.5. If G is effective, then it is jointly effective at every unit. More generally, any unit in an open set of effective points is jointly effective.

For the first assertion, suppose that G is effective, and fix $x \in G^{(0)}$ and $\gamma_1, \dots, \gamma_n \in \mathcal{I}(G)_x \setminus \{x\}$. Fix open bisections B_i in $G \setminus G^{(0)}$ containing γ_i. By shrinking if necessary, we can assume that $W := s(B_i) = s(B_j)$ for all i, j. As G is effective, each $B_i \cap \mathcal{I}(G)$ has empty interior. So, for each i, the set $W_i := s(B_i \setminus \mathcal{I}(G))$ is open and dense in W. Hence, $\bigcap_i W_i$ is open and dense, and in particular non-empty. Now any $y \in \bigcap_i W_i$ satisfies $r(B_i y) \neq y$ for all i.

For the second assertion, suppose only that U is an open subset of $G^{(0)}$ contained in $G^{(0)}_{\mathrm{eff}}$, and fix $x \in U$. As $G^{(0)}_{\mathrm{eff}}$ is invariant, $V := r(GU)$ is open and invariant with $U \subseteq V \subseteq G^{(0)}_{\mathrm{eff}}$. The first assertion applied to $G|_V$ shows that x is jointly effective in $G|_V$, and hence in G.

It is possible for a groupoid to be effective at a unit but not jointly effective at that unit (see Example 4.8).

This leads us to an analogue of Ara and Lolk's notion of relative strong topological freeness.

Definition 4.6. Let G be a locally compact Hausdorff étale groupoid. We say that G is *jointly effective where it is effective* if G is jointly effective at every point in $G^{(0)}_{\mathrm{eff}}$.

Examples 4.7.

(1) By Remark 4.5, if G is effective then it is jointly effective where it is effective. In particular, if G is principal, then it is jointly effective where it is effective.
(2) Suppose G is a Hausdorff étale group bundle (for example, G is a non-trivial discrete group). As $G^{(0)}$ is clopen, G is effective at $x \in G^{(0)}$ if and only if $G_x = \{x\}$. As G is trivially effective at x when $G_x^x = \{x\}$, it follows that G is jointly effective where it is effective. We have $G^{(0)}_{\mathrm{eff}} = \{x : G_x = \{x\}\}$, and the obstruction ideal is generated by $C_0(\{x : G_x \neq \{x\}\})$.
(3) In particular, Willett's groupoid [28] consists entirely of isotropy, and hence is jointly effective where it is effective. It is not inner-exact. The obstruction ideal is the whole reduced groupoid C^*-algebra.

The next examples show that groupoids need not be jointly effective where they are effective and that the property of being jointly effective where effective does not necessarily pass to reductions to closed invariant subsets. This latter permanence property does hold in groupoids all of whose non-trivial isotropy groups are infinite cyclic (see Subsection 5.1).

Example 4.8 (Exel's cross). Let $X = ([-1,1] \times \{0\}) \cup (\{0\} \times [-1,1])$ and consider the two homeomorphisms φ and ψ on X given by $\varphi(x,y) = (-x,y)$ and $\psi(x,y) = (x,-y)$ for all $(x,y) \in X$. These commuting order-two homeomorphisms define an action $\varphi \oplus \psi : \mathbb{Z}/2 \oplus \mathbb{Z}/2\mathbb{Z} \curvearrowright X$. Let $G_{\varphi\oplus\psi}$ be the transformation groupoid $X \rtimes (\mathbb{Z}/2\mathbb{Z})^2$. To keep notation from getting too confusing, we regard $(\mathbb{Z}/2\mathbb{Z})^2$ as the abelian group with four elements $\{e, a, b, ab\}$ (so the group operation is written multiplicatively), so that $a = (1,0)$ and $b = (0,1)$ are the order-two generators.

In this example, the interior of the isotropy $\mathcal{I}^\circ(G_{\varphi\oplus\psi})$ is

$$(X\times\{e\})\cup(((([-1,0)\cup(0,1])\times\{0\})\times\{b\})\cup((\{0\}\times([-1,0)\cup(0,1]))\times\{a\}),$$

and the only effective unit is $(0,0)\in X$.

Every point in X has non-trivial isotropy (so $G_{\varphi\oplus\psi}$ is not effective). More specifically, the isotropy group of every point that is not the origin is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ while the isotropy group at the origin is isomorphic to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. The origin is the only point that is effective, but it is not jointly effective. Therefore, $G_{\varphi\oplus\psi}$ is not jointly effective where it is effective.

Ara and Lolk [5, section 7] exhibit an example of a partial action that shows that their relative strong topological freeness is not automatic, and their example can be adapted to our groupoid setting.

Example 4.9. We can extend Exel's cross to see that being jointly effective where effective does not pass to closed invariant subgroupoids. To see this, let X be as in Exel's cross, and let $Y = X\times[-1,1]$.

Extend φ and ψ to homeomorphisms $\widetilde{\varphi}$ and $\widetilde{\psi}$ on Y by $\widetilde{\varphi}(x,t)=(\varphi(x),-t)$ and similarly $\widetilde{\psi}(x,t)=(\psi(x),-t)$. Again we regard these as determining an action of $\mathbb{Z}_2\oplus\mathbb{Z}_2=\{e,a,b,ab\}$ on Y.

Neither a nor b fixes any point in $Y\setminus X$ because both invert the t-coordinate. As the only point in X fixed by ab is the point $(0,0)\in X$, the only points in Y fixed by ab are those of the form $((0,0),t)$. So, $G_{\widetilde{\varphi},\widetilde{\psi}} := Y\rtimes(\mathbb{Z}/2\mathbb{Z})^2$ is effective, and in particular jointly effective where it is effective. However, its reduction to the closed invariant set X is Exel's cross, which is not jointly effective where it is effective.

The next lemma is an easy adaptation of [10, Lemma 29.4] from partial actions of groups to groupoids, so we give just a fairly succinct proof.

Lemma 4.10. *Let G be a Hausdorff étale groupoid, let $x\in G^{(0)}$ be a unit, and let B be an open bisection such that $B\cap\mathcal{I}(G)_x=\varnothing$. Let $f\in C_c(G)$ be such that f has support in B. Given $\varepsilon>0$ there exists $h\in C_0(G^{(0)})$ satisfying $0\leqslant h\leqslant 1$, h is constantly 1 on a neighbourhood of x, and $\|hfh\|<\varepsilon$.*

Proof. First suppose that $x\notin s(B)$. By Urysohn's lemma we can find $h\in C_0(G^{(0)},[0,1])$ that is 1 on a neighbourhood of x and vanishes on $\{s(\gamma):|f(\gamma)|\geqslant\varepsilon\}\subseteq s(B)$. As f is supported on a bisection, its C^*-norm agrees with its supremum norm [25, Corollary 9.3.4], and hence $\|hfh\|_{C^*(G)}=\|hfh\|_\infty<\varepsilon$.

Now suppose that $x\in s(B)$. Let γ be the unique element of B with $s(\gamma)=x$. By assumption, $r(\gamma)\neq x$ so we can choose an open set V_1 containing x such that $r(V_1)\cap V_1=\varnothing$. By Urysohn's lemma, there exists $h\in C_0(G^{(0)},[0,1])$ such that $h=1$ on a neighbourhood of x and h vanishes off V_1. In particular, $\operatorname{supp}(h)\cap r(B\operatorname{supp}(h))=\varnothing$, and so $hfh=0$. □

The next two results say that when an inner-exact groupoid G whose full and reduced C^*-algebras coincide is jointly effective where it is effective, its obstruction ideal J^{ob} is the minimal dynamical ideal that contains all purely non-dynamical ideals of $C_r^*(G)$. The proof of the first result closely follows that of [5, Theorem 7.12] (which does not require the weak containment property) with only minor modifications.

Remark 4.11. The hypothesis below that the sequence $0 \to J^{\mathrm{ob}} \to C_r^*(G) \to C_r^*(G_{\mathrm{eff}}) \to 0$ is exact holds if, for example, G is inner-exact (in particular, if it is amenable). However, it also holds trivially if G is effective, and we invoke it in that situation in Proposition 4.15. So, we have stated Theorem 4.12 accordingly.

Theorem 4.12. *Let G be a locally compact Hausdorff étale groupoid that is jointly effective where it is effective. Let J^{ob} be the obstruction ideal in $C_r^*(G)$ and suppose the sequence $0 \to J^{\mathrm{ob}} \to C_r^*(G) \to C_r^*(G_{\mathrm{eff}}) \to 0$ is exact. If I is a purely non-dynamical ideal of $C_r^*(G)$ then $I \subseteq J^{\mathrm{ob}}$.*

Proof. We suppose that $I \nsubseteq J^{\mathrm{ob}}$ and derive a contradiction. Fix $a \in I \setminus J^{\mathrm{ob}}$. In particular, $a^*a \in I \setminus J^{\mathrm{ob}}$. Let $E : C_r^*(G) \to C_0(G^{(0)})$ be the canonical faithful conditional expectation, let $G_{\mathrm{eff}} = G|_{G_{\mathrm{eff}}^{(0)}}$, and let $\pi = \pi_{G^{(0)} \setminus G_{\mathrm{eff}}^{(0)}} : C_r^*(G) \to C_r^*(G_{\mathrm{eff}})$ denote the canonical quotient map. Let E_{eff} be the canonical faithful conditional expectation associated with $C_r^*(G_{\mathrm{eff}})$. Then the diagram

$$\begin{array}{ccc} C_r^*(G) & \xrightarrow{\pi} & C_r^*(G_{\mathrm{eff}}) \\ {\scriptstyle E}\downarrow & & \downarrow{\scriptstyle E_{\mathrm{eff}}} \\ C_0(G^{(0)}) & \xrightarrow[\pi]{} & C_0(G_{\mathrm{eff}}^{(0)}) \end{array}$$

commutes. By hypothesis, $J^{\mathrm{ob}} = \ker(\pi)$, so $\pi(a^*a) \neq 0$ because $a^*a \notin J^{\mathrm{ob}}$. As E_{eff} is faithful, $E_{\mathrm{eff}}(\pi(a^*a)) \neq 0$. Hence, $0 \neq E_{\mathrm{eff}}(\pi(a^*a)) = \pi(E(a^*a))$ by commutativity of the diagram. This means that $f := E(a^*a) \in C_0(G^{(0)})$ is non-zero on $G_{\mathrm{eff}}^{(0)}$. Choose $x_0 \in G_{\mathrm{eff}}^{(0)}$ such that

$$|f(x_0)| = \sup_{x \in G_{\mathrm{eff}}^{(0)}} |f(x)|, \tag{4.2}$$

and let $0 < \varepsilon < \frac{|f(x_0)|}{2}$. The set $V = \{x \in G^{(0)} : |f(x)| < |f(x_0)| + \varepsilon/4\}$ is open in $G^{(0)}$ and contains x_0. By Urysohn's lemma we may pick a function $u \in C_0(G^{(0)})$ such that $0 \leqslant u \leqslant 1$, $u(x_0) = 1$, and u vanishes outside V. Set $z := ua^*a \in I \setminus J^{\mathrm{ob}}$ and observe that $E(z) = uE(a^*a) = uf$, and

$$2\varepsilon < |f(x_0)| \leqslant \|E(z)\| \leqslant |f(x_0)| + \varepsilon/4. \tag{4.3}$$

We claim that there exists $h \in C_0(G^{(0)})$ satisfying $0 \leqslant h \leqslant 1$, $h(x_1) = 1$ and

$$\|E(z)\| < \|hE(z)h\| + \varepsilon; \tag{4.4}$$

$$\|hE(z)h - hzh\| < \varepsilon. \tag{4.5}$$

As $z \in C_r^*(G)$, there exists $g \in C_c(G)$ such that $\|z - g\| < \varepsilon/4$. In particular, $\|E(z) - E(g)\| < \varepsilon/4$. Note that $E(g)$ is supported on $G^{(0)}$ and $g - E(g) \in C_c(G \setminus G^{(0)})$. Choose open bisections $B_1, \ldots, B_k \subseteq G \setminus G^{(0)}$ that cover $\mathrm{supp}(g - E(g))$ and write

$$g - E(g) = \sum_{i=1}^{k} g_i \tag{4.6}$$

with $g_i \in C_0(B_i)$ for each i.

For each i such that $B_i \cap \mathcal{I}(G)_{x_0} = \emptyset$, we can apply Lemma 4.10 to obtain a function $h_i \in C_0(G^{(0)}, [0,1])$ that is identically 1 on an open neighbourhood U_i of x_0 and satisfies $\|h_i g_i h_i\| \leqslant \varepsilon/2k$. Consider the open neighbourhood $U := \{x \in G^{(0)} : |E(g)(x) - E(g)(x_0)| < \varepsilon/4\}$ of x_0. As G is jointly effective at x_0 by hypothesis, there exists a unit $x_1 \in U \cap \bigcap_{B_i \cap \mathcal{I}(G)_{x_0} = \emptyset} U_i$ such that for each i satisfying $B_i \cap \mathcal{I}(G)_{x_0} \neq \emptyset$, we have $r(B_i x_1) \neq x_1$. As $x_1 \in U$, we have

$$|E(g)(x_1) - E(g)(x_0)| < \varepsilon/4, \tag{4.7}$$

and for each i such that $B_i \cap \mathcal{I}(G)_{x_0} = \emptyset$, as $x_1 \in U_i$ we have $h_i(x_1) = 1$.

For each i such that $B_i \cap \mathcal{I}(G)_{x_0} \neq \emptyset$, Lemma 4.10 for B_i at x_1 yields a function $h_i \in C_0(G^{(0)}, [0,1])$ satisfying

$$h_i(x_1) = 1 \qquad \text{and} \qquad \|h_i g_i h_i\| < \frac{\varepsilon}{2k}. \tag{4.8}$$

Altogether we have constructed functions $h_1, \ldots, h_k$ that all satisfy (4.8). Set $h := \prod_{i=1}^{k} h_i \in C_0(G^{(0)})$ and note that $0 \leqslant h \leqslant 1$ and $h(x_1) = 1$.

It remains to verify (4.4) and (4.5); we do this by direct computation. Using (4.3) and the fact that $u(x_0) = 1$, we see that

$$\|E(z)\| - \varepsilon \leqslant |f(x_0)| - 3\varepsilon/4 = |E(z)(x_0)| - 3\varepsilon/4.$$

By first using the choice of g and then the choice of x_1 from (4.7), we find

$$|E(z)(x_0)| - 3\varepsilon/4 < |E(g)(x_0)| - \varepsilon/2 < |E(g)(x_1)| - \varepsilon/4.$$

Remembering that $h(x_1) = 1$, we obtain

$$|E(g)(x_1)| - \varepsilon/4 = |(hE(g)h)(x_1)| - \varepsilon/4 \leqslant \|hE(g)h\| - \varepsilon/4 < \|hE(z)h\|.$$

This means that $\|E(z)\| - \varepsilon < \|hE(z)h\|$ so (4.4) follows. For (4.5), we use the decomposition (4.6) and then (4.8) to see that

$$\|hgh - hE(g)h\| = \Big\| \sum_{i=1}^{k} hg_ih \Big\| \leqslant \sum_{i=1}^{k} \|hg_ih\| < \varepsilon/2.$$

Hence,

$$\|hzh - hE(z)h\| \leqslant \|hzh - hgh\| + \|hgh - hE(g)h\| + \|hE(g)h - hE(z)h\| < \varepsilon,$$

and this proves (4.5).

To complete the proof, consider the canonical quotient map $q : C_r^*(G) \to C_r^*(G)/I$ which is injective on the diagonal, as $I \cap C_0(G^{(0)}) = \{0\}$ by hypothesis. As $z \in I$ we have $q(hE(z)h) = q(hE(z)h - hzh)$ so

$$\|hE(z)h\| = \|q(hE(z)h)\| = \|q(hE(z)h - hzh)\| \leqslant \|hE(z)h - hzh\|.$$

Applying (4.4), the above inequality, and then (4.5), we obtain

$$\|E(z)\| < \|hE(z)h - hzh\| + \varepsilon < 2\varepsilon,$$

This contradicts the estimate $2\varepsilon < \|E(z)\|$ from (4.3). Hence, $I \subseteq J^{\mathrm{ob}}$. □

The lemma below uses the full groupoid C^*-algebra $C^*(G)$. We refer the reader to [29] for a discussion of this C^*-algebra that does not assume second-countability.

Lemma 4.13. *Let G be a locally compact Hausdorff étale groupoid whose full and reduced C^*-algebras coincide. Let J^{ob} be the obstruction ideal in $C_r^*(G)$. There is a *-representation ε of the full groupoid C^*-algebra $C^*(G)$ such that* $\ker(\varepsilon)$ *is purely non-dynamical and such that* $\operatorname{supp}(J^{\mathrm{ob}}) \subseteq \operatorname{supp}(\ker(\varepsilon))$.

Proof. The proof of [8, Proposition 5.2] shows that for each $x \in G^{(0)}$ there is an I-norm bounded *-representation ε_x of $C_c(G)$ on the orbit space $\ell^2([x])$ such that $\varepsilon_x(f)e_y = \sum_{\gamma \in G_y} f(\gamma)e_{r(\gamma)}$. By definition of $C^*(G)$, ε_x extends to a representation of $C^*(G)$. Let $\varepsilon = \bigoplus_{x \in G^{(0)}} \varepsilon_x$ (this representation is also described on [19, p. 330]). Then ε is injective on $C_0(G^{(0)})$, because for $f \in C_0(G^{(0)})$ and for $x \in G^{(0)}$, we have $0 \neq f(x) = (\varepsilon_x(f)e_x \mid e_x) \leqslant \|\varepsilon(f)\|$. Hence, $\ker(\varepsilon)$ is a purely non-dynamical ideal.

To see that $\operatorname{supp}(\ker(\varepsilon))$ contains $\operatorname{supp}(J^{\mathrm{ob}})$, by Lemma 2.1 it suffices to show that $G^{(0)} \setminus G^{(0)}_{\mathrm{eff}} = \operatorname{supp}(J^{\mathrm{ob}} \cap G^{(0)}) \subseteq \operatorname{supp}(\ker(\varepsilon))$. Fix $x \in G^{(0)} \setminus G^{(0)}_{\mathrm{eff}}$ and choose $\gamma \in \mathcal{I}^\circ(G) \setminus G^{(0)}$ such that $s(\gamma) = x$. Take an open bisection neighbourhood $B \subseteq \mathcal{I}^\circ(G) \setminus G^{(0)}$ of γ. Choose a non-zero function $f \in C_c(s(B))$ with $f(x) \neq 0$ and let $\widetilde{f} \in C_c(B)$ be the function given by $\widetilde{f}(\eta) = f(s(\eta))$ for all $\eta \in B$. By extending by zero, both functions can be regarded as elements of $C_c(G)$. Direct calculation on basis elements (see the proof of [8, Proposition 5.5(2)]) shows that $f - \widetilde{f} \in \ker(\varepsilon)$. So, $x \in \operatorname{supp}(\ker(\varepsilon))$. □

Corollary 4.14. *Let G be a locally compact Hausdorff étale groupoid that is jointly effective where it is effective. Suppose the sequence $0 \to J^{\mathrm{ob}} \to C_r^*(G) \to C_r^*(G_{\mathrm{eff}}) \to 0$ is exact and that the full and reduced groupoid C^*-algebras of G coincide. Then there is a purely non-dynamical ideal whose support is equal to that of J^{ob}, and J^{ob} is the minimal dynamical ideal that contains all purely non-dynamical ideals of $C_r^*(G)$.*

Proof. Lemma 4.13 gives a purely non-dynamical ideal I such that $\operatorname{supp}(J^{\mathrm{ob}}) \subseteq \operatorname{supp}(I)$. Theorem 4.12 shows that J^{ob} contains all purely non-dynamical ideals in $C_r^*(G)$, and in particular contains I. Hence, $\operatorname{supp}(I) \subseteq \operatorname{supp}(J^{\mathrm{ob}})$, and we obtain equality. Now suppose that I_U is a dynamical ideal that contains every purely non-dynamical ideal. Then, in particular, $I \subseteq I_U$. Hence, $(G^{(0)} \setminus G^{(0)}_{\mathrm{eff}}) \subseteq \operatorname{supp}(J^{\mathrm{ob}}) = \operatorname{supp}(U) \subseteq \operatorname{supp}(I_U) = G_U$. Thus, Proposition 3.3 implies that $J^{\mathrm{ob}} \subseteq I_U$. □

To finish the section, we observe that our results can be used to recover [8, Proposition 5.5(2)], without the assumption that G is second-countable. This is not new. For example, it can be recovered from a special case of [19, Theorem 7.29]. We include it here only to illustrate how our results relate to effective groupoids.

Proposition 4.15. *Let G be a locally compact Hausdorff étale groupoid.*

(1) *If G is effective, then every non-trivial ideal of $C_r^*(G)$ contains a non-zero element of $C_0(G^{(0)})$.*
(2) *If every non-trivial ideal of the full C^*-algebra $C^*(G)$ contains a non-zero element of $C_0(G^{(0)})$, then the full and reduced C^*-algebras of G coincide and G is effective.*

Proof.

(1) Fix an ideal I of $C_r^*(G)$ that contains no non-zero element of $C_0(G^{(0)})$; we show that $I = \{0\}$. As G is effective, it is jointly effective where it is effective, and J^{ob} is trivial. The sequence $0 \to J^{\mathrm{ob}} \to C_r^*(G) \to C_r^*(G_{\mathrm{eff}}) \to 0$ is then trivially exact, and Theorem 4.12 implies that $I \subseteq J^{\mathrm{ob}} = \{0\}$.
(2) We prove the contrapositive. First suppose that the full and reduced C^*-algebras of G do not coincide. Then the kernel of the regular representation $\lambda : C^*(G) \to C_r^*(G)$ is a non-zero purely non-dynamical ideal. Now suppose that the full and reduced C^*-algebras of G coincide but that G is not effective. Then J^{ob} is non-trivial, and Lemma 4.13 implies that there is a purely non-dynamical ideal I of $C_r^*(G)$ whose support contains that of J^{ob}, and in particular is non-zero. □

Remark 4.16. A mainstay of the theory of étale groupoid C^*-algebras is the *diagonal uniqueness theorem*, dating back to [21]: for amenable effective étale groupoids, any *-homomorphism that is injective on the diagonal is injective (see Proposition 4.15). If G is a groupoid that does not satisfy the conclusion of this theorem, then there is a *-homomorphism ϕ of $C_r^*(G)$ whose kernel is purely non-dynamical. So, if G is also inner-exact Hausdorff étale groupoid whose full and reduced C^*-algebras coincide, then the kernel of ϕ is contained in the obstruction ideal. This justifies the terminology *obstruction ideal*: the obstruction ideal measures how far away a groupoid is from satisfying a diagonal uniqueness theorem.

For example, if G is the groupoid of a higher rank graph in the sense of [17], then the obstruction ideal is zero if and only if the higher rank graph is aperiodic (so its C^*-algebra satisfies the Cuntz–Krieger uniqueness theorem) [24, Proposition 3.6].

5 | EXAMPLES

5.1 | Groupoids from local homeomorphisms

First we consider the groupoid constructed from a local homeomorphisms T on a locally compact Hausdorff space X. The associated semi-direct product groupoid, usually called the Deaconu–Renault groupoid, is

$$G_T = \bigcup_{m,n\in\mathbb{N}} \{(x, m-n, y) \in X \times \{m-n\} \times X : T^m x = T^n y\},$$

where the product of (x, p, y) and (y', q, z) is defined precisely if $y = y'$ in which case $(x, p, y)(y, q, z) = (x, p+q, z)$ while inversion is $(x, p, y)^{-1} = (y, -p, x)$. The unit space is naturally identified with X and the range and source maps are then $r(x, p, y) = x$ and $s(x, p, y) = y$.

We first verify that this groupoid is jointly effective where it is effective. For open subsets U and V of X, the sets of the form

$$Z(U, m, n, V) = \{(x, m-n, y) \in G_T : x \in U, y \in V\}$$

comprise a basis for a locally compact Hausdorff étale topology on G_T. The groupoid G_T is amenable, and hence inner-exact [26, section 3].

For the rank-one Deaconu–Renault groupoids, we can describe explicitly the points that are not effective. For $p \in \mathbb{N}_+$, let

$$\mathcal{P}_p = \{x \in X : T^p \text{ pointwise fixes a neighbourhood of } x\} \tag{5.1}$$

and let $\mathcal{P} = \bigcup_{p=1}^{\infty} \mathcal{P}_p$. Then $\mathcal{P}$ is open and invariant in X and the restricted system $(\mathcal{P}, T)$ is reversible.

Lemma 5.1. *For a local homeomorphism T on a locally compact Hausdorff space X, we have*

$$X \setminus X_{\text{eff}} = \{x \in X : \text{orb}_T(x) \cap \mathcal{P} \neq \emptyset\}. \tag{5.2}$$

Proof. Let $V := \{x \in X : \text{orb}_T(x) \cap \mathcal{P} \neq \emptyset\}$ and let $G = G_T$ be the Deaconu–Renault groupoid of T. It is straightforward to verify that V is open and invariant in X. We verify (5.2) one inclusion at a time.

Let $x \in V$ and choose $l \in \mathbb{N}$ such that $x' := T^l(x) \in \mathcal{P}_p$ for some $p \in \mathbb{N}_+$. Pick an open set $U \subseteq X$ all of whose points are p-periodic and consider the open bisection given by

$$B = \{(y, p, y) \in G : y \in U\}. \tag{5.3}$$

Note that $B \subseteq \mathcal{I}^\circ(G) \setminus X$. In particular, $\mathcal{I}^\circ(G)_{x'}$ contains (x', p, x'), so x' is not effective, and by invariance x is not effective.

For the other inclusion, suppose x is not effective. Then $(x, p, x) \in \mathcal{I}^\circ(G)_x$ for some $p \in \mathbb{N}_+$. Fix an open bisection B in $\mathcal{I}^\circ(G) \setminus X$ containing (x, p, x). We may assume that $T^p x = x$, so by shrinking B we may assume that $B \subseteq Z(U, p, 0, U)$ for some open subset U of X. Then $T^p y = y$ for every $y \in s(B)$ because $B \subseteq \mathcal{I}(G)$. Therefore, $x \in V$. □

Next we show that any Deaconu–Renault groupoid G_T is jointly effective where it is effective. The result actually only depends on the non-trivial isotropy being infinite cyclic, so we record this more general result here.

Lemma 5.2. *Any Hausdorff étale groupoid G whose non-trivial isotropy is infinite cyclic is jointly effective where it is effective.*

Proof. Let $x \in G^{(0)}$ be a point with non-trivial isotropy and suppose $B_1, \dots, B_N$ are open bisections in G such that each B_i contains an element $\gamma_i \in \text{Iso}(G)_x \setminus G^{(0)}$. As the isotropy group at x is infinite cyclic there are minimal integers $p_1, \dots, p_N$ such that $\gamma_i^{p_i} = \gamma_j^{p_j}$ for all $i, j = 1, \dots, N$. Put $\gamma := \gamma_i^{p_i}$. Then $B := B_1^{p_1} \cap \cdots \cap B_N^{p_N}$ is an open bisection containing γ.

Assume now that x is effective. So, whenever $U \subseteq G^{(0)}$ is an open neighbourhood of x, there is a point $y \in U$ such that $r(By) \neq y$. Applying this to a neighbourhood basis of x, we can find a sequence $(y_n)_n$ in $G^{(0)}$ such that $y_n \to x$ and $r(By_n) \neq y_n$ for all n. We show that G is jointly effective at x. It suffices to show that for large n, we have $r(B_i y_n) \neq y_n$ for all $i = 1, \dots, N$.

As $B_1^{p_1}$ contains γ, we have $x \in s(B_1^{p_1})$ so $y_n \in s(B_1^{p_1})$ for large n, and as $B \subseteq B_1^{p_1}$ we see that $r(B_1^{p_1} y_n) = r(By_n)$. If $r(B_1 y_n) = y_n$, then

$$y_n = r(B_1 y_n) = r(B_1^{p_1} y_n) = r(By_n),$$

which contradicts our choice of y_n. So, for large n we have $r(B_1 y_n) \neq y_n$ as required. Hence, G is jointly effective at x. □

As an immediate corollary we see that the groupoids built from a local homeomorphism T on a locally compact Hausdorff space X, called rank-one Deaconu–Renault groupoids, are covered by the above result.

Corollary 5.3. *Any rank-one Deaconu–Renault groupoid is jointly effective where it is effective.*

5.2 | Partial actions

Our notion of being jointly effective for groupoids is directly inspired by Ara and Lolk's notion of relative strong topological freeness for partial actions [5, section 7]. A partial action $\theta : \Gamma \curvearrowright X$ of a countable discrete group Γ on a locally compact Hausdorff space X is topologically free at $x \in X$ if whenever $\theta_g(x) = x$ for some $1 \neq g \in \Gamma$, for any open neighbourhood U of x, there exists $y \in U$ such that $\theta_g(y) \neq y$. We say θ is *strongly topologically free* at x if for any finite collection $1 \neq g_1, \dots, g_k \in \Gamma$ such that $\theta_{g_i}(x) = x$ and any neighbourhood U around x, there exists $y \in U$ such that $\theta_{g_i}(y) \neq y$ for all $i = 1, \dots, k$. Finally, θ is *relatively strong topologically free* if it is strongly topologically free at all points at which it is topologically free.

Following [1, section 2], a partial action $\theta : \Gamma \curvearrowright X$ has an associated groupoid

$$G_\theta = \{(x, g, y) \in X \times \Gamma \times X \mid y \in \mathrm{dom}(g),\ \theta_g(y) = x\}$$

whose unit space $G_\theta^{(0)}$ is naturally identified with X. Elements (x, g, y) and (y', g', z) in G_θ are composable if and only if $y = y'$ in which case $(x, g, y)(y', g', z) = (x, gg', z)$. Inversion is given by $(x, g, y)^{-1} = (y, g^{-1}, x)$. The source and range maps $s, r : G_\theta \to X$ are $s(x, g, y) = y$ and $r(x, g, y) = x$. The groupoid G_θ carries a locally compact and Hausdorff étale topology.

Lemma 5.4. *Let $\theta : \Gamma \curvearrowright X$ be a partial action of a countable discrete group Γ on a locally compact Hausdorff space X. Then θ is topologically free at $x \in X$ if and only if G_θ is effective at x. Moreover, θ is strongly topologically free at x if and only if G_θ is jointly effective at x. In particular, θ is relatively strongly topologically free if and only if G_θ is jointly effective where it is effective.*

Proof. Suppose that θ is not strongly topologically free at x. There exist $g_1, \dots, g_k \in \Gamma \setminus \{e\}$ that all fix x, and a neighbourhood U of x such that for every $y \in U$ there exists i such that $\theta_{g_i}(y) = y$.

For each i, define

$$B_i := \{(\theta_{g_i}(y), g_i, y) : y \in U\}.$$

Then each B_i is a bisection containing (x, g_i, x), and there is no $y \in \bigcap_i s(B_i) = U$ such that $r(B_i y) \neq y$ for all i. So, G_θ is not jointly effective at x. Taking $k = 1$ shows that if θ is not topologically free at x then G_θ is not effective at x.

Now suppose that G_θ is not jointly effective at x. Fix elements $\gamma_1, \dots, \gamma_k \in \mathcal{I}(G_\theta)_x \setminus \{x\}$ and open bisections B_i containing γ_i such that for each $y \in \bigcap_i s(B_i)$ there exists i such that $r(B_i y) = y$. By definition of G_θ, each $\gamma_i = (x, g_i, x)$ for some $g_i \in \Gamma \setminus \{e\}$. By definition of the topology on G_θ, for each i there is an open neighbourhood U_i of x such that $\{(\theta_{g_i}(y), g_i, y) : y \in U_i\} \subseteq B_i$. Now $U = \bigcap_i U_i$ is a neighbourhood of x and for each $y \in U$ there exists i such that $r(B_i y) = y$. That is, $\theta_{g_i}(y) = y$. So, θ is not strongly topologically free at x. Again, taking $k = 1$ throughout shows that if G_θ is not effective at x then θ is not topologically free at x.

The final statement follows by definition. □

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