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# Hermitians in matrix algebras with operator norm and associated Lie algebras

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This is a continuation of Crabb et al. (2021) [3] and Duncan and McGregor (2022) [4] in which we began an investigation of the real space $H$ of Hermitian matrices in $M_n(\mathbb{C})$ with respect to norms on $\mathbb{C}^n$. Here, we introduce a family of Lie algebras derived from the classical real Lie algebras, with the aim of determining whether or not they are Lie isomorphic to ideals of some $H$.



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## 1. Introduction

This is a continuation of [3] and [4], except that the focus here is on Lie algebra aspects of the problem. The setting is $M_n(\mathbb{C})$. Given a linear norm $\|\cdot\|$ on $\mathbb{C}^n$, the numerical range $V(T)$ of a matrix $T$ is defined by

$$V(T) = \{f(Tx) : x \in \mathbb{C}^n, f \in \mathbb{C}^{n\prime}, ||x|| = ||f||' = f(x) = 1\}$$

and $T$ is Hermitian if $V(T) \subset \mathbb{R}$. We study the Hermitians $H$ in $M_n(\mathbb{C})$. Basic facts about Hermitians can be found in [1] and [2]; in particular $T$ is Hermitian if and only if $|\exp(itT)| = 1$ $(t \in \mathbb{R})$ where $|\cdot|$ is the operator norm. Each $H$ is a real Lie algebra with Lie product $A \bullet B = i(AB - BA)$. In standard expositions of real Lie algebras in $M_n(\mathbb{C})$ the Lie product is taken to be $A \circ B = AB - BA$. Fortunately there is a trivial way to move between the two different Lie products. Let $\phi(A) = -iA$ for $A \in M_n(\mathbb{C})$, so that $\phi$ is a real linear transformation. Let $\mathcal{L}$ be any real Lie algebra in $M_n(\mathbb{C})$ with product $\circ$. Then we have

$$\phi(A \circ B) = \phi(A) \bullet \phi(B)$$

so that $-i\mathcal{L}$ is a real Lie algebra under $\bullet$ and $\phi$ is a Lie isomorphism between $(\mathcal{L}, \circ)$ and $(-i\mathcal{L}, \bullet)$, with inverse map $\psi$ given by $\psi(B) = iB$.

Perforce we have $I \in H$ and so it is useful to have the following notation. Given a real Lie algebra $(\mathcal{L}, \bullet)$ in $M_n(\mathbb{C})$ with $I \notin \mathcal{L}$, let $\mathcal{L}^1$ be the real Lie algebra generated by $\mathcal{L}$ and $I$; thus $\mathcal{L}^1 = \{A + rI : A \in \mathcal{L}, r \in \mathbb{R}\}$. As a first step we should try to identify all simple real Lie algebras $\mathcal{L}$ for which $\mathcal{L}^1$ is $H$ for some norm.

The classical real Lie algebras $\mathcal{A}_\nu$, $\mathcal{B}_\nu$, $\mathcal{C}_\nu$, $\mathcal{D}_\nu$ (see, for example [6]) are simple except for $\mathcal{D}_1$ and $\mathcal{D}_2$, but they have the wrong Lie product for Hermitians. For any such $\mathcal{K}$ we may replace it with the isomorphic copy $(-i\mathcal{K}, \bullet)$. Since every such $\mathcal{K}$ contains nilpotents the same is clearly true for $(-i\mathcal{K}, \bullet)$ which thus always contains non-Hermitians. Our first project is to modify the classical real Lie algebras into 'self-adjoint' variants (in the sense that all matrices, now with complex entries allowed, are self-adjoint). Self-adjoint matrices have real eigenvalues and hence are natural candidates to be Hermitians. We denote the 'self-adjoint' variants by $sa\mathcal{A}_\nu$, $sa\mathcal{B}_\nu$, $sa\mathcal{C}_\nu$, $sa\mathcal{D}_\nu$. With the exception of $sa\mathcal{D}_1$ and $sa\mathcal{D}_2$ these are all simple real Lie algebras. The case $sa\mathcal{A}_\nu$ is essentially the well understood $C^*$ case. The orthogonal variants, $sa\mathcal{B}_\nu$, $sa\mathcal{D}_\nu$ are isomorphic to a one parameter family of simple Lie algebras which appeared in [3] and [4], and are denoted here by $\mathcal{L}_n$. It is shown in [4] that every $\mathcal{L}_n^1$ is $H$ for some norm. The symplectic variants $sa\mathcal{C}_n$ have a very surprising property. Any norm that makes $sa\mathcal{C}_n$ all Hermitian makes all self-adjoints Hermitian, so that $sa\mathcal{C}_n^1$ is not $H$ for any norm. These simple real Lie algebras are the $(-i\mathcal{L}, \bullet)$ for some of the simple real Lie algebras $(\mathcal{L}, \circ)$ which appear in [5], but our construction of them is entirely elementary and does not require a working knowledge of real forms of simple complex Lie algebras regarded as Lie algebras over the reals.

Some of our results may be deduced easily by using the substantial technical machinery in Lie algebra theory. Our aim is to provide proofs by entirely elementary arguments.

Here we introduce some notation and some computations that will be used in later proofs. In any application the dimension of each of these matrices will be clear from the context. For any $j, k \in \mathbb{N}$, we write $E_{jk}$ for the usual elementary matrix. For $j \neq k$, we define

$$F_{jk} = E_{jk} + E_{kj}, \quad G_{jk} = iE_{jk} - iE_{kj}.$$

Note that $F_{kj} = F_{jk}$, $G_{kj} = -G_{jk}$. For square matrices $A, B$ recall that $A \bullet B = i(AB - BA)$. If and only if $\{j, k\} \cap \{u, v\} = \emptyset$, we have

$$F_{jk} \bullet F_{uv} = O, \quad F_{jk} \bullet G_{uv} = O, \quad G_{jk} \bullet G_{uv} = O. \tag{1.1}$$

With $m \neq k$, we also have

$$F_{jk} \bullet F_{jm} = G_{km}, \quad F_{jk} \bullet G_{jm} = -F_{km}, \quad G_{jk} \bullet G_{jm} = G_{km}. \tag{1.2}$$

Other equations are easily derived by switching the order of the suffices. Finally we have

$$F_{jk} \bullet G_{jk} = 2(E_{jj} - E_{kk}). \tag{1.3}$$

## 2. Self-adjoint variants of the classical Lie algebras

We now define the self-adjoint variants of the classical real Lie algebras and prove first that each is a real Lie algebra. Unless otherwise stated we use the bases for the classical Lie algebras as given in [6, pp. 2-3] but with the following minor alterations: in the basis for $\mathcal{B}_\nu$ replace

$$B = E_{1,\nu+j+1} - E_{j+1,1} \text{ with } -B \quad (1 \leq j \leq \nu).$$

These changes ensure that the non-diagonal basis matrices all occur as pairs $B, B'$ where $B'$ is the transpose of $B$. Let $\mathcal{K}$ stand for any of the classical Lie algebras and let $\mathcal{W}$ be its corresponding basis. Let $\Delta$ denote the diagonal matrices. Define

$$\mathcal{X} = \{A \in \mathcal{W} : A \in \Delta\} \quad \text{and} \quad \mathcal{Y} = \mathcal{W} \setminus \mathcal{X}.$$

Then $\mathcal{W} = \mathcal{X} \cup \mathcal{Y}$. For $B \in \mathcal{Y}$ define

$$B^+ = B + B' \quad \text{and} \quad B^- = i(B - B').$$

Let $\mathcal{Y}_0$ be a subset of $\mathcal{Y}$ consisting of exactly one of each pair $B, B'$ in $\mathcal{Y}$. The choice of $B$ or $B'$ does not affect the following definition of $sa\mathcal{K}$, but once we have established in

Lemma 2.1 that $sa\mathcal{K}$ is a real Lie algebra we make specific choices to define a standard basis for each case. Define

$$\mathcal{Y}^+ = \{B^+ : B \in \mathcal{Y}_0\} \quad \text{and} \quad \mathcal{Y}^- = \{B^- : B \in \mathcal{Y}_0\}.$$

Since elements of $\mathcal{Y}_0$ never overlap in their support, it follows that, for $B_1, B_2 \in \mathcal{Y}_0$, if either $B_1^+ = B_2^+$ or $B_1^- = B_2^-$ then we have $B_1 = B_2$. Hence $\mathcal{E} = \mathcal{X} \cup \mathcal{Y}^+ \cup \mathcal{Y}^-$ consists of self-adjoint (complex) matrices and $|\mathcal{E}| = |\mathcal{W}|$. It is straightforward to verify that the elements of $\mathcal{E}$ are linearly independent. We define $sa\mathcal{K}$ to be the real linear span of $\mathcal{E}$. Let $B \in \mathcal{Y}$. Then $B'^+ = B^+$ and $B'^- = -B^-$. Since $B$ or $B'$ is in $\mathcal{Y}_0$ it follows that $B^+ \in \mathcal{Y}^+$ while $B^- \in \pm\mathcal{Y}^-$. Hence

$$B^+ \in sa\mathcal{K} \quad \text{and} \quad B^- \in sa\mathcal{K} \quad (B \in \mathcal{Y}). \tag{2.1}$$

The following Lemma proves that $sa\mathcal{K}$ is a real Lie algebra with Lie product $\bullet$. We note that $sa\mathcal{K}$ and $\mathcal{K}$ have the same real dimension. Recall that $\circ$ denotes the Lie product in a classical Lie algebra and $\bullet$ denotes the Lie product in the self-adjoint variant.

**Lemma 2.1.** *Let $A, A_1, A_2 \in \mathcal{X}$ and $B, B_1, B_2 \in \mathcal{Y}$. Then $U \bullet V \in sa\mathcal{K}$ for each of the following pairs $(U, V)$:*

$$(A_1, A_2),\ (A, B^+),\ (A, B^-),\ (B_1^+, B_2^+),\ (B_1^-, B_2^-),\ (B_1^+, B_2^-).$$

**Proof.** Let $R \in \mathcal{K}$. Then $R = \Sigma_j \alpha_j P_j + \Sigma_k \beta_k Q_k$ where the $\alpha_j$ and $\beta_k$ are real and the $P_j$ and $Q_k$ are in $\mathcal{X}$ and $\mathcal{Y}$, respectively. Hence, using (2.1),

$$R + R' = 2\sum_j \alpha_j P_j + \sum_k \beta_k (Q_k + Q_k') \in sa\mathcal{K} \tag{2.2}$$

and

$$i(R - R') = \sum_k \beta_k i(Q_k - Q_k') \in sa\mathcal{K}. \tag{2.3}$$

(1) $A_1 \bullet A_2 = O \in sa\mathcal{K}$.

(2) $$\begin{aligned} A \bullet B^+ &= i(AB^+ - B^+A) = i(A(B + B') - (B + B'))A) \\ &= i(AB + AB' - BA - B'A) = i(AB - BA) - i(B'A - AB') \\ &= i((A \circ B) - (A \circ B)') \in sa\mathcal{K} \quad (\text{using } (2.3)). \end{aligned}$$

The remaining cases are similar, making use of (2.2) and (2.3); we present the details for one more example and we give only the final formula for the others.

(3) $A \bullet B^- = (A \circ B') + (A \circ B')' \in sa\mathcal{K}$.

(4) $$\begin{aligned} B_1^+ \bullet B_2^+ &= i(B_1^+ \circ B_2^+) = i((B_1 + B_1')(B_2 + B_2') - (B_2 + B_2')(B_1 + B_1') \\ &= i(B_1B_2 + B_1B_2' + B_1'B_2 + B_1'B_2' - B_2B_1 - B_2B_1' - B_2'B_1 - B_2'B_1') \\ &= i((B_1 \circ B_2) - (B_1 \circ B_2)') + i((B_1 \circ B_2') - (B_1 \circ B_2')') \in sa\mathcal{K}. \end{aligned}$$

(5) $B_1^- \bullet B_2^- = -i((B_1 \circ B_2) - (B_1 \circ B_2)') + i((B_1 \circ B_2') - (B_1 \circ B_2')') \in sa\mathcal{K}$.
(6) $B_1^+ \bullet B_2^- = -((B_1 \circ B_2) + (B_1 \circ B_2)') - ((B_1 \circ B_2') + (B_1 \circ B_2')') \in sa\mathcal{K}$. □

We now list the standard bases $\mathcal{E}_\mathcal{K}$ for $sa\mathcal{K}$ where $\mathcal{K} = \mathcal{A}_\nu, \mathcal{B}_\nu, \mathcal{C}_\nu, \mathcal{D}_\nu$. Unless otherwise stated, when we refer to a basis for $sa\mathcal{K}$, we mean $\mathcal{E}_\mathcal{K}$.

**Definition 2.2.** The self-adjoint variant of Lie algebra $(\mathcal{A}_\nu, \circ)$ is $(sa\mathcal{A}_\nu, \bullet)$ with basis consisting of $\nu^2 + 2\nu$ matrices in $M_{\nu+1}(\mathbb{C})$:

$$\mathcal{E}_{\mathcal{A}_\nu} : \begin{cases} E_{jj} - E_{j+1,j+1} \quad (1 \le j \le \nu), \\ F_{jk}, G_{j,k} \quad (1 \le j \ne k \le \nu + 1). \end{cases} \tag{2.4}$$

**Definition 2.3.** The self-adjoint variant of Lie algebra $(\mathcal{B}_\nu, \circ)$ is $(sa\mathcal{B}_\nu, \bullet)$ with basis consisting of $2\nu^2 + \nu$ matrices in $M_{2\nu+1}(\mathbb{C})$:

$$\mathcal{E}_{\mathcal{B}_\nu} : \begin{cases} F_{1,k+1} - F_{1,\nu+k+1}, \quad G_{1,k+1} + G_{1,\nu+k+1} \quad (1 \le k \le \nu), \\ \begin{bmatrix} 0 & O \\ O & W \end{bmatrix} \quad (W \in \mathcal{E}_{\mathcal{D}_\nu}). \end{cases} \tag{2.5}$$

**Definition 2.4.** The self-adjoint variant of Lie algebra $(\mathcal{C}_\nu, \circ)$ is $(sa\mathcal{C}_\nu, \bullet)$ with basis consisting of $2\nu^2 + \nu$ matrices in $M_{2\nu}(\mathbb{C})$. The additional labels for matrices in $\mathcal{E}_{\mathcal{C}_\nu}$ are used in Theorem 3.8.

$$\mathcal{E}_{\mathcal{C}_\nu} : \begin{cases} \left.\begin{array}{l} A_j := E_{jj} - E_{\nu+j,\nu+j}, \\ B_j := F_{j,\nu+j}, \\ C_j := G_{j,\nu+j} \end{array}\right\} (1 \le j \le \nu), \\ \left.\begin{array}{l} P_{jk} := F_{jk} - F_{\nu+j,\nu+k}, \\ Q_{jk} := G_{jk} + G_{\nu+j,\nu+k}, \\ R_{jk} := F_{j,\nu+k} + F_{k,\nu+j}, \\ S_{jk} := G_{j,\nu+k} + G_{k,\nu+j} \end{array}\right\} (1 \le j < k \le \nu). \end{cases} \tag{2.6}$$

**Definition 2.5.** The self-adjoint variant of Lie algebra $(\mathcal{D}_\nu, \circ)$ is $(sa\mathcal{D}_\nu, \bullet)$ with basis consisting of $2\nu^2 - \nu$ matrices in $M_{2\nu}(\mathbb{C})$:

$$\mathcal{E}_{\mathcal{D}_\nu} : \begin{cases} E_{jj} - E_{\nu+j,\nu+j} \quad (1 \le j \le \nu), \\ \left.\begin{array}{l} F_{jk} - F_{\nu+j,\nu+k}, \quad G_{jk} + G_{\nu+j,\nu+k}, \\ F_{j,\nu+k} - F_{k,\nu+j}, \quad G_{j,\nu+k} - G_{k,\nu+j} \end{array}\right\} (1 \le j < k \le \nu). \end{cases} \tag{2.7}$$

## 3. Simplicity of self-adjoint variants

All Lie algebras $sa\mathcal{K}$ are simple except for $sa\mathcal{D}_1$ and $sa\mathcal{D}_2$. We leave to the reader the verification for $sa\mathcal{A}_1$, $sa\mathcal{B}_1$, $sa\mathcal{C}_1$ and $sa\mathcal{D}_1$.

We prove that $sa\mathcal{A}_\nu$ ($\nu \geq 2$) is simple directly from the definition in two stages: we show first that any non-zero ideal contains a basis element and then that it contains all basis elements. We use the standard basis $\mathcal{E}_{\mathcal{A}_\nu}$ given in (2.4):

$$\begin{array}{ll} \text{1-parameter elements} & A_p = E_{pp} - E_{p+1,p+1} \quad (1 \leq p \leq \nu), \\ \text{2-parameter elements} & F_{j,k}, G_{j,k} \quad (1 \leq j < k \leq \nu + 1). \end{array}$$

Lie products of 2-parameter basis elements appear at the end of Section 1. Other products, involving 1-parameter basis elements, are as follows.

| For ... | $A_p \bullet F_{jk}$ | $A_p \bullet G_{jk}$ |
|---|---|---|
| $1 \leq j < p,\ k = p$ | $-G_{jk}$ | $F_{jk}$ |
| $1 \leq j < p,\ k = p+1$ | $G_{jk}$ | $-F_{jk}$ |
| $j = p,\ p+1 < k \leq \nu+1$ | $G_{jk}$ | $-F_{jk}$ |
| $j = p+1,\ p+1 < k \leq \nu+1$ | $-G_{jk}$ | $F_{jk}$ |
| $j = p,\ k = p+1$ | $2G_{jk}$ | $-2F_{jk}$ |
| Otherwise | $O$ | $O$ |

**Lemma 3.1.** *Let $X = \sum_{r=1}^{\nu} \alpha_r A_j$ where not all $\alpha_r$ are zero. Then, for some $p$ with $1 \leq p \leq \nu$, $X \bullet G_{p,p+1}$ is a non-zero multiple of $F_{p,p+1}$.*

**Proof.** Note that we include the case where only one $\alpha_r$ is non-zero. We have $X \bullet G_{p,p+1} = \beta_p F_{p,p+1}$ where

$$\beta_p = \begin{cases} -2\alpha_1 + \alpha_2 & \text{if } p = 1, \\ \alpha_{p-1} - 2\alpha_p + \alpha_{p+1} & \text{if } 2 \leq p \leq \nu - 1, \\ \alpha_{\nu-1} - 2\alpha_\nu & \text{if } p = \nu. \end{cases}$$

If all $\beta_p = 0$ ($1 \leq p \leq \nu - 1$) then $\alpha_r = r\alpha_1$ ($1 \leq r \leq \nu$) and if, in addition, $\beta_\nu = 0$ then all $\alpha_r$ are zero, contradicting the given hypothesis. So at least one $\beta_p$ is non-zero and the result follows. □

**Lemma 3.2.** *Let $\mathcal{I}$ be a non-zero ideal in $sa\mathcal{A}_\nu$. Then $\mathcal{I}$ contains some basis element of $sa\mathcal{A}_\nu$.*

**Proof.** Every non-zero $X \in \mathcal{I}$ has a non-zero number of basis elements in its support, $\operatorname{supp} X$. Denote this number by $|X|$. Let $m$ be the minimum of $|X|$ for all non-zero $X$ in $\mathcal{I}$ and let $Y \in \mathcal{I}$ with $|Y| = m$. If $m = 1$, we are done. Let $m \geq 2$.

If $\operatorname{supp} Y$ contains only 1-parameter basis elements then Lemma 3.1 applies to give some $F_{jk} \in \mathcal{I}$.

If $\operatorname{supp} Y$ contains a 2-parameter basis element, let $F_{g,k}$ or $G_{g,k}$ be such an element where $g$ is the least first parameter. Then $Z = A_g \bullet Y \in \mathcal{I}$. By Lemma 3.1, $Z \neq O$ and $|Z| \leq |Y|$ so that, by minimality, $|Z| = m$, and further, $\operatorname{supp} Z$ contains no 1-parameter basis element and only 2-parameter basis elements with first parameter $g$ or $g+1$.

Case 1. Let $\operatorname{supp} Z$ contain $F_{g,u}$ and $H_{g,v}$ with $u < v$ where $\{H_{g,v}, K_{g,v}\} = \{F_{g,v}, G_{g,v}\}$. Then

$$F_{g,u} \bullet F_{g,u} = O \quad \text{and} \quad G_{g,u} \bullet H_{g,v} = \lambda K_{u,v}$$

for some $\lambda \neq 0$. Again using Lemma 3.1, $W = F_{g,u} \bullet Z \neq O$, $W \in \mathcal{I}$ and $|W| < m$ giving a contradiction. A similar argument applies when $\operatorname{supp} Z$ contains $G_{g,u}$ and $H_{g,v}$ or with $g$ replaced with $g+1$.

Case 2. If $|Z| \geq 3$ then we can always choose two elements of $\operatorname{supp} Z$ which satisfy Case 1. So we are left to consider $|Z| = 2$, with one element having first parameter $g$ and the other, $g+1$. Let $\operatorname{supp} Z = \{F_{g,u}, H_{g+1,v}\}$ where $H_{g+1,v} = F_{g+1,v}$ or $G_{g+1,v}$. [$\operatorname{supp} Z = \{G_{g,u}, H_{g+1,v}\}$ can be dealt with similarly.] If $\{g, u\} \cap \{g+1, v\} \neq \emptyset$ then either $u = v$ in which case $|F_{g,u} \bullet Z| = 1$, or $u = g+1$ in which case $|F_{g,g+1} \bullet Z| = 1$, and the minimality of $m$ is contradicted. If $\{g, u\} \cap \{g+1, v\} = \emptyset$ then

$$F_{g,u} \bullet G_{g,u} = 2(E_{gg} - E_{uu}) \quad \text{and} \quad H_{g+1,v} \bullet G_{g,u} = O$$

so that $\operatorname{supp}(Z \bullet G_{g,u})$ contains only 1-parameter basis elements and Lemma 3.1 concludes the proof. □

**Theorem 3.3.** *For $\nu \geq 2$, $sa\mathcal{A}_\nu$ is simple.*

**Proof.** Let $\mathcal{I}$ be a non-zero ideal in $sa\mathcal{A}_\nu$. By Lemmas 3.1 and 3.2, $\mathcal{I}$ contains a 2-parameter basis element. Then product rules (1.2) and (1.3) in Section 1 guarantee that $\mathcal{I}$ contains all basis elements. Hence $\mathcal{I} = sa\mathcal{A}_\nu$ as required. □

We establish the simplicity, or otherwise, of $saB_\nu$ and $sa\mathcal{D}_\nu$ ($\nu \geq 2$) by comparing them with the Lie algebras which appeared in [3, pp 287–289] and [4, Theorem 4.1] as subalgebras of examples of $H$, and for which we need to provide further results. For $n \geq 2$, let $\mathcal{L}_n \subset M_n(\mathbb{C})$ denote the real Lie algebra with basis $G_{jk}$ for $1 \leq j < k \leq n$ and Lie product $\bullet$. It is an easy exercise to verify that $\mathcal{L}_3$ is a simple real Lie algebra. The three-dimensional simple real Lie algebras $\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1$ are Lie isomorphic to each other but not to $\mathcal{L}_3$. It is easy to show that $\mathcal{A}_1$ over $\mathbb{C}$ is Lie isomorphic to $\mathcal{L}_3$ over $\mathbb{C}$, thus showing a difference between complex Lie algebras and real Lie algebras. To see the non-isomorphism, recall the usual basis for $\mathcal{A}_1$ given by

$$P = E_{11} - E_{22}, \quad Q = E_{12}, \quad R = E_{21}.$$

Now let

$$x = \tfrac{1}{2}P, \quad y = \tfrac{1}{2}(Q - R), \quad z = \tfrac{1}{2}(Q + R)$$

so that $\{x, y, z\}$ is also a basis with

$$x \circ y = z, \quad y \circ z = x, \quad z \circ x = -y.$$

For $\mathcal{L}_3$ take the usual basis $\{a, b, c\}$, where $a = G_{12}, b = G_{13}, c = G_{23}$, so that

$$a \bullet b = c, \quad b \bullet c = a, \quad c \bullet a = b.$$

Suppose, towards a contradiction, that $\phi$ is a Lie isomorphism from $\mathcal{L}_3$ to $\mathcal{A}_1$ determined by

$$\phi(a) = \lambda_1 x + \lambda_2 y + \lambda_3 z, \quad \phi(b) = \mu_1 x + \mu_2 y + \mu_3 z, \quad \phi(c) = \nu_1 x + \nu_2 y + \nu_3 z,$$

where $\lambda_j, \mu_j, \nu_j \in \mathbb{R}$. With respect to our bases, $\phi$ has corresponding matrix

$$M = \begin{bmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{bmatrix}.$$

The equations

$$\phi(c) = \phi(a) \circ \phi(b), \quad \phi(a) = \phi(b) \circ \phi(c), \quad \phi((b) = \phi(c) \circ \phi(a)$$

give nine equations relating the entries of $M$. In particular we have

$$\lambda_1 = \mu_2\nu_3 - \mu_3\nu_2, \quad \mu_1 = \nu_2\lambda_3 - \nu_3\lambda_2, \quad \nu_1 = \lambda_2\mu_3 - \lambda_3\mu_2$$

and

$$\lambda_2 = \mu_1\nu_3 - \mu_3\nu_1, \quad \mu_2 = \nu_1\lambda_3 - \nu_3\lambda_1, \quad \nu_2 = \lambda_1\mu_3 - \lambda_3\mu_1.$$

Expand $\det M$ by the first two rows to give

$$\det M = \lambda_1^2 + \mu_1^2 + \nu_1^2 = -(\lambda_2^2 + \mu_2^2 + \nu_2^2).$$

Since $M$ is invertible, no rows are zero and we conclude with the contradiction $\det M = 0$.

The Lie algebra $\mathcal{L}_4$ is not simple since it is the direct sum of two ideals $\mathcal{I}_1, \mathcal{I}_2$ where $\mathcal{I}_1$ has basis

$$\tfrac{1}{2}(G_{12} + G_{34}), \quad \tfrac{1}{2}(G_{14} + G_{23}), \quad \tfrac{1}{2}(G_{24} - G_{13}),$$

and $\mathcal{I}_2$ has basis

$$\tfrac{1}{2}(G_{12} - G_{34}), \quad \tfrac{1}{2}(G_{14} - G_{23}), \quad \tfrac{1}{2}(G_{13} + G_{24}).$$

The Lie product table for each of these bases, taken in the above order, matches exactly that of the standard basis for $\mathcal{L}_3$ in order $G_{12}, G_{13}, G_{23}$. The ideals $\mathcal{I}_1, \mathcal{I}_2$ are thus each Lie isomorphic to $\mathcal{L}_3$. The next two Lemmas show that $\mathcal{L}_n$ is simple for $n \geq 5$.

**Lemma 3.4.** *Let $n \geq 5$ and let $\mathcal{I}$ be a non-zero ideal of $\mathcal{L}_n$. Then $\mathcal{I}$ contains some $G_{jk}$.*

**Proof.** Let $X = \sum_{r=1}^{m} \eta_r Z_r \in \mathcal{I}$ with $m \geq 2$, where $Z_r$ are distinct basis elements and $\eta_r \neq 0$.

(i) If $m = 2$ then, since $n \geq 5$, we can find a basis element $Z_0$ (say) such that $Z_0 \bullet X \in \mathcal{I}$ is a non-zero multiple of a single basis element.

(ii) If $m \geq 3$ and every pair of basis elements $Z_r$ has zero Lie product then suppose, without loss, that $Z_1 = G_{pq}$ and $Z_2 = G_{uv}$ with $p < u$. Using (2.2) it follows that $G_{pu} \bullet X \in \mathcal{I}$ is a linear combination of two basis elements and (i), above, applies.

(iii) If $m \geq 3$ and two basis elements, without loss $Z_1$ and $Z_2$ (say), have non-zero Lie product then (1.2) guarantees that $Y = Z_1 \bullet Z_2 \neq \pm Z_1 \bullet Z_r$ for any $r \geq 3$. Hence $Y \in \mathcal{I}$ is non-zero and is a linear combination of at most $m - 1$ basis elements. We can repeat this process until we have a single basis element in $\mathcal{I}$ or we have case (i) or (ii), above. □

**Lemma 3.5.** *Let $\mathcal{I}$ be an ideal of $\mathcal{L}_n$ that contains one of the $G_{jk}$. Then $\mathcal{I}$ contains every $G_{jk}$.*

**Proof.** Suppose that $G_{jk} \in \mathcal{I}$ and let $G_{uv}$ be any other basis element. Either $j \neq u$ or $k \neq v$. If $j \neq u$, using (1.2), we have

$$(G_{jk} \bullet G_{uk}) \bullet G_{jv} = G_{ju} \bullet G_{jv} = G_{uv}.$$

If $k \neq v$ we have

$$(G_{jk} \bullet G_{jv}) \bullet G_{ku} = G_{kv} \bullet G_{ku} = -G_{uv}.$$

Hence $G_{uv} \in \mathcal{L}_n$. □

Combining Lemmas 3.4 and 3.5 we have the following result.

**Theorem 3.6.** *For $n \geq 5$, $\mathcal{L}_n$ is simple.*

We now construct Lie isomorphisms from appropriate $\mathcal{L}_n$ to $sa\mathcal{D}_\nu$ and $sa\mathcal{B}_\nu$. We begin with the case $sa\mathcal{D}_\nu$, with $\nu \geq 2$. Our proof is simplified by the introduction of another basis $\mathcal{E}$ for $\mathcal{L}_{2\nu}$.

$$\mathcal{E}: \begin{cases} G_{j,\nu+j} \quad (1 \le j \le \nu), \\ \left.\begin{array}{ll} G_{jk} + G_{\nu+j,\nu+k}, & G_{jk} - G_{\nu+j,\nu+k}, \\ G_{j,\nu+k} + G_{k,\nu+j}, & G_{j,\nu+k} - G_{k,\nu+j} \end{array}\right\} (1 \le j < k \le \nu). \end{cases} \tag{3.1}$$

For any invertible $2\nu \times 2\nu$ matrix $T$, the similarity $\phi(Z) = T^{-1}ZT$ gives a Lie monomorphism. We take

$$T = \begin{bmatrix} J & -J \\ I & I \end{bmatrix} \tag{3.2}$$

where $J = \operatorname{diag}(i, -i, i, -i, \cdots) = i \operatorname{diag}((-1)^{r+1})$ and $I$ is the identity matrix. Note that

$$2T^{-1} = \begin{bmatrix} -J & I \\ J & I \end{bmatrix}$$

and

$$E_{j,\nu+k} = \begin{bmatrix} O & E_{jk} \\ O & O \end{bmatrix}, \quad E_{\nu+j,k} = \begin{bmatrix} O & O \\ E_{jk} & O \end{bmatrix}, \quad E_{\nu+j,\nu+k} = \begin{bmatrix} O & O \\ O & E_{jk} \end{bmatrix}.$$

For $j, k = 1, 2, \ldots, n$ matrix computation gives

$$\begin{aligned} \phi(E_{jk}) &= \tfrac{1}{2}(-1)^{j+k}(E_{jk} - E_{j,\nu+k} - E_{\nu+j,k} + E_{\nu+j,\nu+k}), \\ \phi(E_{j,\nu+k}) &= \tfrac{1}{2}(-1)^{j}\, i(E_{jk} + E_{j,\nu+k} - E_{\nu+j,k} - E_{\nu+j,\nu+k}), \\ \phi(E_{\nu+j,k}) &= \tfrac{1}{2}(-1)^{k}\, i(-E_{jk} + E_{j,\nu+k} - E_{\nu+j,k} + E_{\nu+j,\nu+k}), \\ \phi(E_{\nu+j,\nu+k}) &= \tfrac{1}{2}(E_{jk} + E_{j,\nu+k} + E_{\nu+j,k} + E_{\nu+j,\nu+k}). \end{aligned}$$

Hence, for $1 \le j < k \le \nu$, we have

$$\begin{aligned} \phi(G_{jk}) &= \tfrac{1}{2}(-1)^{j+k}(G_{jk} - G_{j,\nu+k} + G_{k,\nu+j} + G_{\nu+j,\nu+k}), \\ \phi(G_{\nu+j,\nu+k}) &= \tfrac{1}{2}(G_{jk} + G_{j,\nu+k} + G_{j,\nu+j} - G_{\nu+j,\nu+k}) \end{aligned}$$

and for $1 \le j, k \le \nu$,

$$\phi(G_{j,\nu+k}) = \tfrac{1}{2}(-1)^{j}(-F_{jk} - F_{j,\nu+k} + F_{j,\nu+k} + F_{\nu+j,\nu+k}).$$

For $\phi(G_{k,\nu+j})$ interchange $j$ and $k$ in the above line.

It remains to consider the action of $\phi$ on the basis $\mathcal{E}$. Recall the basis $\mathcal{E}_{\mathcal{D}_\nu}$ listed in (2.7). For $1 \le j \le \nu$, we have

$$\phi(G_{j,\nu+j}) = (-1)^{j}(-E_{jj} + E_{\nu+j,\nu+j}) \in \pm\mathcal{E}_{\mathcal{D}_\nu}.$$

For $1 \leq j < k \leq \nu$,

$$\phi(G_{jk} + G_{\nu+j,\nu+k}) = \tfrac{1}{2}(-1)^{j+k}(G_{jk} - G_{j,\nu+k} + G_{k,\nu+j} + G_{\nu+j,\nu+k}) \\ +\tfrac{1}{2}(G_{jk} + G_{j,\nu+k} - G_{k,\nu+j} + G_{\nu+j,\nu+k}).$$

Hence $\phi(G_{jk} + G_{\nu+j,\nu+k})$ is $(G_{jk} + G_{\nu+j,\nu+k})$ if $j+k$ is even, and is $(G_{j,\nu+k} - G_{k,\nu+j})$ if $j+k$ is odd. Similarly, $\phi(G_{jk} - G_{\nu+j,\nu+k})$ is $-(G_{jk} - G_{\nu+j,\nu+k})$ if $j+k$ is even, and is $-(G_{j,\nu+k} + G_{k,\nu+j})$ if $j+k$ is odd. We next find that $\phi(G_{j,\nu+k} + G_{j,\nu+k})$ is $\mp(F_{jk} - F_{\nu+j,\nu+k})$ if $j, k$ are both even, or both odd, and is $\mp(F_{j,\nu+k} - F_{k,\nu+j})$ if the parities are opposite. A similar result holds for $\phi(G_{j,\nu+k} - G_{k,\nu+j})$. Hence $\phi(\mathcal{E})$ is in $\pm\mathcal{E}_{\mathcal{D}_\nu}$, is one-one on $\mathcal{E}$, and all of $\mathcal{E}_{\mathcal{D}_\nu}$ appears. Thus the similarity $\phi$ is a monomorphism from $\mathcal{L}_{2\nu}$ onto $sa\mathcal{D}_\nu$.

**Theorem 3.7.** *(1) For $\nu \geq 3$, $sa\mathcal{D}_\nu$ is Lie isomorphic to $\mathcal{L}_{2\nu}$ and hence is simple.*
*(2) $sa\mathcal{D}_2$ is Lie isomorphic to $\mathcal{L}_4$ and hence is not simple.*

**Proof.** (1) We have shown, above, that $\phi : \mathcal{L}_{2\nu} \to sa\mathcal{D}_\nu$ is a Lie isomorphism. Since, by Theorem 3.6 ($n = 2\nu$), $\mathcal{L}_{2\nu}$ is simple, it follows that $sa\mathcal{D}_\nu$ is simple.

(2) The Lie isomorphism $\phi : \mathcal{L}_{2\nu} \to sa\mathcal{D}_\nu$ is defined for $\nu = 2$ but, as was previously shown, $\mathcal{L}_4$ is not simple. □

We show next that $\mathcal{L}_{2\nu+1}$ is Lie isomorphic to $sa\mathcal{B}_\nu$ when $\nu \geq 2$. Note that from here up to the statement of Theorem 3.8 all $F_{jk}, G_{jk}$ explicitly mentioned are in $M_{2\nu+1}(\mathbb{C})$. We use the basis $\mathcal{E}$ for $\mathcal{L}_{2\nu}$ listed at (3.1) to define another basis $\mathcal{E}^+$ for $\mathcal{L}_{2\nu+1}$.

$$\mathcal{E}^+ : \begin{cases} G_{1,k+1} & (1 \leq k \leq 2\nu), \\ \begin{bmatrix} 0 & O \\ O & Z \end{bmatrix} & (Z \in \mathcal{E}). \end{cases} \tag{3.3}$$

The basis $\mathcal{E}_{\mathcal{B}_\nu}$ for $sa\mathcal{B}_\nu$ listed at (2.5) can be written as $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ where

$$\mathcal{E}_1 = \{F_{1,k+1} - F_{1,\nu+k+1} : (1 \leq k \leq \nu)\},$$
$$\mathcal{E}_2 = \{G_{1,k+1} + G_{1,\nu+k+1} : (1 \leq k \leq \nu)\},$$
$$\mathcal{E}_3 = \left\{ \begin{bmatrix} 0 & O \\ O & W \end{bmatrix} : (W \in \mathcal{E}_{\mathcal{D}_\nu}) \right\}.$$

Let

$$U = \begin{bmatrix} \sqrt{2} & O \\ O & T \end{bmatrix} \quad \text{where } T \text{ is defined by (3.2)}, \tag{3.4}$$

and define the similarity $\psi(V) = U^{-1}VU$. Straightforward computations show that

$$\psi(G_{1,k+1}) \in \pm\tfrac{1}{\sqrt{2}}\mathcal{E}_1 \quad (1 \le k \le \nu),$$
$$\psi(G_{1,k+1}) \in \tfrac{1}{\sqrt{2}}\mathcal{E}_2 \quad (\nu+1 \le k \le 2\nu)$$

and, for $V = \begin{bmatrix} 0 & O \\ O & W \end{bmatrix}$ with $W \in \mathcal{E}_{\mathcal{D}_\nu}$, $\psi(V) \in \pm\mathcal{E}_3$. Also we have that $\psi$ is one-to-one on $\mathcal{E}^+$ and all of $\mathcal{E}_{\mathcal{B}_\nu}$ appears. It follows that the similarity $\phi : \mathcal{L}_{2\nu} \to sa\mathcal{D}_\nu$ is a Lie isomorphism. This, together with Theorem 3.6 ($n = 2\nu + 1$), establishes the following result.

**Theorem 3.8.** *For $\nu \ge 2$, $sa\mathcal{B}_\nu$ is Lie isomorphic to $\mathcal{L}_{2\nu+1}$ and hence is simple.*

We show next that $sa\mathcal{C}_\nu$ ($\nu \ge 2$) is simple, but without a ready made simple Lie algebra like $\mathcal{L}_n$ for comparison we revert to the method used for $sa\mathcal{A}_\nu$. We use the labels given in (2.6) for the elements in the basis $\mathcal{E}_{\mathcal{C}_\nu}$. There are three types parametrized by $j$:

$$A_j, \quad B_j, \quad C_j \quad (1 \le j \le \nu)$$

and four types parametrized by $(j, k)$:

$$P_{jk}, \quad Q_{jk}, \quad R_{jk}, \quad S_{jk} \quad (1 \le j < k \le \nu).$$

For our proof of the theorem it is essential to have many zero Lie products. Any products of $A_j, B_j, C_j$ with $j' \ne j''$ give zero product. Any products of 2-parameter basis elements with $\{j', k'\} \cap \{j'', k''\} = \emptyset$ give zero product. We give below a partial Lie product table for $1 \le j < k \le \nu$. In fact the products hold equally with $j > k$, and this enables us to calculate products such as $A_k \bullet P_{jk}$, since $P_{jk} = P_{kj}$, $Q_{jk} = -Q_{jk}$, etc.

Partial Lie Product Table for $sa\mathcal{C}_\nu$ Basis ($0 \le j < k \le \nu$)

| $\bullet$ | $B_j$ | $C_j$ | $P_{jk}$ | $Q_{jk}$ | $R_{jk}$ | $S_{jk}$ |
|---|---|---|---|---|---|---|
| $A_j$ | $2C_j$ | $-2B_j$ | $Q_{jk}$ | $-P_{jk}$ | $S_{jk}$ | $-R_{jk}$ |
| $B_j$ | | $2A_j$ | $-S_{jk}$ | $-R_{jk}$ | $Q_{jk}$ | $P_{jk}$ |
| $C_j$ | | | $R_{jk}$ | $-S_{jk}$ | $-P_{jk}$ | $Q_{jk}$ |
| $P_{jk}$ | | | | $2(A_j - A_k)$ | $2(C_j + C_k)$ | $-2(B_j + B_k)$ |
| $Q_{jk}$ | | | | | $-2(B_j - B_k)$ | $-2(C_j - C_k)$ |
| $R_{jk}$ | | | | | | $2(A_j + A_k)$ |

**Lemma 3.9.** *Let $\nu \ge 2$ and let $\mathcal{I}$ be a non-zero ideal of $sa\mathcal{C}_\nu$. Then $\mathcal{I}$ contains some basis element of $sa\mathcal{C}_\nu$.*

**Proof.** Let $X \in \mathcal{I}, X \ne O$. Let $u$ be the smallest $j$-parameter which appears amongst all the basis elements in $X$. Replace $X$ by $X_1 = A_u \bullet X$. Then $X_1 \in \mathcal{I}$. By the product formulas in the above table, $X_1 \neq O$, all basis elements in $X_1$ have $j$-parameter $u$ but $X_1$ does not include $A_u$. If $X_1$ has no $(u,k)$-parameter basis elements then $X_1 = pB_u + qC_u$ with $p, q$ not both zero so that $B_u \bullet X_1$ or $C_u \bullet X_1$ (or both) give $A_u$ in $\mathcal{I}$. If $X_1$ does have a $(u,k)$-parameter basis element then $X_2 = (A_u \bullet (B_u \bullet X_1)) \in \mathcal{I}$ has only $(u,k)$-parameter basis elements and is non-zero. Let $v$ be the largest $k$-parameter which appears in $X_2$. Then $X_3 = A_v \bullet X_2 \in \mathcal{I}$, is non-zero and has only $(u,v)$-parameter basis elements. For some $a, b, c, d \in \mathbb{R}$, not all zero, we have four elements in $\mathcal{I}$:

$$\begin{aligned} X_3 &= \phantom{-}aP_{uv} + bQ_{uv} + cR_{uv} + dS_{uv}, \\ A_u \bullet X_3 &= -bP_{uv} + aQ_{uv} - dR_{uv} + cS_{uv}, \\ B_u \bullet X_3 &= \phantom{-}dP_{uv} + cQ_{uv} - bR_{uv} - aS_{uv}, \\ C_u \bullet X_3 &= -cP_{uv} + dQ_{uv} + aR_{uv} - bS_{uv}, \end{aligned}$$

by the above product table. We shall have $P_{uv} \in \mathcal{I}$ if we can solve

$$\begin{bmatrix} a & -b & d & -c \\ b & a & c & d \\ c & -d & -b & a \\ d & c & -a & -b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. But the rows of the matrix are orthogonal vectors, all with length $\sqrt{(a^2 + b^2 + c^2 + d^2)} \neq 0$. Thus the matrix is a non-zero real multiple of an orthogonal matrix, which completes the proof. □

**Lemma 3.10.** *Let $\nu \geq 2$, and let $\mathcal{I}$ be an ideal of $sa\mathcal{C}_\nu$ that contains one of the basis elements. Then $I$ contains every basis element.*

**Proof.** If $\mathcal{I}$ contains $A_j$ it follows from the above product table that $\mathcal{I}$ contains every basis element with a parameter $j$ and then that it contains $2(A_j - A_k)$ for any other $k$. Hence $\mathcal{I}$ contains all basis elements. If $\mathcal{I}$ contains either $B_j$ or $C_j$ then $\mathcal{I}$ contains both and hence contains $2A_j$, and we are done. If $\mathcal{I}$ contains any $(j,k)$ basis element, then it contains all four $(j,k)$ basis elements and hence contains $2(A_j - A_k)$ and $2(A_j + A_k)$, and again we are done. □

Combining Lemmas 3.9 and 3.10 we have the following result.

**Theorem 3.11.** *For $\nu \geq 2$, $sa\mathcal{C}_\nu$ is a simple Lie algebra.*

## 4. The norm problem

We turn now to the question of whether or not, for a simple Lie algebra $sa\mathcal{K} \subset M_n(\mathbb{C})$, there is a norm on $\mathbb{C}^n$ such that $H$ for that norm is $sa\mathcal{K}^1$. For $sa\mathcal{A}_\nu$, when we adjoin $I$ we obtain all self-adjoint matrices and we are in the $C^*$ situation. This follows from the Vidav-Palmer theorem [1, Theorem 6.9] but we give an elementary proof using the property that, for any $t \in \mathbb{R}$ and any Hermitian element $T$, $\exp(itT)$ is an isometry. For $1 \le p < q$,

$$\exp(i\theta G_{pq}) = I + (\cos\theta - 1)(E_{pp} + E_{qq}) + i\sin\theta\, G_{pq}.$$

We take the range of $\tan^{-1}$ to be $(-\pi/2, \pi/2)$.

**Lemma 4.1.** *Let $n \ge 2$ and let $\|\cdot\|$ be a norm for $\mathbb{C}^n$ such that $G_{1k}$ $(k = 1, \dots, n)$ are Hermitian. Then, for $r_1, \dots, r_n > 0$,*

$$\|(r_1, r_2, \dots, r_n)\| = \alpha\sqrt{r_1^2 + r_2^2 + \cdots + r_n^2}$$

*where $\alpha = \|(1, 0, \dots, 0)\|$.*

**Proof.** Let $\rho_1 = \sqrt{r_1^2 + r_2^2 + \cdots + r_n^2}$ and, for $k = 2, \dots, n$, let $\xi_k = \tan^{-1}(r_k/\rho_k)$ where $\rho_k$ is defined recursively by $\rho_k = \sqrt{\rho_{k-1}^2 - r_k^2}$. Let

$$T = \exp(i\xi_n G_{1n})\exp(i\xi_{n-1}G_{1,n-1})\dots\exp(i\xi_2 G_{12}).$$

We consider $z = T(1, 0, \dots, 0)$. Note that, for $k = 2, \dots, n$, $\cos\xi_k = \rho_j/\rho_{k-1}$ and $\sin\xi_k = r_k/\rho_{k-1}$.

(i) The 1st coordinate of $z$ is $\cos\xi_n \cos\xi_{n-1} \dots \cos\xi_2 = r_1/\rho_1$.

(ii) The 2nd coordinate of $z$ is $\sin\xi_2 = r_2/\rho_1$.

(iii) The $j$th coordinate $(3 \le j \le n)$ of $z$ is $\sin\xi_j \cos\xi_{j-1} \dots \cos\xi_2 = r_j/\rho_1$.

Case (iii) does not occur when $n = 2$. Thus $T(1, 0, \dots, 0) = \rho_1^{-1}(r_1, r_2, \dots, r_n)$ and it follows immediately that

$$\|(r_1, r_2, \dots, r_n)\| = \rho_1\|(1, 0, \dots, 0)\|. \quad \square$$

**Theorem 4.2.** *Let $\nu \in \mathbb{N}$ and let $\|\cdot\|$ be a norm for $\mathbb{C}^{\nu+1}$ such that every element of $sa\mathcal{A}_\nu$ is Hermitian. Then there exists $\alpha > 0$ such that $\|\cdot\| = \alpha\|\cdot\|_2$, where $\|\cdot\|_2$ is the $\ell_2$-norm on $\mathbb{C}^{\nu+1}$.*

**Proof.** Let $z = (r_1 e^{i\theta_1}, \dots, r_{\nu+1}e^{i\theta_{\nu+1}})$ where $r_j > 0$ and $\theta_j \in \mathbb{R}$ $(j = 1, 2, \dots, \nu + 1)$. Continuity extends the result to all $r_j \ge 0$. Since $E_{jj} - E_{j+1,j+1}$ $(1 \le j \le \nu)$ are in $sa\mathcal{A}_\nu$ and hence are Hermitian, and $I$ also is Hermitian, it follows that all $E_{jj}$ $(1 \le j \le \nu + 1)$ are Hermitian. Let

$$U = \exp(-i\theta_1 E_{11})\dots\exp(-i\theta_{\nu+1}E_{\nu+1,\nu+1}).$$

Then $Uz = (r_1, \dots, r_{\nu+1})$ and using Lemma 4.1 with $n = \nu + 1$ the result follows. $\square$

For $sa\mathcal{B}_\nu$ and $sa\mathcal{D}_\nu$ ($\nu \geq 2$) we have defined Lie isomorphisms with the $\mathcal{L}_n$ Lie algebras for which a norm formula was given in [4]. Since the isomorphisms are similarities they lead to corresponding formulas for $sa\mathcal{B}_\nu$ and $sa\mathcal{D}_\nu$ which are given below. Although $sa\mathcal{D}_2$ is not simple the arguments do apply to it as well.

Let $\|\cdot\|_{\mathcal{L}}$ be the norm for $\mathbb{C}^{2\nu}$ as defined in [4, Theorem 4.1]. Using $T$ defined by (3.2), $\|v\|_{\mathcal{D}} = \|Tv\|_{\mathcal{L}}$ defines a norm on $\mathbb{C}^{2\nu}$ for which $H = sa\mathcal{D}_\nu^1$. Let

$$S = \Big\{(\alpha_1, \alpha_2, \ldots, \alpha_\nu, \beta_1, \beta_2, \ldots, \beta_\nu) \in \mathbb{R}^{2\nu} : \sum \alpha_j^2 + \sum \beta_j^2 = 1\Big\}.$$

Then, for $v = (z_1, ..., z_\nu, w_1, ..., w_\nu) \in \mathbb{C}^{2\nu}$ and $Tv = u = (x_1, ..., x_\nu, y_1, ..., y_\nu)$,

$$\|v\|_{\mathcal{D}} = \|u\|_{\mathcal{L}} = \sup\Big\{\Big|\sum \alpha_j x_j + \sum \beta_j y_j\Big| : (\alpha_1, \alpha_2, \ldots, \alpha_\nu, \beta_1, \beta_2, \ldots, \beta_\nu) \in S\Big\}.$$

Then

$$\begin{aligned}\Big|\sum \alpha_j x_j + \sum \beta_j y_j\Big| &= \Big|\sum \alpha_j (-1)^{j-1} i(z_j - w_j) + \sum \beta_j (z_j + w_j)\Big| \\ &= \Big|\sum \xi_j z_j + \sum \eta_j w_j\Big|\end{aligned}$$

where each $\xi_j = \beta_j \pm i\alpha_j$ and each $\eta_j = \overline{\xi_j}$. Thus we have

$$\|v\|_{\mathcal{D}} = \sup\Big\{\Big|\sum \xi_j z_j + \sum \overline{\xi_j} w_j\Big| : (\xi_1, \ldots, \xi_\nu) \in P\Big\} \tag{4.1}$$

where

$$P = \Big\{(\xi_1, \ldots, \xi_\nu) \in \mathbb{C}^\nu : \sum |\xi_j|^2 = 1\Big\}.$$

Similarly, if $\|\cdot\|_{\mathcal{L}^+}$ is the norm for $\mathbb{C}^{2\nu+1}$ as defined in [4, Theorem 4.1] then using $U$ defined by (3.4), $\|v\|_{\mathcal{B}} = \|Uv\|_{\mathcal{L}^+}$ defines a norm on $\mathbb{C}^{2\nu+1}$ for which $H = sa\mathcal{B}_\nu^1$. And, for $v = (z_0, z_1, z_2, ..., z_n, w_1, w_2, ..., w_n) \in \mathbb{C}^{2\nu+1}$, we have

$$\|v\|_{\mathcal{B}} = \sup\Big\{\Big|\sum \xi_j z_j + \sum \overline{\xi_j} w_j\Big| : (\xi_0, \xi_1, \ldots, \xi_\nu) \in Q\Big\} \tag{4.2}$$

where

$$Q = \Big\{(\xi_0, \xi_1, \ldots, \xi_\nu) \in \mathbb{R} \times \mathbb{C}^\nu : \sum |\xi_j|^2 = 1\Big\}.$$

The $\sqrt{2}$ in the matrix $U$ is absorbed into the constant $\xi_0$.

This brings us to the surprising case of $sa\mathcal{C}_\nu$; any norm on $\mathbb{C}^{2\nu}$ which makes all of $sa\mathcal{C}_\nu$ Hermitian is essentially the usual inner product norm so that all self-adjoint matrices in $M_{2\nu}(\mathbb{C})$ are Hermitian for that norm. (In fact it is enough to have a minimal set of Lie algebra generators all Hermitian.) We show in the following two lemmas and theorem that such a norm must be the $\ell_2$-norm or a positive multiple of it. As with the similar result for $sa\mathcal{A}_\nu$ we make repeated use of the property that $\exp(itT)$ is an isometry for any $t \in \mathbb{R}$ and any Hermitian element $T$.

Recall that $Q_{1k} = G_{1k} + G_{\nu+1,\nu+k}$ $(k = 1, \dots, \nu)$ are basis elements of $sa\mathcal{C}_\nu$.

**Lemma 4.3.** *Let $n \geq 2$ and let $\|\cdot\|$ be a norm for $\mathbb{C}^{2n}$ such that $Q_{1k}$ $(k = 1, \dots, n)$ are Hermitian. Then, for $r_1, \dots, r_n > 0$,*

$$\|(r_1, r_2, \dots, r_n, 0, \dots, 0)\| = \alpha\sqrt{r_1^2 + r_2^2 + \dots + r_n^2}$$

*where $\alpha = \|(1, 0, \dots, 0)\|$.*

**Proof.** Define $\rho_k$ $(1 \leq k \leq n)$ and $\xi_k$ $(2 \leq k \leq n)$ exactly as in the proof of Lemma 4.1. Let

$$T^+ = \exp(i\xi_n Q_{1n}) \exp(i\xi_{n-1} Q_{1,n-1}) \dots \exp(i\xi_2 Q_{12}).$$

We consider $w = T^+(1, 0, \dots, 0)$. Each $\exp(i\xi_k Q_k)$ can be written as a block matrix

$$\begin{bmatrix} \exp(i\xi_k G_{1k}) & O \\ O & \exp(i\xi_k G_{1k}) \end{bmatrix}.$$

So we can make use of $T$ in the proof of Lemma 4.1 to determine coordinates 1 to $n$ of $w$, and dealing with coordinates $n+1$ to $2n$ is straightforward.

(i) The 1st coordinate of $w$ is $r_1/\rho_1$.

(ii) The 2nd coordinate of $w$ is $r_2/\rho_1$.

(iii) The $j$th coordinate $(3 \leq j \leq n)$ of $w$ is $r_j/\rho_1$.

(iv) The $k$th coordinate $(n+1 \leq k \leq 2n)$ of $w$ is 0.

Thus $T^+(1, 0, \dots, 0) = \rho_1^{-1}(r_1, r_2, \dots, r_n, 0, \dots, 0)$ and the result follows. □

**Lemma 4.4.** *Let $\nu \geq 2$ and let $\|\cdot\|$ be a norm for $\mathbb{C}^{2\nu}$ such that every element of $sa\mathcal{C}_\nu$ is Hermitian. Let $z = (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}, \dots, r_{2\nu} e^{i\theta_{2\nu}})$ where $r_j \geq 0$ and $\theta_j \in \mathbb{R}$ $(j = 1, 2, \dots, 2\nu)$. Then $\|z\| = \alpha\sqrt{r_1^2 + r_2^2 + \ldots + r_{2\nu}^2}$ where $\alpha = \|(1, 0, \dots, 0)\|$.*

**Proof.** We shall assume all $r_j > 0$. Continuity extends the result to all $r_j \geq 0$. Each of the following products is independent of the order of the exponentials. For $j = 1, 2, \dots, \nu$, let $\eta_j = (\theta_{\nu+j} - \theta_j)/2$ and $\lambda_j = (\theta_{\nu+j} + \theta_j)/2$. Recall that $A_j = E_{jj} - E_{\nu+j,\nu+j}$ and $C_j = G_{j,\nu+j}$ are $saC_\nu$ basis elements. Then

$$\Big(\prod_{j=1}^{\nu} \exp(i\eta_j A_j)\Big) z = (r_1 e^{i\lambda_1}, \dots, r_\nu e^{i\lambda_\nu}, r_{\nu+1} e^{i\lambda_1}, \dots, r_{2\nu} e^{i\lambda_\nu}) = u \text{ (say)}.$$

For $j = 1, 2, \dots, \nu$, let $\mu_j = \tan^{-1}(-r_{\nu+j}/r_j)$ (so $\cos\mu_j$ is +ve and $\sin\mu_j$ is −ve). Then

$$\Big(\prod_{j=1}^{\nu} \exp(i\mu_j C_j)\Big)u = \big(e^{i\lambda_1}\sqrt{r_1^2 + r_{\nu+1}^2}, \dots, e^{i\lambda_\nu}\sqrt{r_\nu^2 + r_{2\nu}^2}, 0, \dots, 0\big) = v \text{ (say)}$$

and

$$\Big(\prod_{j=1}^{\nu} \exp(-i\lambda_j A_j)\Big)v = \big(\sqrt{r_1^2 + r_{\nu+1}^2}, \dots, \sqrt{r_\nu^2 + r_{2\nu}^2}, 0, \dots, 0\big) = w \text{ (say)}.$$

Then, using Lemma 4.3 with $n = \nu$, $\|z\| = \|w\| = \alpha\sqrt{r_1^2 + r_2^2 + \cdots + r_{2\nu}^2}$. □

**Theorem 4.5.** *Let $\nu \in \mathbb{N}$ and let $\mathbb{C}^{2\nu}$ have norm $\|\cdot\|$ such that every element of $sa\mathcal{C}_\nu$ is Hermitian.*
*(1) There exists $\alpha > 0$ such that $\|\cdot\| = \alpha\|\cdot\|_2$, where $\|\cdot\|_2$ is the $\ell_2$-norm on $\mathbb{C}^{2\nu}$.*
*(2) Every self-adjoint matrix in $M_{2\nu}(\mathbb{C})$ is Hermitian with respect to $\|\cdot\|$.*
*(3) For $\nu \geq 2$, there is no norm for $\mathbb{C}^{2\nu}$ with respect to which $sa\mathcal{C}_\nu^1$ is the (entire) set of Hermitian matrices.*

**Proof.** (1) If $\nu \geq 2$ then $\|\cdot\|$ satisfies the conditions of Lemmas 4.3 and 4.4, and (1) follows. Let $\nu = 1$ and let $(a, b) \in \mathbb{C}^2$ with $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$. Let

$$\phi = \tfrac{1}{2}(\alpha + \beta), \quad \psi = \tfrac{1}{2}(\beta - \alpha), \quad \theta = \tan^{-1}(-|a|/|b|)$$

and let $U = \exp(i\theta C_1)\exp(i\psi A_1)$. Then $U(a, b) = e^{i\phi}\sqrt{|a|^2 + |b|^2}\,(1, 0)$ so that $\|(a, b)\| = \alpha\|(a, b)\|_2$ where $\alpha = \|(1, 0)\|$.

(2) This follows from (1) since every self-adjoint matrix in $M_{2\nu}(\mathbb{C})$ is $\|\cdot\|_2$-Hermitian and hence $\|\cdot\|$-Hermitian.

(3) This follows from (2) since not every self-adjoint matrix in $M_{2\nu}(\mathbb{C})$ is in $sa\mathcal{C}_\nu^1$. □

**Corollary 4.6.** *The answer to Problem 5.7 in [4] is NO.*

Suppose we have a linear norm for which all $F_{jk}$ are Hermitian. It is then immediate from (1.2) that all $G_{jk}$ are Hermitian and hence that $H$ is all self-adjoint matrices. The above remarkable result for the symplectic case might be regarded as a half-way house between this example and the case when the initial Lie algebra is $\mathcal{L}_n$. Note also that Theorem 4.5 (1) is a stronger version of the Vidav-Palmer theorem for the algebras $M_{2\nu}(\mathbb{C})$.

## 5. Some Lie isomorphisms

We noted earlier that $sa\mathcal{A}_1, sa\mathcal{B}_1, sa\mathcal{C}_1$ are Lie isomorphic to each other. It is well known that $\mathcal{B}_3$ and $\mathcal{C}_2$ are Lie isomorphic, and also $\mathcal{A}_3$ and $\mathcal{D}_3$; the same is true for their self-adjoint variants.

**Theorem 5.1.** *(1) $sa\mathcal{B}_3$ is Lie isomorphic to $sa\mathcal{C}_2$.*
*(2) $sa\mathcal{A}_3$ is Lie isomorphic to $sa\mathcal{D}_3$.*

**Proof.** (1) It is enough to show that $\mathcal{L}_5$ is isomorphic to $sa\mathcal{C}_2$. As a basis for $sa\mathcal{C}_2$ take

$$A_1 + A_2,\ A_1 - A_2,\ B_1 + B_2,\ B_1 - B_2,\ C_1 + C_2,\ C_1 - C_2,\ P_{12},\ Q_{12},\ R_{12},\ S_{12}$$

where the notation is that of (2.6). As a basis for $\mathcal{L}_5$ take

$$G_{12},\ G_{35},\ -G_{25},\ G_{13},\ G_{15},\ G_{23},\ G_{45},\ G_{34},\ G_{14},\ G_{24}$$

each multiplied by 2. With basis elements taken in this order the Lie product tables then match exactly, as required.

(2) It is enough to prove that $\mathcal{L}_6$ is isomorphic to $sa\mathcal{A}_3$. As a basis for $sa\mathcal{A}_3$ take

$$\begin{array}{llllll} \multicolumn{2}{l}{\mathrm{diag}(1,1,-1,-1),} & \multicolumn{2}{l}{\mathrm{diag}(1,-1,1,-1),} & \multicolumn{2}{l}{\mathrm{diag}(1,-1,-1,1),} \\ F_{12}+F_{34}, & F_{12}-F_{34}, & F_{13}+F_{24}, & F_{13}-F_{24}, & F_{14}+F_{23}, & F_{14}-F_{23}, \\ G_{12}+G_{34}, & G_{12}-G_{34}, & G_{13}+G_{24}, & G_{13}-G_{24}, & G_{14}+G_{23}, & G_{14}-G_{23}. \end{array}$$

As a basis for $\mathcal{L}_6$ take

$$G_{56},\ G_{12},\ G_{34},\ G_{13},\ -G_{24},\ G_{45},\ G_{36},\ -G_{26},\ G_{15},\ G_{23},\ G_{14},\ G_{46},\ -G_{35},\ G_{25},\ G_{16},$$

each multiplied by 2. With basis elements taken in this order the Lie product tables then match exactly, as required. □

It is easily checked that neither of the above Lie isomorphisms preserve spectra, and hence neither is a similarity.

In [5, pp. 451-455] Helgason presents the long list (up to isomorphism) of the simple real Lie algebras. For any such $(\mathcal{L}, \circ)$, the mapping defined in Section 1 by $\phi(A) = -iA$ gives the Lie isomorphic $(-i\mathcal{L}, \bullet)$. In this form the self-adjoint variants of the classical Lie algebras appear as follows: $sa\mathcal{A}_\nu$ is $-i\,\mathfrak{su}(\nu+1, 0)$; for $\nu \geq 5, \mathcal{L}_\nu$ is $-i\,\mathfrak{so}(\nu, 0)$; $sa\mathcal{C}_\nu$ is $-i\,\mathfrak{sp}(\nu, 0)$. No other examples (with Lie product $\bullet$) consist entirely of self-adjoint matrices.

## Declaration of competing interest

There is no conflict of interest.

## Data availability

No data was used for the research described in the article.

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