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DO NOT BLAME BELLMAN: IT IS KOOPMANS' FAULT

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We provide a unified approach to stochastic dynamic programming with recursive utility based on an elementary application of Tarski's fixed point theorem. We establish that the exclusive source of multiple values is the presence of multiple recursive utilities consistent with the given aggregator, each yielding a legitimate value of the recursive program. We also present sufficient conditions ensuring a unique value of the recursive program in some circumstances. Overall, acknowledging the unavoidable failure of uniqueness in general, we argue that the greatest fixed point of the Bellman operator should have a privileged position.

KEYWORDS: Stochastic recursive utility, Koopmans operator, Bellman operator, dynamic programming, multiplicity.

1. INTRODUCTION

OVER THE LAST DECADES, RECURSIVE UTILITY has gained increasing interest in macroeconomics and finance. Building on the earlier work of Koopmans (1960), the approach postulates a stationary aggregator as a primitive representation of preferences over current consumption and future uncertain utility and recovers a time-consistent intertemporal utility function recursively. The representation of preferences by means of a stationary Koopmans aggregator preserves the tractability of traditional discounted expected utility while encompassing empirically relevant behavioral features, such as increasing marginal impatience (Lucas and Stokey (1984)), the distinction of risk attitudes from intertemporal substitution (Epstein and Zin (1989)), preference for early resolution of uncertainty (Kreps and Porteus (1978)), ambiguity aversion (Klibanoff, Marinacci, and Mukerji (2005)), and risk sensitivity and robustness (Hansen and Sargent (1995, 2001)). The expanding domain of applications in macroeconomics and finance is demonstrated by, among others, Backus, Routledge, and Zin (2005), Becker and Boyd III (1997), Hansen and Sargent (2008), Hansen, Heaton, Lee, and Roussanov (2007), Miao (2014), and Skiadas (2009).

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The flourishing popularity of recursive preferences has recently fostered a resurgent interest on foundational issues of existence and uniqueness. In fact, the most relevant specifications of the Koopmans aggregator (e.g., Epstein–Zin preferences) fail to satisfy the traditional Blackwell's discounting condition. As a consequence, an appeal to the Banach contraction mapping theorem is of limited scope, and even establishing the existence of a recursive utility function becomes an arduous task. More importantly, uniqueness and convergence obtain only under certain restrictions on the Koopmans aggregator and its domain. Recent developments along these research lines include Marinacci and Montrucchio (2010), who studied abstract value-concave (Thompson) aggregators, and Borovička and Stachurski (2020), Christensen (2022), and Hansen and Scheinkman (2012), who established a connection between certain recursive utility equations and the spectral radius of a related valuation operator.

Despite the substantial progress on recursive utility, less is known about its implications for dynamic programming. The purpose of this paper is to provide a unified treatment of recursive methods under minimal assumptions on the Koopmans aggregator and conventional restrictions on the constraints. In particular, we develop an elementary approach to dynamic programming with recursive utility based on Tarski's fixed point theorem. We establish the existence of a value of the recursive program and argue that, when the Koopmans aggregator identifies utility uniquely, the value of the recursive program is also unique. As an implication, whenever a recursive program admits multiple values, each corresponds to a different recursive utility consistent with the specification of the Koopmans aggregator. In general, without further restrictions, the recursive program yields a least value and a greatest value.

On merely philosophical grounds, our analysis reveals that the multiplicity of fixed points of the Bellman operator is exclusively caused by an underlying multiplicity of utilities generated by the Koopmans operator. In other terms, the Bellman operator is accurate, and it is rather the planning objective that is imprecisely identified by the Koopmans aggregator. On more practical grounds, our analysis suggests that the advantages of recursive utility are forced to coexist with a risk of value multiplicity, and a consistent selection criterion is unavoidable. We submit that, when all utilities generated by the Koopmans aggregator are legitimate planning objectives, the greatest value should have a focal role because of two relevant features: it is upper semicontinuous and it admits a stationary recursive policy, permitting a sequential implementation for the greatest utility consistent with the Koopmans aggregator. Other values might not admit a recursive policy, and might only be approximated by nonstationary policies.

We supplement our approach with a study of additional conditions ensuring a unique value of the recursive program. Indeed, we provide a generalized discounting criterion, consisting in identifying a sort of supergradient to the Bellman operator and verifying that its spectral radius is less than unity. This method is particularly suited to recursive utilities involving time-varying discounting. Unfortunately, the nature of the supergradient is in general case-specific, and this substantially limits the scope of the theory.

Our Tarski-type approach to dynamic programming requires a preliminary determination of an appropriate domain for the recursive program. For completeness, we present a method to identify suitable bounds in frameworks with homogeneous aggregators, so enhancing the applicability of the theory. We use the spectral radius to estimate the expansion of values in the feasible set and relate it to the contractionary force due to impatience implied by the recursive aggregator. We finally illustrate this method by means of an application to optimal investment in a risky asset under long-run risk and persistent shocks to labor income.

The paper is organized as follows. In Section 2, we briefly present a non-exhaustive discussion of some related literature. In Section 3, we introduce an abstract recursive program encompassing all relevant applications in the literature, and present our major theorems. In Section 4, we prove a sort of principle of optimality: the extreme values of the recursive program are implemented by the extreme utility functions generated by the given aggregator, possibly with nonstationary policies. In Section 5, we discuss sufficient conditions ensuring that the value is unique. In Section 6, we present our operational approach to identify an interval on which the recursive program is properly defined. Finally, in Section 7, we show how our theory applies to the widely studied optimal investment program with a risky asset and possibly growing labor income. All proofs, irrespective of their relevance, are collected in Appendix A. Appendix B presents a short introduction to the spectral radius of monotone sublinear operators, and Appendix C provides a known extension of the Feller property to unbounded values.

2. RELATED LITERATURE

A copious literature has studied dynamic programming with unbounded returns under traditional discounting. Boyd III (1990) introduced the Weighted Contraction approach to economic applications, a method further developed by Alvarez and Stokey (1998) and Durán (2000, 2003). An alternative Local Contraction approach was presented by Rincón-Zapatero and Rodríguez-Palmero (2003) and Martins-da-Rocha and Vailakis (2010). Recently, Rincón-Zapatero (2024) extended this method to a stochastic environment. Le Van and Morhaim (2002), Le Van and Vailakis (2005), Kamihigashi (2014), and Wiszniewska-Matyszkiel and Singh (2021) proposed a more primitive transversality condition to identify an appropriate space of candidate value functions. Jaśkiewisz and Nowak (2011) and Matkowski and Nowak (2011) presented a systematic study of all these approaches under uncertainty. Finally, a recent paper by Ma, Stachurski, and Akira Toda (2022) exploited a transformation of the Bellman operator, along with boundedness of the expected reward, to turn unbounded into bounded programs, so that conventional contraction techniques apply.

Dynamic programming with recursive utility was initially approached by Streufert (1990) and Ozaki and Streufert (1996). They introduced a notion of biconvergence, a condition ensuring that utility values can be arbitrarily approximated by increasing and decreasing orbits. Recently, along similar lines, Bich, Drugeon, and Morhaim (2018) provided a study of deterministic recursive programs under minimal assumptions on primitives. Though foundational, all these biconvergence criteria are not fully operational.

Marinacci and Montrucchio (2010, 2019) introduced a contraction-type approach to recursive utility under an assumption of value-concavity of the Koopmans aggregator, an issue further investigated by Becker and Rincón-Zapatero (2021). This property is shared by a large class of aggregators commonly used in applications, beginning with Epstein–Zin preferences. Dynamic programming with a value-concave aggregator was studied by Balbus (2020), Bloise and Vailakis (2018), and Ren and Stachurski (2021). This promising approach is frustrated by the fact that uniqueness only obtains in the interior of the domain, or subject to some appropriate boundary condition. In many relevant applications, this sort of boundary condition is unnatural.

Recent literature has studied recursive utility in a stochastic environment, with a specific focus on Epstein–Zin preferences. The approach was initially proposed by Hansen and Scheinkman (2012) and consists in establishing a connection between the recursive utility equation and the spectral radius of a related valuation operator. Borovička and

Stachurski (2020) established that Hansen and Scheinkman (2012)'s spectral radius condition is necessary and sufficient for the existence of Epstein–Zin recursive utility. Finally, Christensen (2022) further developed this method, providing a characterization of recursive utility under risk sensitivity, ambiguity, and Epstein–Zin preferences. Uniqueness obtains only under certain restrictions on the state space and the Markov transition. None of these papers studies dynamic programming with recursive utility.

3. ABSTRACT RECURSIVE PROGRAM

Let X and Z be complete separable metric spaces, and let G be a correspondence from X to Z. We interpret X as the state space, whereas Z is the action space. Feasibility is embedded in the correspondence $G : X \twoheadrightarrow Z$, that is, $G(x) \subset Z$ is the set of admissible actions at state x in X. We use $\Gamma \subset X \times Z$ to denote the graph of the feasibility correspondence. If needed, we also consider a (measurable) Markov transition $\Pi : \Gamma \to \Delta(X)$ governing the evolution of the state over time, that is, $\Pi(x, z)$ is a probability measure on the state space X, endowed with its Borel algebra.

We let $\mathcal{E}$ be the space of all maps $f : X \to \mathbb{R}$, endowed with the product topology and the natural ordering. We introduce the complete lattice $\mathcal{F}$ of $\mathcal{E}$ given by

$$\mathcal{F} = \{f \in \mathcal{E} : \underline{f} \leq f \leq \bar{f}\},$$

where $\underline{f} : X \to \mathbb{R}$ and $\bar{f} : X \to \mathbb{R}$ are given bounds. We let $\mathcal{V}$ be the class of measurable maps $v : X \to \mathbb{R}$ in $\mathcal{F}$, that is,

$$\mathcal{V} = \{v \in \mathcal{F} : v \text{ is (Borel) measurable}\}.$$

Notice that $\mathcal{V}$ is *not* a complete lattice, though it is embedded in a complete lattice to ensure the applicability of Tarski's fixed point theorem (see Aliprantis and Border (2006, Theorem 1.11)).

The objective of the planner is given as an aggregator $W : \Gamma \times \mathcal{V} \to \mathbb{R}$. The nature of this aggregator will depend on the application of the theory. The most traditional example is given by

$$W(x, z, v) = (1 - \delta)u(x, z) + \delta \int v(y)\Pi(x, z)(dy),$$

where δ in $(0, 1) \subset \mathbb{R}^+$ is the discount factor and $u : \Gamma \to \mathbb{R}$ is the return, or reward, function. Our abstract formulation, inspired by Bertsekas (2018), encompasses many other instances of recursive preferences. A large variety of applications is discussed in, among others, Christensen (2022), Marinacci and Montrucchio (2010), and Ren and Stachurski (2021).

We impose restrictions on fundamentals that are satisfied in typical applications. In particular, Assumption 2 requires that the lower and upper bounds define a suitable invariant interval. Assumption 3 ensures the applicability of the maximum theorem (see Aliprantis and Border (2006, Section 17.5)), in addition to requiring monotonicity. Finally, Assumption 4 reproduces the logic of Levi's convergence theorem. We notice that, under Assumption 4, the property in Assumption 3(b) is weaker than the canonical Feller property in conventional bounded dynamic programming. In Appendix C, we provide more primitive conditions enforcing this sort of weak Feller property when values are unbounded.

ASSUMPTION 1—Semicontinuous bounds: *Both $\underline{f}$ and $\bar{f}$ in $\mathcal{F}$ are upper semicontinuous.*

ASSUMPTION 2—Invariance: *For every v in $\mathcal{V}$,*

$$\underline{f}(x) \le \inf_{z \in G(x)} W(x, z, v) \le \sup_{z \in G(x)} W(x, z, v) \le \bar{f}(x).$$

ASSUMPTION 3—Basic properties: (a) *The feasible correspondence $G : X \twoheadrightarrow Z$ is upper hemicontinuous with nonempty compact values.* (b) *If v in $\mathcal{F}$ is measurable* (*respectively, upper semicontinuous*), *the map $(x, z) \mapsto W(x, z, v)$ is measurable* (*respectively, upper semicontinuous*) *on Γ.* (c) *The aggregator is monotone in $\mathcal{V}$, that is,*

$$v' \ge v'' \quad \textit{implies} \quad W\big(x, z, v'\big) \ge W\big(x, z, v''\big).$$

ASSUMPTION 4—Monotone convergence: *For any sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{V}$ monotonically converging to v in $\mathcal{V}$,*

$$\lim_{n\to\infty} W(x, z, v_n) = W(x, z, v).$$

A major obstacle in stochastic dynamic programming is due to the difficulty in preserving measurability (see Miao (2014), Stachurski (2009), and Stokey, Lucas, and Prescott (1989)). To avoid these complicated issues, we innocuously extend the utility aggregator to possibly non-measurable values.[1] Formally, for any f in $\mathcal{F}$, we define

$$W^*(x, z, f) = \sup_{v \in \mathcal{V}} \big\{W(x, z, v) : v \le f\big\}.$$

Notice that, by monotonicity, $W^*(x, z, v) = W(x, z, v)$ whenever v is an element of the original domain $\mathcal{V}$. It is simple to verify that the extended utility aggregator $W^* : \Gamma \times \mathcal{F} \to \mathbb{R}$ satisfies the invariance property (Assumption 2) and monotonicity (Assumption 3(c)). To simplify, with some abuse of notation, we denote the extension itself as $W : \Gamma \times \mathcal{F} \to \mathbb{R}$.

The recursive program is described by the Bellman operator $T : \mathcal{V} \to \mathcal{F}$ given as

$$(Tv)(x) = \sup_{z \in G(x)} W(x, z, v).$$

Notice that, due to the mentioned issues of measurability, the Bellman operator returns values in the extended space $\mathcal{F}$. A *value of the recursive program* (or, simply, a *value*) is a fixed point of the Bellman operator, that is, an element v of $\mathcal{V}$ such that $v = (Tv)$. By means of Tarski's fixed point theorem (see Aliprantis and Border (2006, Theorem 1.11)), we establish the existence of values and provide a basic computational approach. Remarkably, upper semicontinuity of the greatest value emerges independently of the monotone convergence assumption (Assumption 4).

PROPOSITION 1—Existence: *Under Assumptions 1–4, the recursive program admits a least value $\underline{v}$ in $\mathcal{V}$ and a greatest value $\bar{v}$ in $\mathcal{V}$. Furthermore, the greatest value $\bar{v}$ in $\mathcal{V}$ is upper semicontinuous. Finally,*

$$\underline{v} = \lim_{n\to\infty} \big(T^n \underline{f}\big) \quad \textit{and} \quad \bar{v} = \lim_{n\to\infty} \big(T^n \bar{f}\big).$$

[1]This extension is innocuous because it involves no substantial alteration of the original preferences. In fact, it will be exploited only in intermediate steps of the analysis, so permitting the application of Tarski's fixed point theorem, and in the end all values will be consistent with the unextended utility aggregator.

Endowed with this basic existence theorem, we present our major contribution: the multiplicity of values only occurs because of an ambiguous identification of the planning objective. A policy is a measurable map $g : X \to Z$ such that $g(x)$ lies in $G(x)$. Let $\mathcal{G}$ be the space of all such policies. For a policy g in $\mathcal{G}$, consider the Koopmans operator $T_g : \mathcal{V} \to \mathcal{V}$ given by

$$(T_g v)(x) = W\big(x, g(x), v\big).$$

We assume that utility values are unambiguously identified by the policy, that is, the Koopmans operator admits a unique fixed point. Subject to this property, the value of the recursive program is certainly unique.

ASSUMPTION 5—Unambiguous identification: *For every policy g in $\mathcal{G}$, there exists a unique v_g in $\mathcal{V}$ such that $v_g = (T_g v_g)$.*

PROPOSITION 2—Uniqueness: *Under Assumptions 1–5, the recursive program admits a unique value v in $\mathcal{V}$.*

We interpret Proposition 2 as establishing that the only source of potential multiplicity of values is the misrepresentation of preferences: the aggregator is consistent with different utilities and, as a consequence, the Bellman operator returns multiple values. As we clarify with an example, multiple values cannot be discarded in general, even with bounded returns and a compact state space.

EXAMPLE 1—Multiplicity: In a deterministic framework, set $X = [0, 1]$, $Z = \mathbb{R}^+$, and $G(x) = [0, x] \subset \mathbb{R}^+$. The utility aggregator is

$$W(x, z, v) = \min\big\{x, \delta z + v(x - z)\big\}, \quad \text{with } \delta \in (0, 1) \subset \mathbb{R}^+.$$

One fixed point is $v(x) = \delta x$, with indeterminate optimal policy $g(x) = z$ for any z in $G(x)$. Another fixed point is $v(x) = x$, with optimal policy $g(x) = 0$.

We conclude with a conventional argument which, under further continuity restrictions, establishes uniform convergence of the iterates of the Bellman operator. Continuity reflects a sort of Feller property when the utility aggregator is bounded. We remark that, under monotone convergence (Assumption 4), Assumption 7(b) implies the upper semicontinuity property in Assumption 3(b).

ASSUMPTION 6—Continuous bounds: *Both $\underline{f}$ and $\bar{f}$ in $\mathcal{F}$ are continuous.*

ASSUMPTION 7—Basic properties strengthened: (a) *The feasible correspondence $G : X \twoheadrightarrow Z$ is continuous with nonempty compact values.* (b) *If v in $\mathcal{V}$ is continuous, the map $(x, z) \mapsto W(x, z, v)$ is continuous on Γ.*

PROPOSITION 3—Uniform convergence: *Under Assumptions 1–7, the only value v^* in $\mathcal{V}$ of the recursive program is continuous. Furthermore, given any v_0 in $\mathcal{V}$, on every compact set $K \subset X$,*

$$\lim_{n\to\infty} \sup_{x\in K} \big|\big(T^n v_0\big)(x) - v^*(x)\big| = 0.$$

Throughout the rest of this paper, we postulate Assumptions 1–4, except when explicitly stated otherwise. Thus, in general, we cannot prevent the occurrence of multiple values. We notice that Assumptions 1–4 are maintained for narrative convenience, even though weaker restrictions would be sufficient to establish some of our claims in the following analysis.

4. PRINCIPLE OF OPTIMALITY

We argue that the extreme fixed points of the Bellman operator correspond to the sequential values of the planning program for extreme utility functions generated by the given aggregator. In particular, the greatest fixed point is the value for the greatest utility function defined on stationary policies, whereas the least fixed point returns the value for the least utility function defined on possibly nonstationary policies. By means of an example, we prove that this gap cannot be closed and nonstationary policies are in general needed to implement the least fixed point. We also show that the optimal recursive policy of the least value, even when it exists, might be misleading. This suggests that the greatest value should have a *focal role*, even when computing the least value would seem more natural, as explained in Remark 2.

A (stationary) utility function $U : \mathcal{G} \times X \to \mathbb{R}$ is recursively generated whenever

$$U_g(x) = W\big(x, g(x), U_g\big).$$

Thus, we interpret $U_g(x)$ in $\mathbb{R}$ as the utility obtained adopting policy g in $\mathcal{G}$ evaluated beginning from state x in X. A recursively generated utility function determines a *sequential value* v in $\mathcal{F}$, namely,

$$v(x) = \sup_{g \in \mathcal{G}} U_g(x).$$

In principle, there could be multiple recursively generated utility functions and, hence, multiple associated sequential values.

We select the greatest among such utilities, $\bar{U} : \mathcal{G} \times X \to \mathbb{R}$, that is, for every policy g in $\mathcal{G}$, $\bar{U}_g$ in $\mathcal{V}$ is the greatest fixed point $U_g = (T_g U_g)$. This utility function can be computed as

$$\bar{U}_g(x) = \lim_{n \to \infty} \big(T_g^n \bar{f}\big)(x).$$

We establish that the greatest fixed point of the Bellman operator is indeed the sequential value for the greatest recursively generated utility.

PROPOSITION 4—Upper value: *The greatest value of the recursive program $\bar{v}$ in $\mathcal{V}$ satisfies*

$$\bar{v}(x) = \sup_{g \in \mathcal{G}} \bar{U}_g(x).$$

We also argue that the least fixed point of the Bellman operator corresponds to the value for the least nonstationary utility function. A nonstationary policy $\gamma = (g_n)_{n \in \mathbb{N}}$ in $\mathcal{G}^{\mathbb{N}}$ describes a time-varying rule of behavior, where g_n in $\mathcal{G}$ is the policy adopted in period n in $\mathbb{N}$. An extended utility function $U : \mathcal{G}^{\mathbb{N}} \times X \to \mathbb{R}$ is recursively generated whenever

$$U_{(g,\gamma)}(x) = W\big(x, g(x), U_\gamma\big),$$

where $U_\gamma(x)$ in $\mathbb{R}$ is the utility obtained adopting policy γ in $\mathcal{G}^{\mathbb{N}}$ evaluated beginning from state x in X. The least utility function on nonstationary policies can be computed, given γ in $\mathcal{G}^{\mathbb{N}}$, as

$$\underline{U}_\gamma(x) = \lim_{n\to\infty} (T_{g_1} \circ T_{g_2} \circ \cdots \circ T_{g_n} \underline{f})(x).$$

This limit is well-defined because it is taken with respect to an increasing sequence. We show that the least fixed point of the Bellman operator is indeed the sequential value corresponding to this least utility function defined on possibly nonstationary policies.

PROPOSITION 5—Lower value: *The least value of the recursive program* $\underline{v}$ *in* $\mathcal{V}$ *satisfies*

$$\underline{v}(x) = \sup_{\gamma\in\mathcal{G}^{\mathbb{N}}} \underline{U}_\gamma(x).$$

Our general theory ensures that the greatest value is upper semicontinuous and, hence, that a stationary optimal policy exists. Furthermore, the value is achieved by the greatest recursive utility under this policy. Both properties are not shared by the least value. In general, we cannot establish that the least value is upper semicontinuous and, even when it is, the corresponding recursive policy might not be enforcing the least value. Proposition 5 only ensures that the least utility can approximate the least value by means of nonstationary policies. As the characterization of the optimal policy is a major advantage of recursive methods, these are rather disturbing features of the least value. We illustrate these drawbacks by means of simple examples.

EXAMPLE 2—Misrepresentation of the policy: In a deterministic framework, let $X = [0, 1]$, $Z = \mathbb{R}^+$, and $G(x) = [0, x]$. The aggregator is

$$W(x, z, v) = z + \min\left\{v(x - z), 1 + \frac{1}{2}v(x - z)\right\}.$$

The bounds are given by $\underline{f}(x) = 0$ and $\bar{f}(x) = 2 + x$. Our theory applies to this simple framework, and we argue that $\underline{v}(x) = x$. Indeed, under this conjecture, the recursive program becomes

$$\max_{z\le x} z + (x - z) = x,$$

thus confirming our claim. The value is continuous and yields an optimal policy $\underline{g}(x) = 0$. However, the least recursively generated utility function cannot achieve this value under this given policy. Indeed, as can be straightforwardly verified, $\underline{U}_{\underline{g}}(x) = 0$.

EXAMPLE 3—Lack of stationary policy: We construct a trivial example to show that nonstationary policies are necessary to sustain the least value. Let $X = \{0\}$, $Z = \{\alpha, \beta, \gamma\}$, and $G(x) = Z$. Also, consider the bounds given by $\underline{f} = -1$ and $\bar{f} = 1$. The aggregator is described by

$$W(x, \alpha, v) = \sqrt{v^+} - \sqrt{v^-},$$
$$W(x, \beta, v) = \sqrt{v + 1} - 1,$$
$$W(x, \gamma, v) = -\frac{1}{5} + \frac{1}{3}v.$$

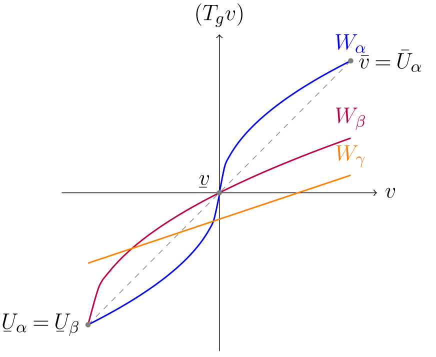

FIGURE 1.—Lack of stationary policy.

All of our assumptions are satisfied under the stated conditions. The Bellman operator corresponds to the upper envelope of the curves plotted in Figure 1. By direct inspection, $\underline{v} = 0$, and the two associated optimal actions are α and β. However, $\underline{U}_\alpha = \underline{U}_\beta = -1$. Hence, by Proposition 5, the least value can be achieved under the least utility only by means of a nonstationary policy.

REMARK 1—Bertsekas (2018)'s noncontractive models: Our implementation of extreme values mirrors Bertsekas (2018, Chapter 4)'s characterization of optimal policies in noncontractive models. Bertsekas (2018) moved from a primitive planning objective generated as

$$\tilde{U}_\gamma(x) = \liminf_{n\to\infty} (T_{g_1} \circ T_{g_2} \circ \cdots \circ T_{g_n} \tilde{f})(x),$$

where γ in $\mathcal{G}^{\mathbb{N}}$ is a nonstationary policy and $\tilde{f}$ in $\mathcal{V}$ is given exogenously. The value of the sequential program for the given planning objective is determined as

$$\tilde{v}(x) = \sup_{\gamma \in \mathcal{G}^{\mathbb{N}}} \tilde{U}_\gamma(x).$$

Bertsekas (2018) argued that, under certain regularity conditions, $\tilde{v}$ in $\mathcal{V}$ is a fixed point of the Bellman operator, $\tilde{v} = (T\tilde{v})$. Furthermore, he remarked that this value is not in general implementable by stationary policies, that is, $\tilde{v}(x) > \sup_{g\in\mathcal{G}} \tilde{U}_g(x)$. In particular, he proved that a restriction to stationary policies is feasible whenever $(T\tilde{f}) \leq \tilde{f}$ (see Bertsekas (2018, Proposition 4.3.9) and the discussion thereafter), and provided streamlined examples of failure of implementability by stationary policies otherwise. This is consistent with our findings in Propositions 4–5.

REMARK 2—Selection: To clarify the role of our selection criterion, consider Epstein–Zin recursive utility. Uniqueness is established only under certain restrictions on primitives and, therefore, a risk of multiplicity persists in general. As the Epstein–Zin aggregator takes values in $\mathbb{R}^+$, $\underline{f} = 0$ is a natural lower bound (and, for empirically plausible elasticity of intertemporal substitution, is not a trivial value of the recursive program). A suitable upper bound $\bar{f}$, instead, needs to be determined depending on specific features of the feasible set. As a consequence, it would seem more parsimonious to confine attention to $\underline{v} = \lim_{n\to\infty}(T^n \underline{f})$. Furthermore, this least value could be interpreted as the limit of finite-time truncations of the infinite-time planning horizon. Instead, we find that attention should be prudentially reserved for the greatest value $\bar{v} = \lim_{n\to\infty}(T^n \bar{f})$, as upper semicontinuity might fail, and a stationary policy might not exist, for the least value.

5. UNIQUENESS

We propose a generalized discounting condition ensuring that the value of the recursive program is unique. The method consists in verifying that the spectral radius of a supergradient to the Bellman operator is less than unity. In the case of time-additive preferences, this property can be verified immediately and operationally. In general, it requires further elaboration in order to identify a suitable supergradient (if any) to the Bellman operator. Some progress along these lines can be found in the recent work of Christensen (2022). In our application to dynamic programming, however, the supergradient is only *sublinear*, so bringing an additional degree of complexity in the analysis. For completeness, as this is a non-conventional technique, we provide a short treatment of the spectral radius of monotone sublinear operators in Appendix B.[2]

Consider the space

$$\mathcal{L} = \{v \in \mathcal{E} : |v| \leq \lambda(\bar{f} - \underline{f}) \text{ for some } \lambda \in \mathbb{R}^+\},$$

endowed with the conventional norm

$$\|v\| = \inf\{\lambda \in \mathbb{R}^+ : |v| \leq \lambda(\bar{f} - \underline{f})\}.$$

Given a monotone sublinear operator $D : \mathcal{L} \to \mathcal{L}$, let $\rho(D)$ in $\mathbb{R}^+$ be its spectral radius, that is,

$$\rho(D) = \lim_{n\to\infty} \sqrt[n]{\|D^n\|}.$$

It is a conventional exercise to show that the spectral radius indeed exists (see Appendix B).

We establish that, when the spectral radius of the supergradient is less than unity at some value of the recursive program, this is necessarily the greatest value. Thus, when the spectral radius condition is satisfied at the least fixed point, the Bellman operator admits a unique (upper semicontinuous) fixed point. In other terms, multiplicity necessarily requires the failure of some sort of discounting in a neighborhood of the least fixed point.

PROPOSITION 6—Spectral radius: *Suppose that a value of the recursive program $\tilde{v}$ in $\mathcal{V}$ admits a monotone sublinear operator $D : \mathcal{L} \to \mathcal{L}$ satisfying, for every v in $\mathcal{V}$,*

$$(Tv) - (T\tilde{v}) \leq D(v - \tilde{v});$$

then, provided that $\rho(D) < 1$, $\tilde{v}$ in $\mathcal{V}$ is the greatest value of the recursive program.

[2]Importantly, even for linear operators, we are not aware of any Perron–Frobenius theorem suitable for our analysis, except under certain restrictive assumptions (e.g., in the recent economic literature, Chattopadhyay (2018, Section 3) and Christensen (2017, Section 2.3)). Therefore, the spectral radius might not be an eigenvalue of the operator. Due to this potential failure, we present an approximation method (Claim 6 in Appendix B).

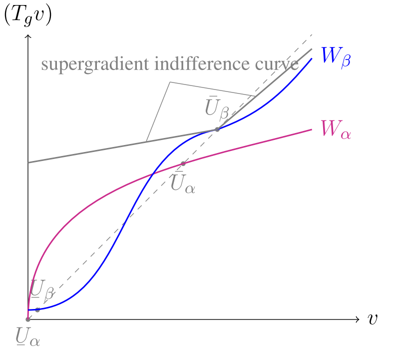

FIGURE 2.—Unique value with multiple utilities.

The supergradient approach is in fact an extension of the Weighted Contraction Approach for traditional linear aggregators. Unfortunately, this method is not fully operational: Establishing the existence of a suitable supergradient is in general a model-specific task. We provide an example inspired by the recent literature on state-contingent discounting (Stachurski and Zhang (2021)).

REMARK 3—Linear supergradient: Christensen (2022) introduced a supergradient (or subgradient) approach to study recursive utility. Given a policy g in $\mathcal{G}$, he postulated the existence of a monotone *linear* operator $D_g : \mathcal{L} \to \mathcal{L}$ satisfying, for every v in $\mathcal{V}$,

$$(T_g v) - (T_g \tilde{v}) \leq D_g(v - \tilde{v}).$$

The Koopmans operator admits a unique fixed point whenever the spectral radius of the supergradient is less than unity. Christensen (2022, Proposition 3.1)'s characterization can be invoked in order to prove, by our Proposition 2, that the value of the recursive program is unique. Our *sublinear* approach via the Bellman operator mimics instead the traditional condition in the established literature on the Weighted Contraction Approach. In particular, the existence of a *sublinear* supergradient is implied by Christensen (2022)'s *linear* restriction for all policies. In general, however, the Bellman operator might admit a suitable sublinear supergradient even when no linear supergradient to the Koopmans operator exists (see Figure 2).

EXAMPLE 4—State-contingent discounting: Consider the aggregator given by

$$W(x, z, v) = u(x, z) + \int \delta(x, y) v(y) \Pi(x, z)(dy),$$

where $\delta : X \times X \to \mathbb{R}^+$ is a continuous state-contingent discounting and $u : \Gamma \to \mathbb{R}$ is a continuous reward. By simple manipulations, we obtain

$$\begin{aligned} W(x, z, v) - W(x, z, \tilde{v}) &= \int \delta(x, y)(v - \tilde{v})(y) \Pi(x, z)(dy) \\ &\leq \sup_{z \in G(x)} \int \delta(x, y) |v - \tilde{v}|(y) \Pi(x, z)(dy) \\ &= D(v - \tilde{v})(x). \end{aligned}$$

Assuming that $D(\mathcal{L}) \subset \mathcal{L}$, $D : \mathcal{L} \to \mathcal{L}$ is in fact a suitable supergradient and our theory applies whenever $\rho(D) < 1$. By an appeal to Claim 6 in Appendix B, uniqueness obtains whenever ρ in $(0, 1) \subset \mathbb{R}^+$ satisfies, for some f in the interior of $\mathcal{L}$,

$$D(f) \leq \rho f,$$

which resembles the discounting condition in the Weighted Contraction Approach (e.g., Stachurski (2009, Condition 12.11)).

6. BOUNDS

Our theory heavily relies on the existence of suitable bounds for a well-defined recursive program (Assumption 2). For operational purposes, we provide a general method to identify such bounds in applications with homogeneous utility aggregators. In particular, in order to ensure a finite value of the recursive program, we compare the expansionary capacity of the feasible set with the contractionary force due to time impatience. We present applications to well-studied recursive programs in the literature. To avoid further complications of independent nature, we confine our analysis to the existence of simply measurable bounds.

We adopt a conventional separation of the attitude towards intertemporal substitution from uncertainty aversion. More precisely, we consider aggregators of the form

$$W(x, z, v) = V\big(u(x, z), I(x, z, v)\big),$$

where $V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and strictly increasing certainty aggregator, $u : \Gamma \to \mathbb{R}^+$ is a continuous current return, and $I : \Gamma \times \mathcal{M}^+ \to \bar{\mathbb{R}}^+$ is a certainty equivalent operator fulfilling monotonicity on $\mathcal{M}^+$ as well as the monotone convergence property (Assumption 4). Here, $\mathcal{M}$ is the space of measurable maps on X with values in $\mathbb{R}$, whereas $\bar{\mathbb{R}}$ is the field of extended reals. We also assume that the map $(x, z) \mapsto I(x, z, v)$ is measurable whenever v lies in $\mathcal{M}^+$. A large variety of recursive aggregators admit this tractable decomposition. Finally, we postulate that aggregator $V : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is homogeneous, that is, for every λ in $\mathbb{R}^+$,

$$\lambda V(u, I) = V(\lambda u, \lambda I).$$

In turn, certainty equivalent $I : \Gamma \times \mathcal{M}^+ \to \bar{\mathbb{R}}^+$ is subhomogeneous, that is, for every λ in $[0, 1] \subset \mathbb{R}^+$,

$$\lambda I(x, z, v) \leq I(x, z, \lambda v).$$

Traditional recursive aggregators are typically homogeneous, so as to generate homothetic preferences. Concave (positive) certainty equivalent operators are subhomogeneous.

We estimate the expansionary tendency of values subject to the feasible set. To this end, we directly assume the existence of some ρ in $\mathbb{R}^+$ satisfying, for some map f in $\mathcal{M}^+$,

$$\sup_{z \in G(x)} I(x, z, f) \leq \rho f(x). \tag{U1}$$

Intuitively, ρ in $\mathbb{R}^+$ captures the rate of growth of future values, adjusted by risk aversion, permitted by the feasible set of the recursive program. The bound needs to be sufficiently permissive to accommodate the growth of current returns, that is,

$$\sup_{z \in G(x)} u(x, z) \leq f(x). \tag{U2}$$

We finally postulate that, for some $t > 1$ in $\mathbb{R}^+$,

$$V(1, \rho t) \le t. \tag{U3}$$

Along with homogeneity, this property ensures that utility does not explode when returns grow at gross rate ρ in $\mathbb{R}^+$. Under conditions (U1)–(U3), an upper bound for the recursive program exists.

CLAIM 1—Bounds: *Under the stated conditions* (U1)–(U3), $\underline{f} = 0$ *and* $\bar{f} = tf$ *in* $\mathcal{M}^+$ *are suitable bounds satisfying*

$$\underline{f} \le (T\underline{f}) \le (T\bar{f}) \le \bar{f}.$$

Our condition (U1) accurately reflects the effects of risk aversion on value growth. For practical purposes, however, it is often convenient to overestimate value growth by considering a risk-neutral benchmark. Indeed, assume that the certainty equivalent satisfies

$$I(x, z, v) \le \int v(y)\Pi(x, z)(dy), \tag{CE}$$

where $\Pi : \Gamma \to \Delta(X)$ is a Markov transition. This property requires that the expected value of a lottery dominates its certainty equivalent, and it is shared by several typical operators. When the certainty equivalent satisfies (CE), root ρ in $\mathbb{R}^+$ can be operationally computed as the spectral radius of a monotone sublinear operator (see Claim 6 in Appendix B).

CLAIM 2—Operational criterion: *Under property* (CE), ρ *in* $\mathbb{R}^+$ *satisfies condition* (U1) *provided that, for some map* f *in* $\mathcal{M}^+$,

$$D(f)(x) \le \rho f(x), \tag{U1*}$$

where

$$D(f)(x) = \sup_{z \in G(x)} \int f(y)\Pi(x, z)(dy).$$

We illustrate the nature of our conditions for canonical Epstein–Zin preferences. We also compare these conditions with Hansen and Scheinkman (2012) and the related literature. It is worth observing that condition (U3) pertains only to the aggregator, whereas conditions (U1)–(U2) involve the feasible set and, thus, depend on the specific recursive program. We shall later present an extended application to the investment in a risky asset with time-varying returns and labor income.

EXAMPLE 5—Epstein–Zin recursive utility: Epstein–Zin preferences correspond to

$$V(u, I) = \left(u^{1-\sigma} + \delta I^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

and

$$I(x, z, v) = \left(\int v(y)^{1-\gamma}\Pi(x, z)(dy)\right)^{\frac{1}{1-\gamma}},$$

where σ in $\mathbb{R}^+$ is the reciprocal of the elasticity of intertemporal substitution and γ in $\mathbb{R}^+$ is the measure of relative risk aversion. We show that condition (U3) is satisfied by any sufficiently large t in $\mathbb{R}^+$ if

$$\delta\rho^{1-\sigma} < 1.$$

Indeed, notice that, when $\sigma < 1$,

$$\frac{V(1, \rho t)}{t} \leq 1 \quad \text{if and only if} \quad \frac{1}{t^{1-\sigma}} + \delta\rho^{1-\sigma} \leq 1,$$

which can be satisfied by some large t in $\mathbb{R}^+$ if and only if $\delta\rho^{1-\sigma} < 1$. On the other hand, when $\sigma > 1$,

$$\frac{V(1, \rho t)}{t} \leq 1 \quad \text{if and only if} \quad \frac{1}{t^{1-\sigma}} + \delta\rho^{1-\sigma} \geq 1,$$

which is always satisfied by a sufficiently large t in $\mathbb{R}^+$.

REMARK 4—Comparison with the previous literature: Despite first appearance, our conditions are consistent with Hansen and Scheinkman (2012) and Christensen (2022). For the purpose of comparison, as the consumption process is given exogenously in that line of literature, consider a fixed policy g in $\mathcal{G}$, and further suppose that their eigenvalue equation is satisfied, that is,

$$\int \tilde{f}(y)\Pi\big(x, g(x)\big)(dy) = \tilde{\rho}\tilde{f}(x).$$

This equation corresponds to Hansen and Scheinkman (2012, Equation (4)) and Christensen (2022, Equation (22)), though it is implicitly posed in terms of consumption levels rather than consumption growth rates. Setting $\tilde{f} = f^{1-\gamma}$ and $\tilde{\rho} = \rho^{1-\gamma}$, the eigenvalue equation becomes

$$\left(\int f(y)^{1-\gamma}\Pi\big(x, g(x)\big)(dy)\right)^{\frac{1}{1-\gamma}} = I(x, z, f) = \rho f(x) = \tilde{\rho}^{\frac{1}{1-\gamma}} f(x),$$

which corresponds to our condition (U1). Therefore, as illustrated in Example 5, condition (U3) is satisfied whenever

$$\delta\rho^{1-\sigma} < 1 \quad \text{or, equivalently,} \quad \delta\tilde{\rho}^{\frac{1-\sigma}{1-\gamma}} < 1,$$

which is consistent with requirement (c) in Hansen and Scheinkman (2012)'s Proposition 6 as well as with restriction (25) in Christensen (2022)'s Theorem 6.1 and Corollary 6.1.

We finally present conditions implying that the recursive program admits no value. To this end, we assume the existence of some ρ in $\mathbb{R}^{++}$ satisfying, for some policy g in $\mathcal{G}$ and some (non-zero) map f in $\mathcal{M}^+$,

$$\rho f(x) \leq I\big(x, g(x), f\big). \tag{L1}$$

Furthermore, we suppose that this lower bound is dominated by the current returns to the given policy, that is,

$$f(x) \leq u\big(x, g(x)\big). \tag{L2}$$

We finally postulate the existence of some η in $(0, 1) \subset \mathbb{R}^+$ such that, for every t in $\mathbb{R}^+$,

$$t < \eta V(1, \rho t). \tag{L3}$$

For homogeneous aggregators and certainty equivalent operators, restrictions (L1)–(L3) imply that a feasible policy yields an infinite value, thus preventing the existence of a solution to the recursive program.

CLAIM 3—Failure of existence: *Under the stated conditions* (L1)–(L3) *for homogeneous aggregators and certainty equivalent operators, the recursive program admits no value.*

7. APPLICATION

We consider a conventional optimal saving program with a safe asset and a risky asset. The exogenous state is governed by an irreducible Markov transition $\Pi : S \to \Delta(S)$, where S is a metric space endowed with its Borel σ-algebra $\mathcal{S}$. Let $X \subset S \times \mathbb{R}^+$, with typical element (s_t, w_t), where s_t in S is the Markov state and w_t in $\mathbb{R}^+$ is the accumulated wealth. The transition is given by

$$w_{t+1} = \big(R_f + \alpha_t\big(R_{t,t+1}(s_t) - R_f\big)\big)(w_t - c_t) + e_{t+1}(s_{t+1}),$$

where R_f in $\mathbb{R}^{++}$ is the safe return, $R_{t,t+1}$ in $\mathbb{R}^{++}$ is the uncertain return on the risky asset, α_t in $[0, 1] \subset \mathbb{R}^+$ is the wealth share invested in the risky asset, c_t in $[0, w_t] \subset \mathbb{R}^+$ denotes current consumption, and e_t in $\mathbb{R}^+$ is labor income, depending on state s_t in S. Thus, the current action is $z_t = (c_t, \alpha_t)$ in $G(x_t) = [0, w_t] \times [0, 1]$. The state space X contains all (s_t, w_t) in $S \times \mathbb{R}^+$ such that $e_t(s_t) \leq w_t$.

We study Epstein–Zin utility with possibly preference shocks. In particular, the utility aggregator is given by

$$V(c_t, I_t) = \big((1-\delta)c_t^{1-\sigma} + \delta I_t^{1-\sigma}\big)^{\frac{1}{1-\sigma}},$$

with certainty equivalent

$$I_t = \big(\mathbb{E}_t v_{t+1}^{1-\gamma}\big)^{\frac{1}{1-\gamma}},$$

where σ in $\mathbb{R}^+$ is the elasticity of intertemporal substitution, γ in $\mathbb{R}^+$ is the coefficient of relative risk aversion, and δ in $(0, 1) \subset \mathbb{R}^+$ is a discount factor. In Example 7, we also consider risk-sensitive preferences of the form

$$V(c_t, I_t) = (1-\delta)c_t + \delta I_t,$$

with certainty equivalent

$$I_t = -\frac{1}{\theta}\log \mathbb{E}_t \exp(-\theta v_{t+1}),$$

where θ in $\mathbb{R}^{++}$ is the risk sensitivity coefficient. In alternative specifications of this optimal saving program, we verify conditions for existence, non-existence, and uniqueness of a value as implied by our previous analysis. The calculations for all these examples are collected in Appendix A.

EXAMPLE 6—Campbell and Viceira (1999): With no labor income, we verify whether our conditions are satisfied whenever the expected risky return is determined by an autoregressive process, as in Campbell and Viceira (1999). In particular, the state variable is governed by

$$s_{t+1} = (1-\phi)\mu + \phi s_t + \epsilon_{t+1},$$

where ϕ lies in $(0,1) \subset \mathbb{R}^+$ and μ in $\mathbb{R}^+$. This state variable affects the risky return, which is given by

$$\log R_{t,t+1} = \log R_f + s_t + \eta_{t+1},$$

where innovations are normally distributed with means $\mathbb{E}\epsilon = \mathbb{E}\eta = 0$ and standard deviations σ_ϵ and σ_η in $\mathbb{R}^+$. To simplify our analysis, we further assume that the innovations are independently distributed. A major difficulty in this specification is that the expected risky return might grow unboundedly. For Epstein–Zin utility without preference shocks, we find that our conditions (U1)–(U3) are satisfied whenever

$$\delta R_f^{1-\sigma} \exp\left(\mu + \frac{\sigma_\eta^2}{2} + \frac{\sigma_\epsilon^2}{(1-\phi)^2}\Phi\left(\frac{\sigma_\epsilon}{(1-\phi)}\right)\right)^{1-\sigma} < 1,$$

where $\Phi : \mathbb{R} \to [0,1]$ is the standard normal distribution. It is worth noticing that estimated standard deviations in Campbell and Viceira (1999, Table 1) are rather small, so that a calibrated condition for existence is approximated by $\delta(R_f \exp(\mu))^{1-\sigma} < 1$.

EXAMPLE 7—Campbell and Viceira (1999) with risk-sensitive preferences: In the environment of Example 6 with risk-sensitive preferences, we establish that a *unique* value exists whenever

$$\delta R_f \exp\left(\mu + \frac{\sigma_\eta^2}{2} + \frac{\sigma_\epsilon^2}{(1-\phi)^2}\Phi\left(\frac{\sigma_\epsilon}{(1-\phi)}\right)\right) < 1.$$

To this end, building on an intuition in Bäurle and Jaśkiewicz (2018, Proposition 1), we exploit the fact that increasing transformations of independent random variables are positively correlated. Thanks to this comonotonicity property, the Bellman operator is contractive on the relevant space, as it would be for discounted expected utility.

EXAMPLE 8—Weil (1993): We next consider an economy with a constant safe return, no risky asset, and a labor income growing according to

$$e_{t+1} = \phi e_t + \epsilon_{t+1},$$

where ϕ lies in $\mathbb{R}^+$ and the innovation is identically and independently distributed with values in an interval in the interior of $\mathbb{R}^+$ and mean μ in $\mathbb{R}^+$. As in Weil (1993, Assumptions 1 and 3), we assume that $R_f > \phi$ and $R_f > 1$. In this framework, we prove that conditions (U1)–(U3) are satisfied and, hence, a value exists whenever $\delta R_f^{1-\sigma} < 1$. Furthermore, conditions (L1)–(L3) are verified and, hence, no value exists in the opposite case when $\delta R_f^{1-\sigma} > 1$.

EXAMPLE 9—Albuquerque, Eichenbaum, Luo, and Rebelo (2016): We finally consider an economy with a constant safe return, no risky asset, no labor income, and preference shocks. As in Albuquerque et al. (2016), the certainty equivalent is replaced by

$$I_t = \left(\mathbb{E}_t\left(\frac{\xi_{t+1}}{\xi_t}\right)^{\frac{1-\gamma}{1-\sigma}} v_{t+1}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}.$$

The preference shock ξ_t in $\mathbb{R}^+$ evolves according to $\xi_{t+1} = \exp(s_{t+1})\xi_t$, with the state fulfilling

$$s_{t+1} = \phi s_t + \epsilon_{t+1},$$

where ϕ lies in $[0, 1) \subset \mathbb{R}^+$ and the innovation is normally distributed with mean $\mathbb{E}\epsilon = 0$ and standard deviation σ_ϵ in $\mathbb{R}^{++}$. In this framework, assuming that σ lies in $(0, 1) \subset \mathbb{R}^+$, we verify existence whenever

$$\delta R_f^{1-\sigma} \exp\left(\frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\sigma)(1-\phi)^2}\right) < 1.$$

It is worth noticing that, dissipating previous doubts in the literature (e.g., Stachurski and Zhang (2021, Section 6.2.4)), our condition for existence is satisfied by a calibrated experiment based on Albuquerque et al. (2016, Table IV: Benchmark Model) given the historical safe interest rate.

8. CONCLUSION

We have studied dynamic programming with recursive utility under conventional restrictions. Our theory suggests that it is an ill-posed effort to search for the application of a suitable fixed point theorem ensuring a unique value of the recursive program. Indeed, uniqueness only fails because of an ambiguous specification of the planning objectives, and when this occurs, no fixed point theorem will be able to restore uniqueness, unless it acts as an implicit selection criterion. We have also argued that, when multiplicity cannot be avoided, the upper value exhibits relevant properties of regularity and should be preferred to any other value.

APPENDIX A: PROOFS

PROOF OF PROPOSITION 1: We enlarge $\mathcal{V}$ to its complete lattice closure $\mathcal{F}$. We also extend the Bellman operator $T : \mathcal{F} \to \mathcal{F}$ as

$$(Tf)(x) = \sup_{z \in G(x)} W^*(x, z, f).$$

The extended operator is monotone and, by Tarski's fixed point theorem, the least and greatest fixed point in $\mathcal{F}$ exist. We show that the least fixed point is indeed in $\mathcal{V}$ and that $\underline{v} = \lim_{n\to\infty}(T^n \underline{f})$.

Let $\underline{v}$ in $\mathcal{F}$ be the least fixed point. By monotonicity, we have $\underline{v}_n = (T^n \underline{f}) \leq \underline{v}$ for every n in $\mathbb{N}$. As each $\underline{v}_n$ is upper semicontinuous (and, hence, measurable) by the maximum theorem, and the sequence is increasing in $\mathcal{V}$, its limit v remains in $\mathcal{V}$ (see Aliprantis and Border (2006, Theorem 4.27)). Furthermore, $\underline{v}_{n+1} = (T\underline{v}_n) \leq (Tv)$ implies $v \leq (Tv)$. Assume that, for some x in X,

$$v(x) < (Tv)(x) = \sup_{z \in G(x)} W(x, z, v).$$

We can find z in $G(x)$ such that

$$v(x) < W(x, z, v).$$

By Monotone Convergence (Assumption 4), for every sufficiently large n in $\mathbb{N}$,

$$\sup_{z \in G(x)} W(x, z, \underline{v}_n) = \underline{v}_{n+1}(x) \leq v(x) < W(x, z, \underline{v}_n),$$

a contradiction. Hence, v in $\mathcal{V}$ is a fixed point and, as $v \leq \underline{v}$, it is the least fixed point, so proving the claim.

We now show that $\bar{v}$ in $\mathcal{F}$ is upper semicontinuous and, hence, measurable, thus an element of $\mathcal{V}$. Let v in $\mathcal{V}$ denote the upper semicontinuous envelope of $\bar{v}$ in $\mathcal{F}$, that is, the least upper semicontinuous v in $\mathcal{F}$ such that $\bar{v} \leq v$. By monotonicity, $\bar{v} = (T\bar{v}) \leq (Tv)$ and, by the maximum theorem, (Tv) in $\mathcal{F}$ is upper semicontinuous and, therefore, in $\mathcal{V}$. Hence, $v \leq (Tv)$. Invoking again Tarski's fixed point theorem, applied to the complete sublattice $\{f \in \mathcal{F} : v \leq f\}$, we conclude that $v \leq \bar{v}$, which proves that $\bar{v} = v$. So, $\bar{v}$ in $\mathcal{F}$ is upper semicontinuous and, hence, an element of $\mathcal{V}$.

Let $\bar{v}_n = (T^n \bar{f})$ in $\mathcal{F}$. By monotonicity, we have $\bar{v} \leq \bar{v}_n$ and, by the maximum theorem, each $\bar{v}_n$ in $\mathcal{V}$ is upper semicontinuous. Let v in $\mathcal{V}$ be the limit of this decreasing orbit, which is upper semicontinuous as it is the limit of a decreasing sequence of upper semicontinuous maps. By monotonicity, we have $(Tv) \leq (T\bar{v}_n) = \bar{v}_{n+1}$, so that $(Tv) \leq v$. Suppose that, for some x in X,

$$v(x) > \max_{z \in G(x)} W(x, z, v).$$

Notice that, for every n in $\mathbb{N}$, there exists z_n in $G(x)$ such that

$$\bar{v}_{n+1}(x) = W(x, z_n, \bar{v}_n).$$

By compactness, we can assume that a subsequence $(z_{n(j)})_{j \in \mathbb{N}}$ converges to z in $G(x)$. By monotone convergence (Assumption 4), we obtain

$$W(x, z, v) = \lim_{n \to \infty} W(x, z, \bar{v}_n).$$

By upper semicontinuity,

$$W(x, z, v) \geq \lim_{n \to \infty} \Big(\limsup_{j \to \infty} W(x, z_{n(j)}, \bar{v}_n) \Big).$$

By the fact that the orbit is decreasing, along with monotonicity,

$$W(x, z, v) \geq \lim_{n \to \infty} \Big(\limsup_{j \to \infty} W(x, z_{n(j)}, \bar{v}_{n(j)}) \Big).$$

By our choice of the sequence,

$$W(x,z,v) \geq \lim_{n\to\infty}\Big(\limsup_{j\to\infty} \bar{v}_{n(j)+1}(x)\Big) = v(x).$$

This shows that v in $\mathcal{V}$ is also a fixed point of $T:\mathcal{V}\to\mathcal{F}$ and, as $\bar{v}\leq v$, it is the greatest fixed point. *Q.E.D.*

PROOF OF PROPOSITION 2: Consider again the Bellman operator $T:\mathcal{V}\to\mathcal{F}$. Suppose that $\bar{v}$ is the greatest fixed point in $\mathcal{V}$ and let $\underline{v}$ be the least fixed point in $\mathcal{V}$. Let g in $\mathcal{G}$ be the policy corresponding to the greatest fixed point, which exists by the maximum theorem as the greatest fixed point is upper semicontinuous (Proposition 1). In particular, measurability of the policy is implied by Brown and Purves (1973, Corollary 1). Notice that

$$(T_g\underline{v})(x) = W\big(x, g(x), \underline{v}\big) \leq \sup_{z\in G(x)} W(x,z,\underline{v}) = (T\underline{v})(x) = \underline{v}(x).$$

By Assumption 4 (Monotone convergence), operator $T_g:\mathcal{V}\to\mathcal{V}$ admits a fixed point v_g in $\{v\in\mathcal{V}: v\leq\underline{v}\}$ given by

$$v_g = \lim_{n\to\infty}\big(T_g^n\underline{v}\big).$$

In fact, sequence $(T_g^n\underline{v})_{n\in\mathbb{N}}$ in $\mathcal{V}$ is decreasing, so that measurability is preserved in the limit. Furthermore, for every n in $\mathbb{N}$,

$$\big(T_g^{n+1}\underline{v}\big)(x) = W\big(x, g(x), \big(T_g^n\underline{v}\big)\big),$$

revealing that the limit is a fixed point. We then conclude that $v_g\leq\underline{v}\leq\bar{v}$ and, as $\bar{v}=(T_g\bar{v})$, we must have that $\bar{v}=\underline{v}$ by Assumption 5 (Unambiguous identification). *Q.E.D.*

PROOF OF PROPOSITION 3: Using the notation in the proof of Proposition 1, by the maximum theorem, $\underline{v}_n = (T^n\underline{f})$ and $\bar{v}_n = (T^n\bar{f})$ are both continuous maps in $\mathcal{V}$. The only fixed point v^* of operator $T:\bar{\mathcal{V}}\to\mathcal{F}$ is continuous because it is, at the same time, the limit of an increasing sequence of continuous maps and the limit of a decreasing sequence of continuous maps. Thus, by Dini's theorem, convergence is uniform. Observing that

$$\big|\big(T^n v_0\big)(x) - v^*(x)\big| \leq \max\big\{\big|\underline{v}_n(x) - v^*(x)\big|, \big|\bar{v}_n(x) - v^*(x)\big|\big\},$$

the claim is proved. *Q.E.D.*

PROOF OF PROPOSITION 4: Let $\bar{v}^*$ in $\mathcal{F}$ be given by

$$\bar{v}^*(x) = \sup_{g\in\mathcal{G}} \bar{U}_g(x).$$

Consider again the extended Bellman operator $T:\mathcal{F}\to\mathcal{F}$. We first argue that $(T\bar{v}^*)\geq\bar{v}^*$. Indeed, by monotonicity, notice that

$$\bar{v}^*(x) = \sup_{g\in\mathcal{G}}\bar{U}_g(x) = \sup_{g\in\mathcal{G}} W\big(x, g(x), \bar{U}_g\big) \leq \sup_{g\in\mathcal{G}} W\big(x, g(x), \bar{v}^*\big) = \big(T\bar{v}^*\big)(x).$$

By Tarski's fixed point theorem, the Bellman operator admits a fixed point on the sublattice $\{v\in\mathcal{F}: v\geq\bar{v}^*\}$. This shows that $\bar{v}\geq\bar{v}^*$, where $\bar{v}$ in $\mathcal{V}$ is the greatest fixed point of the

Bellman operator. By upper semicontinuity of $\bar{v}$ in $\mathcal{V}$ (Proposition 1), there exists a policy g in $\mathcal{G}$ such that

$$\bar{v} = (T\bar{v}) = (T_g\bar{v}).$$

This in turn shows that $\bar{v} \le \bar{U}_g \le \bar{v}^*$, so establishing coincidence. *Q.E.D.*

PROOF OF PROPOSITION 5: Let $\underline{v}^*(x)$ in $\mathcal{F}$ be given as

$$\underline{v}^*(x) = \sup_{\gamma \in \mathcal{G}^{\mathbb{N}}} \underline{U}_\gamma(x).$$

Notice that the least fixed point of the Bellman operator is $\underline{v} = \lim_{n\to\infty}(T^n\underline{f})$ (Proposition 1). For any nonstationary policy γ in $\mathcal{G}^{\mathbb{N}}$, we have

$$(T_{g_1} \circ T_{g_2} \circ \cdots \circ T_{g_n}\underline{f}) \le (T^n\underline{f}),$$

which implies that $\underline{U}_\gamma \le \underline{v}$ and, thus, $\underline{v}^* \le \underline{v}$. For any n in $\mathbb{N}$, observe that

$$(T^n\underline{f}) = (T \circ T^{n-1}\underline{f}) = (T_g \circ T^{n-1}\underline{f}).$$

The existence of such a policy g in $\mathcal{G}$ is ensured, through the maximum theorem, by the upper semicontinuity of $(T^{n-1}\underline{f})$ in $\mathcal{V}$. By induction, this shows the existence of a nonstationary policy γ_n in $\mathcal{G}^{\mathbb{N}}$ such that

$$(T^n\underline{f}) = (T_{g_1} \circ T_{g_2} \circ \cdots \circ T_{g_n}\underline{f}) \le \underline{U}_{\gamma_n} \le \underline{v}^*,$$

as required to establish the claim. *Q.E.D.*

PROOF OF PROPOSITION 6: To prove the first statement, let $\bar{v}$ in $\mathcal{V}$ be the greatest fixed point and notice that

$$\bar{v} - \tilde{v} = (T\bar{v}) - (T\tilde{v}) \le D(\bar{v} - \tilde{v}).$$

By the sublinearity of the monotone operator $D : \mathcal{L} \to \mathcal{L}$, this implies

$$\bar{v} - \tilde{v} \le D^n(\bar{v} - \tilde{v}) \le D^n(\bar{f} - \underline{f}) \le \|D^n\|(\bar{f} - \underline{f}) \le \rho^n(\bar{f} - \underline{f}),$$

where $\rho(D) < \rho < 1$ and n in $\mathbb{N}$ is sufficiently large. As $\lim_{n\to\infty}\rho^n = 0$, we conclude $\tilde{v} = \bar{v}$, as claimed. *Q.E.D.*

PROOF OF CLAIM 1: We show that $(T\bar{f}) \le \bar{f}$. Choose any x in X, and assume that $f(x) > 0$. Notice that

$$\begin{aligned}
(T\bar{f})(x) &= \sup_{z\in G(x)} V\big(u(x,z), I(x,z,\bar{f})\big) \\
&\le V\Big(\sup_{z\in G(x)} u(x,z), \sup_{z\in G(x)} I(x,z,\bar{f})\Big) \\
&\le V\big(f(x), \rho\bar{f}(x)\big) \\
&= f(x)V\bigg(1, \rho\frac{\bar{f}(x)}{f(x)}\bigg) \\
&\le \bar{f}(x).
\end{aligned}$$

Monotonicity delivers the first inequality; the second inequality is an implication of condition (U1) and the bound on the current reward (U2); the third equality is due to homogeneity; and the last inequality follows from the boundary condition on the utility aggregator (U3). When $f(x) = 0$, the claim follows from the fact that, by homogeneity, $V(0, 0) = 0$. *Q.E.D.*

PROOF OF CLAIM 3: Notice that, by condition (L3) evaluated at $t = 0$, $(T_g 0) \geq \lambda f$ for some λ in $\mathbb{R}^{++}$ and, at no loss of generality, we can suppose that $\lambda = 1$. Assuming that $(T_g^n f) \geq \eta^{-n} f$ for n in $\mathbb{Z}^+$, it follows that

$$\begin{aligned}
\big(T_g^{n+1} f\big)(x) &= V\big(u\big(x, g(x)\big), I\big(x, g(x), \big(T_g^n f\big)\big)\big) \\
&\geq V\big(f(x), I\big(x, g(x), \eta^{-n} f\big)\big) \\
&\geq V\big(f(x), \rho\eta^{-n} f(x)\big) \\
&\geq \eta^{-(n+1)} f(x).
\end{aligned}$$

Homogeneity and monotonicity are used throughout these inequalities; property (L2) justifies the first inequality; the second inequality is implied by (L1), whereas the last follows from (L3). Supposing the existence of a value v in $\mathcal{V}$ of the recursive program, we obtain $v = (Tv) \geq (T_g v) \geq (T_g 0) \geq f$, so implying

$$v \geq \big(T_g^n v\big) \geq \big(T_g^n f\big) \geq \eta^{-n} f.$$

As η lies in $(0, 1) \subset \mathbb{R}^+$, and $f(x) > 0$ for some x in X, this delivers a contradiction, thus proving our claim. *Q.E.D.*

CALCULATIONS FOR EXAMPLE 6: To verify conditions (U1)–(U3), consider the monotone sublinear operator given by

$$(Df)(s_t, w_t) = \sup_{\alpha_t \in [0,1]} \mathbb{E}_t f(s_{t+1}, w_{t+1})$$

subject to

$$w_{t+1} = \big(R_f + \alpha_t (R_{t,t+1} - 1) R_f\big) w_t.$$

Also, let map f in $\mathcal{M}^+$ be given by

$$f(s_t, w_t) = e^{(\frac{1}{1-\phi}) s_t^+} w_t \geq w_t,$$

so that condition (U2) is trivially satisfied. By direct computation, we obtain

$$\begin{aligned}
(Df)(s_t, w_t) &= \sup_{\alpha \in [0,1]} \mathbb{E}_t \big(R_f + \alpha\big(e^{s_t + \eta_{t+1}} - 1\big) R_f\big) e^{(\frac{1}{1-\phi})((1-\phi)\mu + \phi s_t + \epsilon_{t+1})^+} w_t \\
&\leq \sup_{\alpha \in [0,1]} \big(R_f + \alpha\big(e^{s_t^+ + \frac{\sigma_\eta^2}{2}} - 1\big) R_f\big) \mathbb{E}_t e^{(\frac{1}{1-\phi})((1-\phi)\mu + \phi s_t^+ + \epsilon_{t+1}^+)} w_t \\
&= R_f e^{s_t^+ + \frac{\sigma_\eta^2}{2}} \mathbb{E}_t e^{(\frac{1}{1-\phi})((1-\phi)\mu + \phi s_t^+ + \epsilon_{t+1}^+)} w_t \\
&= R_f e^{\mu + \frac{\sigma_\eta^2}{2}} \mathbb{E}_t e^{(\frac{1}{1-\phi}) \epsilon_{t+1}^+} e^{(\frac{1}{1-\phi}) s_t^+} w_t \\
&= \rho f(s_t, w_t),
\end{aligned}$$

where

$$\rho = R_f e^{\mu+\frac{\sigma_\eta^2}{2}} \mathbb{E} e^{(\frac{1}{1-\phi})\epsilon^+} = R_f e^{\mu+\frac{\sigma_\eta^2}{2}+\frac{\sigma_\epsilon^2}{(1-\phi)^2}\Phi(\frac{\sigma_\epsilon}{(1-\phi)})}.$$

Hence, condition (U1*) is verified and, as clarified in Example 5, condition (U3) is fulfilled provided that $\delta\rho^{1-\sigma} < 1$. *Q.E.D.*

CALCULATIONS FOR EXAMPLE 7: Given the map f in $\mathcal{M}^+$ described in the calculations for Example 6, it is immediate to verify that $\bar{f} = tf$ in $\mathcal{M}^+$ is a suitable upper bound for a sufficiently large t in $\mathbb{R}^+$. The extreme values of the recursive program are increasing in both state variables (s_t, w_t) in $S \times \mathbb{R}^+$, because the return on the risky asset increases with s_t in S. Furthermore, conditional on the state (s_t, w_t) in $S \times \mathbb{R}^+$, and given any policy (c_t, α_t) in $[0, w_t] \times [0, 1]$, (s_{t+1}, w_{t+1}) in $S \times \mathbb{R}^+$ are independently distributed random variables. We establish uniqueness by means of a simplified version of Proposition 6 (Spectral radius). To this end, consider the minimum λ in $\mathbb{R}^{++}$ such that $\bar{v} \le \underline{v} + \lambda f$. For any given policy, we obtain that

$$\begin{aligned}
V\big(c_t, I(\bar{v}_{t+1})\big) - V\big(c_t, I(\underline{v}_{t+1})\big) &= -\frac{\delta}{\theta}\log\left(\frac{\mathbb{E}_t e^{-\theta\bar{v}_{t+1}}}{\mathbb{E}_t e^{-\theta\underline{v}_{t+1}}}\right) \\
&\le -\frac{\delta}{\theta}\log\left(\frac{\mathbb{E}_t e^{-\theta\underline{v}_{t+1}} e^{-\theta\lambda f_{t+1}}}{\mathbb{E}_t e^{-\theta\underline{v}_{t+1}}}\right) \\
&\le -\frac{\delta}{\theta}\log\big(\mathbb{E}_t e^{-\theta\lambda f_{t+1}}\big) \\
&\le \delta\lambda\mathbb{E}_t f_{t+1} \\
&\le \delta\rho\lambda f_t.
\end{aligned}$$

We use monotonicity for the first inequality and comonotonicity for the second inequality;[3] hence, we apply Jensen's inequality and our characterization of the bound. By a canonical argument, this shows that $\bar{v} \le \underline{v} + \delta\rho\lambda f$ and, as $\delta\rho < 1$, this delivers a contradiction. *Q.E.D.*

CALCULATIONS FOR EXAMPLE 8: To verify conditions (U1)–(U3) for $\rho = R_f$, we consider the monotone linear operator given by

$$(Df)(s_t, w_t) = \mathbb{E}_t f(s_{t+1}, w_{t+1}),$$

where wealth evolves according to

$$w_{t+1} = R_f w_t + e_{t+1}.$$

[3]More precisely, for the second inequality, we exploit the following version of Harris–FKG Inequality (Fortuin, Kasteleyn, and Ginibre (1971)): Let X and Y be independently distributed real-valued random variables, and let $h_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a weakly decreasing map. We have

$$\mathbb{E}h_1(X, Y)h_2(X, Y) \ge \mathbb{E}h_1(X, Y)\mathbb{E}h_2(X, Y).$$

In our application, $h_1 = (e^{-\theta\underline{v}_{t+1}}|\mathcal{F}_t)$ and $h_2 = (e^{-\theta\lambda f_{t+1}}|\mathcal{F}_t)$.

Setting

$$f(s_t, w_t) = w_t + \left(\frac{\phi}{R_f - \phi}\right)e_t + \left(\frac{1}{R_f - 1}\right)\left(\frac{R_f}{R_f - \phi}\right)\mu \geq w_t,$$

direct computations reveal that

$$\begin{aligned}(Df)(s_t, w_t) &= R_f w_t + \left(\frac{R_f}{R_f - \phi}\right)\mathbb{E}_t e_{t+1} + \left(\frac{1}{R_f - 1}\right)\left(\frac{R_f}{R_f - \phi}\right)\mu \\ &= R_f w_t + R_f\left(\frac{\phi}{R_f - \phi}\right)e_t + R_f\left(\frac{1}{R_f - 1}\right)\left(\frac{R_f}{R_f - \phi}\right)\mu \\ &= \rho f(s_t, w_t),\end{aligned}$$

so that condition (U1*) is satisfied.

Turning to conditions (L1)–(L3), let η in $(0, 1) \subset \mathbb{R}^+$ and consider the policy $c_t = \eta w_t$, together with the map $f(s_t, w_t) = \eta w_t$. Condition (L2) obtains immediately. To verify condition (L1), setting $\rho = (1 - \eta)R_f$, notice that

$$\begin{aligned}(If)(s_t, w_t) &= \left(\mathbb{E}_t(\eta w_{t+1})^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\ &= \left(\mathbb{E}_t\left(R_f(1-\eta)\eta w_t + \eta e_{t+1}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\ &\geq \left(\mathbb{E}_t\left(R_f(1-\eta)\eta w_t\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\ &= (1-\eta)R_f f(s_t, w_t) \\ &= \rho f(s_t, w_t).\end{aligned}$$

Observing that $\delta R_f^{1-\sigma} > 1$ implies $\delta\rho^{1-\sigma} > 1$ for all sufficiently small η in $(0, 1) \subset \mathbb{R}^+$, this establishes our claim of non-existence. *Q.E.D.*

CALCULATIONS OF EXAMPLE 9: The condition for existence is of the form $\delta\rho^{1-\sigma} < 1$, where

$$\rho = R_f\left(\mathbb{E}\exp\left(\left(\frac{1}{(1-\sigma)(1-\phi)}\right)\epsilon\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} = R_f \exp\left(\frac{(1-\gamma)\sigma_\epsilon^2}{2(1-\sigma)^2(1-\phi)^2}\right).$$

To verify conditions (U1)–(U3), we consider an arbitrarily small χ in $\mathbb{R}^-$ such that the condition for existence is still satisfied by the truncation

$$\rho_\chi = R_f\left(\mathbb{E}\exp\left(\left(\frac{1}{(1-\sigma)(1-\phi)}\right)\epsilon \vee (1-\phi)\chi\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}},$$

where we use short notation $\epsilon \vee (1-\phi)\chi = \max\{\epsilon, (1-\phi)\chi\}$. To our end, we introduce the map f in $\mathcal{M}^+$ given by

$$f(s_t, w_t) = e^{\left(\frac{\phi}{(1-\sigma)(1-\phi)}\right)(s_t - \chi)^+} w_t \geq w_t.$$

Notice that $(\epsilon - \chi)^+ = \epsilon \vee \chi - \chi$ and

$$(\phi s + \epsilon - \chi)^+ \leq \phi(s - \chi)^+ + \left(\epsilon - (1-\phi)\chi\right)^+$$

$$= \phi(s-\chi)^+ + \epsilon \vee (1-\phi)\chi - (1-\phi)\chi.$$

Direct computations reveal

$$\begin{aligned}
(If)(s_t, w_t) &= \left(\mathbb{E}_t\left(\left(\left(\frac{\xi_{t+1}}{\xi_t}\right)^{\frac{1}{1-\sigma}} f(s_{t+1}, w_{t+1})\right)^{1-\gamma}\right)\right)^{\frac{1}{1-\gamma}} \\
&= \left(\mathbb{E}_t\left(e^{(\frac{1}{1-\sigma})s_{t+1}} R_f w_t e^{(\frac{\phi}{(1-\sigma)(1-\phi)})(s_{t+1}-\chi)^+}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\
&\leq \left(\mathbb{E}_t\left(e^{(\frac{1}{1-\sigma})((s_{t+1}-\chi)^+ + \chi)} R_f w_t e^{(\frac{\phi}{(1-\sigma)(1-\phi)})(s_{t+1}-\chi)^+}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\
&= \left(\mathbb{E}_t\left(R_f w_t e^{(\frac{1}{(1-\sigma)(1-\phi)})((s_{t+1}-\chi)^+ + (1-\phi)\chi)}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\
&\leq \left(\mathbb{E}_t\left(R_f w_t e^{(\frac{1}{(1-\sigma)(1-\phi)})(\phi(s_t-\chi)^+ + \epsilon_{t+1}\vee(1-\phi)\chi)}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\
&= R_f \left(\mathbb{E}_t\left(e^{(\frac{1}{(1-\sigma)(1-\phi)})\epsilon_{t+1}\vee(1-\phi)\chi}\right)^{1-\gamma}\right)^{\frac{1}{1-\gamma}} e^{(\frac{\phi}{(1-\sigma)(1-\phi)})(s_t-\chi)^+} w_t \\
&= \rho_\chi f_t(s_t, w_t),
\end{aligned}$$

so that condition (U1) is satisfied. This establishes our claim. *Q.E.D.*

APPENDIX B: SPECTRAL RADIUS

We present a spectral radius theory for monotone sublinear operators. Consider the space $\mathcal{L} = \mathcal{L}(f)$ for some f in $\mathcal{E}^+$,

$$\mathcal{L}(f) = \left\{v \in \mathcal{E} : |v| \leq \lambda f \text{ for some } \lambda \in \mathbb{R}^+\right\},$$

endowed with the conventional supremum norm,

$$\|v\| = \inf\left\{\lambda \in \mathbb{R}^+ : |v| \leq \lambda f\right\}.$$

The norm of a sublinear operator $D : \mathcal{L} \to \mathcal{L}$ is given by

$$\|D\| = \sup_{\|v\|\leq 1} \left\|D(v)\right\|.$$

This norm is finite if the operator is bounded.

CLAIM 4—Boundedness: *Any monotone sublinear operator $D : \mathcal{L} \to \mathcal{L}$ is bounded*.

PROOF: Indeed, for some λ in $\mathbb{R}^+$, $D(f) \leq \lambda f$ and sublinearity implies

$$-D(-f) \leq D(f) \leq \lambda f.$$

Notice that $\|v\| \leq 1$ if and only if $-f \leq v \leq f$, which by monotonicity implies

$$-\lambda f \leq D(-f) \leq D(v) \leq D(f) \leq \lambda f,$$

thus proving that $\|D\| \leq \lambda$. *Q.E.D.*

The *spectral radius* of a monotone sublinear operator is defined as

$$\rho(D) = \lim_{n\to\infty} \sqrt[n]{\|D^n\|}.$$

Reproducing conventional arguments, we show that the spectral radius exists. We complement the mere existence with an operational criterion under a sort of property of monotone convergence. This criterion replaces the potential absence of an eigenprocess associated with the spectral radius. In fact, more than the spectral radius itself, the application of our theory requires the supplemental existence of an element $\bar{f}$ in the interior of $\mathcal{L}$ such that, for some ρ in $\mathbb{R}^+$,

$$D(\bar{f}) \leq \rho \bar{f}.$$

Claim 6 ensures that such a condition is verified for an approximated spectral radius.

CLAIM 5—Spectral radius: *A monotone sublinear operator $D : \mathcal{L} \to \mathcal{L}$ admits a spectral radius $\rho(D)$ in $\mathbb{R}^+$.*

PROOF: We first show that $\|D^n\| \leq \|D\|^n$. Indeed, by homogeneity, we have that

$$\|D(v)\| \leq \|D\|\|v\|.$$

Therefore, by a reiterated application of this principle,

$$\|D^n(v)\| \leq \|D\|^n\|v\|,$$

which implies

$$\|D^n\| \leq \|D\|^n.$$

Endowed with this property, and reproducing the arguments in the proof of Aliprantis and Border (2006, Lemma 20.15), it is simple to establish that

$$\rho(D) = \inf_{n\in\mathbb{N}} \sqrt[n]{\|D^n\|},$$

thus confirming existence. *Q.E.D.*

ASSUMPTION 8—Monotone convergence: *For every monotonically increasing sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{L}$ converging pointwise to v in $\mathcal{L}$,*

$$\lim_{n\to\infty} D(v_n)(x) = D(v)(x).$$

CLAIM 6—Operational criterion: *Under Assumption 8, given any $\bar{\rho}$ in $\mathbb{R}^{++}$, a monotone sublinear operator $D : \mathcal{L} \to \mathcal{L}$ admits a spectral radius $\rho(D)$ in $[0, \bar{\rho}) \subset \mathbb{R}^+$ if and only if there exists $\bar{f}$ in the interior of $\mathcal{L}^+$ such that, for some ρ in $(0, \bar{\rho}) \subset \mathbb{R}^+$,*

$$D(\bar{f}) \leq \rho \bar{f}.$$

PROOF: To verify sufficiency, notice there exist λ_l and λ_u in $\mathbb{R}^{++}$ such that $\lambda_l f \leq \bar{f} \leq \lambda_u f$. Monotonicity delivers, for all n in $\mathbb{N}$,

$$\lambda_l D^n(f) \leq D^n(\bar{f}) \leq \rho^n \bar{f} \leq \rho^n \lambda_u f.$$

This in turn implies

$$\sqrt[n]{\|D^n\|} \le \rho \sqrt[n]{\frac{\lambda_u}{\lambda_l}},$$

so proving that $\rho(D) \le \rho$. We now verify necessity.

Fix any ρ in $(\rho(D), \bar{\rho}) \subset \mathbb{R}^+$ and notice that, for every sufficiently large n in $\mathbb{N}$, $D^n(f) \le \|D^n\| f \le \rho^n f$. Consider

$$\bar{f}_n = f + \left(\frac{1}{\bar{\rho}}\right) D(f) + \cdots + \left(\frac{1}{\bar{\rho}}\right)^n D^n(f).$$

The sequence $(\bar{f}_n)_{n\in\mathbb{N}}$ in $\mathcal{L}^+$ is increasing and, as $D^n(f) \le \rho^n f$ eventually, it monotonically converges pointwise to $\bar{f}$ in $\mathcal{L}^+$. Furthermore, by sublinearity,

$$f + \left(\frac{1}{\bar{\rho}}\right) D(\bar{f}_n) \le f + \left(\frac{1}{\bar{\rho}}\right) D(f) + \cdots + \left(\frac{1}{\bar{\rho}}\right)^{n+1} D^{n+1}(f) = \bar{f}_{n+1} \le \bar{f}.$$

Invoking the monotone convergence hypothesis (Assumption 8), we obtain

$$f + \left(\frac{1}{\bar{\rho}}\right) D(\bar{f}) \le \bar{f}.$$

Noticing that $(\bar{\rho} - \rho)\bar{f} \le \bar{\rho} f$ for some sufficiently large ρ in $(0, \bar{\rho}) \subset \mathbb{R}^+$, this proves our claim. *Q.E.D.*

In many applications, the monotone sublinear operator involves an expectation and, hence, it is defined as $D^* : \mathcal{L}^* \to \mathcal{L}$, where $\mathcal{L}^*$ is the space of measurable maps in $\mathcal{L}$. To deal with this inconsistency, assuming that f in $\mathcal{E}$ is measurable, we implicitly extend the operator as

$$D(v)(x) = \inf_{v^* \in \mathcal{L}^*} \{ D^*(v^*)(x) : v \le v^* \}.$$

This extension does not affect the action of the operator on measurable maps and preserves monotone sublinearity. Indeed, letting $v_1^* \ge v_1$ and $v_2^* \ge v_2$, with v_j^* in $\mathcal{L}^*$ and v_j in $\mathcal{L}$, by the sublinearity of the original operator, we obtain

$$D^*(v_1^*) + D^*(v_2^*) \ge D^*(v_1^* + v_1^*) \ge D(v_1 + v_2),$$

where the right inequality is implied by the nature of the extension. Taking the infimum over each $v_j^* \ge v_j$, we conclude that

$$D(v_1) + D(v_2) \ge D(v_1 + v_2),$$

as claimed.

APPENDIX C: EXTENDED FELLER PROPERTY

The purpose of this appendix is to provide primitive assumptions enforcing the sort of Feller property with possibly unbounded values appearing in Assumption 3(b). To simplify notation, we consider a Markov transition $\Pi : X \to \Delta(X)$, where X is endowed with its

Borel algebra. We let $C(X)$ be the space of continuous functions $f : X \to \mathbb{R}$, and let $C_b(X)$ be its restriction to uniformly bounded functions. We assume that the Markov transition satisfies the canonical Feller property.

ASSUMPTION 9—Canonical Feller property: *For any f in $C_b(X)$, (Πf) is also in $C_b(X)$, where*

$$(\Pi f)(x) = \int f(y)\Pi(x)(dy).$$

We preliminarily present a well-known fact: the Feller property preserves upper semicontinuity. We let $U(X)$ be the space of upper semicontinuous functions $f : X \to \mathbb{R}$, and let $U_b(X)$ be its restriction to uniformly bounded functions. We so obtain this simple property.

CLAIM 7—Upper semicontinuity: *Under the Feller property (Assumption 9), for any f in $U_b(X)$, (Πf) is also in $U_b(X)$.*

PROOF: By Aliprantis and Border (2006, Theorem 3.13), f in $U_b(X)$ is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions in $C_b(X)$. By the Feller property (Assumption 9), along with the monotonicity of the integral, $(\Pi f_n)_{n\in\mathbb{N}}$ is a decreasing sequence of functions in $C_b(X)$. By Aliprantis and Border (2006, Lemma 2.41), its pointwise limit is upper semicontinuous. By the monotone convergence lemma (see Aliprantis and Border (2006, Lemma 13.36)), the sequence pointwise converges to (Πf) in $U_b(X)$, thus proving our claim. *Q.E.D.*

We extend the Feller property to unbounded values. To this purpose, we exploit a theorem presented in Feinberg, Kasyanov, and Zadoianchuk (2014, Theorem 1.1) that we restate for a more convenient and direct application to our analysis.

CLAIM 8—Theorem 1.1 in Feinberg, Kasyanov, and Zadoianchuk (2014): *Under the Feller property (Assumption 9), given an upper semicontinuous negative function $f : X \to \mathbb{R}^-$,*

$$\limsup_{x_n \to x} (\Pi f)(x_n) \leq (\Pi f)(x).$$

We consider a positive continuous map $\bar{f}$ in $C(X)$ and construct the restricted space of upper semicontinuous functions limited by this upper bound, that is,

$$\bar{U}(X) = \{ f \in U(X) : |f| \leq \lambda \bar{f} \text{ for some } \lambda > 0 \}.$$

We then add a primitive assumption on the Markov transition, complementing the Feller property, and show that the Markov transition preserves class $\bar{U}(X)$. Notice that, by this same assumption, $(\Pi f)(x)$ exists for any f in $\bar{U}(X)$.

ASSUMPTION 10—Upper bound: *Positive continuous map $\bar{f}$ in $C(X)$ is such that $(\Pi \bar{f})$ is also a positive continuous map in $C(X)$.*

CLAIM 9—Extended Feller property: *Under the Feller property (Assumption 9), complemented by a suitable upper bound (Assumption 10), for any f in $\bar{U}(X)$, (Πf) is also in $\bar{U}(X)$.*

PROOF: We have to show that, for every f in $\bar{U}(X)$,

$$\limsup_{x_n \to x} (\Pi f)(x_n) \le (\Pi f)(x).$$

Notice that, at no loss of generality, $h = f - \bar{f}$ is a negative upper semicontinuous function and, by Claim 8,

$$\limsup_{x_n \to x} (\Pi h)(x_n) \le (\Pi h)(x).$$

Furthermore, for any n in $\mathbb{N}$, $(\Pi h)(x_n) = (\Pi f)(x_n) - (\Pi \bar{f})(x_n)$ and, by Assumption 10,

$$\lim_{x_n \to x} (\Pi \bar{f})(x_n) = (\Pi \bar{f})(x),$$

which suffices to confirm our claim. *Q.E.D.*

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