# ON $P$-ALGEBRAS AND THEIR DUALS 

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#### Abstract

The notion of $P$-algebra due to Margolis, building on work of Moore \& Peterson, was motivated by the case of the Steenrod algebra at a prime and its modules. We develop aspects of this theory further, focusing especially on coherent modules and finite dimensional modules. We also discuss the dual Hopf algebra of $P$-algebra and its comodules. One of our aims is provide a collection of techniques for calculating cohomology groups over $P$-algebras and their duals, in particular giving vanishing results. Much of our work is implicit in that of Margolis and others but we are unaware of systematic discussions in the literature. We give some examples illustrating topological applications which follow easily from our results.


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## Introduction

This paper is a rewrite of the algebraic part of [Bak21] and we intend writing a companion paper on the topological applications. Shortly after posting the earlier version, we discovered some significant gaps in the calculations and this led to some substantial reworking of the algebra which we feel is of enough interest to present it independently. Indeed, as far as we are aware, although there has been significant use of properties of $P$-algebras in work involving the Steenrod algebra, there does not appear to be much discussion of the dual notion so it seems

[^0]worthwhile recording some basic results and in particular the existence of certain spectral sequences for their cohomology.

The theory of $P$-algebras of Margolis [Mar83] which builds on work of Moore \& Peterson [MP73], provides an important theoretical framework for understanding modules over the Steenrod algebra at a prime with many applications in Algebraic Topology. In this paper we review some basic results on $P$-algebras and their modules, particularly emphasising properties of coherent modules and their relationship with finite dimensional modules; we also mention some properties of pseudo-coherent modules. Next we consider the dual Hopf algebra of a $P$-algebra and its comodules. In order to set up some Cartan-Eilenberg spectral sequences we discuss the homological algebra required to make a bivariant derived functor of the homomorphism for comodules with the aim of obtaining three such spectral sequences, one of which involves dualising to modules over the $P$-algebra itself because comodules over Hopf algebras do not always have resolutions by projective objects (although recent work of Salch related to [Sal16] sheds light on this for connective graded comodules over a connective graded Hopf algebra). After reviewing some properties of the mod 2 Steenrod algebra and its dual, we discuss doubling and then describe some families of subHopf algebras and their duals. Finally we give some applications of our results on coherent modules to show vanishing of homotopy groups of maps between some spectra, rederiving and extending results of Lin, Margolis and Ravenel.

An important motivation for our work is to rework some the algebraic machinery used in Ravenel's seminal paper [Rav84] and we will apply this in the planned sequel. In [Bak22] we have also developed an ungraded analogue of the theory of $P$-algebras which sheds light on some aspects of it.

## 1. $P$-algebras and their modules

We will make use material on $P$-algebras and their modules contained in Mar83, chapter 13]. Here is a summary of our assumptions, conventions and notations, some of which differ slightly from those of Margolis.

- We will often suppress explicit mention of internal grading in cohomology and just write $M$ for $M^{*}$ when discussing a module over $A=A^{*}$. When working in homology we will usually write $M_{*}$ or $A_{*}$.
- We will work with graded vector spaces over a field $\mathfrak{k}$ and in particular, $\mathfrak{k}$-algebras and their (co)modules. In our topological applications, $\mathfrak{k}=\mathbb{F}_{2}$ although similar results for odd primes are easily found.

For a finite type graded vector space $V_{*}$ we think of $V_{n}$ as dual to $V^{n}=\operatorname{Hom}_{\mathfrak{k}}\left(V_{n}, \mathbb{k}\right)$, so $V^{*}$ is the cohomologically graded degree-wise dual, and bounded below means that $V^{*}$ is bounded below. We can also start with a finite type graded vector space $V^{*}$ and form $V_{*}$ where $V_{n}=\operatorname{Hom}_{\mathfrak{k}}\left(V^{n}, \mathbb{k}\right)$; the double dual of $V_{*}$ is canonically isomorphic to $V_{*}$, and vice versa. We will denote the positive degree part of a graded vector space by $V^{+}$; of course for a graded $\mathbb{k}$-algebra $R, R^{+}$is an ideal which is maximal if $R$ is also connected, making $R$ local.

A graded vector space which is finite dimensional will be referred to as a finite; when $\mathbb{k}$ is finite this terminology agrees with the use of finite for a module over a $P$-algebra by Margolis.

- In this paper, a $P$-algebra $A$ will always be a $P$-Hopf algebra, i.e., a strictly increasing union of connected finite dimensional cocommutative Hopf algebras $A(n) \subset A(n+1) \subset$ $A$. Thus each $A(n)$ is a Poincaré duality algebra and we will denote its highest nontrivial degree by $\operatorname{pd}(n)$ as it is also the Poincare duality degree; this number satisfies the inequality $\operatorname{pd}(n)<\operatorname{pd}(n+1)$. We also stress that each $A(n)$ is a local graded ring and $A(n)^{\mathrm{pd}(n)}=A(n)^{0}=\mathbb{k}$. Other important properties are that $A$ is free as a left or right $A(n)$-module and $A$ is a coherent $\mathbb{k}$-algebra. Although in general Margolis does not require $P$-algebras to be of finite type, all the examples we consider have that property and it is required in some of our homological results so we will assume it holds.

It is important to note that $P$-algebras are coherent (but not Noetherian) and their coherent modules play an important rôle in this paper. For general properties of coherence and pseudo-coherence we refer the reader to Bourbaki [Bou80, ex. §3.10], and also Cohen [Coh69] for an account aimed at an algebraic topology audience.

In Algebraic Topology the important examples of $P$-algebras are the Steenrod algebra $\mathcal{A}$ at a prime $p$ as well as infinite dimensional sub and quotient Hopf algebras.
We will use the following basic result stated on page 195 of [Mar83] but left as an exercise.
Proposition 1.1. Suppose that $A$ is a P-algebra. Let $M$ be a finite $A$-module and $F$ a bounded below free $A$-module. Then

$$
\operatorname{Ext}_{A}^{*}(M, F)=0 .
$$

Proof. By [Mar83, theorem 13.12], bounded below projective $A$-modules are also injective, so $\operatorname{Ext}_{A}^{s}(M, F)=0$ when $s>0$, therefore we only have to show that $\operatorname{Hom}_{A}(M, F)=0$. It suffices to prove this for the case $F=A$.

Suppose that $0 \neq \theta \in \operatorname{Hom}_{A}(M, A)$. The image of $\theta$ contains a simple submodule in its top degree, so let $\theta(x) \neq 0$ be in this submodule; then $a \theta(x)=0$ for every positive degree element $a \in A$. Now for some $n, \theta(x) \in A(n)^{k}$ where $k<\operatorname{pd}(n)$, and by Poincaré duality for $A(n)$ there exists $z \in A(n)$ for which $0 \neq z \theta(x) \in A(n)^{\mathrm{pd}(n)}$. This gives a contradiction, hence no such $\theta$ can exist.

For a different proof, see Lorentz [Lor18, proposition 10.6], which shows that an infinite dimensional Hopf algebra has no non-trivial finite dimensional left or right ideals.

A particular concern for us will be the situation where we have a pair of $P$-algebras $B \subseteq A$ with $B$ a subHopf algebra of $A$.

Proposition 1.2. For a pair of P-algebras $B \subseteq A$,

$$
\operatorname{Ext}_{A}^{*}\left(A \otimes_{B} \mathbb{k}, A\right) \cong \operatorname{Ext}_{B}^{*}(\mathbb{k}, A)=\operatorname{Hom}_{B}(\mathbb{k}, A)=0 .
$$

Proof. First we recall a classic result of Milnor \& Moore [MM65, proposition 4.9]: $A$ is free as a left or right $B$-module. This guarantees the change of rings isomorphism (which is valid for any subalgebra where $A$ is a flat $B$-module) and the second isomorphism follows from Proposition 1.1.

Since a $P$-algebra $A$ is coherent, its finitely presented modules are its coherent modules, and they form a full abelian subcategory $\operatorname{Mod}_{A}^{\text {coh }}$ of $\operatorname{Mod}_{A}$ with finite limits and colimits (in
particular it has kernels, cokernels and images). A coherent $A$-module $M$ admits a finite presentation

$$
A^{\oplus k} \xrightarrow{\pi} A^{\oplus \ell} \longrightarrow M \longrightarrow 0
$$

which can be defined over some $A(n)$, i.e., there is a finite presentation

$$
A(n)^{\oplus k} \xrightarrow{\pi^{\prime}} A(n)^{\oplus \ell} \longrightarrow M^{\prime} \longrightarrow 0
$$

of $A(n)$-modules and a commutative diagram of $A$-modules

with exact rows. It is standard that every coherent $A$-module admits a resolution by finitely generated free modules. It is also true that a homomorphism between $M \rightarrow N$ coherent modules is induced from a homomorphism between finitely generated modules over some $A(m)$.

For a $P$-algebra we also have injective resolutions by finitely generated free modules.
Proposition 1.3. Let $M$ be a coherent module over a P-algebra $A$. Then $M$ admits an injective resolution by finitely generated free modules.

Proof. By [Mar83, theorem 13.12], bounded below projective $A$-modules are injective.
For some $n, M \cong A \otimes_{A(n)} M^{\prime}$ where $M^{\prime}$ is a finitely generated $A(n)$-module. Since $A(n)$ is a Poincaré duality algebra, it is standard that $M^{\prime}$ admits a monomorphism $M^{\prime} \rightarrow J^{\prime}$ into a finitely generated free $A(n)$-module which is also injective (this is obvious for a simple module and can be proved by induction on dimension). By flatness the composition

$$
M \xlongequal[\cong]{\cong} A \otimes_{A(n)} M^{\prime} \longrightarrow A \otimes_{A(n)} J^{\prime}
$$

is a monomorphism of $A$-modules into a finitely generated free module with coherent cokernel. Iterating this we can build a resolution of the stated form and since $A$ is self-injective this is an injective resolution.

Our next result summarises the properties of coherent modules.
Proposition 1.4. The coherent modules over a P-algebra A form a full subcategory $\mathbf{M o d}_{A}^{\mathrm{coh}}$ of $\mathbf{M o d}_{A}$ with enough projectives and injectives, and all finite limits and colimits.

We can generalise Proposition 1.1.
Proposition 1.5. Let $M$ be a finite $A$-module and $N$ a coherent $A$-module. Then

$$
\operatorname{Ext}_{A}^{*}(M, N)=0 .
$$

Proof. The left exact functor $\operatorname{Hom}_{A}(M,-)$ has the right derived functors Ext** $(M,-)$. By Proposition 1.1, for each finitely generated free module $F, \operatorname{Ext}_{A}^{*}(M, F)=0$. This means that $F$ is $\operatorname{Hom}_{A}(M,-)$-acyclic, and it is well-known that these derived functors can be computed using resolutions by such modules; see [Wei94] on $F$-acyclic objects and dimension shifting.

Now every coherent $A$-module $N$ admits a resolution $0 \rightarrow N \rightarrow J^{*}$ where each $J^{s}$ is a finitely generated free module and these are $\operatorname{Hom}_{A}(M,-)$-acyclic. Therefore $\operatorname{Ext}_{A}^{*}(M, N)=0$.

For example, every left $A$-module of the form

$$
A / / A(n)=A \otimes_{A(n)} \mathbb{k} \cong A / A A(n)^{+}
$$

is coherent so it admits such an injective resolution and then $\operatorname{Ext}_{A}^{s}(\mathbb{k}, A / / A(n))=0$.
Here is a more general statement.
Proposition 1.6. Let $A$ be a $P$-algebra and suppose that $B \subseteq A$ is a subalgebra which is also a $P$ algebra and that $A$ is a free left $B$-module. Let $L$ be a finite $B$-module and $N$ a coherent $A$-module. Then

$$
\operatorname{Ext}_{A}^{*}\left(A \otimes_{B} L, N\right)=0 .
$$

Proof. We have

$$
\operatorname{Ext}_{A}^{*}\left(A \otimes_{B} L, N\right) \cong \operatorname{Ext}_{B}^{*}(L, N) .
$$

By Proposition 1.3, $N$ admits an injection resolution

$$
0 \rightarrow N \rightarrow J^{*}
$$

by finitely generated free $A$-modules. For each $s \geqslant 0$,

$$
\operatorname{Hom}_{A}\left(A \otimes_{B} L, J^{s}\right) \cong \operatorname{Hom}_{B}\left(L, J^{s}\right)=0
$$

since a $B$-module homomorphism $L \rightarrow J^{s}$ must factor through finitely many $B$-summands and therefore it is trivial. Therefore the cohomology $\operatorname{Ext}_{B}^{*}(L, N)$ is trivial. A similar argument also applies to the case where $N$ is a coproduct of coherent $A$-modules.

Pseudo-coherent modules over a coherent ring. In Bourbaki [Bou80, ex. §3.10], as well as coherent modules, pseudo-coherent modules are considered, where a module is pseudo-coherent if every finitely generated submodule is finitely presented (so over a coherent ring it is coherent). The reader is warned that pseudo-coherent is sometimes used in a different sense; for example, over a coherent ring the definition of Weibel [Wei13, example II.7.1.4] corresponds to our coherent.

Examples of pseudo-coherent modules over a coherent ring $A$ include coproducts of coherent modules such as the finitely related modules defined by Lam [Lam99, chapter 2§4]. However pseudo-coherent modules do not form a full abelian subcategory of $\mathbf{M o d}_{A}$ since cokernels of homomorphisms between pseudo-coherent modules need not be pseudo-coherent. Nevertheless, pseudo-coherent modules do occur quite commonly when working with coherent modules over $P$-algebras, for example as tensor products over a Hopf algebra.

Lemma 1.7. Let $A$ be a ring and $M$ a pseudo-coherent $A$-module. Then $M$ is the union of its coherent submodules and therefore their colimit.

Proof. Every element $m \in M$ generates a cyclic submodule which is coherent.
Lemma 1.8. Let $A$ be a coherent ring and $B$ an Artinian subring where $A$ is flat as a right $B$-module. Then every extended $A$-module $A \otimes_{B} N$ is pseudo-coherent.

Proof. Let $U \subseteq A \otimes_{B} N$ be a finitely generated submodule. Taking a finite generating set and expressing each element as a sum of basic tensors we find that $U \subseteq A \otimes_{B} U^{\prime}$ where $U^{\prime} \subseteq N$ is a finitely generated submodule. As $B$ is Artinian and so Noetherian, $U^{\prime}$ is finitely presented, so by flatness of $A, A \otimes_{B} U^{\prime}$ is a finitely presented $A$-module. Since $A$ is coherent, $U$ is also finitely presented.

An important special case of this occurs when $A$ is a coherent algebra over a field $\mathbb{k}$ and $B$ is a finite dimensional subalgebra. If $N$ is a $B$-module then it is locally finite, i.e., every element is contained in a finite dimensional submodule.

In general, the tensor product of two coherent modules over a Hopf algebra that is a $P$ algebra is not a coherent module. However, as we will soon see, it turns out that it is pseudocoherent.

Assumption 1.9. We will assume from now on that $A$ is a cocommutative Hopf algebra over a field $\mathbb{k}$, and $B \subseteq A$ a subHopf algebra where $A$ is $B$-flat as a left and right $B$-module.

Now given two left $A$-modules $L$ and $M$, their tensor product $L \otimes M$ is a left $A$-module with the diagonal action given by

$$
a \cdot(\ell \otimes m)=\sum_{i} a_{i}^{\prime} \ell \otimes a_{i}^{\prime \prime} m
$$

where the coproduct on $a$ is

$$
\psi a=\sum_{i} a_{i}^{\prime} \otimes a_{i}^{\prime \prime}
$$

In particular, given a left $A$-module $M$ and a left $B$-module $N$, the tensor product of $M$ and $A \otimes_{B} N$ is a left $A$-module $M \otimes\left(A \otimes_{B} N\right)$. There is an isomorphism of $A$-modules

$$
\begin{equation*}
M \otimes\left(A \otimes_{B} N\right) \xrightarrow[\cong]{\cong} A \otimes_{B}(M \otimes N) ; \quad m \otimes(a \otimes n) \mapsto \sum_{i} a_{i}^{\prime} \otimes\left(\overline{a_{i}^{\prime \prime}} m \otimes n\right) \tag{1.1}
\end{equation*}
$$

where $\bar{x}=\chi(x)$ and $M \otimes N$ is a left $B$-module with the diagonal action.
A particular instance of this is

$$
\begin{equation*}
(A / / B) \otimes(A / / B) \cong A \otimes_{B}(A / / B) \tag{1.2}
\end{equation*}
$$

where

$$
A / / B=A / A B^{+} \cong A \otimes_{B} \mathbb{k} .
$$

We will be especially interested in the case where the $B$-module $A / / B$ is a coproduct of finitely generated modules.

Lemma 1.10. Suppose that $A$ is coherent and $B$ is finite dimensional. Let $M$ be a left $A$-module and $N$ a left $B$-module. Then the $A$-module $M \otimes\left(A \otimes_{B} N\right)$ is pseudo-coherent.

Proof. Using (1.1),

$$
M \otimes\left(A \otimes_{B} N\right) \cong A \otimes_{B}(M \otimes N)
$$

which is pseudo-coherent by Lemma 1.8
Corollary 1.11. Suppose that $A$ is a $P$-algebra and that $M_{1}$ and $M_{2}$ are two coherent $A$-modules. Then $M_{1} \otimes M_{2}$ is pseudo-coherent.

Proof. Every coherent $A$-module is induced from a finitely generated $A(n)$-module for some $n$. By choosing a large enough $n$ we can assume that $M_{1} \cong A \otimes_{A(n)} M_{1}^{\prime}$ for some finitely generated $A(n)$-module $M_{1}^{\prime}$. Then

$$
M_{1} \otimes M_{2} \cong A \otimes\left(M_{1}^{\prime} \otimes M_{2}\right)
$$

so it is a pseudo-coherent $A$-module.
Pseudo-coherence itself is probably of limited practical use, but stronger versions are perhaps more likely to be of use for computational purposes.

## Definition 1.12

- A module over a coherent ring is strongly pseudo-coherent if it is a coproduct of coherent modules.
- A module $M$ over a $P$-algebra $A$ is $n$-strongly pseudo-coherent if

$$
M \cong A \otimes_{A(n)} M^{\prime}
$$

where the $A(n)$-module $M^{\prime}$ is a coproduct of finitely generated $A(n)$-modules.
Over a coherent ring every free module is strongly pseudo-coherent as is every finitely related module since it is the sum of a coherent module and a free module.

Of course an $n$-strongly pseudo-coherent module over a $P$-algebra $A$ is strongly pseudocoherent. For a given $n$, the cyclic $A$-module $A / / A(n)=A / A A(n)^{+} \cong A \otimes_{A(n)}$ k can be viewed as an $A(n)$-module, and by (1.2),

$$
A / / A(n) \otimes A / / A(n) \cong A \otimes_{A(n)} A / / A(n)
$$

If $A / / A(n)$ is a coproduct of finitely generated (i.e., finite dimensional) $A(n)$-modules then $A / / A(n) \otimes A / / A(n)$ is $n$-strongly pseudo-coherent. This occurs when $A$ is the Steenrod algebra for a prime $p$.

Example 1.13. When $A=\mathcal{A}$ is the $\bmod 2$ Steenrod algebra it is known that there is a coproduct decomposition of the $\mathcal{A}(n)$-module $\mathcal{A} / / \mathcal{A}(n)$ into finite dimensional modules which are generalised Brown-Gitler modules. For details see Behrens et al [BHHM08, corollary 5.6].

## 2. The dual of a $P$-algebra and its comodules

The theory of $P$-algebras can be dualised: given a finite type $P$-algebra $A$, we define its (graded) dual $A_{*}$ by setting $A_{n}=\operatorname{Hom}_{\mathfrak{k}}\left(A^{n}, \mathbb{k}\right)$ and making this a commutative Hopf algebra by dualising the structure maps of $A$ (doing this the finite type condition is essential). We will refer to the dual Hopf algebra of a (finite type) $P$-algebra as a $P_{*}$-algebra; although this is not standard terminology it seems appropriate and convenient.

When working with comodules over a $P_{*}$-algebra $A_{*}$ we will use homological grading. For left $A_{*}$-comodules which are bounded below and of finite type there is no significant difference between working with them or their (degree-wise) duals as $A$-modules. In particular,

$$
\operatorname{Cohom}_{A_{*}}\left(M_{*}, N_{*}\right) \cong \operatorname{Hom}_{A}\left(N^{*}, M^{*}\right),
$$

where $M^{n}=\operatorname{Hom}_{\mathbb{F}_{2}}\left(M_{n}, \mathbb{F}_{2}\right)$ and $M^{*}$ is made into a left $A$-module using the antipode. More generally,

$$
\operatorname{Coext}_{A_{*}}^{s_{3}^{*}}\left(M_{*}, N_{*}\right) \cong \operatorname{Ext}_{A}^{s^{s, *}}\left(N^{*}, M^{*}\right),
$$

where $\operatorname{Coext}_{A_{*}}^{s_{*}^{*}}\left(M_{*},-\right)$ denotes the right derived functor of $\operatorname{Cohom}_{A_{*}}\left(M_{*},-\right)$, which can be computed using extended comodules which are injective comodules here since we are working over a field.

Here are the dual versions of Propositions 1.1 and 1.2 Recall that a cofree or extended $A_{*}$ comodule is one of the form $A_{*} \otimes W_{*}$ where $W_{*}$ is a graded vector space.

Proposition 2.1. Suppose that $A_{*}$ is a $P_{*}$-algebra. Let $L_{*}$ a bounded below cofree $A_{*}$-comodule and let $M_{*}$ be a finite $A_{*}$-comodule. Then

$$
\operatorname{Coext}_{A_{*}}^{*}\left(L_{*}, M_{*}\right)=0 .
$$

Proposition 2.2. For a surjective morphism of $P_{*}$-algebras $A_{*} \rightarrow B_{*}$,

$$
\operatorname{Coext}_{A_{*}}^{* *}\left(A_{*}, A_{*} \square_{B_{*}} \mathbb{k}\right) \cong \operatorname{Coext}_{B_{*}}^{* *}\left(A_{*}, \mathbb{k}\right)=\operatorname{Cohom}_{B_{*}}\left(A_{*}, \mathbb{k}\right)=0
$$

Definition 2.3. If $A_{*}$ is a $P_{*}$-algebra then an $A_{*}$-comodule $M_{*}$ is coherent if its dual $M^{*}$ is a coherent $A$-module; this is equivalent to the existence of an exact sequence of $A_{*}$-comodules

$$
0 \rightarrow M_{*} \rightarrow A_{*} \otimes U_{*} \rightarrow A_{*} \otimes V_{*}
$$

where $U_{*}, V_{*}$ are finite $\mathbb{k}$-vector spaces.
Here is a dual version of Proposition 1.3 ,
Proposition 2.4. Let $M_{*}$ be a coherent comodule over a $P_{*}$-algebra $A_{*}$. Then $M_{*}$ admits a projective resolution by finitely generated cofree comodules.

Proof. Take an injective resolution of $M^{*}$ as in Proposition 1.3 and then take duals to obtain a projective resolution.

Notice that since $A_{*}$ is an injective comodule we have

$$
\operatorname{Coext}_{A_{*}}^{*}\left(A_{*}, A_{*}\right) \cong \operatorname{Hom}_{\mathfrak{k}}\left(A_{*}, \mathbb{k}\right) \cong A .
$$

Proposition 2.5. Let $M_{*}$ be a coherent comodule over a $P_{*}$-algebra $A_{*}$ and let $N_{*}$ be a finite $P_{*}$ comodule. Then

$$
\operatorname{Coext}_{A_{*}}^{*}\left(M_{*}, N_{*}\right)=0 .
$$

Proof. Let $P_{\bullet, *} \rightarrow M_{*} \rightarrow 0$ be a resolution of $M_{*}$ by cofree comodules. Then by Proposition 2.1, for each $s \geqslant 0$ we have

$$
\operatorname{Cohom}_{A_{*}}\left(P_{s, *}, N_{*}\right)=0,
$$

and the result follows.
Remark 2.6. For bounded below finite type comodules over a $P_{*}$-algebra $A_{*}$ dual to a $P$-algebra $A$, taking degree-wise duals defines an equivalence of categories

$$
\operatorname{Comod}_{A_{*}}^{\text {b,f.t. }} \underset{(-)_{*}}{\stackrel{(-)^{*}}{\rightleftarrows}}\left(\operatorname{Mod}_{A}^{\text {b,f.t. }}\right)^{\text {op }}
$$

between the $A_{*}$-comodule and the $A$-module categories. Moreover, these functors are exact, so this equivalence identifies injective comodules (which are retracts of extended comodules) with bounded below projective modules. By [Mar83, theorem 13.12], bounded below projective $A$-modules are injective so it also identifies bounded below injective comodules as projective objects (this is not true in general of course). In fact this equivalence fits into a bigger diagram

where $\operatorname{Mod}_{A}^{\text {t.f.t. }}$ denotes the category of finite type bounded below homologically graded $A$ modules (with $A$ acting by decreasing degree), Mod ${ }_{A}^{\text {b,f.t. }}$ denotes the category of finite type bounded below cohomologically graded $A$-modules and $\operatorname{Mod}_{A}$ denoting the category of all $A$-modules. All of the functors here are exact.

Remark 2.7. Each object $M_{*}$ of Mod ${ }_{A}^{\text {t, f.t. }}$ is a locally finite $A$-module, i.e., it is a union of finite modules. It is convenient to regrade $M_{*}$ so that $M_{n}$ is given cohomological degree $-n$ and then multiplication by a positive degree element of $A$ increases this cohomological degree. Then by Proposition 1.5 we have for any coherent $A$-module $N, \operatorname{Ext}_{A}^{*}\left(M_{*}, N\right)=0$.

For a fixed $A_{*}$-comodule $M_{*}$, the functor

$$
\operatorname{Cohom}_{A_{*}}\left(M_{*},-\right)=\operatorname{Comod}_{A_{*}}^{\text {b,f.t. }}\left(M_{*},-\right) \rightarrow \operatorname{Mod}_{\mathfrak{k}}^{\text {b,f.t. }}
$$

is left exact and has right derived functors Coext $_{A_{*}}^{*}\left(M_{*},-\right)$. Since

$$
\operatorname{Cohom}_{A_{*}}\left(M_{*},-\right) \cong \operatorname{Hom}_{A}\left((-)^{*}, M^{*}\right)=\operatorname{Mod}_{A}^{\text {b, f.t. }}\left((-)^{*}, M^{*}\right)=\operatorname{Mod}_{A}\left((-)^{*}, M^{*}\right)
$$

and injective comodules are sent to projective modules, we also have

$$
\begin{equation*}
\operatorname{Coext}_{A_{*}}^{*}\left(M_{*},-\right) \cong \operatorname{Ext}_{A}^{*}\left((-)^{*}, M^{*}\right) . \tag{2.2}
\end{equation*}
$$

The contravariant functor $\operatorname{Comod}{ }_{A_{+}}^{\text {b,f.t. }} \rightarrow \operatorname{Mod}_{A}^{\mathrm{op}}$ allows us to define cohomological invariants of comodules using injective resolutions in $\operatorname{Mod}_{A}$ as a substitute for projective resolutions in Comod ${ }_{A_{*}}^{\text {b,f.t. }}$. In effect for a comodule $N_{*}$ we define

$$
\operatorname{Coext}_{A_{*}}^{*}\left(-, N_{*}\right)=\operatorname{Ext}_{A}\left(N^{*},(-)^{*}\right) .
$$

Of course this is calculated using injective resolutions of $A$-modules; since $\operatorname{Ext}_{A}(-,-)$ is a balanced functor, (2.2) implies that $\operatorname{Coext}_{A_{t}}^{*}(-,-)$ is too, whenever we can use projective comodule resolutions in the first variable. For example, if we restrict to the subcategory of coherent comodules we obtain balanced bifunctors

$$
\operatorname{Coext}_{A_{*}}^{s}(-,-):\left(\operatorname{Comod}_{A_{*}}^{\mathrm{coh}}(-,-)\right)^{\mathrm{op}} \otimes \operatorname{Comod}_{A_{*}}^{\mathrm{coh}}(-,-) \rightarrow \operatorname{Mod}_{\mathbb{k}}^{\mathrm{b}, \text { f.t. }} .
$$

Given a surjection of $P_{*}$-algebras $A_{*} \rightarrow B_{*}$ there are adjunction isomorphisms of the form

$$
\begin{align*}
& \operatorname{Cohom}_{A_{*}}(-,-) \cong \operatorname{Cohom}_{B_{*} \backslash A_{*}}\left(\left(B_{*} \backslash \backslash A_{*}\right) \square_{A_{*}}(-),-\right),  \tag{2.3}\\
& \operatorname{Cohom}_{B_{*}}(-,-) \cong \operatorname{Cohom}_{A_{*}}\left(-, A_{*} \square_{B_{*}}(-)\right), \tag{2.4}
\end{align*}
$$

where $B_{*} \backslash \backslash A_{*}=\mathbb{k}_{\square_{B_{*}}}$. Later we will use these adjunctions to construct composite functor spectral sequences.

Remark 2.8. Since writing this we became aware of the revised version of Salch [Sal16], where it is shown that the category of graded connected comodules over a graded connected Hopf algebra over a field has enough projectives. As our results are stronger but more restricted, we feel it worthwhile presenting them despite the greater generality of Salch's result.

## 3. Some homological algebra

In this section we describe some Cartan-Eilenberg spectral sequences for comodules over a commutative Hopf algebra over a field. Some of these are similar to other examples in the literature such as that for computing Cotor for Hopf algebroids in [Rav86].

To ease notation, in this section we suppress internal gradings and assume that all our objects are connective and of finite type over a field $\mathbb{k}$. We refer to the classic [MM65] as well as the more recent [MP12] for notation and basic ideas about graded Hopf algebras.

Before discussing cohomology for comodules, we will recall the dual theory to modules over algebras, where there are classical Cartan-Eilenberg spectral sequences of [CE99] for a normal sequence of Hopf algebras over a field $\mathbb{k}$,

$$
\begin{equation*}
R \rightarrow S \rightarrow S / / R \tag{3.1}
\end{equation*}
$$

where $S$ is a free $R$-module. Then for a left $S / / R$-module $L$ and a left $S$-module $M$ there is a spectral sequence of the form

$$
\begin{equation*}
\mathrm{E}_{2}^{s, t}=\operatorname{Ext}_{S / / R}^{s}\left(L, \operatorname{Ext}_{R}^{t}(\mathbb{k}, M)\right) \Longrightarrow \operatorname{Ext}_{S}^{s+t}(L, M) \tag{3.2}
\end{equation*}
$$

There is another similar spectral sequence for a left $S$-module $M$ and a left $S / / R$-module $N$ which has the form

$$
\begin{equation*}
\mathrm{E}_{2}^{s, t}=\operatorname{Ext}_{S / / R}^{s}\left(\operatorname{Tor}_{R}^{t}(\mathbb{k}, M), N\right) \Longrightarrow \operatorname{Ext}_{S}^{s+t}(M, N) \tag{3.3}
\end{equation*}
$$

Since Ext and Tor are balanced functors, one approach to setting these spectral sequences is by resolving both variables and using double complex spectral sequences. However, they can instead be viewed as composite functor spectral sequences obtained using injective or projective resolutions of the $S$-module $M$. When the algebras and modules are graded $\mathbb{k}$ vector spaces, these spectral sequences are tri-graded; also, in topological applications, (3.1) is often a sequence of cocommutative Hopf algebras.

Now suppose we have a sequence of homomorphisms of connected commutative graded Hopf algebras over $\mathbb{k}$,

$$
K \backslash \backslash H \mapsto H \rightarrow K
$$

where in the notation of [MM65, definition 3.5],

$$
K \backslash \backslash H=\mathbb{k}_{\square_{K}} H=H \square_{K} \mathbb{k} \subseteq H
$$

We also assume given a left $K \backslash \backslash H$-comodule $M$ and a left $H$-comodule $N$. Of course $M$ and $N$ inherit structures of $H$-comodule and $K$-comodule respectively, where $M$ is trivial as a $K$ comodule. Our aim is to calculate $\operatorname{Coext}_{H}^{*}(M, N)$, the right derived functor of

$$
\operatorname{Cohom}_{H}(M,-): \operatorname{Comod}_{K \backslash \backslash H} \rightarrow \operatorname{Mod}_{\mathfrak{k}} ; \quad N \mapsto \operatorname{Cohom}_{H}(M, N) .
$$

Following Hovey [Hov04], we will write $U \stackrel{H}{\wedge} V$ to indicate the tensor product of two $H$ comodules $U \otimes V=U \otimes_{\mathbb{k}} V$ with the diagonal coaction given by the composition

$$
U \otimes V \xrightarrow[\mu \otimes \mu]{\longrightarrow}(H \otimes U) \otimes(H \otimes V) \xrightarrow[\cong]{\cong}(H \otimes H) \otimes(U \otimes V) \underset{\varphi \otimes \mathrm{Id}}{\longrightarrow} H \otimes(U \otimes V) .
$$

For a vector space $W$, the notation $U \otimes W$ will be used to denote $H$-comodule with coaction

$$
U \otimes W \xrightarrow[\mu \otimes \mathrm{Id}]{\longrightarrow}(H \otimes U) \otimes W \longrightarrow H \otimes(U \otimes W)
$$

carried on the first factor alone.
If $L$ is a left $H$-comodule, then there is a well-known isomorphism of left $H$-comodules

$$
\begin{equation*}
K \backslash \backslash H \stackrel{H}{\wedge} L=\left(H \square_{K} \mathbb{k}\right) \stackrel{H}{\wedge} L \cong H \square_{K} L, \tag{3.4}
\end{equation*}
$$

We can also regard $K \backslash \backslash H=\mathbb{k} \square_{K} H$ as a right $H$-comodule to form the left $K \backslash \backslash H$-comodule

$$
\begin{equation*}
K \backslash \backslash H \square_{H} L=\left(\mathbb{k} \square_{K} H\right) \square_{H} L \cong \mathbb{k}_{\mathbf{k}} \square_{K} L ; \tag{3.5}
\end{equation*}
$$

in particular, if $L$ is a trivial $K$-comodule then as left $K \backslash \backslash H$-comodules,

$$
\begin{equation*}
K \backslash \backslash H \square_{H} L \cong L \tag{3.6}
\end{equation*}
$$

We will use two more functors

$$
\operatorname{Comod}_{H} \rightarrow \operatorname{Comod}_{K \backslash \backslash H} ; \quad N \mapsto K \backslash \backslash H \square_{H} N=\left(\mathbb{k}_{K} H\right) \square_{H} N \cong \mathbb{k}_{K} N
$$

and

$$
\operatorname{Comod}_{K \backslash \backslash H} \rightarrow \operatorname{Mod}_{\mathfrak{k}} ; \quad N \mapsto \operatorname{Cohom}_{K \backslash \backslash H}(M, N) .
$$

Notice that there is a natural isomorphism

$$
\operatorname{Cohom}_{K \backslash \backslash H}\left(M, K \backslash \backslash H \square_{H}(-)\right) \cong \operatorname{Cohom}_{H}(M,-)
$$

and for an injective $H$-comodule $J, K \backslash \backslash H \square_{H} J$ is an injective $K \backslash \backslash H$-comodule. This means we are in a situation where we have a Grothendieck composite functor spectral sequence which in this case is a form of Cartan-Eilenberg spectral sequence; for details see [Wei94, section 5.8] for example.

Proposition 3.1. Let $M$ be a left $K \backslash \backslash H$-comodule and $N$ a left $H$-comodule. Then there is a first quadrant cohomologically indexed spectral sequence with

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Coext}_{K \backslash \backslash H}^{s}\left(M, \operatorname{Cotor}_{K}^{t}(\mathbb{k}, N)\right) \Longrightarrow \operatorname{Coext}_{H}^{s+t}(M, N) .
$$

If $N$ is a trivial $K$-comodule then

$$
\mathrm{E}_{2}^{s, t} \cong \operatorname{Coext}_{K \backslash \backslash H}^{s}\left(M, \operatorname{Cotor}_{K}^{t}(\mathbb{k}, \mathbb{k}) \stackrel{K \backslash \backslash H}{\Lambda} N\right)
$$

There is another spectral sequence that we will use whose construction requires that one of the Hopf algebras involved is a $P_{*}$-algebra. The reason for this is discussed in Remark 2.6. in the category of finite type connected comodules, extended comodules are projective objects.

Proposition 3.2. Assume that $H$ and $K \backslash \backslash H$ are $P_{*}$-algebras. Let $M$ be a left $H$-comodule which admits a projective resolution and let $N$ be a bounded below left $K \backslash H$-comodule. Then there is a first quadrant cohomologically indexed spectral sequence with

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Coext}_{K \backslash \backslash H}^{s}\left(\operatorname{Cotor}_{K}^{t}(\mathbb{k}, M), N\right) \Longrightarrow \operatorname{Coext}_{H}^{s+t}(M, N) .
$$

If $M$ is a trivial $K$-comodule then

$$
\mathrm{E}_{2}^{s, t} \cong \operatorname{Coext}_{K \backslash \backslash H}^{s}\left(\operatorname{Cotor}_{K}^{t}(\mathbb{k}, \mathbb{k}) \stackrel{K \backslash \backslash H}{\wedge} M, N\right)
$$

Proof. The construction is similar to the other one, and involves expressing $\operatorname{Cohom}_{H}(-, N)$ as a composition

$$
\operatorname{Cohom}_{K \backslash \backslash H}(-, N) \circ\left(K \backslash \backslash H \square_{H}(-)\right)=\operatorname{Cohom}_{K \backslash \backslash H}\left(K \backslash \backslash H \square_{H}(-), N\right) \cong \operatorname{Cohom}_{H}(-, N)
$$

The functor $K \backslash \backslash H \square_{H}(-):$ Comod $_{H} \rightarrow \mathbf{C o m o d}_{K \backslash \backslash H}$ sends projective objects to projective objects (see Remark 2.6), so the standard construction can be used to give a spectral sequence.

Of course the condition that $M$ admits a projective resolution is crucial; in the case of $P_{*}-$ algebras this is satisfied if $M$ is a coherent comodule.

## 4. Finite comodule filtrations

In this section we will give some results based on a particular kind of comodule filtration.
We will make use of the adjoint coaction dual to the adjoint action which is given a thorough treatment in Singer [Sin06, chapter 4]; the dualisation to the comodule setting is straightforward and details are left to the reader. For our purposes we need to know the following: If $B \subseteq A$ is a conormal subHopf algebra of a commutative Hopf algebra over a field $\mathbb{k}$, then there is a (left) adjoint coaction $A \rightarrow A \otimes A$ which is the composition of the solid arrows in the following commutative diagram, where T is the switch map.


By [Sin06, proposition 4.24], the adjoint coaction makes $A$ and any conormal subHopf algebra $B$ into an $A$-comodule Hopf algebra; furthermore, for any left $A$-comodule $M$ (which also becomes a left $A / / B$-comodule through the projection $A \rightarrow A / / B)$, there is an induced left $A$ coaction on $\operatorname{Cotor}_{A / / B}(\mathbb{k}, M)$ that factors through a left $B$-coaction.


Now let $C$ be a cocommutative coalgebra over a field $\mathbb{k}$. A $C$-comodule $M$ is unipotent if it has a finite length descending filtration by subcomodules

$$
M=M^{\ell} \supset M^{\ell-1} \supset \cdots \supset M^{1} \supset M^{0}=0
$$

where each quotient comodule $M^{i+1} / M^{i}$ has trivial coaction. We will refer to such a filtration as a unipotent filtration of length $\ell$. Every comodule $M$ contains a primitive sequence of subcomodules

$$
M \supseteq \cdots \supseteq M^{[i+1]} \supseteq M_{12}^{[i]} \supseteq \cdots \supseteq M^{[2]} \supseteq M^{[1]} \supseteq M^{[0]}=0
$$

defined recursively by

$$
M^{[i]}=\pi_{i-1}^{-1} \operatorname{Prim}_{C}\left(M / M^{[i-1]}\right)
$$

where $\pi_{i-1}: M \rightarrow M / M^{[i-1]}$ is the quotient homomorphism, so $\pi_{i-1}$ induces an isomorphism

$$
M^{[i]} / M^{[i-1]} \xrightarrow{\cong} \operatorname{Prim}_{C}\left(M / M^{[i-1]}\right) .
$$

In general this need not be exhaustive or become stable, but if it has both these properties then it is a unipotent filtration of $M$ and we will say that $M$ has a finite primitive filtration.

Lemma 4.1. Suppose that the comodule $M$ is unipotent. Then the primitive sequence of $M$ is finite.
Proof. Suppose that

$$
M=M^{\ell} \supset M^{\ell-1} \supset \cdots \supset M^{1} \supset M^{0}=0
$$

is a unipotent filtration. Then we clearly have $M^{1} \subseteq M^{[1]}$. Now suppose that for some $k \geqslant 1$, $M^{k} \subseteq M^{[k]}$. There is a commutative diagram of comodules

from which it follows that $M^{k+1} \subseteq M^{[k+1]}$. By Induction we find that $M^{n} \subseteq M^{[n]}$ for all $n \geqslant 1$ and in particular $M=M^{\ell} \subseteq M^{[\ell]}$, so $M^{[\ell]}=M$.

Lemma 4.2. Let $L, M, N$ be $C$-comodules fitting into a short exact sequence

$$
0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 .
$$

Then $M$ is unipotent if and only if $L$ and $N$ are unipotent.
Proof. Suppose that $M$ and $N$ are unipotent. A unipotent filtration of $N$ pulls back to a comodule filtration of $M$ where each stage contains the image of $L$, and this can be extended to a unipotent filtration of $M$ using a unipotent filtration of $L$.
Suppose that $M$ has a finite primitive filtration

$$
M=M^{[\ell]} \supset M^{[\ell-1]} \supset \cdots \supset M^{[1]} \supset M^{[0]}=0
$$

Set $L^{1}=\operatorname{Prim}_{C}(L)=L \cap M^{[1]}$. Then

$$
\operatorname{Prim}_{C}\left(L / L^{1}\right) \cong \operatorname{Prim}_{C}\left(L+M^{[1]} / M^{[1]}\right)=\left(\left(L+M^{[1]}\right) \cap M^{[2]}\right) / M^{[1]} .
$$

Now for each $1 \leqslant k \leqslant \ell$ define $L^{k}=L \cap M^{[k]}$. Notice that $L^{k}=L \cap M^{[k]} \subseteq=L^{k+1}$ and $L^{\ell}=L$. Also,

$$
L^{k+1} / L^{k} \cong\left(L^{k+1}+M^{[k]}\right) / M^{[k]} \subseteq\left(\left(L+M^{[k]}\right) \cap M^{[k+1]}\right) / M^{[k]} \subseteq M^{[k+1]} / M^{[k]}
$$

so the coaction of $L^{k+1} / L^{k}$ is trivial. Now taking $N=M / L$ define

$$
N^{k}=\left(L+M^{[k]}\right) / L \cong M^{[k]} / L^{k} \cap M^{[k]}
$$

so that

$$
N^{k+1} / N^{k} \cong\left(L+M^{[k+1]}\right) /\left(L+M^{[k]}\right)
$$

which is easily seen to have trivial coaction.

Remark 4.3. Suppose that $M$ is a unipotent $C$-comodule. Then $M$ is also unipotent as a $D$ comodule where $C \rightarrow D$ is a surjective morphism of coalgebras. Also, if the coaction factors through a $C^{\prime}$-coaction $M \rightarrow C_{0} \otimes M$ where $C^{\prime} \subseteq C$ is a subcoalgebra, then $M$ is a unipotent $C_{0}$-comodule.

Our main use of such unipotent filtrations is to situations described in the next result.
Proposition 4.4. Let $B \subseteq A$ be a conormal subHopf algebra of a commutative Hopf algebra over a field $\mathbb{k}$. Suppose that $M$ is a unipotent left $A$-comodule and that for every $k \geqslant 0, \operatorname{Cotor}_{A / / B}^{k}(\mathbb{k}, \mathbb{k})$ is a unipotent left $A$-comodule. Then each $\operatorname{Cotor}_{A / / B}^{k}(\mathbb{k}, M)$ is a unipotent $A$-comodule.

Proof. We remark that the long exact sequence for $\operatorname{Cotor}_{A / B}^{k}(\mathbb{k},-)$ used below is one of $A$ comodules; the dual module case follows from the details given in Singer [Sin06, chapter 4].

We prove this by induction on the length $\ell$ of the unipotent filtration on $M$. When $\ell=1$,

$$
\operatorname{Cotor}_{A / / B}^{k}(\mathbb{k}, M) \cong \operatorname{Cotor}_{A / / B}^{k}(\mathbb{k}, \mathbb{k}) \otimes M
$$

and the result holds. Now suppose that it holds for $\ell<n$ and let $\ell=n$. Consider the short exact sequence

$$
\begin{equation*}
0 \rightarrow M^{[n-1]} \rightarrow M \rightarrow M / M^{[n-1]} \rightarrow 0 \tag{4.3}
\end{equation*}
$$

where we know that for all $k \geqslant 0$,

$$
\operatorname{Cotor}_{A / B}^{k}\left(\mathbb{k}, M^{[n-1]}\right), \quad \operatorname{Cotor}_{A / / B}^{k}\left(\mathbb{k}, M^{[n]} / M^{[n-1]}\right)
$$

are unipotent $A$-comodules. On applying $\operatorname{Cotor}_{A / B}^{k}(\mathbb{k},-)$ to (4.3) we obtain a long exact sequence where in degree $k$ we have

$$
\cdots \longrightarrow \operatorname{Cotor}_{A / / B}^{k}\left(\mathbb{k}, M^{[n-1]}\right) \longrightarrow \operatorname{Cotor}_{A / B}^{k}(\mathbb{k}, M) \longrightarrow \operatorname{Cotor}_{A / B}^{k}\left(\mathbb{k}, M^{[n-1]}\right) \longrightarrow \cdots
$$

so by Lemma 4.2 there is a short exact sequence

$$
0 \rightarrow N_{k}^{\prime} \rightarrow \operatorname{Cotor}_{A / B}^{k}(\mathbb{k}, M) \rightarrow N_{k}^{\prime \prime} \rightarrow 0
$$

for unipotent comodules $N_{k}^{\prime}$ and $N_{k}^{\prime \prime}$. Again using Lemma 4.2 we find that $\operatorname{Cotor}_{A / / B}^{k}(\mathbb{k}, M)$ is unipotent.

Remark 4.5. Although we have stated the last result in terms the $A$-comodule structure, in our applications we will use it by passing to a conormal quotient Hopf algebra of $A$.

Proposition 4.6. Let $B \subseteq A$ be a conormal subHopf algebra of a commutative Hopf algebra over a field $\mathbb{k}$ where $B$ is cocommutative. Let $M$ be a left $A$-comodule such that for every $k \geqslant 0, \operatorname{Cotor}_{A / B}^{k}(\mathbb{k}, M)$ is a unipotent left $B$-comodule. Then each $\operatorname{Cotor}_{A}^{k}(\mathbb{k}, M)$ is a unipotent $B$-comodule.

Proof. There is a Cartan-Eilenberg spectral sequence with

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Cotor}_{B}^{s}\left(\mathbb{k}, \operatorname{Cotor}_{A / B}^{t}(\mathbb{k}, M)\right) \Longrightarrow \operatorname{Cotor}_{A}^{s+t}(\mathbb{k}, M) .
$$

When $B$ is cocommutative this is a first quadrant cohomological spectral sequence of left $B$ comodules, and the differentials are homomorphisms of unipotent $B$-comodules; this is verified by noting that the functor $\operatorname{Cotor}_{A / B}^{t}(\mathbb{k},-) \cong \operatorname{Cotor}_{A}^{t}(B,-)$ takes values in the category of left $B$-comodules, and when $B$ is cocommutative and $U$ is finite dimensional, $\operatorname{Cohom}_{B}(U, V)$ is naturally a left comodule. By Lemma 4.2, each $\mathrm{E}_{\infty}^{s, t}$ is a unipotent $B$-comodule and the same is true of each $\operatorname{Cotor}_{A}^{n}(\mathbb{k}, M)$.

In order to state a special case which will be used later, we recall a standard result which can be found in [MP12, corollary 21.24]. Let $B \subseteq A$ and $C \subseteq A$ be conormal subHopf algebras of a commutative Hopf algebra over a field $\mathbb{k}$ with $C \subseteq B$. Then there is an induced normal inclusion $B / / C \rightarrow A / / B$ and an isomorphism of Hopf algebras

$$
\begin{equation*}
(A / / C) / /(B / / C) \cong A / / B \tag{4.4}
\end{equation*}
$$

Corollary 4.7. Suppose that $B / / C$ is cocommutative and $M$ is a left $A / / C$-comodule such that for every $k \geqslant 0, \operatorname{Cotor}_{A / B}^{k}(\mathbb{k}, M)$ is a unipotent left $A / / C$-comodule. Then each cohomology group $\operatorname{Cotor}_{A / / C}^{k}(\mathbb{k}, M)$ is a unipotent B//C-comodule.

Proof. Use the Cartan-Eilenberg spectral sequence in the proof of Proposition 4.6,

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Cotor}_{B / / C}^{s}\left(\mathbb{k}, \operatorname{Cotor}_{(A / / C) / /(B / / C)}^{t}(\mathbb{k}, M)\right) \Longrightarrow \operatorname{Cotor}_{A / / C}^{s+t}(\mathbb{k}, M) .
$$

where by (4.4),

$$
\operatorname{Cotor}_{(A / / C) / /(B / / C)}^{t}(\mathbb{k}, M) \cong \operatorname{Cotor}_{A / / B}^{t}(\mathbb{k}, M) .
$$

Our motivation for developing these ideas is the following result.
Proposition 4.8. Suppose that $A_{*}$ is a $P_{*}$-algebra and $M_{*}$ a unipotent left $A_{*}$-comodule. Then

$$
\operatorname{Cohom}_{A_{*}}\left(A_{*}, M_{*}\right)=0
$$

Proof. A non-trivial comodule homomorphism $f: A_{*} \rightarrow M_{*}$ factors through some subcomodule $M_{*}^{k} \subseteq M_{*}$ in a unipotent filtration where and $k$ is minimal. Now choose a non-zero element $x \in M_{*}^{k} \cap \operatorname{im} f$ and choose a $\mathbb{k}$-linear map $M_{*}^{k} / M_{*}^{k-1} \rightarrow \mathbb{k}$ so that $x \mapsto 1$ under the composition. The resulting composition

$$
A_{*} \xrightarrow[f]{\longrightarrow} M_{*}^{k} \longrightarrow M_{*}^{k} / M_{*}^{k-1} \longrightarrow \mathbb{k}
$$

is a non-trivial comodule homomorphism which cannot exist by Proposition [2.1.
Corollary 4.9. For $s \geqslant 0$,

$$
\operatorname{Coext}_{A_{*}}^{s}\left(A_{*}, M_{*}\right)=0
$$

Proof. This can be proved by induction of the length of a unipotent filtration for $M_{*}$.
We will use this repeatedly in what follows to show that certain Coext groups vanish.

## 5. Recollections on the Steenrod algebra and its dual

The theory of $P$-algebras applies to many situations involving sub and quotient Hopf algebras of the Steenrod algebra and its dual for a prime. We will focus attention on the prime 2 but the methods are applicable for all primes.

To illustrate this, here is a simple application involving the mod 2 Steenrod algebra; this result appears in [Rav84, corollary 4.10]. We denote the mod 2 Eilenberg-Mac Lane spectrum by $H=H \mathbb{F}_{2}$ and recall that

$$
\mathcal{A}_{*}=H_{*}(H)=\mathbb{F}_{2}\left[\xi_{1}, \ldots, \xi_{n}, \ldots\right]=\mathbb{F}_{2}\left[\zeta_{1}, \ldots, \zeta_{n}, \ldots\right]
$$

where $\zeta_{n}=\chi\left(\xi_{n}\right) \in \mathcal{A}_{2^{n}-1}$ and the coproduct is given by the equivalent formulae

$$
\psi \xi_{n}=\sum_{0 \leqslant i \leqslant n} \xi_{n-i}^{2^{i}} \otimes \xi_{i}, \quad \psi \zeta_{n}=\sum_{0 \leqslant i \leqslant n} \zeta_{i} \otimes \zeta_{n-i}^{2^{i}} .
$$

Doubling. The operation of doubling has been used frequently in studying $\mathcal{A}$-modules. The reader is referred to the account of Margolis [Mar83, section 15.3] which we will use as background.

Since the dual $\mathcal{A}_{*}$ is a commutative Hopf algebra, it admits a Frobenius endomorphism $\mathcal{A}_{*} \rightarrow \mathcal{A}_{*}$ which doubles degrees and has Hopf algebra cokernel

$$
\mathcal{E}_{*}=\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}=\Lambda_{\mathbb{F}_{2}}\left(\bar{\zeta}_{s}: s \geqslant 1\right),
$$

where $\mathcal{A}_{*}^{(1)}=\mathbb{F}_{2}\left[\zeta_{s}^{2}: s \geqslant 1\right]$. Dually, there is a Verschiebung $\mathcal{A} \rightarrow \mathcal{A}$ which halves degrees and satisfies

$$
\mathrm{Sq}^{r} \mapsto\left\{\begin{array}{cl}
\mathrm{Sq}^{r / 2} & \text { if } r \text { is even }, \\
0 & \text { if } r \text { is odd. }
\end{array}\right.
$$

The kernel of this Verschiebung is the ideal generated by the Milnor primitives $\mathrm{P}_{t}^{0}(t \geqslant 1)$, hence there is a degree-halving isomorphism of Hopf algebras $\mathcal{A} / / \mathcal{E} \stackrel{\cong}{\rightrightarrows} \mathcal{A}$, where $\mathcal{E} \subseteq \mathcal{A}$ is the subHopf algebra generated by the primitives $\mathrm{P}_{t}^{0}$ and dual to the exterior quotient Hopf algebra $\mathcal{E}_{*}$.

Given a left (graded) $\mathcal{A}$-module $M$, we can induce an $\mathcal{A} / / \mathcal{E}$-module $M_{(1)}$ where

$$
M_{(1)}^{n}= \begin{cases}M^{n / 2} & \text { if } n \text { is even }, \\ 0 & \text { if } n \text { is odd },\end{cases}
$$

and we write $x_{(1)}$ to indicate the element $x \in M$ regarded as an element of $M_{(1)}$; the module structure is given by

$$
\mathrm{Sq}^{r}\left(x_{(1)}\right)= \begin{cases}\left(\mathrm{Sq}^{r / 2} x\right)_{(1)} & \text { if } r \text { is even, } \\ 0 & \text { if } r \text { is odd },\end{cases}
$$

Using this construction, the category of left $\mathcal{A}$-modules $\operatorname{Mod}_{\mathcal{A}}$ admits an additive functor to the category of evenly graded $\mathcal{A} / / \mathcal{E}$-modules,

$$
\Phi: \operatorname{Mod}_{\mathcal{A}} \rightarrow \operatorname{Mod}_{\mathcal{A} / / \mathcal{E}}^{\mathrm{ev}} ; \quad M \mapsto M_{(1)}
$$

which is an isomorphism of categories. The quotient homomorphism $\rho: \mathcal{A} \rightarrow \mathcal{A} / / \mathcal{E}$ also induces an additive isomorphism of categories $\rho^{*}: \operatorname{Mod}_{\mathcal{A} / / \mathcal{E}}^{\mathrm{ev}} \rightarrow \operatorname{Mod}_{\mathcal{A}}^{\mathrm{ev}}$ and it is often useful to consider the composition $\rho^{*} \circ \Phi: \operatorname{Mod}_{\mathcal{A}} \rightarrow \operatorname{Mod}_{\mathcal{A}}^{\mathrm{ev}}$.

By iterating $\Phi^{(1)}=\Phi$ we obtain isomorphisms

$$
\Phi^{(s)}=\Phi \circ \Phi^{(s-1)}: \operatorname{Mod}_{\mathcal{A}} \rightarrow \operatorname{Mod}_{\mathcal{A} / / \mathcal{E}^{(s-1)}}^{(s)} ; \quad M \mapsto M_{(s)}
$$

where the codomain is the category of $\mathcal{A} / / \mathcal{E}^{(s-1)}$-modules concentrated in degrees divisible by $2^{s}$ and $\mathcal{E}^{(s-1)} \subseteq \mathcal{A}$ is the subHopf algebra multiplicatively generated by the elements

$$
\begin{equation*}
\mathrm{P}_{b}^{a} \quad(s-1 \geqslant a \geqslant 0, b \geqslant 1), \tag{5.1}
\end{equation*}
$$

and $\mathcal{E}^{(0)}=\mathcal{E}$.
By doubling all three of the variables involved the following homological result is immediate for $e \geqslant 1$ and two $\mathcal{A}$-modules $M, N$ :

$$
\begin{equation*}
\operatorname{Ext}_{\mathcal{A}_{(e)}}^{s, 2^{e} t}\left(M_{(e)}, N_{(e)}\right) \cong \operatorname{Ext}_{\mathcal{A}}^{s, t}(M, N) . \tag{5.2}
\end{equation*}
$$

Because doubling is induced using a grade changing Hopf algebra endomorphism, the double $\mathcal{A}_{(1)}$ is also a Hopf algebra isomorphic to the quotient Hopf algebra $\mathcal{A} / / \mathcal{E}$ and dual to the subHopf algebra of squares $\mathcal{A}_{*}^{(1)} \subseteq \mathcal{A}_{*}$ which is also given by

$$
\mathcal{A}_{*}^{(1)}=\mathcal{A}_{*} \square_{\mathcal{A}^{\prime} / / / \mathcal{A}_{*}^{(1)}} \mathbb{F}_{2}=\mathbb{F}_{2} \square_{\mathcal{A}_{*} / / / \mathcal{A}_{*}^{(1)}} \mathcal{A}_{*}=\left(\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}\right) \backslash \backslash \mathcal{A}_{*} .
$$

More generally, for any $s \geqslant 1, \mathcal{A}_{(s)}$ is isomorphic to the quotient Hopf algebra of $\mathcal{A} / / \mathcal{E}^{(s)}$ dual to the subalgebra of $2^{s}$-th powers

$$
\mathcal{A}_{*}^{(s)}=\left(\mathcal{A}_{*} / / \mathcal{A}_{*}^{(s)}\right) \backslash \backslash \mathcal{A}_{*} \subseteq \mathcal{A}_{*} .
$$

In many ways, doubling is more transparent when viewed in terms of comodules. For an $\mathcal{A}_{*}$-comodule $M_{*}$, we can define a $\mathcal{A}_{*}^{(1)}$-coaction $\mu^{(1)}: M_{*}^{(1)} \rightarrow \mathcal{A}_{*}^{(1)} \otimes M_{*}^{(1)}$ where $M_{*}^{(1)}$ denotes $M_{*}$ with its degrees doubled; this is given on elements by the composition


By iterating we also obtain a $\mathcal{A}_{*}^{(s)}$-coaction $\mu^{(s)}: M_{*}^{(s)} \rightarrow \mathcal{A}_{*}^{(s)} \otimes M_{*}^{(s)}$.
Then the comodule analogue of (5.2) is

$$
\begin{equation*}
\operatorname{Coext}_{\mathcal{A}_{*}^{(e)}}^{\left.{ }^{(e,}\right)^{e} t}\left(M_{*}^{(e)}, N_{*}^{(e)}\right) \cong \operatorname{Coext}_{\mathcal{A}_{*}}^{s, t}\left(M_{*}, N_{*}\right) . \tag{5.3}
\end{equation*}
$$

We can use iterated doubling combined with Proposition 1.2 to show that for any $d \geqslant 1$,

$$
\begin{equation*}
\operatorname{Coext}_{\mathcal{A}_{*}}^{s_{\mathcal{L}} t}\left(\mathcal{A}_{*}, \mathcal{A}_{*}^{(d)}\right) \cong \operatorname{Ext}_{\mathcal{A}}^{s, t}\left(\mathcal{A}_{(d)}, \mathcal{A}\right)=0 \tag{5.4}
\end{equation*}
$$

By doubling all three of the variables involved here we can also prove that for $e \geqslant 0$,

$$
\begin{equation*}
\operatorname{Coext}_{\mathcal{A}_{*}^{(e)}}^{\mathrm{s},{ }^{2} t}\left(\mathcal{A}_{*}^{(e)}, \mathcal{A}_{*}^{(d+e)}\right) \cong \operatorname{Coext}_{\mathcal{A}_{*}}^{\mathrm{s}, t}\left(\mathcal{A}_{*}, \mathcal{A}_{*}^{(d)}\right)=0 . \tag{5.5}
\end{equation*}
$$

Some families of quotient $P_{*}$-algebras of $\mathcal{A}_{*}$. We will begin by describing some quotients of the dual Steenrod algebra $\mathcal{A}_{*}$. For any $n \geqslant 1,\left(\zeta_{1}, \ldots, \zeta_{n}\right) \triangleleft \mathcal{A}_{*}$ is a Hopf ideal so there is a quotient Hopf algebra $\mathcal{A}_{*} /\left(\zeta_{1}, \ldots, \zeta_{n}\right)$ together with the subHopf algebra

$$
\mathcal{P}(n)_{*}=\mathcal{A}_{*} \square_{\mathcal{A}_{*} /\left(\zeta_{1}, \ldots, \zeta_{n}\right)} \mathbb{F}_{2}=\mathbb{F}_{2}\left[\zeta_{1}, \ldots, \zeta_{n}\right] \subseteq \mathcal{A}_{*}
$$

and in fact

$$
\mathcal{A}_{*} / / \mathcal{P}(n)_{*}=\mathcal{A}_{*} /\left(\zeta_{1}, \ldots, \zeta_{n}\right) .
$$

Similarly, for any $s \geqslant 0$, the ideal $\left(\zeta_{1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right) \triangleleft \mathcal{A}_{*}$ is a Hopf ideal and there is a quotient Hopf algebra

$$
\mathcal{A}_{*} / / \mathcal{P}(n)_{*}^{(s)}=\mathcal{A}_{*} /\left(\zeta_{1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right)
$$

with associated subHopf algebra

$$
\mathcal{P}(n)_{*}^{(s)}=\mathcal{A}_{*} \square \square_{\mathcal{A}^{\prime} / / \mathcal{P}(n)^{(s)}} \mathbb{F}_{2}=\mathbb{F}_{2}\left[\zeta_{1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right] \subseteq \mathcal{A}_{*}
$$

For each $t \geqslant 0$ there is a finite quotient Hopf algebra

$$
\mathcal{P}(n)_{*}^{(s)} /\left(\zeta_{1}^{2^{s+t}}, \zeta_{2}^{2^{s+t-1}}, \ldots, \zeta_{t}^{2^{s+1}}, \zeta_{t+1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right)
$$

and we have

$$
\mathcal{P}(n)_{*}^{(s)}=\lim _{t} \mathcal{P}(n)_{*}^{(s)} /\left(\zeta_{1}^{2^{s+t}}, \zeta_{17}^{2^{s+t-1}}, \ldots, \zeta_{t}^{2^{s+1}}, \zeta_{t+1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right)
$$

where the limit is computed degree-wise. The graded dual Hopf algebra

$$
\mathcal{P}(n)_{(s)}=\left(\mathcal{P}(n)_{*}^{(s)}\right)^{*}=\operatorname{Hom}\left(\mathcal{P}(n)_{*}^{(s)}, \mathbb{F}_{2}\right)
$$

is the colimit of the finite dual Hopf algebras

$$
\operatorname{Hom}\left(\mathcal{P}(n)_{*}^{(s)} /\left(\zeta_{1}^{2^{s+t}}, \zeta_{2}^{2^{s+t-1}}, \ldots, \zeta_{t}^{2^{s+1}}, \zeta_{t+1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right), \mathbb{F}_{2}\right)
$$

i.e.,

$$
\mathcal{P}(n)_{(s)}=\underset{t}{\operatorname{colim}} \operatorname{Hom}\left(\mathcal{P}(n)_{*}^{(s)} /\left(\zeta_{1}^{2^{s+t}}, \zeta_{2}^{2^{s+1-1}}, \ldots, \zeta_{t}^{2^{s+1}}, \zeta_{t+1}^{2^{s}}, \ldots, \zeta_{n}^{2^{s}}\right), \mathbb{F}_{2}\right) .
$$

Therefore $\mathcal{P}(n)_{(s)}$ is a $P$-algebra and $\mathcal{P}(n)_{*}^{(s)}$ is a $P_{*}$-algebra.

## 6. Some comodules and their cohomology

Later we will need to determine various cohomology groups such as

$$
\operatorname{Coext}_{\mathcal{A}_{*}}^{* * *}\left(\mathcal{A}_{*}^{(1)}, \mathbb{F}_{2}\right)
$$

using the Cartan-Eilenberg spectral sequence of Proposition 3.1,

$$
\begin{equation*}
\mathrm{E}_{2}^{s, t}=\operatorname{Coext}_{\mathcal{A}_{*}^{(1)}}^{s}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right) \Longrightarrow \operatorname{Coext}_{\mathcal{A}_{*}}^{s+t}\left(\mathcal{A}_{*}^{(1)}, \mathbb{F}_{2}\right), \tag{6.1}
\end{equation*}
$$

where we have suppressed the internal grading. In fact, $\mathcal{A}_{*}^{(1)}$ is a projective $\mathcal{A}_{*}^{(1)}$-comodule, so $\mathrm{E}_{2}^{s, t}=0$ when $s>0$, therefore we only need to consider

$$
\mathrm{E}_{2}^{0, t}=\operatorname{Cohom}_{\mathcal{A}_{t}^{(1)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right) .
$$

Here the Cotor term is bigraded with

$$
\operatorname{Cotor}_{\mathcal{A}_{2} / / \mathcal{A}_{\lambda}^{(1)}}^{*, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)=\mathbb{F}_{2}\left[q_{n}: n \geqslant 0\right]
$$

for $q_{n}=\left[\overline{\zeta_{n+1}}\right] \in \operatorname{Cotor}^{1,2^{n+1}-1}$ represented in the cobar construction by the residue class of $\overline{\zeta_{n+1}} \in \mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}$. We need to understand the comodule structure on this and similar Cotor groups.

Now we can consider the adjoint coaction for $\mathcal{A}_{\star}$.
Lemma 6.1. The left adjoint coaction of $\mathcal{A}_{*}$ is given by

$$
\mu: \mathcal{A}_{*} \rightarrow \mathcal{A}_{*} \otimes \mathcal{A}_{*} ; \quad \mu\left(\zeta_{n}\right)=\sum_{i \geqslant 0} \sum_{j \geqslant 0} \zeta_{i} \xi_{n-i-j}^{2^{i+j}} \otimes \zeta_{j}^{2^{i}} .
$$

Proof. This follows by iterating the coaction and using the formulae

$$
\psi \zeta_{n}=\sum_{0 \leqslant k \leqslant n} \zeta_{k} \otimes \zeta_{n-k}^{2^{k}}, \quad \xi_{r}=\chi\left(\zeta_{r}\right) .
$$

The left coaction on $q_{n}$ can be deduced from that on $\zeta_{n+1}$ where we can ignore all terms in the sum for $\mu\left(\zeta_{n+1}\right)$ with $i>0$, thus giving

$$
\begin{equation*}
\mu q_{n}=\sum_{0 \leqslant j \leqslant n} \xi_{n-j}^{\xi^{j+1}} \otimes q_{j} . \tag{6.2}
\end{equation*}
$$

This extends to polynomials in the $q_{n}$ using multiplicativity.
This coaction is visibly defined over $\mathcal{A}_{*}^{(1)} \subseteq \mathcal{A}_{*}$, and in practise we will only require the coaction over the quotient $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(3)}$ so we will denote this coaction by

$$
\bar{\mu}: \operatorname{Cotor}_{\mathcal{A}_{*} / / / \mathcal{A}_{*}^{(1)}}^{*, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right) \rightarrow \mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(3)} \otimes \operatorname{Cotot}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{*, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right),
$$

where for $n \geqslant 2$,

$$
\bar{\mu} q_{n}=\overline{\xi_{n}^{2}} \otimes q_{0}+\overline{\xi_{n-1}^{4}} \otimes q_{1}+1 \otimes q_{n}
$$

and

$$
\bar{\mu} q_{1}=\overline{\xi_{1}^{2}} \otimes q_{0}+1 \otimes q_{1} .
$$

In what follows we will make use of this $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(3)}$-comodule structure and also the induced $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$-comodule structure.

Proposition 6.2. For $k \geqslant 0$ there are no non-trivial $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$-comodule homomorphisms

$$
\mathcal{A}_{*}^{(1)} \rightarrow \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)
$$

Hence

$$
\operatorname{Cohom}_{\mathcal{A}_{*}^{(1)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)=\operatorname{Cohom}_{\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)=0
$$

Before giving the proof, we will define for each $k \geqslant 0$, an increasing filtration of the $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$ comodule

$$
\operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)=\mathbb{F}_{2}\left\{q_{0}^{r_{0}} q_{1}^{r_{1}} \cdots q_{\ell}^{r_{\ell}}: \sum_{0 \leqslant i \leqslant \ell} r_{i}=k\right\}
$$

by setting

$$
\mathrm{F}^{k, s}=\mathbb{F}_{2}\left\{q_{0}^{r_{0}} q_{1}^{r_{1}} \cdots q_{\ell}^{r_{\ell}}: r_{0} \geqslant k-s, \sum_{0 \leqslant i \leqslant \ell} r_{i}=k\right\} .
$$

Each $\mathrm{F}^{k, s}$ is a subcomodule of $\operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$ and

$$
\mathrm{F}^{k, 0}=\mathbb{F}_{2}\left\{q_{0}^{k}\right\}, \quad \mathrm{F}^{k, k}=\mathbb{F}_{2}\left\{q_{1}^{r_{1}} \cdots q_{\ell}^{r_{\ell}}: \sum_{0 \leqslant i \leqslant \ell} r_{i}=k\right\} .
$$

Now suppose that $f: M_{*} \rightarrow \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)$ is a non-trivial comodule homomorphism. Choose $s_{0}$ to be minimal so that $\operatorname{im} f \subseteq \mathrm{~F}^{k, s_{0}}$. Then the composition

is a non-trivial comodule homomorphism, where $\mathrm{F}^{k, s_{0}} / \mathrm{F}^{k, s_{0}-1}$ has the trivial coaction. Now we may choose any non-trivial linear map $\mathrm{F}^{k, s_{0}} / \mathrm{F}^{k, s_{0}-1} \rightarrow \mathbb{F}_{2}[d]$ which is non-trivial on im $\bar{f}$ and so obtain a non-trivial comodule homomorphism $M_{*} \rightarrow \mathbb{F}_{2}[d]$. This is the key observation required for our proof.

Proof of Proposition6.2 Such a comodule homomorphism leads to a non-trivial $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$ comodule homomorphism $\mathcal{A}_{*}^{(1)} \rightarrow \mathbb{F}_{2}[d]$ for some $d$. By the Milnor-Moore theorem, $\mathcal{A}_{*}^{(1)}$ is an extended $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$-comodule, so there must be a non-trivial $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$-comodule homomorphism $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)} \rightarrow \mathbb{F}_{2}\left[d^{\prime}\right]$ for some $d^{\prime}$. But since $\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}$ is a $P_{*}$-algebra, this contradicts Proposition 2.5 .

Since the natural homomorphism

$$
\operatorname{Cohom}_{\mathcal{A}_{*}^{(1)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right) \rightarrow \operatorname{Cohom}_{\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)
$$

is injective, the last statement follows.

Corollary 6.3. The $\mathrm{E}_{2}$-term of the spectral sequence (6.1) is trivial, hence

$$
\operatorname{Coext}_{\mathcal{A}_{\star}}^{* * *}\left(\mathcal{A}_{*}^{(1)}, \mathbb{F}_{2}\right)=0
$$

A generalisation of these results is
Proposition 6.4. For any $n \geqslant 0$,

$$
\operatorname{Coext}_{\mathcal{A}_{*}, *}^{* *}\left(\mathcal{A}_{*}^{(1)}, \mathcal{P}(n)_{*}^{(1)}\right)=0
$$

Proof. We need to deal with the case $n \geqslant 1$. By Proposition 3.1 there is a Cartan-Eilenberg spectral sequence of form

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Coext}_{\mathcal{A}_{*}^{(1)}}^{s}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{( }\left(\mathbb{F}_{2}, \mathcal{P}(n)_{*}^{(1)}\right)\right) \Longrightarrow \operatorname{Coext}_{\mathcal{A}_{*}}^{s+t}\left(\mathcal{A}_{*}^{(1)}, \mathbb{F}_{2}\right) .
$$

Since the $\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}$-coaction on $\mathcal{P}(n)_{*}^{(1)}$ is trivial,

$$
\operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{\left(\mathbb{F}_{2}, \mathcal{P}(n)_{*}^{(1)}\right) \cong \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)_{\wedge}^{\mathcal{A}_{\star}^{(1)}} \mathcal{P}(n)_{*}^{(1)} . . .}
$$

As left $\mathcal{A}_{*}^{(1)}$-comodules

$$
\mathcal{P}(n)_{*}^{(1)} \cong \mathcal{A}_{*}^{(1)} \square_{\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)_{*}^{(1)}} \mathbb{F}_{2}
$$

so by a standard 'change of rings' isomorphism,

$$
\mathrm{E}_{2}^{s, t} \cong \operatorname{Coext}_{\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)_{*}^{(1)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right) .
$$

Again the Milnor-Moore theorem shows that $\mathcal{A}_{*}^{(1)}$ is a cofree $\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)_{*}^{(1)}$-comodule and since this is a $P_{*}$-algebra, $\mathrm{E}_{2}^{s, t}=0$ for $s>0$. Now a similar argument to that in the proof of Proposition 6.2] shows that $\mathrm{E}_{2}^{0, t}=0$.

Before passing on to discuss other examples, we note that in $\mathcal{A}_{*}$ there is a subcomodule ${ }^{\leqslant k} \mathcal{A}_{*}$ spanned by the monomials in the $\xi_{i}$ of degree at most $k$. Under the doubling isomorphism this corresponds to a subcomodule ${ }^{\leqslant k} \mathcal{A}_{*}^{(1)}$ of $\mathcal{A}_{*}^{(1)}$ spanned by monomials in the $\xi_{i}^{2}$ of degree at most $k$.

Proposition 6.5. For $k \geqslant 0$ there is an isomorphism of $\mathcal{A}_{*}^{(1)}$-comodules

$$
\operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{k, *}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right) \xrightarrow{\cong} \leqslant k \mathcal{A}_{*}^{(1)} ; \quad q_{0}^{r_{0}} q_{1}^{r_{1}} \cdots q_{\ell}^{r_{\ell}} \leftrightarrow \xi_{1}^{2 r_{1}} \cdots \xi_{\ell}^{2 r_{\ell}} .
$$

The following result is an analogue of Proposition 6.2 whose proof can be adapted using the filtration of ${ }^{\leqslant k} \mathcal{A}_{*}$ based on polynomial degree.

Proposition 6.6. For $k \geqslant 0$,

$$
\operatorname{Coext}_{\mathcal{A}_{*}}\left(\mathcal{A}_{*}, \leqslant k \mathcal{A}_{*}\right)=0
$$

We will also require some other vanishing results.
Proposition 6.7. For $n \geqslant 0$,

$$
\operatorname{Coext}_{\mathcal{A}_{*}}^{*_{*}^{*}}\left(\mathcal{A}_{*}^{(1)}, \mathcal{P}(n)_{*}^{(2)}\right)=0
$$

and

$$
\operatorname{Coext}_{\mathcal{A}_{*}^{* * *}}^{*}\left(\mathcal{A}_{*}^{(1)}, \mathcal{A}_{*}^{(2)}\right)=0
$$

Proof. By setting $\mathcal{P}(\infty)_{*}^{(2)}=\mathcal{A}_{*}^{(2)}$ we can present the proofs of these in a uniform faahion.
There is a Cartan-Eilenberg spectral sequence

$$
\mathrm{E}_{2}^{s, t}=\operatorname{Coext}_{\mathcal{A}_{*}^{(1)}}^{s}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathcal{P}(n)_{*}^{(2)}\right)\right) \Longrightarrow \operatorname{Coext}_{\mathcal{A}_{*}}^{s+t}\left(\mathcal{A}_{*}^{(1)}, \mathcal{P}(n)_{*}^{(2)}\right) .
$$

Here

$$
\operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathcal{P}(n)_{*}^{(2)}\right) \cong \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)^{\mathcal{A}_{4}^{(1)}} \mathcal{P}(n)_{*}^{(2)}
$$

and

$$
\mathcal{P}(n)_{*}^{(2)} \cong \mathcal{A}_{*}^{(1)} \square_{\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)_{*}^{(2)}} \mathbb{F}_{2}
$$

so

$$
\mathrm{E}_{2}^{s, t} \cong \operatorname{Coext}_{\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)^{(2)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)
$$

Since $\mathcal{A}_{*}^{(1)}$ is a cofree $\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n){ }_{*}^{(2)}$-comodule, $\mathrm{E}_{2}^{s, t}=0$ when $s>0$. Also the change of coalgebra homomorphism
$\operatorname{Cohom}_{\mathcal{A}_{*}^{(1)} / / \mathcal{P}(n)^{(2)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right) \rightarrow \operatorname{Cohom}_{\mathcal{A}_{*}^{(1)} / / \mathcal{A}_{*}^{(2)}}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{*} / / \mathcal{A}_{*}^{(1)}}^{t}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)$ is injective. By Proposition 6.2, the codomain is trivial so $\mathrm{E}_{2}^{0, t}=0$.

## 7. Some topological applications

We illustrate our theory with a few calculations of homotopy groups of spectra. Some of the following examples where found in response to questions of John Rognes. Results of this type were proved by Lin, Margolis, Ravenel and others. The interested reader will be able to give others, especially at odd primes.

For simplicity we assume that all spectra are 2-completed. We are interested in determining when $[X, Y]^{*}=0$ for two connective finite type spectra $X, Y$. Here the Adams spectral sequence

$$
\mathrm{E}_{2}^{s, t}(X, Y)=\operatorname{Ext}_{\mathcal{A}}^{s, t}\left(H^{*}(Y), H^{*}(X)\right)=\operatorname{Coext}_{\mathcal{A}}^{s, t}\left(H_{*}(X), H_{*}(Y)\right) \Longrightarrow[X, Y]^{s-t}
$$

converges by work of Boardman [Boa99], so we are interested in examples where $\mathrm{E}_{2}^{* *}(X, Y)=0$. The general result of Proposition 1.5 applies to many interesting examples.

- For each the spectra $X$ where $H=H \mathbb{F}_{2}, H \mathbb{Z}_{2^{s}}(s \geqslant 2), H \mathbb{Z}, k \mathrm{O}, k \mathrm{U}$, tmf, and $B P\langle n\rangle$ $(n \geqslant 1),[X, S]^{*}=0$ since $H^{*}(X)$ is a coherent $\mathcal{A}$-module and $\mathrm{E}_{2}^{*, *}(X, S)=0$.
- For each of the above spectra $X,[X, B P]^{*}=0$. This follows since

$$
H^{*}(B P) \cong \mathcal{A} \otimes_{\mathcal{E}} \mathbb{F}_{2}
$$

where $\mathcal{E} \subseteq \mathcal{A}$ is the subHopf algebra generated by the Milnor primitives and this is also a $P$-algebra. So by Proposition 1.6,

$$
\mathrm{E}_{2}^{*, *}(X, B P) \cong \operatorname{Ext}_{\mathcal{E}}^{* * *}\left(\mathbb{F}_{2}, H^{*}(X)\right)=0
$$

- The case of $[B P, S]^{*}$ involves more work. We will use the dual version of the Adams spectral sequence,

$$
\mathrm{E}_{2}^{*, *}(B P, S)=\operatorname{Coext}_{\mathcal{A}_{*}}^{s, t}\left(\mathcal{A}_{*}^{(1)}, \mathbb{F}_{2}\right) \Longrightarrow[B P, S]^{s-t} .
$$

We can calculate the $\mathrm{E}_{2}$-term using the Cartan-Eilenberg spectral sequence of (6.1), and by Corollary 6.3 this $\mathrm{E}_{2}$-term

$$
\operatorname{Coext}_{\mathcal{A}_{*}^{(1)}}^{*}\left(\mathcal{A}_{*}^{(1)}, \operatorname{Cotor}_{\mathcal{A}_{4} / / \mathcal{A}_{*}^{(1)}}^{*}\left(\mathbb{F}_{2}, \mathbb{F}_{2}\right)\right)
$$

is trivial. Therefore $\mathrm{E}_{2}^{*, *}(B P, S)=0$ and $[B P, S]^{*}=0$.

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