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Coulomb branches have symplectic singularities

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Abstract
We show that Coulomb branches for 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories have symplectic singularities. This confirms a conjecture of Braverman–Finkelberg–Nakajima.

1 Introduction

Let G be a complex reductive algebraic group and N a finite-dimensional representation of G. A mathematical definition of the Coulomb branch (of cotangent type) $\mathcal{M}_C(G, N)$ of a 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory associated to (G, N) was introduced in the seminal papers [5, 9]. They showed that Coulomb branches have a number of remarkable properties. Of relevance to us is the fact that they are irreducible normal Poisson varieties, where the Poisson structure is non-degenerate on the smooth locus. Therefore, it is natural to conjecture, as they do, that Coulomb branches have symplectic singularities in the sense of Beauville [1].

Using partial resolutions of singularities constructed from flavor symmetries, it was shown by Weekes [11] that most Coulomb branches arising from quiver gauge theories have symplectic singularities. In this note, we extend that result by showing that all Coulomb branches have symplectic singularities.

Theorem 1.1 *$\mathcal{M}_C(G, N)$ has symplectic singularities.*

This confirms the "optimistic conjecture" of Braverman–Finkelberg–Nakajima [5, 3(iv)]. As an immediate corollary, we note that:

Corollary 1.2 *$\mathcal{M}_C(G, N)$ has finitely many symplectic leaves.*

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In the case of quiver gauge theories for finite type quivers, the symplectic leaves of $\mathcal{M}_C(G, N)$ have been explicitly described in [7]. See [11] for other consequences of the main theorem.

Our proof relies on an elementary observation about varieties with symplectic singularities. Namely, if there is a birational Poisson morphism $X \to Y$ between normal affine varieties and Y is known to have symplectic singularities then so too does X. We apply this observation twice—first in the case where G is a (connected) torus to allow us to reduce to the case where the Coulomb branch can be identified with a toric hyper-Kähler manifold and secondly to reduce from the case of a Coulomb branch for a general reductive group to one for a torus. In both cases, the birational Poisson morphism we require was already constructed by Braverman–Finkelberg–Nakajima [5].

2 The proof

2.1 An elementary observation

Throughout, variety will mean a integral, separated scheme of finite type over the complex numbers. We recall, following [1], that a variety X has symplectic singularities if it is a normal variety whose smooth locus admits a symplectic form ω such that for some (any) resolution of singularities $q\colon Z \to X$, $q^*\omega$ extends to a regular 2-form on Z.

The following elementary lemma is the key to the proof of the main theorem.

Lemma 2.1 *Let X, Y be complex normal Poisson varieties. Assume that Y has symplectic singularities and the Poisson structure on the smooth locus of X is non-degenerate. If there exists a generically étale Poisson morphism $f\colon X \to Y$, then X has symplectic singularities.*

Proof The only thing to check is that the pull-back to some resolution of singularities of the symplectic form ω on the smooth locus of X is regular. Let ω_0 denote the symplectic form on the smooth locus of Y.

We choose a resolution of singularities $p\colon W \to Y$. Let C denote the (unique) irreducible component of $W \times_Y X$ dominating both Y and X. By base change, $C \to X$ is a proper generically étale map. Taking a resolution of singularities $Z \to C$, we form the commutative diagram

$$\begin{tikzcd} Z \arrow[dr] \arrow[drr, "q"] \arrow[ddr, "g"'] & & \\ & C \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & W \arrow[r, "p"'] & Y. \end{tikzcd} \tag{2.1}$$

Since all the maps f, g, p, q are generically étale, there exists a dense open subset U of Y such that the restrictions $p^{-1}(U) \to U$, $f^{-1}(U) \to U$ and $g^{-1}(p^{-1}(U)) \to$

$p^{-1}(U), q^{-1}(f^{-1}(U)) \to f^{-1}(U)$ are étale. We check that $q^*\omega$ extends to a regular form on Z. This means that there exists some regular 2-form (necessarily unique) on Z whose restriction to some dense open subset, over which q is étale, agrees with $q^*\omega$. Since f is assumed Poisson, $f^*(\omega_0|_U) = \omega|_{f^{-1}(U)}$. Therefore,

$$q^*(\omega|_{f^{-1}(U)}) = q^*(f^*(\omega_0|_U)) = g^*(p^*(\omega_0|_U)).$$

Since Y is assumed to have symplectic singularities, there exists a regular 2-form η on W whose restriction to $p^{-1}(U)$ agrees with $p^*(\omega_0|_U)$. Thus, $q^*(\omega|_{f^{-1}(U)})|_V$ equals $g^*(\eta)|_V$, where $V = g^{-1}(p^{-1}(U)) \cap q^{-1}(f^{-1}(U))$ and $g^*(\eta)$ is a regular 2-form on Z. $\square$

Instead of using Y to deduce that X has symplectic singularities, one can ask if we can use X to deduce that Y has symplectic singularities. As shown in the result below, the answer is yes, provided the morphism is also assumed proper; see also [1, Proposition 2.4] or [3, Lemma 6.12]. The result is not required in this paper, but we provide a proof for completeness.

Proposition 2.2 *Let X, Y be complex normal Poisson varieties. Assume that X has symplectic singularities and the Poisson structure on the smooth locus of Y is non-degenerate. If there exists a generically étale proper Poisson morphism $f : X \to Y$, then Y has symplectic singularities.*

The outline of the proof is the same as that of Lemma 2.1. The difference is that we now have a meromorphic form $p^*\omega_0$ on W that we wish to show is regular. Diagram (2.1) implies that $g^*(p^*\omega_0) = q^*(f^*\omega)$ is regular on Z. We deduce from the key lemma below that $p^*\omega_0$ is regular.

Lemma 2.3 *Let $g : Z \to W$ be a proper, generically étale morphism between smooth complex varieties. Then a meromorphic k-form ω on W is regular if and only if $g^*\omega$ is regular.*

Proof Our assumptions imply that g is surjective. First, we note that the locus where ω is not regular is a divisor on W; locally we can pick $w_1, \dots, w_n$ such that $dw_1, \dots, dw_n$ are a basis of Ω^1_W. Then ω can be uniquely expressed as $\sum_{\mathbf{i}} a_{\mathbf{i}} dw_{\mathbf{i}}$, and the non-regular locus of ω is the union of the non-regular loci of the meromorphic functions $a_{\mathbf{i}}$.

Next, we claim that the locus (on W) where g is finite has complement of codimension at least two. Since g is assumed proper, Stein factorization says that we can factor $g = \phi \circ h$, where $h : Z \to T$ has connected fibers and $\phi : T \to W$ is finite. It suffices then to show that the locus of points t on T where $\dim h^{-1}(t) = 0$ has complement C of codimension at least two. But if c is a generic point of an irreducible component C_0 of C, then $\dim C_0 + \dim h^{-1}(c) < \dim Z$ since $h^{-1}(C_0)$ is a proper closed subset of Z. Since $\dim h^{-1}(c) \geq 1$, this implies that $\dim C_0 < \dim T - 1$.

Therefore, we may assume that g is a finite morphism. If $U \subset W$ is any open set such that g is étale on $g^{-1}(U)$ then it is clear that $\omega|_U$ is regular if and only if $g^*(\omega|_U)$ is regular. Thus, we just need to consider a generic point $w \in g(R_g) \subset W$, where R_g is the ramification divisor of g. Let $D \subset g(R_g)$ be an irreducible component, and $E \subset R_g$ an irreducible component of R_g mapping onto D. We may assume that $w_1 = 0$ is a local equation for D at w. Since Z and W are smooth, the local rings $\mathcal{O}_{W,D}$ and $\mathcal{O}_{Z,E}$ are (noetherian) regular local rings of dimension one; that is, they are discrete valuation rings. We have an embedding $g^* : \mathcal{O}_{W,D} \to \mathcal{O}_{Z,E}$, with $\mathcal{O}_{Z,E}$ finite over $\mathcal{O}_{W,D}$. The function w_1 is a uniformizer for $\mathcal{O}_{W,D}$, and choosing a uniformizer t for $\mathcal{O}_{Z,E}$, we have $g^*(w_1) = t^\ell u$ for some unit $u \in \mathcal{O}_{Z,E}^\times$. Here ℓ is the ramification index of D. The module $\Omega^1_{\mathcal{O}_{Z,E}}$ has basis $dt, dg^*(w_2), \dots, dg^*(w_n)$. Therefore, if $\omega_{\mathbf{i}} = a_{\mathbf{i}} dw_1 \wedge dw_{i_1} \wedge \cdots \wedge dw_{i_{k-1}}$ is a summand of ω, for some $1 < i_1 < \cdots < i_{k-1} \le n$, then

$$\begin{aligned} g^*\omega_{\mathbf{i}} &= \ell u g^*(a_{\mathbf{i}}) t^{\ell-1} dt \wedge dg^*(w_{i_1}) \wedge \\ &\quad \cdots \wedge dg^*(w_{i_{k-1}}) + g^*(a_{\mathbf{i}}) t^\ell du \wedge dg^*(w_{i_1}) \wedge \cdots \wedge dg^*(w_{i_{k-1}}). \end{aligned}$$

If $a_{\mathbf{i}} = w_1^{-r} s$ for some unit $s \in \mathcal{O}_{W,D}$ and $r \ge 1$, then $\ell u g^*(a_{\mathbf{i}}) t^{\ell-1} = \ell u^{1-r} g^*(s) t^{-\ell(r-1)-1}$ has a pole of order $\ell(r-1)+1 \ge 1$. Since $du \in \bigoplus_{i \ge 2} \mathcal{O}_{Z,E}\, dg^*(w_i)$, we deduce that $\omega_{\mathbf{i}}$ is regular if and only if $g^*\omega_{\mathbf{i}}$ is regular. $\square$

2.2 Toric hyper-Kähler manifolds

Consider a short exact sequence

$$0 \to \mathbb{Z}^k \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0. \tag{2.2}$$

We assume that no row of B is zero. Write $T := \mathbb{C}^\times$ for the one torus. The above sequence encodes an action of T^d on $\mathbb{C}^n$ via

$$(t_1, \dots, t_d) \cdot x_i = t_1^{a_{i,1}} \cdots t_d^{a_{i,d}} x_i.$$

Since (2.2) is exact, the stabilizer of any $x \in \mathbb{C}^n$, with $x_i \neq 0$ for all i, is trivial. In particular, the action is effective. The induced action on $T^*\mathbb{C}^n$ is Hamiltonian, and we write $\mu : T^*\mathbb{C}^n \to \mathfrak{t}_d^*$ for the associated moment map. Explicitly,

$$\mu(x, y) = \left(\sum_i a_{i,j} x_i y_i \right)_{j=1}^d.$$

If θ denotes a rational character of T^d and $\zeta \in \mathfrak{t}_d^*$, then we can take Hamiltonian reduction

$$\mathcal{M}_H(\theta, \zeta) := \mu^{-1}(\zeta)^\theta /\!\!/ T^d.$$

Here $\mu^{-1}(\zeta)^\theta$ denotes the open subset of θ-semistable points in $\mu^{-1}(\zeta)$. The variety $\mathcal{M}_H(\theta, \zeta)$ is a *toric hyper-Kähler manifold* (also called a *hypertoric variety* in the literature) and is a Higgs branch for the gauge theory $(T^d, \mathbb{C}^n)$. The fact that these varieties have symplectic singularities is well-known, but the proofs in the literature, [2, Proposition 4.11] or [8, Theorem 2.16], always assume that the matrix A is unimodular (so that the variety admits a symplectic resolution given by variation of GIT). Since we will need to consider matrices A that are not unimodular, we explain how to extend this result to general toric hyper-Kähler manifolds.

Lemma 2.4 *Choose θ, θ' such that $\mu^{-1}(\zeta)^{\theta'} \subset \mu^{-1}(\zeta)^{\theta}$. Then there exists a projective birational Poisson morphism $\mathcal{M}_H(\theta', \zeta) \to \mathcal{M}_H(\theta, \zeta)$.*

Proof Since we have assumed that no row of B is zero, $\mu^{-1}(\zeta)$ is a reduced, irreducible complete intersection [2, Lemma 4.7]. Moreover, as shown in [2, Proposition 4.11] when A is unimodular and in [10] in general, the quotient $\mathcal{M}_H(\theta, \zeta)$ is normal. Since the variety is constructed as a Hamiltonian reduction, the Poisson bracket on $\mathcal{O}_{T^*\mathbb{C}^n}$ descends to a Poisson bracket on $\mathcal{O}_{\mathcal{M}_H(\theta,\zeta)}$.

The fact that there is a projective Poisson morphism $\pi : \mathcal{M}_H(\theta', \zeta) \to \mathcal{M}_H(\theta, \zeta)$ is a direct consequence of Hamiltonian reduction; see for instance the proof of [3, Lemma 2.4]. We need to check that it is birational.

Let $\mu^{-1}(\zeta)^{\theta\,\mathrm{st}}$ denote the set of θ-stable points in $\mu^{-1}(\zeta)$ and $\mathcal{M}_H(\theta, \zeta)^{\theta\,\mathrm{st}}$ its image in $\mathcal{M}_H(\theta, \zeta)$. The map π is bijective over $\mathcal{M}_H(\theta, \zeta)^{\theta\,\mathrm{st}}$. Hence, we need to show that $\mathcal{M}_H(\theta, \zeta)^{\theta\,\mathrm{st}}$ (or equivalently, $\mu^{-1}(\zeta)^{\theta\,\mathrm{st}}$) is non-empty. Since $\mu^{-1}(\zeta)^{0\,\mathrm{st}}$ is contained in $\mu^{-1}(\zeta)^{\theta\,\mathrm{st}}$, it suffices to show that $\mu^{-1}(\zeta)^{0\,\mathrm{st}} \neq \emptyset$. In other words, there exists a closed orbit in $\mu^{-1}(\zeta)$ with finite (in fact trivial) stabilizer. Let $U \subset T^*V$ consist of all points (x, y) with $x_i, y_i \neq 0$ for all $1 \leq i \leq n$. As noted previously, the stabilizer of any point in U is trivial. We claim that (a) every orbit in U is closed in T^*V, and (b) $\mu^{-1}(\zeta) \cap U \neq \emptyset$. Thus, (a) and (b) would imply $\emptyset \neq \mu^{-1}(\zeta) \cap U \subset \mu^{-1}(\zeta)^{0\,\mathrm{st}}$.

Let $(p, q) \in U$. If $p_i q_i =: \lambda_i \in \mathbb{C}^\times$, then the equation $x_i y_i = \lambda_i$ holds for all points in $\overline{T^d \cdot (p, q)}$. But this forces $x_i, y_i \neq 0$ for all points (x, y) in $\overline{T^d \cdot (p, q)}$. That is, $\overline{T^d \cdot (p, q)} \subset U$. Since all orbits in U are free, we have $\overline{T^d \cdot (p, q)} = T^d \cdot (p, q)$ proving (a).

For (b), the exactness of (2.2) implies that the rank of A is d. Therefore, permuting the x_i, we may assume that the first $d \times d$ block of A has non-zero determinant. Applying an automorphism to T^d corresponds to multiplying A on the left by a unimodular $d \times d$ matrix U. Therefore, replacing A by UA, we may assume that A is in Hermite form. In particular, the moment map relations $\mu(x, y) = \zeta$ become

$$x_i y_i = a_{i,i}^{-1} \zeta_i - \sum_{j>i} a_{i,i}^{-1} a_{j,i} x_j y_j. \tag{2.3}$$

Making further substitutions (and replacing $a_{i,i}^{-1} \zeta_i$ by some ζ_i'), we may assume $a_{j,i} = 0$ for $j \leq d$ in the relations (2.3). The fact that no row of B is zero translates into the fact that for each $1 \leq i \leq d$ there exists some $j > d$ with $a_{j,i} \neq 0$. This means that for generic $(x_{d+1}, \ldots, x_n, y_{d+1}, \ldots, y_n)$ with $x_j, y_j \neq 0$ the relations (2.3) can be satisfied, but only with $x_i y_i \neq 0$ for $1 \leq i \leq d$ too. Thus, $\mu^{-1}(\zeta) \cap U \neq \emptyset$. □

Proposition 2.5 *The toric hyper-Kähler manifold $\mathcal{M}_H(\theta, \zeta)$ has symplectic singularities.*

Proof As noted in the proof of Lemma 2.4, the quotient $\mathcal{M}_H(\theta, \zeta)$ is a normal Poisson variety. Moreover, since T^d acts freely, with closed orbits, on the non-empty open set $\mu^{-1}(\zeta)^{\theta\text{ st}} \cap U$, the Poisson structure on $\mathcal{M}_H(\theta, \zeta)$ is generically non-degenerate.

Choose a generic θ' such that $\mu^{-1}(\zeta)^{\theta'} \subset \mu^{-1}(\zeta)^{\theta}$. Then Lemma 2.4 says that there exists a projective birational Poisson morphism $\mathcal{M}_H(\theta', \zeta) \to \mathcal{M}_H(\theta, \zeta)$. If $\mathcal{M}_H(\theta', \zeta)$ admits symplectic singularities, then [3, Lemma 6.12] implies that $\mathcal{M}_H(\theta, \zeta)$ will also admit symplectic singularities. Thus, we may assume that θ is generic.

To check that $\mathcal{M}_H(\theta, \zeta)$ has symplectic singularities, it suffices to check étale locally. As explained in the proof of [6, Proposition 6.2], the fact that θ is generic means that the stabilizer under T^d of each point in $\mu^{-1}(\zeta)^{\theta}$ is finite. Therefore, the (étale) symplectic slice theorem, e.g., [3, Theorem 3.8],[1] says that étale locally $\mathcal{M}_H(\theta, \zeta)$ is isomorphic (as a Poisson variety) to the quotient of a symplectic vector space by a finite (abelian) group acting symplectically. In particular, it has symplectic singularities by [1, Proposition 2.4] and hence the Poisson structure is non-degenerate on the whole of the smooth locus of $\mathcal{M}_H(\theta, \zeta)$. □

While this note was in preparation, the above statement also appeared as [4, Proposition 5.1].

2.3 Coulomb branches

Coulomb branches are normal varieties whose smooth locus admits a symplectic form [5]. Let G° be the connected component of the identity in G. Then, as noted in [5, Remarks 2.8(3)], $\mathcal{M}_C(G, N) \cong \mathcal{M}_C(G^\circ, N)/(G/G^\circ)$ as Poisson varieties. Hence, $\mathcal{M}_C(G, N)$ will have symplectic singularities by [1, Proposition 2.4] if we can show that $\mathcal{M}_C(G^\circ, N)$ has symplectic singularities. Therefore, we may assume that G is connected. We first consider the abelian case.

Lemma 2.6 *Assume $G = T^k$ is a torus. Then $\mathcal{M}_C(G, N)$ has symplectic singularities.*

Proof The action of $G = T^k$ on $N = \mathbb{C}^m$ is encoded in an integral $k \times m$ matrix B_0. Namely,

$$(t_1, \ldots, t_k) \cdot x_i = t_1^{b_{1,i}} \cdots t_d^{b_{k,i}} x_i.$$

If we decompose $N = N_0 \oplus N^G$, then [5, 3(vii)] says that $\mathcal{M}_C(G, N) = \mathcal{M}_C(G, N_0)$. Therefore, we may assume that $N = N_0$. In other words, no row of B_0 is zero; this will be important later.

The idea of course is to identify the Coulomb branch with a toric hyper-Kähler manifold and apply Proposition 2.5. However, this identification only holds if there is sufficient matter in the theory, specifically if the representation N is assumed to be a faithful T-module.

Let $N' = N \oplus \mathbb{C}^k = \mathbb{C}^n$, where T^k acts on $\mathbb{C}^k$ in the natural way (so that the weights are encoded by the identity matrix) and $n = m + k$. By [5, 4(vi)], there is a

[1] This is stated and proved for Nakajima quiver varieties, but both the statement and proof go through without change for any reductive group acting symplectically on a symplectic vector space.

birational Poisson morphism $\mathcal{M}_C(T^k, N) \to \mathcal{M}_C(T^k, N')$. Then Lemma 2.1 says that $\mathcal{M}_C(T^k, N)$ will have symplectic singularities if we can show that $\mathcal{M}_C(T^k, N')$ has symplectic singularities. The action of T^k on N' is encoded in the matrix $B = \begin{pmatrix} B_0 \\ \mathrm{Id} \end{pmatrix}$ and we may form a short exact sequence

$$0 \to \mathbb{Z}^k \xrightarrow{B} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^d \to 0,$$

where $A = (\mathrm{Id}| - B_0^T)$. We note that no row of B is zero. In this situation, it is noted in [5, 4(iv)] that $\mathcal{M}_C(T, N')$ is isomorphic to the affine toric hyper-Kähler manifold $\mathcal{M}_H((T^d)^\vee, N')$. By Proposition 2.5, the latter has symplectic singularities. □

Now we return to the general situation, where G is a connected reductive group. Let T be a maximal torus of G and W the associated Weyl group. It is shown in [5, Lemma 5.9, Lemma 5.10] that there exists a birational Poisson morphism $\mathcal{M}_C(G, N) \to \mathcal{M}_C(T, N|_T)/W$. Lemma 2.1 implies that $\mathcal{M}_C(T, N|_T)$ has symplectic singularities. It follows from [1, Proposition 2.4] that $\mathcal{M}_C(T, N|_T)/W$ also has symplectic singularities. Therefore, Theorem 1.1 is a consequence of Lemma 2.1.

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References

1. Beauville, A.: Symplectic singularities. Invent. Math. **139**(3), 541–549 (2000)
2. Bellamy, G., Kuwabara, T.: On deformation quantizations of hypertoric varieties. Pac. J. Math. **260**(1), 89–127 (2012)
3. Bellamy, G., Schedler, T.: Symplectic resolutions of quiver varieties. Sel. Math. (N.S.) **27**(3), 36 (2021)
4. Bielawski, R., Foscolo, L.: Hypertoric varieties, W-Hilbert schemes, and Coulomb branches. arXiv arXiv:2304.08125v1 (2023)
5. Braverman, A., Finkelberg, M., Nakajima, H.: Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories. II. Adv. Theor. Math. Phys. **22**(5), 1071–1147 (2018)
6. Hausel, T., Sturmfels, B.: Toric hyperKähler varieties. Doc. Math. **7**, 495–534 (2002). (**electronic**)
7. Muthiah, D., Weekes, A.: Symplectic leaves for generalized affine Grassmannian slices. Ann. Sci. Éc. Norm. Supér. **56**(1), 287–298 (2023)
8. Nagaoka, T.: The universal Poisson deformation of hypertoric varieties and some classification results. Pac. J. Math. **313**(2), 459–508 (2021)
9. Nakajima, H.: Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories. I. Adv. Theor. Math. Phys. **20**(3), 595–669 (2016)
10. Špenko, Š, Van den Bergh, M.: Tilting bundles on hypertoric varieties. Int. Math. Res. Not. IMRN **2**, 1034–1042 (2021)

11. Weekes, A.: Quiver gauge theories and symplectic singularities. arXiv arXiv:2005.01702v4 (2020)

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