A Letters Journal Exploring the Frontiers of Physics

August 2023

EPL, **143** (2023) 35002
doi: 10.1209/0295-5075/ace8b7

www.epljournal.org

On lines of constant polarisation in structured light beams

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received 15 June 2023; accepted in final form 19 July 2023
published online 28 July 2023

Abstract – We show that skyrmion field lines, constructed from the local Stokes parameters, trace out lines of constant optical polarisation.

Structured light beams are characterised by an engineered spatial variation of amplitude, phase and polarisation [1–4]. Important examples of these include beams carrying orbital angular momentum [5–9], helicity lattices [10–15], and the vector vortex beams [16–19]. Some of these beams, in particular those with spatially varying polarisation, have been shown to exhibit skyrmionic structure [20,21]. Typically, these have a polarisation pattern in the transverse plane that, at its centre, has one polarisation but at the outer reaches of the plane has the orthogonal polarisation. In general, all possible polarisations appear at some point in this transverse plane in a winding pattern, and skyrmions are characterised by a corresponding winding number, the skyrmion number [20–26]. There exist numerous variations on this theme [22–26]. What has not yet been identified, however, is the physical significance of the skyrmion field itself: we rectify that in this letter.

We present an unexpected property of skyrmion field lines that has application whether or not a structured light beam has a non-zero skyrmion number, as long as the polarisation pattern covers the whole Poincaré sphere continuously [27]. Put simply, it is that skyrmion field lines trace out contours of constant polarisation. Moreover, all such lines of constant polarisation are skyrmion field lines. Several important properties of structured light beams then follow from the mathematical properties of the skyrmion field. The central theme of our paper is the application of these ideas to paraxial light beams, but we conclude with a brief discussion of these ideas in other fields of physics, including electron [28–30] and neutron [31] optics and also gravitational waves [32].

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Skyrmion field lines for paraxial light beams are defined in terms of the normalised Stokes parameters, S_1, S_2 and S_3 [33]. The i-th component of the skyrmion field is [20,21]

$$\Sigma_i = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{pqr}S_p\frac{\partial S_q}{\partial x_j}\frac{\partial S_r}{\partial x_k}, \tag{1}$$

where ε_{ijk} is the alternating or Levi-Civita symbol and we employ the summation convention in which a summation is implied over repeated indices. The specific form of Σ_i is crucial to an appreciation of the link with lines of constant polarisation. For this reason it is worth writing explicitly one of the Cartesian components of $\mathbf{\Sigma}$,

$$\begin{aligned}\Sigma_z &= \frac{1}{2}\varepsilon_{pqr}S_p\left(\frac{\partial S_q}{\partial x}\frac{\partial S_r}{\partial y} - \frac{\partial S_r}{\partial x}\frac{\partial S_q}{\partial y}\right)\\ &= S_1\left(\frac{\partial S_2}{\partial x}\frac{\partial S_3}{\partial y} - \frac{\partial S_3}{\partial x}\frac{\partial S_2}{\partial y}\right)\\ &\quad + S_2\left(\frac{\partial S_3}{\partial x}\frac{\partial S_1}{\partial y} - \frac{\partial S_1}{\partial x}\frac{\partial S_3}{\partial y}\right)\\ &\quad + S_3\left(\frac{\partial S_1}{\partial x}\frac{\partial S_2}{\partial y} - \frac{\partial S_2}{\partial x}\frac{\partial S_1}{\partial y}\right). \end{aligned}\tag{2}$$

Note that this z-component depends on the variation of the Stokes parameters, and therefore of the polarisations, only in the x- and y-directions. Each term, moreover, depends on all three Stokes parameters.

The skyrmion number associated with our structured beam is readily obtained by integration over the plane transverse to the direction of propagation. If we take this direction to define our z-axis, then the skyrmion number is

$$n = \frac{1}{4\pi}\int \Sigma_z \,\mathrm{d}x\mathrm{d}y, \tag{3}$$

where the integral runs over the whole transverse plane. This value is typically an integer, although structures with non-integer skyrmion number can be constructed [20]. Our present concerns do not involve skyrmions or the skyrmion number explicitly, but rather focus on the skyrmion field. This exists wherever there is a continuously spatially varying polarisation apart from a few special cases, such as where there is a polarisation variation only in one direction. We note that the skyrmion field is a transverse field in that

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} = 0, \tag{4}$$

and, therefore, the integral over any closed surface is zero $\oint \boldsymbol{\Sigma} \cdot \mathrm{d}\,\mathbf{S} = 0$. The only exception to this condition will occur if there are lines along which the polarisation is undefined.

Let us turn to the properties of lines of constant polarisation. As is well known, structured light beams are threaded by lines of constant polarisation. The most studied example is the C-lines along which the polarisation is purely left- or right-handed circularly polarised [1,34]. There is nothing in this context, however, that is specific to circular polarisation, and we can trace such contours of constant polarisation for any chosen polarisation. Consider a point p in a structured paraxial light beam, as depicted in fig. 1. From this point there extends a line (in two directions) along which the polarisation is the same as at p. Note that the amplitude and phase will not, in general, remain the same along this line. Let us introduce a local right-handed coordinate system at $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ at p, in which the line of constant polarisation extends in the direction $\mathbf{u}$. As the polarisation in the direction $\mathbf{u}$ (and $-\mathbf{u}$) is unchanged, it follows that the direction of the Stokes vector $\mathbf{S}$ is also unchanged:

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{S} = \frac{\partial}{\partial u}\mathbf{S} = 0, \tag{5}$$

where $\mathbf{u}$ is a unit vector in the direction of the coordinate u. We can write the components of the skyrmion field at p in the u, v, w basis and find

$$\begin{aligned} \Sigma_u &= \frac{1}{2}\varepsilon_{pqr} S_p \left(\frac{\partial S_q}{\partial v}\frac{\partial S_r}{\partial w} - \frac{\partial S_r}{\partial v}\frac{\partial S_q}{\partial w} \right), \\ \Sigma_v &= \frac{1}{2}\varepsilon_{pqr} S_p \left(\frac{\partial S_q}{\partial w}\frac{\partial S_r}{\partial u} - \frac{\partial S_r}{\partial w}\frac{\partial S_q}{\partial u} \right), \\ \Sigma_w &= \frac{1}{2}\varepsilon_{pqr} S_p \left(\frac{\partial S_q}{\partial u}\frac{\partial S_r}{\partial v} - \frac{\partial S_r}{\partial u}\frac{\partial S_q}{\partial v} \right). \end{aligned} \tag{6}$$

The derivatives of the Stokes parameters with respect to u are zero and it follows that $\Sigma_v = 0 = \Sigma_w$, and, therefore, that the skyrmion field line points in the direction of constant polarisation. This is our principal result.

It is straightforward to confirm that the skyrmion field is independent of the basis used to denote the Stokes vectors and hence the identification of the skyrmion field lines with lines of constant polarisation holds for every possible polarisation. Such a global transformation changes the polarisation at every point in the field but does not alter the skyrmion field, which is associated with lines of constant polarisation, but not the specific polarisation along these lines. Identifying a skyrmion field line does not determine the polarisation along the field line, merely the line itself. More formally, the skyrmion field is invariant under any unitary transformation of the Poincaré sphere and so is not dependent on the basis used to express S_1, S_2 and S_3. In this way, the skyrmion field extends the characterization of lines of constant circular or linear polarisation [35] to every polarisation.

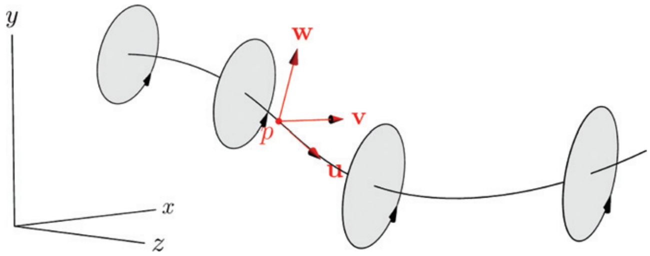

Fig. 1: Plot of a line of constant elliptical polarisation and the local coordinate system $\mathbf{u}, \mathbf{v}, \mathbf{w}$ at p.

The mathematical properties of the skyrmion field allow us to make general statements about lines of constant polarisation. The simplest and most important among these follows from the transverse nature of the skyrmion field, $\boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} = 0$. Like other transverse fields, such as the magnetic induction $\mathbf{B}$ in electromagnetism, skyrmion field lines cannot start or end (no monopoles) and nor can they branch or coalesce. The identification of skyrmion field lines with lines of constant polarisation means that the same properties must hold for lines of constant polarisation. The only exception to this rule occurs along lines at which the polarisation is undefined, where several lines of different polarisation can meet. At such lines, however, the transverse condition on $\boldsymbol{\Sigma}$ will fail.

One remaining subtlety needs to be addressed. This is the fact that lines of constant polarisation do not have a preferred sense of direction: such a line is independent of whichever direction we choose to move along it. The skyrmion field line, however, has a specific direction; if we change the sign of $\boldsymbol{\Sigma}$, then it reverses its direction along the line of constant polarisation. In this sense, at least, there seems to be a significant difference between lines of constant polarisation and skyrmion field lines and we should explain the origin of this difference.

We have seen that the skyrmion field lines do not determine the local polarisation, merely the local direction along which the polarisation does not vary. For the structured light beam, however, there is a further class of symmetries that leaves the pattern of lines of constant polarisation unchanged. This is to apply the operation of complex conjugation to the polarisations. To be specific, let $\mathbf{e}_\mathrm{H}$ and $\mathbf{e}_\mathrm{V}$ be the real unit vectors corresponding to horizontal and vertical polarisation, so that left- and right-circular polarisations are $\mathbf{e}_\mathrm{L} = (\mathbf{e}_\mathrm{H} + i\mathbf{e}_\mathrm{V})/\sqrt{2}$ and $\mathbf{e}_\mathrm{R} = (\mathbf{e}_\mathrm{H} - i\mathbf{e}_\mathrm{V})/\sqrt{2}$. If we apply the complex conjugation operation then the left- and right-circular polarisations switch, but the horizontal and vertical are unchanged, as are all the other possible linear polarisations, and we arrive at an alternative (but physically allowed) polarisation pattern with the same lines of constant polarisation. This transformation is antiunitary in nature [36]. Such transformations are familiar from the study of time-reversal and *CP* symmetries in particle physics [37]. The complex conjugate transformation coupled with rotations provides a second set of symmetries under which the polarisation changes but the lines of constant polarisation do not. The skyrmion field lines, however, switch direction under the antiunitary transformation as their value is based on a right-handed coordinate system for the Poincaré sphere. The complex conjugation operation applied to polarisation, however, changes a right-handed arrangement of the Stokes parameters into a left-handed one and, in doing so, flips the sign of the skyrmion field.

It is interesting to ask if there is always a skyrmion field in a structured beam. The answer follows from the discussion in the preceding paragraph; if, for example, the field is unchanged by performing complex conjugation on the polarisation then the skyrmion field is everywhere equal to its negative and therefore is equal to zero. Simple examples are the radially and azimuthally polarised beams [16] for which the polarisation is everywhere linear. For such beams we have *planes* of constant polarisation rather than lines.

The connection between the skyrmion field and lines of constant polarisation has been established here for paraxial structured light beams, but we may expect it to have wider applications. For electrons and neutrons, a similar association will hold for the skyrmion field lines and lines along which the particle spin does not change. For non-paraxial optical fields there exists a variety of features that can be associated with skyrmions and, by extension, with a skyrmion field. It will be interesting to see how these are related to the spatial arrangement of spin-related properties of the electromagnetic field. Finally, gravitational waves have two orthogonal polarisations and so we would expect skyrmion field lines to be associated, also, with spatial variations of the polarisation of these fields.

In summary, we have shown that lines of constant polarisation in any structured paraxial light beam are identified with skyrmion field lines. It follows that there is a more intimate relationship between structured light beams and skyrmion fields than simply whether or not a particular beam has an associated skyrmion number or skyrmionic structures.

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This work was supported by a Royal Society Research Professorship, grant No. RP150122, and the UK Engineering and Physical Sciences Research Council, grant Nos. EP/R008264/1 and EP/V048449/1.

Data availability statement: No new data were created or analysed in this study.

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