PURE BRAID GROUP PRESENTATIONS VIA LONGEST ELEMENTS

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Abstract. This paper gives a new, simplified presentation of the classical pure braid group. The generators are given by the squares of the longest elements over connected subgraphs, and we prove that the only relations are either commutators or certain palindromic length 5 box relations. This presentation is motivated by twist functors in algebraic geometry, but the proof is entirely Coxeter-theoretic. We also prove that the analogous set does not generate for all Coxeter arrangements, which in particular answers a question of Donovan and Wemyss.

1. Introduction

The classical pure braid group, equivalently the pure braid group of type A, is a fundamental object in algebra, geometry and topology. The purpose of this paper is to give a new, and simplified, presentation of this group using only squares of longest elements over connected subgraphs, and to then use this to answer questions motivated from algebraic geometry. In the process we place this presentation in the context of other fundamental groups. Our methods are algebraic, and are independent of the geometric motivation.

1.1. **The new presentation.** As recalled in Definition 2.1, the classical pure braid group PBr_{A_n} is the kernel of the natural surjection

$$\mathrm{Br}_{n+1} \twoheadrightarrow \mathfrak{S}_{n+1},$$

and can also be viewed as π_1 of the type A (complexified) hyperplane arrangement. Generators typically involve a choice of looping around hyperplanes. Unfortunately, these choices often lead to non-symmetric and often unpleasant presentations.

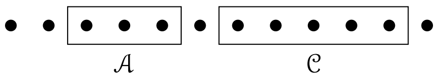
Both the need to give a nice presentation, and our geometric purposes, require a different generating set. Consider the A_n Dynkin graph numbered

$$\underset{1}{\bullet} - \underset{2}{\bullet} - \cdots - \underset{n-1}{\bullet} - \underset{n}{\bullet}$$

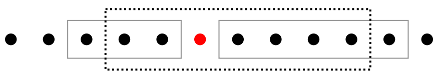
Then for any a connected subgraph $\mathcal{A} \subseteq A_n$, consider $\ell_{\mathcal{A}}^2$, where $\ell_{\mathcal{A}}$ is the longest element over $\mathcal{A}$. The following is our first result.

Proposition 1.1 (2.5)**.** *The set* $\{\ell_{\mathcal{A}}^2 \mid \mathcal{A} \subseteq A_n,\ \mathcal{A}$ *connected* $\}$ *generates* PBr_{A_n}.

By slight abuse of notation, write $\mathcal{A} := \ell_{\mathcal{A}}^2$. Leading to our main result, consider connected subgraphs $\mathcal{A}$ and $\mathcal{C}$ of A_n, then by the distance $d(\mathcal{A}, \mathcal{C})$ we mean the number of edges between $\mathcal{A}$ and $\mathcal{C}$. The case $d(\mathcal{A}, \mathcal{C}) = 2$ corresponds to when $\mathcal{A}$ and $\mathcal{C}$ are precisely one node apart, namely



Given such a pair, we say that a subgraph $\mathcal{B}$ is compatible with $(\mathcal{A}, \mathcal{C})$ if $\mathcal{B}$ is a connected subgraph of the following dotted area, containing the red node.



The following is our main result.

Theorem 1.2 (3.10)**.** *The pure braid group* PBr_{A_n} *has a presentation with generators given by connected subgraphs* $\mathcal{A} \subseteq A_n$*, subject to the relations*

(1) $\mathcal{A} \cdot \mathcal{B} = \mathcal{B} \cdot \mathcal{A}$ *if* $d(\mathcal{A}, \mathcal{B}) \geq 2$*, or* $\mathcal{A} \subseteq \mathcal{B}$*, or* $\mathcal{B} \subseteq \mathcal{A}$*.*

(2) *For all* $\mathcal{A}$ *and all* $\mathcal{C}$ *such that* $d(\mathcal{A},\mathcal{C})=2$*, then*

$$(\mathcal{A}\cup\mathcal{B})^{-1}\cdot(\mathcal{A}\cdot\mathcal{B}\cdot\mathcal{C})\cdot(\mathcal{B}\cup\mathcal{C})^{-1}=(\mathcal{C}\cup\mathcal{B})^{-1}\cdot(\mathcal{C}\cdot\mathcal{B}\cdot\mathcal{A})\cdot(\mathcal{B}\cup\mathcal{A})^{-1}$$

for all $\mathcal{B}$ *with compatible* $(\mathcal{A},\mathcal{C})$.

We also give a presentation in terms of double-indices in Corollary 3.11, which then makes it easier to compare to the other presentations in the literature. We remark that the above presentation is symmetric, and furthermore there are precisely $\binom{n+2}{5}$ non-commutator relations, each of which is palindromic and has degree 5. The above presentation is different to the presentations in [BB, A, FV, DG], and the geometric presentations in [MM]. Using the exact sequence [GW, (4), p150], Theorem 1.2 also independently recovers the presentation of the pure mapping class group of the punctured 2-sphere discovered in the recent work of Hirose–Omori [HO, 3.1].

To prove that the set $\{\ell^2_{\mathcal{A}}\}$ generates, it suffices to show that the standard generators in [A, FV] can be written as a product of the elements in the set $\{\ell^2_{\mathcal{A}}\}$. The $\{\ell^2_{\mathcal{A}}\}$ are symmetric whereas the standard relations for PBr_{A_n} ([A, FV], see e.g [BB]) come from the existence of a split short exact sequence

$$1\to \mathrm{F}_{n-1}\to \mathrm{PBr}_{A_n}\to \mathrm{PBr}_{A_{n-1}}\to 1,$$

where $\mathrm{F_{n-1}}$ is a free group, using an inductive argument. To prove results on the relations in the new presentation, we track the standard relations in [A, FV] under a homomorphism ϕ that is expressed in Proposition 2.4, and show that the standard relations are mapped to the identity in the new presentation. This part of the argument is much harder, since the splitting involves choice, whereas the new relations are symmetric.

We finally show in Corollary 3.15 that the pure braid groups of other Coxeter arrangements are not in general generated by squares of longest elements, and so the above is largely a type A phenomena. We further explain how this relates to monodromy around high codimension walls in the corresponding hyperplane arrangement, and thus answer a question of Donovan-Wemyss [DW3].

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2. Preliminaries

2.1. **Classical Presentation.** The classical Artin braid group is defined to be

$$\mathrm{Br}_n:=\left\langle s_1,\dots,s_{n-1}\;\middle|\;\begin{array}{l} s_is_j=s_js_i \text{ if } |i-j|\geq 2,\\ s_is_{i+1}s_i=s_{i+1}s_is_{i+1} \text{ for all } i=1,2,\dots,n-2\end{array}\right\rangle.$$

Definition 2.1. *The kernel of* $\mathrm{Br}_{n+1}\to\mathfrak{S}_{n+1}$ *sending* s_i *to the permutation* $(i,i+1)$ *is defined to be the* pure braid group, *and will be written* PBr_{A_n}.

For any i,j satisfying $1\leq i<j\leq n+1$, set

$$\sigma_{i,j}=\sigma_{j,i}=(s_{j-1}\dots s_{i+1})s_i(s_{i+1}^{-1}\dots s_{j-1}^{-1})\in \mathrm{Br}_{n+1}.$$

According to [A, FV], a presentation of the pure braid group can be given as follows. As generators, $\mathrm{PBr}_{A_n}=\langle A_{i,j}=A_{j,i}=\sigma^2_{i,j}\rangle$, subject to the relations

$$A_{r,s}^{-1}A_{i,j}A_{r,s}$$
$$=\begin{cases} A_{i,j} & \text{if } r<s<i<j \text{ or } i<r<s<j,\\ A_{r,j}A_{i,j}A_{r,j}^{-1} & \text{if } r<s=i<j,\\ (A_{i,j}A_{s,j})(A_{i,j})(A_{i,j}A_{s,j})^{-1} & \text{if } r=i<s<j,\\ (A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1})A_{i,j}(A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1})^{-1} & \text{if } r<i<s<j.\end{cases}$$

2.2. **Generation via longest elements squared.** A connected subgraph of A_n is determined by its leftmost vertex i, and its rightmost vertex j, where $i \leq j$. To such a subgraph is an associated longest element in the corresponding parabolic subgroup of the symmetric group generated by the subgraph. The standard lift of this element to $\mathrm{Br}_{A_n} = \mathrm{Br}_{n+1}$, will be written $\ell_{i,j}$ (see e.g [BT, p4], [ECHLPT, Lemma 9.1.10] and [G, p2]).

Reversing words in the Artin generators, that is to say reading words backwards gives an antiautomorphism $\mathrm{Br}_{n+1} \to \mathrm{Br}_{n+1}$ which we will write as $g \mapsto \bar{g}$ (see e.g [FDSM, G]).

Lemma 2.2. *$\ell_{i,i} = s_i$ and further if $i < j$ then*

$$\begin{aligned}\ell_{i,j} &= (s_i)(s_{i+1}s_i)(s_{i+2}s_{i+1}s_i)\dots(s_{j-1}\dots s_i)(s_j\dots s_i) = (s_i\dots s_j)\ell_{i,j-1} = \ell_{i,j-1}(s_j\cdots s_i)\\ &= (s_j)(s_{j-1}s_j)(s_{j-2}s_{j-1}s_j)\dots(s_{i+1}\dots s_j)(s_i\dots s_j) = (s_j\dots s_i)\ell_{i+1,j} = \ell_{i+1,j}(s_i\dots s_j).\end{aligned}$$

Proof. The first equality is standard (see e.g [D2]). The second equality holds by regrouping using $s_is_j = s_js_i$ whenever $|i-j| \geq 2$, to bring forward certain elements as follows

$$(s_is_{i+1})\ (s_i)(s_{i+2}s_{i+1}s_i)(s_{i+3}s_{i+2}s_{i+1}s_i)\dots(s_j\dots s_i).$$

The third equality follows, since $\ell_{i,j-1} = (s_i)(s_{i+1}s_i)(s_{i+2}s_{i+1}s_i)\dots(s_{j-1}\dots s_i)$. Applying the antiautomorphism $g \to \bar{g}$, which fixes $\ell_{i,j-1}$, gives the third equality. The second line is simillar. $\square$

Corollary 2.3. *If $i < j$ then*

$$\begin{aligned}\ell_{i,j}^2 &= (s_j\dots s_i)(s_i\dots s_j)\ell_{i,j-1}^2 = (s_i\dots s_j)(s_j\dots s_i)\ell_{i+1,j}^2\\ \ell_{i,j}^2 &= \ell_{i,j-1}^2(s_j\dots s_i)(s_i\dots s_j) = \ell_{i+1,j}^2(s_i\dots s_j)(s_j\dots s_i).\end{aligned}$$

Proof. This follows by repeated application of Lemma 2.2. Indeed,

$$\begin{aligned}\ell_{i,j}^2 &= (s_j\dots s_i)\ell_{i+1,j}(s_i\dots s_j)\ell_{i,j-1}\\ &= (s_j\dots s_i)\ell_{i,j}\ell_{i,j-1}\\ &= (s_j\dots s_i)(s_i\dots s_j)\ell_{i,j-1}^2\end{aligned}$$

and

$$\begin{aligned}\ell_{i,j}^2 &= (s_i\dots s_j)\ell_{i,j-1}(s_j\dots s_i)\ell_{i+1,j}\\ &= (s_i\dots s_j)\ell_{i,j}\ell_{i+1,j}\\ &= (s_i\dots s_j)(s_j\dots s_i)\ell_{i+1,j}^2.\end{aligned}$$

Similarly, $\ell_{i,j}^2 = \ell_{i,j-1}^2(s_j\dots s_i)(s_i\dots s_j) = \ell_{i+1,j}^2(s_i\dots s_j)(s_j\dots s_i)$. $\square$

The above allows us to exhibit a new generating set for PBr_{A_n}. In what follows, to obtain a unified statement we adopt the convention that $\ell_{i,j} = 1$ whenever $j < i$. As calibration in the statement below, this means that $A_{i,i+1} = \ell_{i,i}^2$.

Proposition 2.4. *For all $i < j$, $A_{i,j} = \ell_{i,j-2}^{-2} \cdot \ell_{i,j-1}^2 \cdot \ell_{i+1,j-2}^2 \cdot \ell_{i+1,j-1}^{-2}$.*

Proof. Consider $\ell^2_{i,j-2} \cdot A_{i,j} \cdot \ell^2_{i+1,j-1}$. By definition, this equals

$$
\begin{aligned}
&= \ell^2_{i,j-2} \cdot \sigma^2_{i,j} \cdot \ell^2_{i+1,j-1} \\
&= \ell^2_{i,j-2} \cdot (s_{j-1} \dots s_{i+1}) \cdot s_i^2 \cdot (s^{-1}_{i+1} \dots s^{-1}_{j-1}) \cdot \ell^2_{i+1,j-1} && \text{(by definition)} \\
&= \ell^2_{i,j-2} \cdot (s_{j-1} \dots s_{i+1}) \cdot s_i^2 \cdot (s^{-1}_{i+1} \dots s^{-1}_{j-1}) \cdot (s_{j-1} \dots s_{i+1}) \cdot \ell_{i+2,j-1} \cdot \ell_{i+1,j-1} \\
& && \text{(Lemma 2.2)} \\
&= \ell^2_{i,j-2} \cdot (s_{j-1} \dots s_{i+1}) \cdot s_i^2 \cdot \ell_{i+2,j-1} \cdot \ell_{i+1,j-1} \\
&= \ell_{i,j-2} \cdot \ell_{i,j-2} \cdot (s_{j-1} \dots s_i) \cdot s_i \cdot \ell_{i+2,j-1} \cdot (s_{i+1} \dots s_{j-1}) \ell_{i+1,j-2} \\
&= \ell_{i,j-2} \cdot \ell_{i,j-1} \cdot s_i \cdot \ell_{i+1,j-1} \cdot \ell_{i+1,j-2} && \text{(Lemma 2.2)} \\
&= \ell_{i,j-2} \cdot (s_{j-1} \dots s_i) \cdot \ell_{i+1,j-1} \cdot s_i \cdot (s_{i+1} \dots s_{j-1}) \cdot \ell_{i+1,j-2} \cdot \ell_{i+1,j-2} \\
&= \ell_{i,j-1} \cdot \ell_{i+1,j-1} \cdot (s_i \dots s_{j-1}) \cdot \ell_{i+1,j-2} \cdot \ell_{i+1,j-2} \\
&= \ell_{i,j-1} \cdot \ell_{i,j-1} \cdot \ell_{i+1,j-2} \cdot \ell_{i+1,j-2} \\
&= \ell^2_{i,j-1} \cdot \ell^2_{i+1,j-2}.
\end{aligned}
$$

□

Corollary 2.5. *The set* $\{\ell^2_{\mathcal{A}} \mid \mathcal{A} \subseteq A_n, \mathcal{A} \text{ connected } \} \leq \mathrm{Br}_{A_n}$ *generates the pure braid group* PBr_{A_n}.

Proof. Proposition 2.4 shows that each element $A_{i,j}$ of the generating set of the pure braid group in [A, FV] can be written as the product of elements in the set $\{\ell^2_{\mathcal{A}} \mid \mathcal{A} \subseteq A_n, \mathcal{A} \text{ connected}\}$. Since each element $\ell^2_{\mathcal{A}} \in \mathrm{PBr}_{A_n}$, the result follows. □

3. The new relations

In this section, we first show in §3.1 that certain commutator and box relations hold, then in §3.2 we prove that these suffice to give a full presentation of PBr_{A_n}.

3.1. **Box and commutator relations.** As notation, set $A_n :=$ ●●··●●. By ●●··●○ we mean the connected subgraph starting at 1 and ending at $n-1$.

Lemma 3.1. *If $\mathcal{K} \subseteq A_n$ is connected, then the following statements hold*

(1) $(s_n \dots s_1)(s_1 \dots s_n)$ *commutes with* $\ell_{\mathcal{K}}$ *for all* $\mathcal{K} \subseteq$ ●●··●○
(2) $(s_1 \dots s_n)(s_n \dots s_1)$ *commutes with* $\ell_{\mathcal{K}}$ *for all* $\mathcal{K} \subseteq$ ○●··●●

Proof. Say $\mathcal{K}$ starts at vertex i and ends at vertex j, with $i \leq j < n$. Then

$$
\begin{aligned}
\ell_{\mathcal{K}}(s_n \dots s_1)(s_1 \dots s_n) &= \ell_{i,j}(s_n \dots s_1)(s_1 \dots s_n) \\
&= s_n \dots s_{j+2}(\ell_{i,j}\, s_{j+1} \dots s_i) s_{i-1} \dots s_1 s_1 \dots s_n \\
& \qquad \text{(commutativity of braids)} \\
&= s_n \dots s_{j+2}(s_{j+1} \dots s_i\, \ell_{i+1,j+1}) s_{i-1} \dots s_1 s_1 \dots s_n \\
& \qquad \text{(Lemma 2.2, both brackets equal to } \ell_{i,j+1}) \\
&= s_n \dots s_i\, \ell_{i+1,j+1}(s_{i-1} \dots s_1 s_1 \dots s_{i-1}) s_i \dots s_n \\
&= s_n \dots s_i (s_{i-1} \dots s_1 s_1 \dots s_{i-1}) \ell_{i+1,j+1}\, s_i \dots s_n \\
& \qquad \text{(commutativity of braids)} \\
&= s_n \dots s_1 s_1 \dots s_{i-1}(\ell_{i+1,j+1}\, s_i \dots s_{j+1}) s_{j+2} \dots s_n \\
&= s_n \dots s_1 s_1 \dots s_{i-1}(s_i \dots s_{j+1}\, \ell_{i,j}) s_{j+2} \dots s_n \\
& \qquad \text{(Lemma 2.2, both brackets equal to } \ell_{i,j+1}) \\
&= (s_n \dots s_1)(s_1 \dots s_n) \ell_{i,j} \qquad \text{(commutativity of braids)} \\
&= (s_n \dots s_1)(s_1 \dots s_n) \ell_{\mathcal{K}}.
\end{aligned}
$$

The other statement is similar. □

Lemma 3.2. *If $i < j$ then* $\ell^2_{i,j} = (s_j \dots s_i)^{(j-i)+2} = (s_i \dots s_j)^{(j-i)+2}$.

Proof. We will prove the case when $i = 1$ and $j = n$ since the notation for this is clearer. Recall by Lemma 2.2 that $\ell_{1,n} = (s_n \dots s_1)(s_n \dots s_2)(s_n \dots s_3) \dots (s_n s_{n-1}) s_n$, and also that $\ell_{1,n} = (s_1)(s_2s_1)(s_3s_2s_1)\dots(s_{n-1}\dots s_1)(s_n\dots s_1)$. Given this,

$$\begin{aligned}
&\ell_{1,n}^2\\
&= [(s_n\dots s_1)(s_n\dots s_2)(s_n\dots s_3)\dots(s_ns_{n-1})s_n\cdot(s_1)(s_2s_1)\dots(s_{n-1}\dots s_1)(s_n\dots s_1)\\
&= [(s_n\dots s_1)(s_n\dots s_1)(s_n\dots s_3)\dots(s_ns_{n-1})\cdot(s_2s_1)\dots(s_n\dots s_1)(s_n\dots s_1)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(s_1 \text{ and } s_n \text{ commute through})\\
&= (s_n\dots s_1)(s_n\dots s_2s_1)\dots(s_n\dots s_2s_1)(s_n\dots s_2s_1) \qquad\qquad (\text{repeat the above step})\\
&= (s_n\dots s_2s_1)^{n+1}.
\end{aligned}$$

The general case is similar. ∎

The following technical Lemma will be required later.

Lemma 3.3. *For all $i \geq 1$, and $j \geq i+2$, the following statements hold.*

(1) $(s_j\dots s_i^2\dots s_j)(s_{j-1}\dots s_i^2\dots s_{j-1}) = (s_j\dots s_{i+1})^2s_is_{i+1}(s_{i+1}s_i)(s_{i+2}s_{i+1})\dots(s_{j-1}s_{j-2})(s_js_{j-1})$.
(2) $(s_i\dots s_j^2\dots s_i)(s_{i+1}\dots s_j^2\dots s_{i+1}) = (s_i\dots s_{j-1})^2s_js_{j-1}(s_{j-1}s_j)(s_{j-2}s_{j-1})\dots(s_{i+1}s_{i+2})(s_is_{i+1})$.
(3) $(s_{j-1}\dots s_i^2\dots s_{j-1})(s_j\dots s_i^2\dots s_j) = (s_{j-1}s_j)(s_{j-2}s_{j-1})\dots(s_{i+1}s_{i+2})(s_is_{i+1})s_{i+1}s_i(s_{i+1}\dots s_j)^2$.
(4) $(s_{i+1}\dots s_j^2\dots s_{i+1})(s_i\dots s_j^2\dots s_i) = (s_{i+1}s_i)(s_{i+2}s_{i+1})\dots(s_{j-1}s_{j-2})(s_js_{j-1})s_{j-1}s_j(s_{j-1}\dots s_i)^2$.

Proof. For (1), observe that

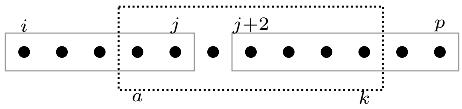
$$\begin{aligned}
(s_j\dots s_i^2\dots s_j)(s_{j-1}\dots s_i^2\dots s_{j-1}) &= (s_j\dots s_{i+1})s_i^2\dots s_{j-1}s_js_{j-1}\dots s_i^2\dots s_{j-1}\\
&= (s_j\dots s_{i+1})s_i^2\dots s_js_{j-1}s_j\dots s_i^2\dots s_{j-1}\\
&= (s_j\dots s_{i+1})s_js_i^2\dots s_{j-2}s_{j-1}s_{j-2}\dots s_i^2\dots(s_js_{j-1}).
\end{aligned}$$

This process is repeated until we achieve the desired expression. The other statements are similar. ∎

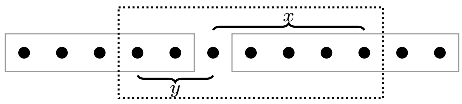
As in the introduction, consider the graph A_n, with connected subgraphs $\mathcal{A}, \mathcal{B}, \mathcal{C}$.

Definition 3.4. *If $\mathcal{A}\cap\mathcal{B} \neq \emptyset$ then define $d(\mathcal{A},\mathcal{B}) = 0$. Else, the distance $d(\mathcal{A},\mathcal{B})$ is defined to be the number of edges between $\mathcal{A}$ and $\mathcal{B}$.*

Notation 3.5. Fix $\mathcal{A},\mathcal{C}$ with $d(\mathcal{A},\mathcal{C}) = 2$ and $\mathcal{B}$ compatible with $(\mathcal{A},\mathcal{C})$ in the sense of the introduction. Equivalently, writing $\mathcal{A} = [i,j]$, $\mathcal{C} = [j+2,p]$ and $\mathcal{B} = [a,k]$, we have



The choice of such $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ translates into the condition $1 \leq i < a \leq j+1 \leq k < p \leq n$. Given $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$, set $x := k-j$ and $y := j+2-a$, which visually are



Then for $b := i+(k-(j+1))$ and $h := a+(p-(j+1))$ define

$$\begin{aligned}
\mathfrak{c}_{a-1,b} &:= (s_{a-1}\dots s_{a+x-2})(s_{a-2}\dots s_{a+x-3})\dots(s_{i+1}\dots s_{b+1})(s_i\dots s_b)\\
\mathfrak{c}_{j+1,x+a-1} &:= (s_{j+1}\dots s_k)(s_j\dots s_{k-1})\dots(s_{a+1}\dots s_{x+a})(s_a\dots s_{x+a-1})\\
\mathfrak{d}_{k+1,h} &:= (s_{k+1}\dots s_{k-y+2})(s_{k+2}\dots s_{k-y+3})\dots(s_{p-1}\dots s_{h-1})(s_p\dots s_h)\\
\mathfrak{d}_{j+1,k-y+1} &:= (s_{j+1}\dots s_a)(s_{j+2}\dots s_{a+1})\dots(s_{k-1}\dots s_{k-y})(s_k\dots s_{k-y+1})
\end{aligned}$$

where for $\mathfrak{c}$ each factor has length x, and for $\mathfrak{d}$ each factor has length y.

Note that $\mathfrak{c}_{\alpha,\beta}$ should be understood as starting at s_α and ending at s_β, where each factor has length x, and the indices decrease by one at each step. The notation $\mathfrak{d}_{\alpha,\beta}$ should be understood similarly, but each factor has length y and each step increases the indices.

The following is a general version of Lemma 3.3, which covered the cases $x = 2$ and $y = 2$ respectively. Recall from §2.2 that $\bar{\mathfrak{c}}_{a-1,b}$ will be the expression for $\mathfrak{c}_{a-1,b}$ read backwards, namely $\bar{\mathfrak{c}}_{a-1,b} = (s_b \dots s_i)(s_{b+1} \dots s_{i+1}) \dots (s_{a+x-3} \dots s_{a-2})(s_{a+x-2} \dots s_{a-1})$ with the obvious variations for $\bar{\mathfrak{d}}$.

Lemma 3.6. *For all $1 \leq i < a \leq j+1 \leq k < p \leq n$, as in Notation 3.5 set $x := k - j$ and $y := j + 2 - a$. Then for $b := i + (k - (j+1))$ and $h := a + (p - (j+1))$, the following hold.*

(1) $\underbrace{(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})}_{x \text{ terms}} = \big(s_k \dots s_a\big)^x \cdot \mathfrak{c}_{a-1,b} \cdot \bar{\mathfrak{c}}_{a-1,b} \cdot \bar{\mathfrak{c}}_{j+1,x+a-1}$

(2) $\underbrace{(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)}_{y \text{ terms}} = \mathfrak{d}_{j+1,k-y+1} \cdot \mathfrak{d}_{k+1,h} \cdot \bar{\mathfrak{d}}_{k+1,h} \cdot \big(s_k \dots s_a\big)^y$

(3) $\underbrace{(s_a \dots s_p^2 \dots s_a) \dots (s_{j+1} \dots s_p^2 \dots s_{j+1})}_{y \text{ terms}} = \big(s_a \dots s_k\big)^y \cdot \mathfrak{d}_{k+1,h} \cdot \bar{\mathfrak{d}}_{k+1,h} \cdot \bar{\mathfrak{d}}_{j+1,k-y+1}$

(4) $\underbrace{(s_{j+1} \dots s_i^2 \dots s_{j+1}) \dots (s_k \dots s_i^2 \dots s_k)}_{x \text{ terms}} = \mathfrak{c}_{j+1,x+a-1} \cdot \mathfrak{c}_{a-1,b} \cdot \bar{\mathfrak{c}}_{a-1,b} \cdot \big(s_a \dots s_k\big)^x.$

Proof. (1) Consider $(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})$, where there are the x products. If $x = 1$ then $k = j+1$ and $b = i$, so

$$\begin{aligned}
\underbrace{(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})}_{x \text{ terms}} &= s_{j+1} \dots s_b^2 \dots s_{j+1} \\
&= (s_{j+1} \dots s_a)(s_{a-1} \dots s_b)(s_b \dots s_{a-1})(s_a \dots s_{j+1}) \\
&= \big(s_{j+1} \dots s_a\big)^1 \cdot \mathfrak{c}_{a-1,b} \cdot \bar{\mathfrak{c}}_{a-1,b} \cdot \bar{\mathfrak{c}}_{j+1,x+a-1}.
\end{aligned}$$

When $x = 2$ then $k = j+2$ and the result follows by Lemma 3.3 since the expression $(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})$ equals

$$\big(s_{j+2} \dots s_a\big)^2 \Big((s_{a-1}s_a)(s_{a-2}s_{a-1}) \dots (s_i s_{i+1})\Big)\Big((s_{i+1}s_i) \dots (s_{a-1}s_{a-2})(s_a s_{a-1})\Big)\Big((s_{a+1}s_a) \dots (s_{j+2}s_{j+1})\Big),$$

since $b = i+1$. Now for general x, repeating the proof of Lemma 3.3, in a similar way the product of x terms $(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})$ equals

$$\big(s_k \dots s_a\big)^x \Big((s_{a-1} \dots s_{a+x-2}) \dots (s_i \dots s_b)\Big)\Big((s_b \dots s_i) \dots (s_{a+x-2} \dots s_{a-1})\Big)\Big((s_{a+x-1} \dots s_a) \dots (s_k \dots s_{j+1})\Big).$$

(2) Consider $(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)$, where there are y products. If $y = 1$, then $a = j+1$ and $h = p$, so

$$\begin{aligned}
\underbrace{(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)}_{y \text{ terms}} &= s_{j+1} \dots s_p^2 \dots s_{j+1} \\
&= (s_{j+1} \dots s_k)(s_{k+1} \dots s_p)(s_p \dots s_{k+1})(s_k \dots s_{j+1}) \\
&= \mathfrak{d}_{j+1,k-y+1} \cdot \mathfrak{d}_{k+1,h} \cdot \bar{\mathfrak{d}}_{k+1,h} \cdot \big(s_k \dots s_{j+1}\big)^1.
\end{aligned}$$

When $y = 2$ then $a = j$ and the result follows since by Lemma 3.3 the statement $(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)$ equals

$$\Big((s_{j+1}s_j)(s_{j+2}s_{j+1}) \dots (s_k s_{k-1})\Big)\Big((s_{k+1}s_k) \dots (s_p s_{p-1})\Big)\Big((s_{p-1}s_p) \dots (s_{k+1}s_{k+2})(s_k s_{k+1})\Big)\big(s_k \dots s_j\big)^2,$$

since $h = p-1$. Now for general y, repeating the proof of Lemma 3.3, in a similar way the product of y terms $(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_i^2 \dots s_a)$ equals

$$\Big((s_{j+1} \dots s_a) \dots (s_k \dots s_{k-y+1})\Big)\Big(s_{k+1} \dots s_{k-y+2}) \dots (s_p \dots s_h)\Big)\Big((s_h \dots s_p) \dots (s_{k-y+2} \dots s_{k+1})\Big)\big(s_k \dots s_a\big)^y.$$

The statements (3) and (4) follow by applying the antiautomorphism $g \mapsto \bar{g}$, that is to say by reading (1) and (2) backwards. $\square$

Leading into the next results, observe that $a + x - 1 = k - y + 1$, as both equal $k - j + a - 1$.

Corollary 3.7. *With notation as in Notation 3.5 $\mathfrak{c}_{j+1,a+x-1} = \mathfrak{d}_{j+1,k-y+1}$ and furthermore $\bar{\mathfrak{c}}_{j+1,a+x-1} = \bar{\mathfrak{d}}_{j+1,k-y+1}$.*

Proof. When $x = y = 1$, or equivalently when $a = k = j+1$, then $\mathfrak{c}_{j+1,a+x-1} = \mathfrak{c}_{j+1,j+1} = s_{j+1} = \mathfrak{d}_{j+1,j+1} = \mathfrak{d}_{j+1,k-y+1}$.

When $x, y \geq 2$, by pulling the first element in each factor to the left as follows

$$\mathfrak{c}_{j+1,a+x-1} = (s_{j+1}\ s_{j+2} \dots s_k)(s_j s_{j+1} \dots s_{k-1}) \dots (s_a \dots s_{a+x-1}),$$

we see that $\mathfrak{c}_{j+1,a+x-1} = (s_{j+1}s_j \dots s_a)(s_{j+2} \dots s_k)(s_{j+1} \dots s_{k-1}) \dots (s_{a-1} \dots s_{a+x-1})$. Repeating, pulling again the following to the left

$$\mathfrak{c}_{j+1,a+x-1} = (s_{j+1}s_j \dots s_a)(s_{j+2}\ \dots s_k)(s_{j+1} \dots s_{k-1}) \dots (s_{a-1} \dots s_{a+x-1}),$$

we see that $\mathfrak{c}_{j+1,a+x-1} = (s_{j+1}s_j \dots s_a)(s_{j+2}s_{j+1} \dots s_{a-1})(s_{j+3} \dots s_k)(s_{j+2} \dots s_{k-1}) \dots (s_{a-2} \dots s_{a+x-1})$. Repeating, $\mathfrak{c}_{j+1,a+x-1}$ can be written as a product, each factor of length y. By definition, this is $\mathfrak{d}_{j+1,a+x-1} = \mathfrak{d}_{j+1,k-y+1}$. The final statement follows by applying the antiautomorphism $g \mapsto \bar{g}$. $\square$

The following result proves various relations hold between the squares of longest elements over connected subgraphs. The relations are either 'far away commutativity' when there exists at least two edges between the subgraphs, 'inclusion commutativity' when one of the subgraphs is contained in the other, or length five palindromic relations which we refer to as the box relations (as explained in Remark 3.12).

Proposition 3.8. *Let $\mathcal{J}, \mathcal{K} \subseteq A_n$ be connected. Then the following hold*

1. $\ell^2_{\mathcal{J}} \ell^2_{\mathcal{K}} = \ell^2_{\mathcal{K}} \ell^2_{\mathcal{J}}$ *if* $d(\mathcal{J}, \mathcal{K}) \geq 2$.
2. $\ell^2_{\mathcal{J}} \ell^2_{\mathcal{K}} = \ell^2_{\mathcal{K}} \ell^2_{\mathcal{J}}$ *if* $\mathcal{J} \subseteq \mathcal{K}$ *or* $\mathcal{K} \subseteq \mathcal{J}$.
3. *There is an equality*

$$\ell^{-2}_{i,k}\, \ell^2_{i,j}\, \ell^2_{a,k}\, \ell^2_{j+2,p}\, \ell^{-2}_{a,p} = \ell^{-2}_{a,p}\, \ell^2_{j+2,p}\, \ell^2_{a,k}\, \ell^2_{i,j}\, \ell^{-2}_{i,k}$$

whenever $1 \leq i < a \leq j+1 \leq k < p \leq n$.

Proof. (1) Since $\ell^2_{\mathcal{J}}$ consists of only s_j with $j \in \mathcal{J}$, and $\ell^2_{\mathcal{K}}$ consists of only s_k with $k \in \mathcal{K}$, the result follows from braid relations, since by assumption s_j commutes with s_k whenever $j \in \mathcal{J}$ and $k \in \mathcal{K}$.
(2) Without loss of generality we can consider the case $\mathcal{J} \subsetneq \mathcal{K}$. The statement follows since $\ell^2_{\mathcal{K}}$ is central in the parabolic subgroup of Br_{A_n} generated by $\mathcal{K}$ (see e.g [G, Theorem 7]).
(3) Recall from Lemma 3.2 that

$$\ell^2_{a,k} = (s_k \dots s_a)^{(k-a)+2} = (s_a \dots s_k)^{(k-a)+2}. \tag{3.A}$$

Using Corollary 2.3 repeatedly, we can factor $\ell^{-2}_{i,k}$ and $\ell^{-2}_{a,p}$ to obtain

$$\ell^{-2}_{i,k} = \Big(\ell^2_{i,j}\, \underbrace{(s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})}_{k-j \text{ terms}}\Big)^{-1} \tag{3.B}$$

$$\ell^{-2}_{a,p} = \Big(\underbrace{(s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)}_{j+2-a \text{ terms}}\ell^2_{j+2,p}\Big)^{-1} \tag{3.C}$$

Substituting in (3.B) and (3.C), then $\ell^{-2}_{i,k}\ell^2_{i,j}\ell^2_{a,k}\ell^2_{j+2,p}\ell^{-2}_{a,p}$ equals

$$\big((s_k \dots s_i^2 \dots s_k) \dots (s_{j+1} \dots s_i^2 \dots s_{j+1})\big)^{-1} \ell^2_{a,k} \big((s_{j+1} \dots s_p^2 \dots s_{j+1}) \dots (s_a \dots s_p^2 \dots s_a)\big)^{-1}.$$

By applying Lemma 3.6 to the outer terms, and (3.A) to the middle term, the above displayed equation equals

$$\big((s_k \dots s_a)^x \cdot \mathfrak{c}_{a-1,b} \cdot \bar{\mathfrak{c}}_{a-1,b} \cdot \bar{\mathfrak{c}}_{j+1,a+x-1}\big)^{-1} \cdot (s_k \dots s_a)^{x+y} \cdot \big(\mathfrak{d}_{j+1,k-y+1} \cdot \mathfrak{d}_{k+1,h} \cdot \bar{\mathfrak{d}}_{k+1,h} \cdot (s_k \dots s_a)^y\big)^{-1},$$

where $b := i + (k - (j+1))$, $h := a + (p - (j+1))$, with x and y as in Notation 3.5.

By the obvious cancellations, it follows that

$$\begin{aligned}
&\ell_{i,k}^{-2}\,\ell_{i,j}^{2}\,\ell_{a,k}^{2}\,\ell_{j+2,p}^{2}\,\ell_{a,p}^{-2}\\
&= \big(\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\cdot\bar{\mathfrak{c}}_{j+1,x+a-1}\big)^{-1}\cdot\big(\mathfrak{d}_{j+1,k-y+1}\cdot\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\big)^{-1}\\
&= \big(\bar{\mathfrak{c}}_{j+1,x+a-1}\big)^{-1}\cdot\big(\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\big)^{-1}\cdot\big(\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\big)^{-1}\cdot\big(\mathfrak{d}_{j+1,k-y+1}\big)^{-1}\\
&= \big(\bar{\mathfrak{c}}_{j+1,x+a-1}\big)^{-1}\cdot\big(\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\big)^{-1}\cdot\big(\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\big)^{-1}\cdot\big(\mathfrak{d}_{j+1,k-y+1}\big)^{-1}\\
&\hspace{20em}\text{(middle terms commute)}\\
&= \big(\bar{\mathfrak{d}}_{j+1,k-y+1}\big)^{-1}\cdot\big(\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\big)^{-1}\cdot\big(\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\big)^{-1}\cdot\big(\mathfrak{c}_{j+1,x+a-1}\big)^{-1}\\
&\hspace{20em}\text{(Corallary 3.7)}\\
&= \big(\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\cdot\bar{\mathfrak{d}}_{j+1,k-y+1}\big)^{-1}\cdot\big(\mathfrak{c}_{j+1,x+a-1}\cdot\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\big)^{-1}
\end{aligned}$$

By backward substitution, $\ell_{i,k}^{-2}\,\ell_{i,j}^{2}\,\ell_{a,k}^{2}\,\ell_{j+2,p}^{2}\,\ell_{a,p}^{-2}$ is then equal to

$$\big((s_a\dots s_k)^y\cdot\mathfrak{d}_{k+1,h}\cdot\bar{\mathfrak{d}}_{k+1,h}\cdot\bar{\mathfrak{d}}_{j+1,k-y+1}\big)^{-1}\cdot(s_a\dots s_k)^{x+y}\cdot\big(\mathfrak{c}_{j+1,a+x-1}\cdot\mathfrak{c}_{a-1,b}\cdot\bar{\mathfrak{c}}_{a-1,b}\cdot(s_a\dots s_k)^x\big)^{-1} \tag{3.D}$$

By Lemma 3.6, (3.D) is equal to

$$\big((s_a\dots s_p^2\dots s_a)\dots(s_{j+1}\dots s_p^2\dots s_{j+1})\big)^{-1}\ell_{a,k}^2\big((s_{j+1}\dots s_i^2\dots s_{j+1})\dots(s_k\dots s_i^2\dots s_k)\big)^{-1},$$

which by further backward substitution equals $\ell_{a,p}^{-2}\,\ell_{j+2,p}^{2}\,\ell_{a,k}^{2}\,\ell_{i,j}^{2}\,\ell_{i,k}^{-2}$, as required. □

3.2. **Proof of all relations.** Set $G := \langle A_{i,j} \mid R_1\rangle$ and $H := \langle \mathbb{I}_{i,j} \mid R_2\rangle$, where R_1 are the relations in §2.1 and R_2 are the commutator and box relations in Proposition 3.8 (substituting $\mathbb{I}_{i,j} = \ell_{i,j}^2$). The following is our main technical lemma.

Lemma 3.9. *There is a well defined group homomorphism* $\phi\colon G \to H$ *defined by*

$$A_{i,j} \mapsto \mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1},$$

where as in Proposition 2.4 we adopt the convention that $\mathbb{I}_{i,j} = 1$ *if* $j < i$.

Proof. We show that ϕ is well defined by tracking the relations R_1 to H.

(1) $A_{i,j}A_{r,s} = A_{r,s}A_{i,j}$ if $r < s < i < j$. The left hand side is sent to

$$(\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1})(\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}_{r+1,s-1}^{-1}) \tag{3.E}$$

and the right hand side is sent to

$$(\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}_{r+1,s-1}^{-1})(\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1}). \tag{3.F}$$

Each term in the left bracket of (3.E) commutes with each term in the right bracket, by far-away commutativity. Hence (3.E) equals (3.F).

(2) $A_{i,j}A_{r,s} = A_{r,s}A_{i,j}$ if $i < r < s < j$. The left hand side is still sent to (3.E), and the right hand side to (3.F). Now each term in the left bracket commutes with each term in the right bracket by inclusion commutativity, so (3.E) equals (3.F).

(3) $A_{r,s}^{-1}A_{i,j}A_{r,s} = A_{r,j}A_{i,j}A_{r,j}^{-1}$ if $r < s = i < j$. The left hand side is sent to

$$(\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r+1,s-2}^{-1}\,\mathbb{I}_{r,s-1}^{-1}\,\mathbb{I}_{r,s-2})(\mathbb{I}_{s,j-2}^{-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}_{s+1,j-1}^{-1})(\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}_{r+1,s-1}^{-1}).$$

By inclusion commutativity $[\mathbb{I}_{r+1,s-2}, \mathbb{I}_{r,s-1}] = 1$, and by far-away commutativity both $\mathbb{I}_{r+1,s-2}$ and $\mathbb{I}_{r,s-2}$ commute with each term in the middle bracket, so the above simplifies to

$$(\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r,s-1}^{-1})(\mathbb{I}_{s,j-2}^{-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}_{s+1,j-1}^{-1})(\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-1}^{-1}). \tag{3.G}$$

In a similar way, using $[\mathbb{I}_{r+1,j-2}, \mathbb{I}_{r,j-1}] = 1$ and $\mathbb{I}_{r,j-1}, \mathbb{I}_{r+1,j-1}$ commute with each term in the appropriate middle bracket, the right hand side of the relation gets sent to

$$(\mathbb{I}_{r,j-2}^{-1}\,\mathbb{I}_{r+1,j-2})(\mathbb{I}_{s,j-2}^{-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}_{s+1,j-1}^{-1})(\mathbb{I}_{r+1,j-2}^{-1}\,\mathbb{I}_{r,j-2}). \tag{3.H}$$

By assumption $r < s = i < j$, so inclusion commutativity gives $[\mathbb{I}_{r,j-2}, \mathbb{I}^{-1}_{r+1,s-1}] = 1$. Thus left multiplying both (3.G) and (3.H) by $\mathbb{I}_{r,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}$, and using other inclusion commutativity, it suffices to prove that

$$\begin{aligned}&(\mathbb{I}_{r,j-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}_{s,j-1})\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r+1,s-1}\\ &= (\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}_{s,j-1})\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}.\end{aligned}$$

By applying the box relations to each, this is equivalent to asking that

$$\begin{aligned}&(\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,j-2})\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r+1,s-1}\\ &= (\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}_{r+1,j-2})\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}.\end{aligned}$$

Using $r < s = i < j$ and commutativity, these expressions are indeed equal.

(4) $A^{-1}_{r,s}A_{i,j}A_{r,s} = (A_{i,j}A_{s,j})A_{i,j}(A_{i,j}A_{s,j})^{-1}$ if $r = i < s < j$. The LHS is sent to

$$(\mathbb{I}_{r+1,s-1}\mathbb{I}^{-1}_{r+1,s-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2})(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1})(\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}^{-1}_{r+1,s-1}) \tag{3.I}$$

while the right hand side is sent to

$$\begin{aligned}(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}^{-1}_{s+1,j-1})\,(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,j-1}\,\mathbb{I}_{r+1,j-2}\\ \mathbb{I}^{-1}_{r+1,j-1})(\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s+1,j-2}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2})(\mathbb{I}_{r+1,j-1}\,\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}^{-1}_{r,j-1}\,\mathbb{I}_{r,j-2}).\end{aligned} \tag{3.J}$$

Since $r = i < s < j$, we have that $\mathbb{I}_{r,j-1}$ commutes all through and so $\mathbb{I}_{r,j-1}$ and $\mathbb{I}^{-1}_{r,j-1}$ in (3.J) cancel. Further, left multiply both (3.I) and (3.J) by $\mathbb{I}^{-1}_{r,j-1}$, we can remove all $\mathbb{I}_{r,j-1}$ from (3.I) and (3.J).

By inclusion commutativity $\mathbb{I}_{r+1,s-2}$ commutes with $\mathbb{I}_{r,s-1}, \mathbb{I}^{-1}_{r,s-2}$ and each term in the middle bracket of (3.I) and so $\mathbb{I}_{r+1,s-2}$ and $\mathbb{I}^{-1}_{r+1,s-2}$ cancel in (3.I) . Moreover $\mathbb{I}_{r,j-2}$ commutes with each term in the left bracket of (3.I), so $\mathbb{I}^{-1}_{r,j-2}$ can be brought to the front of (3.I). Similarly $\mathbb{I}_{r+1,j-1}$ commutes with each term in the second last bracket of (3.J) so cancels with the $\mathbb{I}^{-1}_{r+1,j-1}$. Similarly $\mathbb{I}^{-1}_{s+1,j-2}$ and $\mathbb{I}_{s+1,j-2}$ cancel in (3.J). Thus left multiplying (3.I) and (3.J) by $\mathbb{I}_{r,j-2}$, it suffices to show that

$$\begin{aligned}&(\mathbb{I}_{r+1,s-1}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2})(\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1})\,(\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r+1,s-1})\\ &= (\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1})\,(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2})(\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2})(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}).\end{aligned}$$

By inclusion commutativity, $\mathbb{I}^{-1}_{s,j-2}$ commutes with $\mathbb{I}_{r+1,j-2}$, $\mathbb{I}^{-1}_{r+1,j-1}$, $\mathbb{I}^{-1}_{r+1,j-2}, \mathbb{I}_{r,j-2}$ and further $[\mathbb{I}_{r,s-1}, \mathbb{I}^{-1}_{r,s-2}] = 1$. By conjugating the above by $\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{s,j-2}$, the claim becomes that

$$\begin{aligned}&(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{s,j-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{r,s-1})\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1}(\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{r+1,j-2})\\ &= \mathbb{I}^{-1}_{r+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{r,j-2}\end{aligned}$$

By use of the box relations, the top line becomes

$$(\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{s,j-2}\,\mathbb{I}^{-1}_{r+1,j-2})\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1}(\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1})$$

which by inclusion commutativity and obvious cancellations simplifies to

$$\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{r+1,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}.$$

By right multiplying by $\mathbb{I}_{r,s-2}$ and left multiplying by $\mathbb{I}_{r,s-1}$, the claim becomes that

$$\begin{aligned}&\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{r+1,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r,s-1}\\ &= \mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{s+1,j-1}(\mathbb{I}^{-1}_{r+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{r,j-2})\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{r,j-2}.\end{aligned}$$

By the box relations, the bottom line equals

$$\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{s+1,j-1}(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,s-2}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{r+1,j-1})\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{r,j-2}$$

which simplifies to

$$\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{s+1,j-1}(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{r+1,j-1})\mathbb{I}_{r,j-2}.$$

One final application of the box relations, and commutativity, proves the claim.

(5) $A^{-1}_{r,s}A_{i,j}A_{r,s} = (A_{r,j}A_{s,j}A^{-1}_{r,j}A^{-1}_{s,j})A_{i,j}(A_{r,j}A_{s,j}A^{-1}_{r,j}A^{-1}_{s,j})^{-1}$ if $r < i < s < j$.

The factor $(A_{r,j}A_{s,j}A_{r,j}^{-1}A_{s,j}^{-1})$ gets sent to

$$\begin{aligned}(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r,j-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,j-1})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{s+1,j-2}\,\mathbb{I}^{-1}_{s+1,j-1})(\mathbb{I}_{r+1,j-1}\\ \mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}^{-1}_{r,j-1}\,\mathbb{I}_{r,j-2})(\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s+1,j-2}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2}),\end{aligned}$$

which by inclusion commutativity and cancellation equals

$$(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1})(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2})(\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2}). \tag{3.K}$$

Now the left hand side of the relation is sent to

$$(\mathbb{I}_{r+1,s-1}\,\mathbb{I}^{-1}_{r+1,s-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2})(\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1})(\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}^{-1}_{r+1,s-1})$$

whilst using (3.K) the right hand side of the relation is sent to

$$\begin{aligned}(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1})(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2})(\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2})\\ (\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1})(\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1})(\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2})\\ (\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2})(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}).\end{aligned}$$

Multiplying both to the left and right of the last two equations by $\mathbb{I}^{-1}_{r+1,s-1}$ and using inclusion commutativity to commute $\mathbb{I}_{s,j-2}$ through the middle bracket of the right hand side of the relation , it suffices to prove that

$$\begin{aligned}&\mathbb{I}^{-1}_{r+1,s-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2}\,\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\\ &= \mathbb{I}^{-1}_{r+1,s-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,s-2}(\mathbb{I}_{i,j-1}\,\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1})\mathbb{I}^{-1}_{r,s-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-2}\end{aligned} \tag{3.L}$$

equals

$$\begin{aligned}\mathbb{I}^{-1}_{r,j-2}(\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}_{s,j-1})\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\\ \mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2}(\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s+1,j-1}\\ \mathbb{I}_{s,j-2}\,\mathbb{I}_{r+1,s-1}\mathbb{I}^{-1}_{r+1,j-2})\mathbb{I}_{r,j-2}.\end{aligned}$$

By the box relations, the last equation equals

$$\begin{aligned}\mathbb{I}^{-1}_{r,j-2}(\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}_{r+1,j-2})\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\\ \mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,j-2}(\mathbb{I}^{-1}_{r+1,j-2}\,\mathbb{I}_{r+1,s-1}\\ \mathbb{I}_{s,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1})\mathbb{I}_{r,j-2},\end{aligned}$$

which by obvious cancellations simplifies to

$$\begin{aligned}\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}_{r,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\\ \mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,j-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{s,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{r,j-2}.\end{aligned}$$

Inserting $\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r,s-1}=1$ twice and using $[\mathbb{I}_{r,s-1},\mathbb{I}_{r,j-2}]=1$, this equals

$$\begin{aligned}\mathbb{I}^{-1}_{r,j-2}(\mathbb{I}_{s,j-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}_{r,j-2})\mathbb{I}_{r,s-1}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\\ \mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,s-1}(\mathbb{I}^{-1}_{r,j-2}\mathbb{I}_{r,s-1}\\ \mathbb{I}_{s,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{s,j-1})\mathbb{I}_{r,j-2},\end{aligned}$$

which again by the box relations becomes

$$\begin{aligned}\mathbb{I}^{-1}_{r,j-2}(\mathbb{I}_{r,j-2}\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}^{-1}_{s+1,j-1}\cdot\mathbb{I}_{s,j-1})\mathbb{I}_{r,s-1}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\\ \mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}^{-1}_{s+1,j-1}\,\mathbb{I}^{-1}_{r,s-1}(\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s+1,j-1}\\ \mathbb{I}_{s,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r,j-2})\mathbb{I}_{r,j-2}.\end{aligned}$$

By commutativity, this simplifies to

$$\begin{aligned}\mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}^{-1}_{s,j-2}\,\mathbb{I}_{s,j-1}\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}^{-1}_{i,j-2}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}\\ \mathbb{I}^{-1}_{r,s-1}\,\mathbb{I}^{-1}_{s,j-1}\,\mathbb{I}_{s,j-2}\,\mathbb{I}_{r,s-1}.\end{aligned} \tag{3.M}$$

Now, conjugate both (3.L) and (3.M) by $\mathbb{I}_{r,s-1}$, and use that $[\mathbb{I}_{r+1,s-2},\mathbb{I}_{r,s-1}]=1$. Further, left multiply both (3.L) and (3.M) by $\mathbb{I}_{s,j-2}$, and use the fact that $\mathbb{I}_{s,j-2}$ commutes all through (3.L). Then, right multiply both (3.L) and (3.M) by $\mathbb{I}^{-1}_{s,j-2}$. Furthermore, conjugate both (3.L) and (3.M) by $\mathbb{I}_{s,j-1}^{-1}$ and using commutativity, it suffices to show that

$$\mathbb{I}_{r+1,s-2}^{-1}\,\mathbb{I}_{r,s-2}(\mathbb{I}_{i,j-1}\,\mathbb{I}_{s,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r+1,s-2}$$

equals

$$\mathbb{I}_{r,s-1}\,\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{s,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r,s-1}^{-1}. \tag{3.N}$$

Inserting $\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r,j-2}\mathbb{I}_{r,j-2}^{-1}\,\mathbb{I}_{r,s-2}=1$, the top line of the claim becomes

$$\mathbb{I}_{r+1,s-2}^{-1}\,\mathbb{I}_{r,s-2}(\mathbb{I}_{i,j-1}\,\mathbb{I}_{s,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r,j-2})(\mathbb{I}_{r,j-2}^{-1}\,\mathbb{I}_{r,s-2}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{r+1,s-2}$$

which by the box relations equals

$$\mathbb{I}_{r+1,s-2}^{-1}\mathbb{I}_{r,s-2}(\mathbb{I}_{r,j-2}\,\mathbb{I}_{r,s-2}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{s,j-1}^{-1}\,\mathbb{I}_{i,j-1})(\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r,s-2}\,\mathbb{I}_{r,j-2}^{-1})\mathbb{I}_{r,s-2}^{-1}\mathbb{I}_{r+1,s-2},$$

which by inclusion commutativity and cancellations simplifies to

$$\mathbb{I}_{r+1,s-2}^{-1}\,\mathbb{I}_{r,j-2}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r,j-2}^{-1}\,\mathbb{I}_{r+1,s-2}. \tag{3.O}$$

Inserting $\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{s+1,j-1}^{-1}=1$ in (3.O), using commutativity and conjugating (3.O) and (3.N) by $\mathbb{I}_{r,s-1}^{-1}$, the claim becomes

$$\begin{aligned}&\mathbb{I}_{r+1,s-2}^{-1}(\mathbb{I}_{r,j-2}\,\mathbb{I}_{r,s-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{s+1,j-1}^{-1}\,\mathbb{I}_{i,j-1})(\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{r,j-2}^{-1})\mathbb{I}_{r+1,s-2}\\ &=\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{s,j-1}^{-1}\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1}\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}.\end{aligned}$$

Using box relations the top line of the claim equals

$$\mathbb{I}_{r+1,s-2}^{-1}(\mathbb{I}_{i,j-1}\,\mathbb{I}_{s+1,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{r,s-1}^{-1}\,\mathbb{I}_{r,j-2})(\mathbb{I}_{r,j-2}^{-1}\,\mathbb{I}_{r,s-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\mathbb{I}_{r+1,s-2},$$

which simplifies to

$$\mathbb{I}_{r+1,s-2}^{-1}(\mathbb{I}_{i,j-1}\,\mathbb{I}_{s+1,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\mathbb{I}_{r+1,s-2},$$

which after adding $\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r+1,j-2}^{-1}\,\mathbb{I}_{r+1,s-1}=1$ becomes

$$\mathbb{I}_{r+1,s-2}^{-1}(\mathbb{I}_{i,j-1}\,\mathbb{I}_{s+1,j-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{r+1,j-2})(\mathbb{I}_{r+1,j-2}^{-1}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\mathbb{I}_{r+1,s-2}.$$

Thus by the box relations this equals

$$\mathbb{I}_{r+1,s-2}^{-1}(\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{s+1,j-1}^{-1}\,\mathbb{I}_{i,j-1})(\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r+1,j-2}^{-1})\mathbb{I}_{r+1,s-2},$$

which by inclusion commutativity becomes

$$\mathbb{I}_{r+1,s-2}^{-1}\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r+1,s-1}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-1}^{-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}_{r+1,j-2}^{-1}\mathbb{I}_{r+1,s-2}. \tag{3.P}$$

Now conjugate the bottom line of the claim and (3.P) by $\mathbb{I}_{r+1,s-1}$ and use commutativity. We further insert $\mathbb{I}_{s,j-1}\,\mathbb{I}_{s,j-1}^{-2}=1$ in (3.P) and use commutativity so that the claim becomes

$$\begin{aligned}&(\mathbb{I}_{r+1,j-2}\,\mathbb{I}_{r+1,s-2}^{-1}\,\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{s,j-1}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-1}^{-1})\,(\mathbb{I}_{s,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r+1,s-2}\,\mathbb{I}_{r+1,j-2}^{-1})\\ &=\mathbb{I}_{s,j-1}^{-1}\mathbb{I}_{i,j-2}^{-1}\,\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{i+1,j-1}^{-1}\mathbb{I}_{s,j-1}.\end{aligned}$$

By use of the box relations on the the top line of the claim, commutativity, and obvious cancellations, the claim holds. □

As a slight abuse of notation, set $\mathcal{A} := \ell^2_{\mathcal{A}}$. The following is our main result.

Theorem 3.10. *The pure braid group* PBr_{A_n} *has a presentation with generators given by connected subgraphs* $\mathcal{A} \subseteq A_n = \bullet\bullet\cdot\cdot\bullet\bullet$, *subject to the relations*

(1) $\mathcal{A}\cdot\mathcal{B}=\mathcal{B}\cdot\mathcal{A}$ *if* $d(\mathcal{A},\mathcal{B})\geq 2$, *or* $\mathcal{A}\subseteq\mathcal{B}$, *or* $\mathcal{B}\subseteq\mathcal{A}$.

(2) *For all* $\mathcal{A}$ *and all* $\mathcal{C}$ *such that* $d(\mathcal{A},\mathcal{C})=2$, *then*
$$(\mathcal{A}\cup\mathcal{B})^{-1}\cdot(\mathcal{A}\cdot\mathcal{B}\cdot\mathcal{C})\cdot(\mathcal{B}\cup\mathcal{C})^{-1}=(\mathcal{C}\cup\mathcal{B})^{-1}\cdot(\mathcal{C}\cdot\mathcal{B}\cdot\mathcal{A})\cdot(\mathcal{B}\cup\mathcal{A})^{-1}$$
for all $\mathcal{B}$ *compatible with* $(\mathcal{A},\mathcal{C})$.

Proof. Consider the homomorphism $\phi \colon G \to H$ defined in Lemma 3.9. By Proposition 3.8, there is also a homomomorphism $H \twoheadrightarrow \mathrm{PBr}_{A_n}$ sending $\mathbb{I}_{i,j} \mapsto \ell^2_{i,j}$, which is surjective by Corollary 2.5. By Proposition 2.4, by chasing the generators A_{ij} in both directions, the following diagram commutes

$$\begin{tikzcd} & G \arrow[dl, "\phi"'] \arrow[d, "\cong"] \\ H \arrow[r, two heads] & \mathrm{PBr}_{A_n} \end{tikzcd}$$

$_{i+1,j-1}\,\mathbb{I}_{s,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r,s-2}\,\mathbb{I}$

which by inclusion commutativity and cancellations simplifies to

$\mathbb{I}_{r+1,s-2}.$

$\phi(A_{i,j+1}) = \mathbb{I}^{-1}_{i,j}\,\mathbb{I}_{i,j}\,\mathbb{I}_{i+1,j+1}\,\mathbb{I}^{-1}_{i+1,j}.$ $= 1$ in (3.O), using commutativity and conjugating (3.O) and

$\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r,s-1}\,\mathbb{I}^{-1}$

$\mathbb{I}_{s,j-1}\,\mathbb{I}_{r+1,s-1}.$

Using box relations the top line of the claim equals

$\mathbb{I}_{r,s-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}^{-1}$

12 23 34 45 56 67 78 89
13 24 35 $\mathbb{I}_{i+1,j-2}$ 46 57 $\mathbb{I}_{s+1,j-1}$ 68 79 $\mathbb{I}^{-1}$
14 25 36 47 58 69
15 26 37 $\mathbb{I}_{r+1,s-1}$ 48 59 $= 1$ becomes
16 27 38 49
$_{r+1,s-1}\,\mathbb{I}_{r+1,j-2}$ 17 28 39
18 29

Thus by the box relations this equals

$_{s+1,j-1}\,\mathbb{I}_{i,j-1})(\mathbb{I}^{-1}_{i+1,j-1}\,\mathbb{I}_{s+1,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}_{r+1,s-1}\,\mathbb{I}$

which by inclusion commutativity becomes

$\mathbb{I}^{-1}_{r+1,s-2}\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,s-1}\,\mathbb{I}^{-1}$

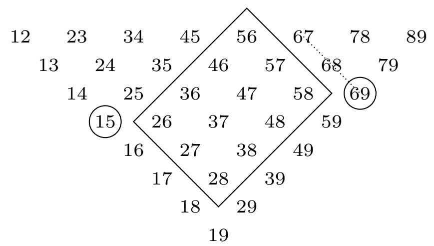
Now conjugate the bottom line of the claim and (3.P) by $\mathbb{I}_{r+1,s-1}$ and use commutativity. We further insert $\mathbb{I}_{s,j-1}\,\mathbb{I}^{-2} \quad = 1$ in (3.P) and use commutativity so that the claim

$= z^{-1}_{xk}(z_{j+1k}z_{xy}z_{ij})z^{-1}_{iy}$

$(\mathbb{I}_{r+1,j-2}\,\mathbb{I}^{-1}_{r+1,s-2}\,\mathbb{I}^{-1}$

$\mathbb{I}_{i,j-1}\,\mathbb{I}_{i+1,j-2}\,\mathbb{I}^{-1}$

By use of the box relations on the the top line of the claim, commutativity, and obvious



This justifies calling the length five relations the box relations, since each relation is characterized by a choice of an element in the box.

Example 3.13. PBr_{A_4} has as generators

$$\begin{array}{ccccccc} 12 & & 23 & & 34 & & 45 \\ & 13 & & 24 & & 35 & \\ & & 14 & & 25 & & \\ & & & 15 & & & \end{array} \quad = \quad \begin{array}{ccccccc} a & & b & & c & & d \\ & e & & f & & g & \\ & & h & & i & & \\ & & & j & & & \end{array}$$

As a slight abuse of notation in the sense of the introduction $\ell_{ij}^2 = [i, j]$. There are 29 commutator relations, together with the 6 box relations

$$\begin{aligned} e^{-1}abcf^{-1} &= f^{-1}cbae^{-1} \\ f^{-1}bcdg^{-1} &= g^{-1}dcbf^{-1} \\ e^{-1}abgi^{-1} &= i^{-1}gbae^{-1} \\ h^{-1}afgi^{-1} &= i^{-1}gfah^{-1} \\ h^{-1}ecdg^{-1} &= g^{-1}dceh^{-1} \\ h^{-1}efdi^{-1} &= i^{-1}dfeh^{-1}. \end{aligned}$$

3.3. **Other Coxeter Types.** This section explains that pure braid groups of other Coxeter arrangements are not in general generated by the analogue of ℓ_{ij}^2. Consider the Dynkin diagram

$$I_n \quad := \quad \circ \overset{\text{n}}{\text{———}} \circ$$

with associated braid group

$$B_{I_n} := \left\langle s_1, s_2 \;\middle|\; \underbrace{s_1 s_2 \dots}_{n} = \underbrace{s_2 s_1 \dots}_{n} \right\rangle$$

and Weyl group

$$W_{I_n} := \left\langle s_1, s_2 \;\middle|\; \begin{array}{l} s_1^2 = s_2^2 = 1 \\ \underbrace{s_1 s_2 \dots}_{n} = \underbrace{s_2 s_1 \dots}_{n} \end{array} \right\rangle \cong D_{2n},$$

where D_{2n} is the dihedral group of order $2n$. The pure braid group PBr_{I_n} associated to the corresponding finite Coxeter group is still the kernel of the natural map $\phi\colon B_{I_n} \twoheadrightarrow W_{I_n}$, and is isomorphic to $\pi_1(\mathbb{C}^n \backslash \bigcup_{i=1}^n (H_i)_{\mathbb{C}})$ where $\mathcal{H} = \{H_i\}_{i=1}^n$ is the corresponding reflection arrangement.

Now, in general consider $\mathrm{PBr}_{\mathcal{H}} := \pi_1(\mathbb{C}^n \backslash \bigcup H_{\mathbb{C}})$ for any Coxeter hyperplane arrangement $\mathcal{H}$, where $\bigcup H_{\mathbb{C}}$ is the union of complexified hyperplanes. By [BMR, Proposition 2.2(2)] the abelianization of $\mathrm{PBr}_{\mathcal{H}}$ is the free abelian group over a set of hyperplanes.

Remark 3.14. In [DW3], it is proved that the derived category of a flopping contraction $X \to \operatorname{Spec} R$ carries an action of a subgroup of the fundamental group of the complexified complement of an associated $\mathcal{H}$, without knowledge of the group presentation. This subgroup K is defined to be generated by monodromy around all walls (including those of high codimension) from any fixed chamber. In type A, this corresponds to the $\ell_{i,j}^2$ from the earlier sections.

Write $H_1, \dots, H_n$ for the n hyperplanes in $\mathbb{R}^2$ associated to the Dynkin diagram I_n. Starting from a given chamber, we make a choice on the generators of the pure braid group PBr_{I_n} by finding the shortest way to loop around each of the hyperplanes; see e.g. [BMR, Proposition 2.2(1)].

Corollary 3.15. *The pure braid group* PBr_{I_n} *has at least* n *generators, and so* $K \neq \mathrm{PBr}_{I_n}$ *whenever* $n \geq 4$.

Proof. The number of generators of PBr_{I_n} is at least the number of generators for its abelianization. By [BMR, Proposition 2.2(2)] this is the number of hyperplanes, which is n. But K has only 3 generators since the arrangement is in $\mathbb{R}^2$ so any chamber has only 3 walls: two of codimension one and one of codimension two. □

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