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SLOPES AND SIGNATURES OF LINKS

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ABSTRACT. We define the *slope* of a colored link in an integral homology sphere, associated to *admissible* characters on the link group. Away from a certain singular locus, the slope is a rational function which can be regarded as a multivariate generalization of the Kojima–Yamasaki η -function. It is the ratio of two Conway potentials, *provided that the latter makes sense*; otherwise, it is a new invariant. The slope is responsible for an extra correction term in the signature formula for the splice of two links, in the previously open exceptional case where both characters are admissible. Using a similar construction for a special class of tangles, we formulate generalized skein relations for the signature.

1. INTRODUCTION

In our previous paper [DFL17], we gave a formula for the signature of the splice of two colored links in terms of the signatures of the summands. One exceptional case, that of the characters vanishing on the components of the links identified by the splice, was left open. The formula does not work in this exceptional case, and the study of the defect term was the original goal of the present paper. To do so, we introduce a new link invariant, called *slope*. This invariant appears to be interesting in its own right, and we present several properties and examples.

As another development, we change the paradigm and extend the splice formula to all, not necessarily rational, characters (see [Vir09] for the extension of the signature function).

1.1. Slopes. The *slope* is an isotopy invariant, defined for an oriented link $K \cup L$, with a distinguished component K , in an integral homology sphere \mathbb{S} . Optionally, L may be given a μ -coloring $\pi_0(L) \rightarrow \{1, \dots, \mu\}$; then, this coloring should be respected by characters ω below. Denoting by T_C a small open tubular neighborhood of a link $C \subset \mathbb{S}$, we consider the inclusion

$$i: \partial T_K \hookrightarrow \mathbb{S} \setminus T_{K \cup L},$$

with a particular interest in the map induced in the first homology with the coefficients twisted by a character $\omega: \pi_1(\mathbb{S} \setminus L) \rightarrow \mathbb{C}^\times$. If ω satisfies the *admissibility* condition $\omega([K]) = 1$, then the vector space $H_1(\partial T_K; \mathbb{C}(\omega))$ is generated by the meridian m and preferred longitude l of K . If the kernel of

$$i_*: H_1(\partial T_K; \mathbb{C}(\omega)) \rightarrow H_1(\mathbb{S} \setminus T_{K \cup L}; \mathbb{C}(\omega))$$

is generated by a single vector $am + bl$, $[a : b] \in \mathbb{P}^1(\mathbb{C})$, we define the *slope* of $K \cup L$ at ω via

$$(K/L)(\omega) := -\frac{a}{b} \in \mathbb{C} \cup \infty$$

(see Definition 3.3). Crucial is the fact that, if ω is unitary and *nonvanishing* (i.e., ω does not take value 1 on any meridian of L), then $(K/L)(\omega)$ is well defined and real, possibly infinite (see Proposition 3.6). This special case is used in our signature formula (see §1.2 below).

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For a link $K \cup L \subset S^3$, the slope can be computed from the link diagram, using the Wirtinger presentation of the group $\pi_1(S^3 \setminus (K \cup L))$ and Fox calculus (see §3.2). In general, the slope is not determined by the combinatorial data, such as linking numbers, depending more deeply on the group (cf. Example 3.11 of the Whitehead link and a number of examples in §3.4).

Admissible nonvanishing characters constitute an algebraic variety $\mathcal{A}^\circ(K/L) \subset (\mathbb{C}^\times \setminus 1)^\mu$. Our first main result, Theorem 3.21, states that, if the Alexander polynomial Δ_L is not zero, the function K/L is rational and finite outside the zero locus of Δ_L . In fact, denoting by \prime the derivative with respect to the variable t corresponding to K , we have

$$(K/L)(\omega) = -\frac{\nabla'_{K \cup L}(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

provided that this ratio makes sense, *i.e.*, is not $0/0$. (Here, ∇ is the Conway potential; we have to use ∇ instead of Δ to avoid the sign ambiguity, cf. Remark 3.23). At the common zeroes of the two polynomials, the slope may still be defined, but its values are less predictable, which makes this invariant interesting. Even if $\Delta_L \equiv 0$, generically the slope is still a rational function (possibly, but not necessarily, identically ∞) whose denominator is the first nonvanishing order $\Delta_{L,r}$ (see Theorem 3.19). Our experiments with the link tables [KAT] reveal the independence of K/L of the higher Fitting ideals (cf. Examples 3.26 and 3.27); moreover, the slope distinguishes some links that are not distinguishable by the higher Alexander polynomials.

As an indirect consequence of Theorem 3.21, it appears that the slope can be regarded as a multivariate generalization of the Kojima–Yamasaki η -function [KY79], defined originally for two-component links with linking number 0 (see Corollary 3.24). Cochran [Coc85] showed that the η -function provides a generating function of a sequence of β -invariants that corresponds to the Sato–Levine invariants [Sat84] of successive derivatives of the links. In particular, he proved that they are integral lifts of certain Milnor-type $\bar{\mu}$ -invariants. In a forthcoming paper [DFL], we study the concordance invariance of the slope and give more details on its relation to the η -function. We also present a number of alternative methods of computing the slope, including those in terms of Seifert surfaces and C -complexes.

1.2. The splice formula. Our first motivation for developing the slope function was an attempt to extend the signature formula for the splice of two colored links to the exceptional case $v' = v'' = 1$ that was left open in [DFL17]. Let $L \subset \mathbb{S}$ be the splice of two links $K^* \cup L^* \subset \mathbb{S}^*$, where $*$ = \prime or $''$. Given a unitary character ω^* on L^* , denote $v^* := \omega^*([K^*])$, see (5.1). Then, as shown in [DFL17], for a pair (ω', ω'') of *rational* unitary characters one has

$$\sigma_L(\omega', \omega'') = \sigma_{K' \cup L'}(v'', \omega') + \sigma_{K'' \cup L''}(v', \omega'') + \delta_{\lambda'}(\omega') \delta_{\lambda''}(\omega''),$$

provided that $(v', v'') \neq (1, 1)$. (See Theorem 5.2 and §2.5 for the precise statement and definition of the defect function δ , which is related to the signature of a generalized Hopf link and depends on the linking numbers only.) In this paper, we consider the signature as a function on all (non necessarily rational) characters (see also [CNT17]) and establish the splice formula in this full generality. We mainly follow Viro’s approach [Vir09] for links in S^3 , extending it to integral homology spheres and filling in a few details. .

More importantly, we extend the splice formula to the exceptional case where $v' = v'' = 1$, *i.e.*, both characters are admissible. Then, we have well defined slopes $\kappa^* := (K^*/L^*)(\omega^*)$, and the formula reads (see Theorem 5.3)

$$\sigma_L(\omega', \omega'') = \sigma_{L'}(\omega') + \sigma_{L''}(\omega'') + \delta_{\lambda'}(\omega') \delta_{\lambda''}(\omega'') + \Delta\sigma(\kappa', \kappa''),$$

where the correction term $\Delta\sigma(\kappa', \kappa'') \in \{0, \pm 1, \pm 2\}$ depends only on the two slopes, see (5.5). (Observe that $\Delta\sigma$ is the only contribution of the knots K^* to the formula.) Geometrically, $\Delta\sigma$ is the sum of two Maslov indices in Wall’s non-additivity theorem [Wal69].

We have similar statements for the *nullity* η_L of the splice: plain additivity if $(v', v'') \neq (1, 1)$ (see [Theorem 5.2](#)) and an extra correction term $\Delta\eta(\kappa', \kappa'') \in \{-1, 0, 1, 2\}$ as the only contribution of K^* in the exceptional case (see [Theorem 5.3](#)).

1.3. Tangles and skein relations. The concept of slope turns out quite fruitful in a number of other applications. As an example, we consider the slope function for tangles in integral homology balls with four labelled loose ends (this time, no component is distinguished). The main application of this slope is an analogue of the skein relations for the signatures.

For a given μ -colored tangle T in \mathbb{B} , with exterior \mathbb{B}_T , and a character ω in $(\mathbb{C}^\times \setminus 1)^\mu$, one can consider the subspace

$$\text{Ker}[\text{inclusion}_* : H_1(\partial(\mathbb{B}_T); \mathbb{C}(\omega)) \rightarrow H_1(\mathbb{B}_T; \mathbb{C}(\omega))].$$

These subspaces were already introduced for colored braids and tangles in [\[GG05, CC\]](#), assuming that the characters have finite order, where certain functorial properties with respect to concatenation were also established (see also [\[CC18\]](#)). In the present paper, we restrict ourselves to tangles with four ends and observe that, for *generic* characters ω , the kernel has dimension one. As in the case of links, it is then determined by a single number $\kappa_T(\omega) \in \mathbb{C} \cup \infty$, which we call *slope*. We study this slope κ_T as a function defined on a dense Zariski open set of $(\mathbb{C}^\times \setminus 1)^\mu$ —including all unitary characters, at which it takes real values. Using the technics similar to previous sections, we show that κ_T is the ratio (assuming that it makes sense) of the Conway potentials of two auxiliary links obtained, roughly, by patching the loose ends of the tangle with an overcrossing and an undercrossing (see [Theorem 6.7](#) for the precise statement).

We define the *sum* $T' \sqcup T'' \subset \mathbb{S}$ of two tangles $T' \subset \mathbb{B}'$, $T'' \subset \mathbb{B}''$, which is a link in an integral homology sphere. Then, given a pair of unitary characters ω' , ω'' properly matching on the boundary, we can define a character $\omega' \sqcup \omega''$ on $T' \sqcup T''$ and, hence, speak about the signature $\sigma_{T' \sqcup T''}(\omega' \sqcup \omega'')$. By Wall's non-additivity theorem [\[Wal69\]](#), the signatures of the three pairwise sums of three tangles $T^i \subset \mathbb{B}^i$, $i \in \mathbb{Z}/3$ are related as follows (see [Theorem 6.9](#))

$$\sum_{i \in \mathbb{Z}/3} \sigma_{T^{i+1} \sqcup T^i}(\omega^{i+1} \sqcup \omega^i) = \text{sign}(\kappa^0, \kappa^1, \kappa^2).$$

Here, $\text{sign}(\kappa^0, \kappa^1, \kappa^2) \in \{0, \pm 1\}$ is a certain skew-symmetric function of the slopes of the tangles involved, see [Corollary 2.22](#). For rational characters, this formula could alternatively be obtained from [\[CC, Theorem 1.1\]](#) and elementary properties of the Maslov index.

Applying the results to certain elementary basic tangles (see [Example 6.2](#)), we obtain a multivariate generalization (and a four-dimensional proof) of the skein relations found in [\[CF08\]](#) (see [Corollary 6.11](#)). Note that our version is also a refinement: it disambiguates the right hand side of the relations when it does not make sense (*i.e.*, becomes 0/0). In this case, slopes should be computed by other means. Furthermore, using the concept of slope, we bridge the gap between the skein relations found in [\[CF08\]](#), in terms of the Conway potentials, and those that could be derived from [\[CC\]](#), in terms of the Maslov index.

1.4. Contents of the paper. In [Section 2](#), we discuss several unrelated well-known subjects used throughout the paper, our principal goal being fixing the terminology and notation. Thus, we introduce (co-)homology with twisted coefficients, characteristic varieties, sign-determined torsion of a *CW*-complex, and twisted Poincaré duality. We also discuss Wall's non-additivity theorem, which is our principal tool in establishing the signature formulas, and recall the terminology concerning colored links and characters on the link group preserving the coloring.

[Section 3](#) contains the construction of the slope. We study its basic properties and, at the end, establish the rationality of the slope and its relation to the Conway potential.

Sections 4 and 5 deal with multivariate signature: we discuss the extension of the signature function to all, not necessarily rational, characters and prove its invariance (Section 4), and then we state and prove the splice formula, both for signature and nullity (Section 5).

Finally, in Section 6, we construct the slope of a colored tangle, discuss its rationality and relation to the Conway potential, and state and prove the generalized skein relations.

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2. PRELIMINARIES

Throughout the paper, all group actions are *right* and by default, all modules are right. Matrices act on *row vectors* by *right multiplication*. We use the notation R^\times to refer to the multiplicative group of a commutative ring R .

In this section, X will mainly be a finite CW -complex with $\pi := \pi_1(X)$. In §2.4 and §2.5, we will consider more specifically compact smooth manifolds with boundary, possibly empty; then, the CW -decomposition is given by any smooth triangulation.

2.1. Twisted (co-)homology. Define the *chain complex* of right $\mathbb{Z}\pi$ -modules $C_*(X; \mathbb{Z}\pi)$ as the complex of cellular chains of the universal covering $\tilde{X} \rightarrow X$, freely generated by (arbitrary) lifts of the cells of X . For the lifts, we retain the same orientation as for the original cells. Recall that the group ring $\mathbb{Z}\pi$ is equipped with a canonical involutive anti-automorphism

$$*: \lambda = \sum n_i g_i \mapsto \lambda^* := \sum n_i g_i^{-1}.$$

(We use $*$ instead of $\bar{}$, reserving the latter for the complex conjugation). Hence, any right $\mathbb{Z}\pi$ -module M gives rise to a left module M^* , with the $\mathbb{Z}\pi$ -action given by $\lambda m := m \lambda^*$.

For a right $\mathbb{Z}\pi$ -module M and a left $\mathbb{Z}\pi$ -module N , we consider the complexes

$$C^*(X; M) := \text{Hom}_{\mathbb{Z}\pi}(C_*(X; \mathbb{Z}\pi), M), \quad C_*(X; N) := C_*(X; \mathbb{Z}\pi) \otimes_{\mathbb{Z}\pi} N$$

and their homology

$$H^*(X; M) := H^*(C^*(X; M)), \quad H_*(X; N) := H_*(C_*(X; N)).$$

A similar construction applies to a CW -pair (X, A) , starting with $C_*(X, A; \mathbb{Z}\pi)$.

An important special case is that of a group ring $M = N = \mathbb{Z}G$, with the $\mathbb{Z}\pi$ -module structure given by an epimorphism $\varphi: \pi \twoheadrightarrow G$. Then, clearly, C_* and C^* are merely the (co-) chain complexes of the G -covering of X defined by φ . If G is a free abelian group with a basis t_1, \dots, t_μ , then $\mathbb{Z}G = \mathbb{Z}[t_1^{\pm 1}, \dots, t_\mu^{\pm 1}]$ is the ring of Laurent polynomials; it is an integral domain, and we can also consider the field of fractions $\mathbb{Q}(G) := \mathbb{Q}(t_1, \dots, t_\mu)$.

Another special case is the ring $\mathbb{C}(\omega)$, which is the field \mathbb{C} regarded as a $\mathbb{Z}\pi$ -module *via* a multiplicative character $\omega: \pi \rightarrow \mathbb{C}^\times$. If ω factors through a finite abelian group, $\omega: \pi \rightarrow G \rightarrow \mathbb{C}^\times$, the homology $H_*(X; \mathbb{C}(\omega))$ can be identified with the ω -isotypical component (eigenspace) of the induced representation of G on the homology $H_*(X_G; \mathbb{C})$ of the corresponding G -covering $X_G \rightarrow X$; in particular, the latter isotypical component does not depend on the choice of an intermediate group G . Alternatively, $\mathbb{C}(\omega)$ can be regarded as a local system on X , and

then $H_*(X; \mathbb{C}(\omega))$ is the ordinary homology of X with coefficients in this local system. Similar statements hold for cohomology and for the (co-)chain complexes. We have an obvious \mathbb{C} -linear isomorphism

$$C^*(X; \mathbb{C}(\omega)) = \text{Hom}_{\mathbb{C}}(C_*(X; \mathbb{C}(\omega)), \mathbb{C})$$

and, hence, a “universal coefficient formula”

$$(2.1) \quad H^*(\cdot; \mathbb{C}(\omega)) = H_*(\cdot; \mathbb{C}(\omega))^\vee,$$

where $^\vee$ stands for the dual vector space. We fix the notation ω^* and $\bar{\omega}$ for the *dual* and *conjugate* characters, respectively:

$$\omega^*: g \mapsto \omega(g)^{-1}, \quad \bar{\omega}: g \mapsto \overline{\omega(g)}.$$

Note that $\mathbb{C}(\omega)^* = \mathbb{C}(\omega^*)$. We say that ω is a *unitary character* if $\bar{\omega} = \omega^*$.

The following simple observations, whose utmost importance was probably first observed by Viro [Vir09], are used throughout the paper without further reference.

Lemma 2.2. *If $\omega \neq 1: H_1(S^1) \rightarrow \mathbb{C}^\times$, then $H_*(S^1; \mathbb{C}(\omega)) = H^*(S^1; \mathbb{C}(\omega)) = 0$.*

Corollary 2.3. *If $X \rightarrow B$ is a circle bundle and a character $\omega: \pi \rightarrow \mathbb{C}^\times$ restricts to a nontrivial character on the fibers, then $H_*(X; \mathbb{C}(\omega)) = H^*(X; \mathbb{C}(\omega)) = 0$.*

Corollary 2.4. *If $X = T^2$ is a torus and $\omega \neq 1$, then $H_*(X; \mathbb{C}(\omega)) = H^*(X; \mathbb{C}(\omega)) = 0$.*

The next corollary, although also straightforward, is used several times in the paper. Consider a 3-manifold X decomposed into the union $X = Y \cup Z$ of two compact submanifolds. Given a character $\omega: \pi_1(X) \rightarrow \mathbb{C}^\times$, we will say that the Z -part is ω -invisible if

- Z is a plumbed 3-manifold, possibly with boundary, which is assumed to be a disjoint union of tori fibered over the boundary of the core surface, away from the nodes;
- $Y \cap Z$ is a union of whole components of the boundary ∂Z ;
- ω restricts nontrivially to each fiber (“meridian”) of each circle bundle constituting Z .

To make the ω -invisible part of X clearly seen, we will sometimes use the notation $X = Y \cup \{Z\}$. Combining Corollaries 2.3 and 2.4 with the Mayer–Vietoris exact sequence, we arrive at the following statement.

Corollary 2.5. *Let $X = Y \cup \{Z\}$ be a 3-manifold and $\omega: \pi_1(X) \rightarrow \mathbb{C}^\times$ a character such that Z is ω -invisible. Then the inclusion $Y \hookrightarrow X$ induces isomorphisms*

$$H_*(Y; \mathbb{C}(\omega)) = H_*(X; \mathbb{C}(\omega)), \quad H^*(X; \mathbb{C}(\omega)) = H^*(Y; \mathbb{C}(\omega)).$$

2.2. Characteristic varieties (see [Lib01]). Let H be a free abelian group with basis t_1, \dots, t_μ , and let $\varphi: \pi \twoheadrightarrow H$ be an epimorphism. Any multiplicative character $\omega: \pi \rightarrow \mathbb{C}^\times$ that factors through H is determined by the sequence $(\omega_1, \dots, \omega_\mu)$, where $\omega_i := \omega(t_i) \in \mathbb{C}^\times$. This identifies the group of such characters with the complex torus $(\mathbb{C}^\times)^\mu$.

The *characteristic varieties* $\mathcal{V}_r(X)$ of X (related to φ) are defined via

$$\mathcal{V}_r(X) := \{\omega \in (\mathbb{C}^\times)^\mu \mid \dim H_1(X; \mathbb{C}(\omega)) \geq r\}, \quad r \geq 0.$$

These are indeed algebraic varieties in $(\mathbb{C}^\times)^\mu$, which are nested ($\mathcal{V}_r \supset \mathcal{V}_{r+1}$) and depend on the fundamental group π of X (and φ) only. In view of (2.9) below, each \mathcal{V}_r is real with respect to the standard real structure $\omega \mapsto \bar{\omega}$.

Let $C_* := C_*(X; \mathbb{Z}H)$. Since the complexes $C_* \otimes_{\mathbb{Z}H} \mathbb{C}(\omega)$ compute the homology $H_*(X; \mathbb{C}(\omega))$ and $H_0(X; \mathbb{C}(\omega)) = 0$ unless $\omega = 1$, we have

$$\mathcal{V}_r(X) \setminus 1 = V_{\mathbb{C}}(E_r(C_1 / \text{Im } \partial_1)) \setminus 1, \quad r \geq 0,$$

where $E_s(M) \subset \mathbb{Z}H$ is the s -th elementary ideal of a $\mathbb{Z}H$ -module M and $V_{\mathbb{C}}(I)$, $I \in \mathbb{Z}H$, stands for the zero locus of the ideal $I \otimes \mathbb{C} \subset \mathbb{C}H$ in $(\mathbb{C}^\times)^\mu = \text{Spec}(\mathbb{C}H)$. In this identity, C_* can be

replaced with any complex of free $\mathbb{Z}H$ -modules computing the homology $H_{\leq 1}(X; \mathbb{Z}H)$ (e.g., the one given by the Fox calculus, cf. §3.2 below). In fact, $C_1/\text{Im } \partial_1 = H_1(X, x_0; \mathbb{Z}H)$, where $x_0 \in X$ is the basepoint. Furthermore, according to [Lib01], there also are isomorphisms

$$\mathcal{V}_r(X) \setminus 1 = V_{\mathbb{C}}(E_{r-1}(H_1(X; \mathbb{Z}H))) \setminus 1, \quad r \geq 1,$$

which are sometimes taken for the definition of $\mathcal{V}_r(X)$, which extends to any finitely generated $\mathbb{Z}H$ -module. In particular, it follows that

$$\mathcal{V}_1(X) \setminus 1 = V_{\mathbb{C}}(\text{Ann } H_1(X; \mathbb{Z}H)) \setminus 1.$$

(Recall that the ideals $E_0(M)$ and $\text{Ann } M$ have essentially the same radical.) The irreducible components of $\mathcal{V}_r(X)$ of codimension ≤ 1 constitute the zero locus of the $(r-1)$ -st order

$$\Delta_{X, r-1} := \text{g.c.d. } E_{r-1}(H_1(X; \mathbb{Z}H)) = \text{g.c.d. } E_r(C_1/\text{Im } \partial_1) \in \mathbb{Z}H;$$

the 0-th order $\Delta_X := \Delta_{X,0}$ is called the *multivariate Alexander polynomial* of X (and φ).

2.3. Sign-determined torsion of CW-complexes (see [Tur02]). Let \mathbb{F} be a field and

$$C_*: \quad C_m \xrightarrow{\partial_{m-1}} C_{m-1} \xrightarrow{\partial_{m-2}} \cdots \xrightarrow{\partial_0} C_0$$

a finite chain complex of finite-dimensional vector spaces over \mathbb{F} . Suppose that C_* is based, i.e., each C_i has a distinguished basis c_i . Suppose that $H_i(C_*)$ are also based, $i = 1, \dots, m$. Let

$$\beta_i(C_*) := \sum_{r \leq i} (-1)^{i-r} \dim H_r(C_*), \quad \gamma_i(C_*) := \sum_{r \leq i} (-1)^{i-r} \dim C_r, \quad |C_*| := \sum_{i=0}^m \beta_i(C_*) \gamma_i(C_*).$$

Lift the distinguished basis for $H_i(C_*)$ to a sequence $h_i \subset C_i$. Let $b_i \subset C_i$ be a sequence of vectors such that $\partial_{i-1}(b_i)$ is a basis of $\text{Im } \partial_{i-1}$. Then, for each i , the concatenation $\partial_i(b_{i+1})h_i b_i$ is a basis of C_i . Denoting by T_i the transition matrix from $\partial_i(b_{i+1})h_i b_i$ to the distinguished basis c_i , the *torsion* of C_* is defined as

$$\tau(C_*) := (-1)^{|C_*|} \prod_{i=0}^m \det T_i^{(-1)^{i+1}} \in \mathbb{F}^\times,$$

The torsion of C_* depends on the given bases in C_i and $H_i(C_*)$, but it does not depend on the choice of h_i and b_i .

Fix now an epimorphism $\varphi: \pi \rightarrow H$ onto a free abelian group H and consider the complex $C_*(X; \mathbb{Q}(H))$ of $\mathbb{Q}(H)$ -vector spaces. It is based (see §2.1), and the *torsion* $\tau_\varphi(X) \in \mathbb{Q}(H)/\pm H$ of X is defined as the torsion of $C_*(X; \mathbb{Q}(H))$, with the extra convention that $\tau_\varphi(X) = 0$ if the complex is not acyclic, i.e., $H_*(X; \mathbb{Q}(H)) \neq 0$. Here, the group $\pm H \subset \mathbb{Q}(H)^\times$ acts on $\mathbb{Q}(H)$ by the multiplication, and the ambiguity is due to the non-uniqueness in the choice of the bases; modulo this ambiguity, $\tau_\varphi(X)$ is invariant under simple homotopy equivalences and subdivisions.

The sign of the torsion can be refined if X is *homologically oriented*, i.e., equipped with a distinguished orientation v of the space $H_*(X; \mathbb{R}) = \bigoplus_{i \geq 0} H_i(X; \mathbb{R})$. The orientation and order of the cells in \tilde{X} induce an orientation and order of those of X , and thus a distinguished bases for $C_*(X; \mathbb{R})$. Choose bases h_i of $H_i(X; \mathbb{R})$ so that the basis $h_0 h_1 \cdots h_{\dim X}$ of $H_*(X; \mathbb{R})$ is positive with respect to v , let $\tau_0(X) = \pm 1$ be the sign of the torsion $\tau(C_*(X; \mathbb{R})) \in \mathbb{R}^\times$, and set

$$\tau_\varphi(X, v) = \tau_0(X) \cdot \tau_\varphi(X).$$

The *sign-determined torsion* $\tau_\varphi(X, v) \in \mathbb{Q}(G)/H$ depends only on (X, v) and φ . In the forthcoming sections, X is a 3-manifold and its torsion is a topological and simple homotopy invariant.

2.4. Poincaré duality (see [Wal99]). Throughout this section, X is a smooth compact connected oriented manifold. Then, according to [Wal99, §2], $(X, \partial X)$ is a *simple Poincaré pair* of dimension $n := \dim X$ in the following sense: there is a simple chain homotopy equivalence

$$D_X : C^*(X; \mathbb{Z}\pi) \rightarrow C_{n-*}(X, \partial X; \mathbb{Z}\pi),$$

well defined up to chain homotopy; in particular, for any integer r and any right $\mathbb{Z}\pi$ -module M , there is a canonical *Poincaré duality isomorphism*

$$(2.6) \quad D_X : H^r(X; M) \xrightarrow{\cong} H_{n-r}(X, \partial X; M^*).$$

Furthermore, each connected component of the boundary ∂X is a simple Poincaré complex of dimension $(n - 1)$, and the following diagram commutes

$$(2.7) \quad \begin{array}{ccc} H^r(X; M) & \xrightarrow{i^*} & H^r(\partial X; M) \\ D_X \downarrow \cong & & D_{\partial X} \downarrow \cong \\ H_{n-r}(X, \partial X; M^*) & \xrightarrow{\partial} & H_{n-1-r}(\partial X; M^*), \end{array}$$

where $i : \partial X \hookrightarrow X$ is the inclusion and, for each component $Y \subset \partial X$, we regard M as a $\mathbb{Z}\pi_1(Y)$ -module *via* the inclusion homomorphism $\pi_1(Y) \rightarrow \pi$. (We ignore the technicality related to the choice of the basepoints as we will mainly work over the commutative rings of the form $\mathbb{Z}H_1(X)$.)

If $M = \mathbb{C}(\omega)$ for a character $\omega : \pi \rightarrow \mathbb{C}^\times$, then, in view of (2.1), the Poincaré duality in X and ∂X can be restated in the form of isomorphisms

$$(2.8) \quad \begin{aligned} H_{n-r}(X, \partial X; \mathbb{C}(\omega^*)) &= H_r(X; \mathbb{C}(\omega))^\vee, \\ H_{n-1-r}(\partial X; \mathbb{C}(\omega^*)) &= H_r(\partial X; \mathbb{C}(\omega))^\vee, \end{aligned}$$

and (2.7) means that the map $\partial : H_{n-r}(X, \partial X; \mathbb{C}(\omega^*)) \rightarrow H_{n-1-r}(\partial X; \mathbb{C}(\omega^*))$ is the adjoint of the inclusion homomorphism $i_* : H_r(\partial X; \mathbb{C}(\omega)) \rightarrow H_r(X; \mathbb{C}(\omega))$.

If $n = 2r$ is even and the character ω is unitary, the first homomorphism in (2.8) composed with the \mathbb{C} -anti-linear isomorphism

$$(2.9) \quad H_*(\cdot; \mathbb{C}(\omega)) \rightarrow H_*(\cdot; \mathbb{C}(\bar{\omega}))$$

gives rise to a sesquilinear form

$$(2.10) \quad \circ : \alpha \otimes \beta \mapsto \langle \alpha, \bar{\beta} \rangle \quad \text{on} \quad H_r(X; \mathbb{C}(\omega)),$$

called the *intersection index*; it is Hermitian if r is even and skew-Hermitian if r is odd. In the former case, the signature $\text{sign}^\omega(X)$ of (2.10) is called the *twisted signature of X* . Certainly, in the special case $\omega = 1$ we obtain the ordinary signature $\text{sign}(X) = \text{sign}^1(X)$. The notion of signature extends also to open manifolds of the form $\bar{X} \setminus \partial \bar{X}$, where \bar{X} is compact.

By (2.7), the kernel $\text{Ker } \partial$ of the intersection index form is

$$H_r(X; \mathbb{C}(\omega))^\perp = \text{Im}[i_* : H_r(\partial X; \mathbb{C}(\omega)) \rightarrow H_r(X; \mathbb{C}(\omega))].$$

In particular, if $\partial X = \emptyset$ or, more generally, $H_*(\partial X; \mathbb{C}(\omega)) = 0$, this form is nondegenerate.

Consider now the case of $n = 2r + 1$ odd and let

$$(2.11) \quad \begin{aligned} Z_r(X; \mathbb{C}(\omega)) &:= \text{Ker}[i_* : H_r(\partial X; \mathbb{C}(\omega)) \rightarrow H_r(X; \mathbb{C}(\omega))] \\ &= \text{Im}[\partial : H_{r+1}(X, \partial X; \mathbb{C}(\omega)) \rightarrow H_r(\partial X; \mathbb{C}(\omega))], \end{aligned}$$

where the equality follows from the exact sequence of pair $(X, \partial X)$. Since ∂X is a closed manifold, Poincaré duality induces a perfect pairing

$$(2.12) \quad H_r(\partial X; \mathbb{C}(\omega)) \otimes H_r(\partial X; \mathbb{C}(\omega^*)) \rightarrow \mathbb{C}$$

which, composed with (2.9), coincides with the intersection index

$$(2.13) \quad \circ: \alpha \otimes \beta \mapsto \langle \alpha, \bar{\beta} \rangle \quad \text{on} \quad H_r(\partial X; \mathbb{C}(\omega))$$

if ω is unitary. Combining these observations, we arrive at the following statement.

Lemma 2.14. *Given a multiplicative character $\omega: \pi \rightarrow \mathbb{C}^\times$, one has*

$$Z_r(X; \mathbb{C}(\omega)) = Z_r(X; \mathbb{C}(\omega^*))^\perp$$

with respect to (2.12). If ω is unitary, then $Z_r(X; \mathbb{C}(\omega)) = Z_r(X; \mathbb{C}(\omega))^\perp$ with respect to (2.13). In particular, in this case one has $\dim Z_r(X; \mathbb{C}(\omega)) = \frac{1}{2} \dim H_r(\partial X; \mathbb{C}(\omega))$.

In the next statement, which is an immediate consequence of Lemma 2.14, we change the notation $X \mapsto W$ and $\partial X \mapsto \partial W := X$.

Corollary 2.15. *Assume that a closed oriented $4k$ -manifold X is the boundary ∂W of a compact $(4k+1)$ -manifold W . Then*

$$\text{sign}^\omega(X) = 0$$

for each unitary character ω on X which extends to a unitary character on W .

2.5. Signatures and additivity (see [Wal69]). Consider a compact connected oriented 4-manifold N and assume that $N := N_1 \cup_{X_0} N_2$, where N_1, N_2 are manifolds with boundaries

$$\partial N_1 \cong X_1 \cup_T -X_0, \quad \partial N_2 \cong X_0 \cup_T -X_2$$

and X_0, X_1, X_2 , in turn, are 3-manifolds with common boundary

$$T := \partial X_0 = \partial X_1 = \partial X_2.$$

Consider the \mathbb{C} -vector spaces $A_i := Z_1(X_i; \mathbb{C}) \subset V := H_1(T; \mathbb{C})$, $i = 0, 1, 2$, see (2.11), and let

$$W := \frac{A_0 \cap (A_1 + A_2)}{(A_0 \cap A_1) + (A_0 \cap A_2)}.$$

By Lemma 2.14, these spaces are Lagrangian with respect to the intersection index form \circ of the closed surface T . Hence, \circ induces a nondegenerate Hermitian form f on W : it is given by

$$f(a_0, a'_0) := a_0 \circ a'_0, \quad \text{where } a'_0 + a'_1 + a'_2 = 0 \text{ and } a'_i \in A_i.$$

Theorem 2.16 (Wall [Wal69]). *In the above notation, one has*

$$\text{sign}(N) = \text{sign}(N_1) + \text{sign}(N_2) - \text{sign } f.$$

Remark 2.17. As mentioned in [Wal69] (and follows easily from the proof), the conclusion of Theorem 2.16 holds as well for the twisted signature. Pick a unitary character $\omega: H_1(N) \rightarrow \mathbb{C}^\times$ and denote by the same letter the restriction of ω to the other spaces involved. Then, the signature formula reads

$$\text{sign}^\omega(N) = \text{sign}^\omega(N_1) + \text{sign}^\omega(N_2) - \text{sign } f,$$

where the form f is defined as above, using the Lagrangian subspaces $Z_1(X_i; \mathbb{C}(\omega))$.

In the rest of this section, we discuss various forms of Wall's correction term $\text{sign } f$ in the important special case where $\dim V = 2$. Let $\varphi: V \otimes V \rightarrow \mathbb{C}$ be a nondegenerate skew-Hermitian form; we abbreviate $\varphi(a, b) = a \circ b$ and $\varphi(a, a) = a^2$. The form $i\varphi$ is nondegenerate Hermitian; hence, $\text{sign}(i\varphi)$ takes values ± 2 or 0. In the former case, V has no nontrivial isotropic vectors; thus, we assume that $\text{sign}(i\varphi) = 0$. In this case, V has a *standard symplectic basis*, i.e., a basis m, l with the property that

$$m^2 = l^2 = 0, \quad m \circ l = -1.$$

Fixing such a basis, we can parametrize the directions (1-subspaces) in V by assigning to a subspace $\mathbb{C}(am + bl)$ its *slope*

$$(2.18) \quad \kappa := -\frac{a}{b} \in \mathbb{C} \cup \infty = \mathbb{P}^1(\mathbb{C}).$$

The following statement is immediate.

Lemma 2.19. *A direction in V is isotropic if and only if its slope is real: $\kappa \in \mathbb{R} \cup \infty$.*

In other words, isotropic directions constitute a circle $\mathbb{P}^1(\mathbb{R})$ in the sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{P}(V)$ of all directions. This circle $\mathbb{R} \cup \infty$ has a canonical orientation, *viz.* the one that restricts to the order on \mathbb{R} . (Since the group $SL(2, \mathbb{C})$ of isometries of V is connected, this orientation does not depend on the choice of a standard basis.) The Lagrangian subspaces in V have dimension 1; hence, the correction term $\text{sign } f$ in Wall's formula becomes a function of three isotropic directions $\mathbb{C}a_i$, $i = 0, 1, 2$, or, equivalently, three slopes $\kappa_i \in \mathbb{R} \cup \infty$. We will use the notation

$$\text{sign } f = \text{sign}(a_0, a_1, a_2) = \text{sign}(\kappa_0, \kappa_1, \kappa_2) \in \{0, \pm 1\}.$$

Lemma 2.20. *One has*

$$\text{sign}(a_0, a_1, a_2) = \text{sg}[(a_0 \circ a_1)(a_1 \circ a_2)(a_2 \circ a_0)].$$

In particular, the function $\text{sg}(a_0, a_1, a_2)$ is skew-symmetric.

Proof. Note that both sides of the identity in the statement remain unchanged if any of a_i is replaced with $\alpha_i a_i$, $\alpha_i \in \mathbb{C}^\times$. If $A_1 = A_2$, both sides vanish. Otherwise, the pair (a_1, a_2) can be rescaled to a standard symplectic basis for V , *i.e.*, we can assume that $a_1 \circ a_2 = -1$. Then W is generated by $a_0 = \beta_1 a_1 + \beta_2 a_2$ and, assuming β_1, β_2 real, one has $f(a_0, a_0) = -\beta_1 \beta_2$, which is equal to the product in the right hand side. \square

Corollary 2.21. *One has $\text{sign}(a_0, a_1, a_2) = 0$ if and only if at least two of the three directions coincide. Otherwise, $\text{sign}(a_0, a_1, a_2) = 1$ if and only if the cyclic order (a_0, a_1, a_2) agrees with the canonical orientation of the circle of isotropic directions.*

Corollary 2.22. *Given a triple $\kappa_0, \kappa_1, \kappa_2 \in \mathbb{R} \cup \infty$, one has*

$$\begin{aligned} \text{sign}(\kappa_0, \kappa_1, \kappa_2) &= \text{sg}[(\kappa_0 - \kappa_1)(\kappa_1 - \kappa_2)(\kappa_2 - \kappa_0)] \quad \text{if } \kappa_0, \kappa_1, \kappa_2 \neq \infty, \\ \text{sign}(\infty, \kappa_1, \kappa_2) &= \text{sign}(\kappa_1, \kappa_2, \infty) = \text{sign}(\kappa_2, \infty, \kappa_1) = \text{sg}(\kappa_2 - \kappa_1). \end{aligned}$$

To make the last formula valid even if one or both of $\kappa_{1,2}$ is ∞ , we extend the sg function via

$$\text{sg } x = \begin{cases} 0, & \text{if } x = 0 \text{ or } \infty, \\ 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0 \end{cases}$$

and agree to disambiguate $\infty - \infty$ to 0 in its argument.

In a sense, the last formula in [Corollary 2.22](#) agrees with the general expression, which gives us $\text{sg}[-\infty^2(\kappa_1 - \kappa_2)]$; but then, it disagrees with our definition of sg . For this reason, we prefer to disambiguate expressions involving ∞ explicitly.

2.6. Colored links. Typically, given an oriented link $L \subset \mathbb{S}$ in an integral homology sphere \mathbb{S} , we denote by $T_L := \text{Tub } L$ a small open tubular neighborhood of L and let $X := \mathbb{S} \setminus T_L$. For a component $C \subset L$, we denote by $\partial_C X$ the intersection of ∂X with the closure of T_C . The group $H_1(\partial_C X) = H_1(\partial T_C)$ is generated by a meridian m_C and preferred longitude ℓ_C , *viz.* the one unlinked with C ; we call ℓ_C a *Seifert longitude*.

The meridian m_C is oriented so that $m_C \circ \ell_C = 1$ *with respect to the orientation of ∂T_C induced from T_C .*

The group $H_1(X)$ is the free abelian group generated by the classes m_C of the meridians of the components $C \subset L$, and the coloring gives rise to an epimorphism

$$(2.23) \quad \varphi: \pi_1(X) \longrightarrow H_1(X) \longrightarrow H := \bigoplus_{i=1}^{\mu} \mathbb{Z}t_i$$

sending m_C to t_i whenever $C \subset L_i$. Thus, we usually consider the character torus $(\mathbb{C}^\times)^\mu$, confining ourselves to the characters $\omega = (\omega_1, \dots, \omega_\mu)$ that factor through H .

Definition 2.24. A character $\omega = (\omega_1, \dots, \omega_\mu)$ is *nonvanishing* if $\omega_i \neq 1$ for all i .

The *characteristic varieties*, *orders*, and *Alexander polynomial* of the colored link L are defined as those of X and φ , see §2.2, and we use the notation $\mathcal{V}_r(L) := \mathcal{V}_r(X)$, $\Delta_{L,r} := \Delta_{X,r}$, *etc.*

Since $H_*(\partial X; \mathbb{C}(\omega)) = 0$ for any nonvanishing character ω , one has $H_*(X, \partial X; \mathbb{C}(\omega)) = H_*(X; \mathbb{C}(\omega))$; hence, by Poincaré duality (2.8), the *restricted characteristic variety*

$$\mathcal{V}_r^\circ(L) := \mathcal{V}_r(L) \cap (\mathbb{C}^\times \setminus 1)^\mu$$

is invariant under the automorphism $\omega \mapsto \omega^*$. It follows that, modulo units $(\mathbb{Z}H)^\times$, the greatest common divisor Δ_r° of the defining ideal of the closure of $\mathcal{V}_{r+1}^\circ(L)$ is invariant under the involutive automorphism $(t_1, \dots, t_\mu) \mapsto (t_1^{-1}, \dots, t_\mu^{-1})$; hence, so is the order $\Delta_{L,r}$, which differs from Δ_r° by a number of factors of the form $(t_i - 1)$.

Let, further, v be the orientation of $H_*(X; \mathbb{R})$ given by the basis consisting of $1 \in H_0(X; \mathbb{R})$, the meridians $m_C \in H_1(X; \mathbb{R})$ of the components $C \subset L$ (in some order), and the classes $[\partial_C X] \in H_2(X; \mathbb{R})$ of all but the last component (in the same order). Then, according to [Tur02], up to units $(\mathbb{Z}H)^\times$ one has

$$\tau_\varphi(X, v) = \begin{cases} \Delta_L & \text{if } \mu > 1, \\ \Delta_L / (t_1 - 1) & \text{if } \mu = 1. \end{cases}$$

Hence, as shown above,

$$\tau_\varphi(X, v)(t_1^{-1}, \dots, t_\mu^{-1}) = (-1)^n t_1^{\nu_1} \dots t_\mu^{\nu_\mu} \tau_\varphi(X, v)(t_1, \dots, t_\mu)$$

for some integers ν_1, \dots, ν_μ , where n is the number of components of L . The *Conway potential function* of the colored link L is defined as the symmetric renormalization of τ_φ :

$$(2.25) \quad \nabla_L(t_1, \dots, t_\mu) := -t_1^{\nu_1} \dots t_\mu^{\nu_\mu} \tau_\varphi(X, v)(t_1^2, \dots, t_\mu^2).$$

The Conway function of L is that of the maximal coloring; all others are obtained from the maximal one by the specialization $H_1(X) \twoheadrightarrow H$, *cf.* (2.23).

Definition 2.26. The *nullity* of a μ -colored link $L \subset \mathbb{S}$ at a nonvanishing character ω is

$$(2.27) \quad \eta_L(\omega) := \dim H_1(X; \mathbb{C}(\omega)).$$

We extend η to all characters *via* $\eta_L(\omega) := \eta_{L'}(\omega')$, where ω' is obtained from ω by removing all components $\omega_i = 1$, and L' is obtained from L by removing the corresponding components L_i . Occasionally, we consider also the literal extension given by (2.27); then, it is denoted by $\tilde{\eta}_L$.

Remark 2.28. In several other definitions below, we use the same strategy as in [Definition 2.26](#), *i.e.*, we define a certain quantity $q_L(\omega)$ for *nonvanishing* characters and extend it to the whole character torus by patching the components of L on which ω vanishes. The “literal” extension, if any, is then denoted by \tilde{q}_L . The principal reason for this approach is [Corollary 2.5](#): if there are too many boundary components with nontrivial homology, we loose control over the situation (*cf.* [§4.4](#)) and sometimes cannot even assert that the quantity in question is well defined.

A $(1, \mu)$ -colored link is a $(1 + \mu)$ -colored link $K \cup L$ in \mathbb{S} in which the *knot* K is the only component (considered distinguished) given the distinguished color 0. In addition to the space $X = \mathbb{S} \setminus L$ and epimorphism [\(2.23\)](#), we will also consider the complement $\bar{X} := \mathbb{S} \setminus (K \cup L)$ and epimorphism

$$(2.29) \quad \bar{\varphi}: \pi_1(\bar{X}) \longrightarrow H_1(\bar{X}) \longrightarrow \bar{H} := \mathbb{Z}t \oplus H$$

sending the meridian $m := m_K$ to the generator $t := t_0$. We use the notation

$$(2.30) \quad \bar{\ell}k(K, L) := (\lambda_1, \dots, \lambda_\mu) \in \mathbb{Z}^\mu, \quad \lambda_i := \ell k(K, L_i), \quad i = 1, \dots, \mu$$

for the *linking vector* of $K \cup L$.

3. SLOPES

In this section we define the main character of this article: the slope of a link in a homology sphere, which is a function defined on (a part of) the character torus and taking values in $\mathbb{C} \cup \infty$. We start with the definition of this new invariant and its first properties. We then show how to compute it *via* the Fox calculus and, in some special cases, *via* closed braids. We continue the section by proving that the slope is mainly a rational function, whose poles are determined by the Alexander invariants of the link. Then, we show that in a Zariski dense open set, it is determined by certain Conway potential functions. Finally, we present a list of examples.

3.1. Definition and first properties. Consider a $(1, \mu)$ -colored link $K \cup L \subset \mathbb{S}$.

Definition 3.1. A character $\omega \in (\mathbb{C}^\times)^\mu$ of $\pi_1(\mathbb{S} \setminus L)$ is *admissible* if $\omega([K]) = 1$. The variety of admissible characters is denoted

$$\mathcal{A}(K/L) = \{\omega \in (\mathbb{C}^\times)^\mu \mid \omega^\lambda = 1\},$$

where $\lambda := \bar{\ell}k(K, L)$ is the linking vector, see [\(2.30\)](#).

Note that an admissible character ω restricts to the trivial character on $H_1(\partial_K \bar{X})$. If $\lambda = 0$, then $\mathcal{A}(K/L) = (\mathbb{C}^\times)^\mu$; otherwise, letting $N := \text{g.c.d.}(\lambda)$ and $\nu := \lambda/N$, the irreducible over \mathbb{Q} components of $\mathcal{A}(K/L)$ are the zero sets of the cyclotomic polynomials $\Phi_d(\omega^\nu)$, $d \mid N$. Since all varieties are defined over \mathbb{Z} , for each component $\mathcal{A} \subset \mathcal{A}(K/L)$ and each $r \geq 0$, the complement $\mathcal{A} \setminus \mathcal{V}_r(L)$ is either empty or dense in \mathcal{A} .

Example 3.2. If L has one component with $\ell k(K, L) = \lambda$, then $\mathcal{A}(K/L) = \{1, \zeta, \zeta^2, \dots, \zeta^{\lambda-1}\}$, where ζ is any primitive root of 1 of order λ . If L has two components, with $\bar{\ell}k(K, L) = (1, \lambda)$, then $\mathcal{A}(K/L) = \{(\omega^\lambda, \omega^{-1}); \omega \in \mathbb{C}^\times\}$.

We will mainly consider the variety of nonvanishing admissible characters

$$\mathcal{A}^\circ(K/L) := \mathcal{A}(K/L) \cap (\mathbb{C}^\times \setminus 1)^\mu.$$

Let $\omega \in \mathcal{A}^\circ(K/L)$. Since ω is nonvanishing, we have $H_*(\partial_L \bar{X}; \mathbb{C}(\omega)) = 0$ and, since ω is also admissible,

$$H_1(\partial \bar{X}; \mathbb{C}(\omega)) = H_1(\partial_K \bar{X}; \mathbb{C}(\omega)) = H_1(\partial_K \bar{X}; \mathbb{C}).$$

The latter isomorphism is canonical up to a multiplicative constant, commutes with (2.9), and takes (2.12) and (2.13) to forms that are congruent to, respectively, the bilinear and sesquilinear extensions of the intersection index form on $H_1(\partial_K \bar{X})$. Hence, we have well-defined subspaces

$$\mathcal{Z}(\omega) = \mathcal{Z}_{K \cup L}(\omega) := Z_1(\bar{X}; \mathbb{C}(\omega)) \subset H_1(\partial_K \bar{X}; \mathbb{C}),$$

see §2.4, which have the following properties:

$$\mathcal{Z}(\bar{\omega}) = \overline{\mathcal{Z}(\omega)}, \quad \mathcal{Z}(\omega^*) = \mathcal{Z}(\omega)^\perp.$$

Let m, l be the meridian and Seifert longitude of K , forming a basis of $H_1(\partial_K \bar{X}; \mathbb{C}(\omega)) = \mathbb{C}^2$.

Definition 3.3. Let $\omega \in \mathcal{A}^\circ(K/L)$ and assume that $\dim \mathcal{Z}(\omega) = 1$, i.e., $\mathcal{Z}(\omega)$ is generated by a single vector $am + bl$ for some $[a : b] \in \mathbb{P}^1(\mathbb{C})$. Then, the *slope* of $K \cup L$ at ω is the quotient

$$(K/L)(\omega) := -\frac{a}{b} \in \mathbb{C} \cup \infty.$$

In agreement with Remark 2.28, we extend \mathcal{Z} and K/L to all admissible characters $\omega \in \mathcal{A}(K/L)$ by patching the components of L on which ω vanishes. The literal extension makes no sense.

Example 3.4 (generalized Hopf links). Recall that a *generalized Hopf link* $H_{m,n}$ is obtained from the ordinary Hopf link $\bar{V} \cup \bar{U}$ by replacing \bar{V} and \bar{U} by, respectively, m and n close parallel copies. Assume $m > 0$ and take for K one of the V -components. Then, the slope κ of $H_{m,n}$ at any *nonvanishing* character is

$$\kappa = \begin{cases} 0 & \text{if } n = 0 \text{ or } m > 1, \\ \infty & \text{otherwise.} \end{cases}$$

Indeed, if $n = 0$, then l_K bounds a disk, and if $m > 1$, then l_K is homotopic to the longitude of any other V -component, which vanishes in the twisted homology. However, if $m = 1$ and $n > 0$, then the meridian m_K is homotopic to the longitude of any of the U -components.

The following statements are immediate consequences of Poincaré duality.

Proposition 3.5. *If the slope at a character $\omega \in \mathcal{A}(K/L)$ is well defined, then so are the slopes at $\bar{\omega}$, ω^* , and $\bar{\omega}^*$, and one has*

$$(K/L)(\omega^*) = (K/L)(\omega), \quad (K/L)(\bar{\omega}) = (K/L)(\bar{\omega}^*) = \overline{(K/L)(\omega)}.$$

Furthermore, the slope does not change if the orientation of K is reversed.

Proposition 3.6. *If $\omega \in \mathcal{A}(K/L)$ is a unitary character, then the slope $(K/L)(\omega)$ is well defined and, moreover, is real: $(K/L)(\omega) \in \mathbb{R} \cup \infty$.*

Proof. The slope is well defined due to the last statement in Lemma 2.14; it is real due to Proposition 3.5. \square

Proposition 3.7. *The slope at a character $\omega \in \mathcal{A}^\circ(K/L)$ is well defined if and only if the two inclusion homomorphisms $H_1(K; \mathbb{C}(\zeta)) \rightarrow H_1(X; \mathbb{C}(\zeta))$, $\zeta = \omega$ or ω^* , are either both trivial or both nontrivial. The slope is finite, $(K/L)(\omega) \in \mathbb{C}$, if and only if the two homomorphisms are both trivial.*

Proof. By the Mayer–Vietoris exact sequence, the homomorphism $H_1(K; \mathbb{C}(\zeta)) \rightarrow H_1(X; \mathbb{C}(\zeta))$ is trivial if and only if $\mathcal{Z}(\zeta)$ contains an element of the form $l + \kappa m$, $\kappa \in \mathbb{C}$ (and, in particular, $\mathcal{Z}(\zeta) \neq 0$). Thus, the statement follows from the duality given by Lemma 2.14. \square

Corollary 3.8. *The slope is well defined and finite on $\mathcal{A}(K/L) \setminus \mathcal{V}_1(L)$.*

Remark 3.9. As follows from [Proposition 3.7](#), for each character ω , the existence of the slope $(K/L)(\omega)$ and its finiteness depend only on the conjugacy class realized by K in the fundamental group $\pi_1(\mathbb{S} \setminus L)$. However, the slope itself (when finite) is a more subtle invariant of $K \cup L$.

Proposition 3.10. *If K is contained in a ball B disjoint from L , then $(K/L)(\omega) = 0$ for any character $\omega \in \mathcal{A}(K/L)$.*

Proof. Any admissible character restricts to the trivial character on $\pi_1(B \setminus K)$, and the image of l vanishes already in $H_1(B \setminus K; \mathbb{C}(\omega)) = H_1(B \setminus K; \mathbb{C})$. \square

3.2. Fox calculus. We illustrate how the slope can be computed by means of the Fox calculus from a presentation of the fundamental group $\pi_1(\bar{X})$ of the link complement, together with the classes $m, l \in \pi_1(\bar{X})$ of the meridian and Seifert longitude of K . In the case of links in S^3 , both pieces of data can be derived from the link diagram. Indeed, for the group one can choose the Wirtinger presentation, where meridians are the generators. For l , we trace a curve C parallel to K and such that $\ell k(K, C) = 0$; then, starting from the segment corresponding to the chosen meridian of K and moving along C in the positive direction, we write down the corresponding generator (or its inverse) each time when undercrossing positively (respectively, negatively) the diagram of $K \cup L$. Thus, let

$$m, l \in \pi_1(\bar{X}) = \langle x_1, \dots, x_p \mid r_1, \dots, r_q \rangle,$$

$F := \langle x_1, \dots, x_p \rangle$, and let $\Lambda := \mathbb{Z}H$, where H is the abelianization of F . Since we consider abelian coverings only, we can specialize Fox derivatives to maps $\partial/\partial x_i: F \rightarrow \Lambda$. Consider the complex of Λ -modules

$$S_*: S_2 \xrightarrow{\partial_1} S_1 \xrightarrow{\partial_0} S_0 \longrightarrow 0,$$

where

$$S_2 = \bigoplus_{i=1}^q \Lambda r_i, \quad S_1 = \bigoplus_{i=1}^p \Lambda dx_i, \quad S_0 = \Lambda$$

and dx_i stands for a formal generator corresponding to x_i . The ‘‘differential’’ of a word $w \in F$ is

$$dw := \sum_{i=1}^p \frac{\partial w}{\partial x_i} dx_i \in S_1;$$

then, letting

$$\partial_1: r_i \mapsto dr_i, \quad \partial_0: dx_i \mapsto (\text{the image of } x_i \text{ in } H \subset \Lambda),$$

we obtain a complex computing the homology $H_{\leq 1}$ of the H -covering of \bar{X} .

Now, pick an admissible nonvanishing character $\omega \in \mathcal{A}^\circ(K/L)$ and consider the specialization $S_*(\omega) := S_* \otimes_\Lambda \mathbb{C}(\omega)$. Then, it is straightforward that

$$\mathcal{Z}(\omega) = \text{Ker}[H_1(\partial_K X; \mathbb{C}(\omega)) = \mathbb{C}m \oplus \mathbb{C}l \xrightarrow{\text{in}_*} S_1(\omega)/\text{Im } \partial_1(\omega)],$$

where the inclusion homomorphism in_* is the specialization of $m \mapsto dm, l \mapsto dl$. (Note that, by the assumption that $\omega \in \mathcal{A}^\circ(K/L)$, this homomorphism lands into $\text{Ker } \partial_0(\omega)$.) Computing the above kernel in the basis m, l , we can also compute the slope, whenever it is defined.

Example 3.11 (the Whitehead link). Consider the $(1, 1)$ -colored Whitehead link $K \cup L$. Since $\ell k(K, L) = 0$, we have $\mathcal{A}^\circ(K/L) = S^1 \setminus 1$. The standard presentation of $\pi_1(\bar{X})$ (derived from the Wirtinger representation) is

$$\pi_1(\bar{X}) = \langle m, m_1, l \mid [m, l] = 1, l = m_1 m^{-1} m_1^{-1} m m_1^{-1} m^{-1} m_1 m \rangle,$$

where m and m_1 are the meridians of K and L , respectively, and l is a Seifert longitude of K . We can further specialize Λ to the group ring $\mathbb{Z}H_1(\bar{X}) = \mathbb{Z}[t^{\pm 1}, t_1^{\pm 1}]$, sending the generators m, m_1 ,

and l to t , t_1 , and 1 , respectively. Then, denoting by x, y the two relations in the presentation above, we have

$$dx = (t-1)dl, \quad dy = dl - t^{-1}(1-t_1)(1-t_1^{-1})dm - (1-t^{-1})(1-t_1^{-1})dm_1.$$

The specialization at a character $\omega \in \mathcal{A}^\circ(K/L)$ means sending $t_1 \mapsto \omega$ and $t \mapsto 1$, so that the image $\text{Im } \partial_1(\omega)$ is generated by $dl - (1-\omega)(1-\omega^{-1})dm$. We conclude that

$$(K/L)(\omega) = (1-\omega)(1-\omega^{-1}).$$

The algorithm outlined in this section and using the Wirtinger representation of the knot group was implemented in GAP [GAP15] and used to compute uni- and multivariate slopes of all links with up to eleven crossings (see [KAT]). We observed quite a few interesting examples, some of which are mentioned in §3.4.

3.3. Closed braids. As another example we consider a $(1,1)$ -colored link $K \cup L \subset S^3$, where L is the closure of a braid $\beta \in \mathbb{B}_n$ and K is its axis. Since $\ell k(K, L) = n$, nonvanishing admissible characters are n -th roots of unity $\omega \neq 1$.

By the assumption, $S^3 \setminus K$ is fibered over the circle S^1 , and the fibers D_t are open disks that can all be chosen transversal to L . Let $D^\circ := D_0 \setminus L$, so that \mathbb{B}_n acts on the free group $\pi_1(D^\circ)$. We fix a geometric basis $\alpha_1, \dots, \alpha_n$ for $\pi_1(D^\circ)$ and denote by $\sigma_1, \dots, \sigma_{n-1}$ the corresponding Artin generators of \mathbb{B}_n . Let $\text{deg}: \pi_1(D^\circ) \rightarrow \mathbb{Z}$ and $\text{deg}: \mathbb{B}_n \rightarrow \mathbb{Z}$ be the homomorphisms given by the exponent sum with respect to the chosen bases.

Let $\Lambda := \mathbb{Z}[t^{\pm 1}]$ be the ring of integral Laurent polynomials. Recall that the (*reduced*) *Burau representation* is the homomorphism $\mathbb{B}_n \rightarrow GL(n-1, \Lambda)$ given by the induced \mathbb{B}_n -action on the homology $A_n := H_1(\tilde{D}^\circ)$ of the infinite cyclic covering $\tilde{D}^\circ \rightarrow D^\circ$ corresponding to deg , regarded as a Λ -module *via* the deck translation. Algebraically, $A_n = \text{Ker deg} / [\text{Ker deg}, \text{Ker deg}]$, and t acts on the class $[h]$ of an element $h \in \text{Ker deg}$ *via* $[h] \mapsto [\alpha_1 h \alpha_1^{-1}]$. As a Λ -module, A_n is freely generated by the vectors

$$\mathbf{e}_i := [\alpha_{i+1} \alpha_i^{-1}], \quad i = 1, \dots, n-1;$$

letting $\mathbf{e}_j = 0$ for $j \leq 0$ or $j \geq n$, one has

$$\sigma_i: \mathbf{e}_{i-1} \mapsto \mathbf{e}_{i-1} + t\mathbf{e}_i, \quad \mathbf{e}_i \mapsto -t\mathbf{e}_i, \quad \mathbf{e}_{i+1} \mapsto \mathbf{e}_i + \mathbf{e}_{i+1},$$

(It is understood that $\sigma_i: \mathbf{e}_j \mapsto \mathbf{e}_j$ whenever $|i-j| > 1$.)

Since the action of \mathbb{B}_n preserves the degree, for any pair $\alpha \in \pi_1(D^\circ)$ and $\beta \in \mathbb{B}_n$ we have a well defined projection $\{\alpha, \beta\} := (\alpha\beta) \cdot \alpha^{-1} \in A_n$. It has the following simple properties:

$$(3.12) \quad \begin{aligned} \{\alpha, \beta' \beta''\} &= \{\alpha, \beta'\} \beta'' + \{\alpha, \beta''\} && \text{for } \alpha \in \pi_1(D^\circ) \text{ and } \beta', \beta'' \in \mathbb{B}_n, \\ \{\alpha, \beta^{-1}\} &= -\{\alpha, \beta\} \beta^{-1} && \text{for } \alpha \in \pi_1(D^\circ) \text{ and } \beta \in \mathbb{B}_n. \end{aligned}$$

Pick $\omega = \xi_r := \exp(2\pi i r/n) \neq 1$. A standard computation shows that, for $\bar{X} = S^3 \setminus (K \cup L)$ with L the closure of β , we have

$$(3.13) \quad H_1(\bar{X}; \mathbb{C}(\omega)) = (A_n \otimes_\Lambda \mathbb{C}(\omega)) / (\beta - 1).$$

The image $l \in H_1(\bar{X}; \mathbb{C}(\omega))$ of the longitude of K is the projection of the class of the element $\alpha_1 \dots \alpha_n$ homotopic to ∂D° ; it is easily computed by the Reidemeister–Schreier method:

$$(3.14) \quad l = \sum_{1 \leq i \leq j \leq n-1} \omega^j \mathbf{e}_i.$$

To compute the class of the meridian m , we identify $H_1(\bar{X}; \mathbb{C}(\omega))$ with the ω -eigenspace of t in the homology of the n -fold covering $\tilde{X} \rightarrow \bar{X}$. Represent m by a loop $\gamma \in \pi_1(S^3 \setminus K)$, which we identify with its lift to $\pi_1(\tilde{X})$, and pick an element $\alpha \in \pi_1(D^\circ)$, $d := \text{deg } \alpha \neq 0 \pmod n$. The

homology class of γt^d is represented by $\alpha \gamma \alpha^{-1}$. On the other hand, $\gamma^{-1} \alpha \gamma = \alpha \beta \in \pi_1(\bar{X})$. Hence, $(\omega^d - 1)m = \{\alpha, \beta\}$ and, assuming that $\omega^d \neq 1$ and specializing at $t = \omega$,

$$(3.15) \quad m = \frac{1}{\omega^d - 1} \{\alpha, \beta\}(\omega).$$

In particular, it follows that (3.15) is independent of α . Formally, $\{\alpha, \beta\}$ cannot be computed within the framework of the Burau representation: it is an additional piece of data. In practice, we usually let $\alpha = \alpha_1$ and compute $\{\alpha_1, \beta\}$ inductively, using (3.12) and the obvious identities

$$(3.16) \quad \{\alpha_1, \sigma_1\} = t \mathbf{e}_1, \quad \{\alpha_1, \sigma_i\} = 0 \quad \text{for } i > 1.$$

According to Proposition 3.6, in the space $H_1(\bar{X}; \mathbb{C}(\omega))$ given by (3.13) there is exactly one nontrivial relation $am + bl = 0$ between the vectors m, l given by (3.15) and (3.14), respectively. Moreover, this relation can be chosen to have real coefficients. (We do not know an algebraic proof of these facts.) Thus, we have well defined class functions

$$\beta \mapsto \kappa_r(\beta) := -\frac{b}{a} = [(K/L)(\xi_r)]^{-1} \in \mathbb{R} \cup \infty, \quad r = 1, \dots, n-1.$$

(Note that we define κ_r as the *inverse* of the slope. This choice is motivated by the somewhat better properties of this invariant in the realm of closed braids, see, e.g., Proposition 3.18.)

Numeric experiments reveal rather irregular behaviour of κ : it may take rational or irrational values (e.g., $\kappa_r(\beta) = \frac{3}{5} - \frac{1}{5}(\xi_r + \xi_r^{-1})$ for $\beta = \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_1 \sigma_2 \in \mathbb{B}_5$), and it may take value ∞ (e.g., for $\beta = \sigma_2^{-1} \sigma_1 \in \mathbb{B}_3$). Below are a few further observations concerning these class functions.

Corollary 3.17 (of (3.15)). *If β has an invariant element $\alpha \in \pi_1(D^\circ)$ of degree $d > 0$ (e.g., if $\beta \in \mathbb{B}_d \times \mathbb{B}_{n-d} \subset \mathbb{B}_n$), then $\kappa_r(\beta) = 0$ whenever $rd \not\equiv 0 \pmod{n}$.*

Proposition 3.18. *Let $\beta \in \mathbb{B}_n$ and $p \in \mathbb{Z}$. Then, for any $0 < r < n$, one has:*

- (1) $\kappa_r(\beta^p) = p \kappa_r(\beta)$;
- (2) $\kappa_r(\beta \Delta^2) = \kappa_r(\beta) - 1$;
- (3) $\kappa_r((\sigma_1 \dots \sigma_{n-1})^p) = -p/n$.

(Here, $\Delta \in \mathbb{B}_n$ is the Garside element, so that Δ^2 is the generator of the center of \mathbb{B}_n).

Proof. For (1), assume that $p > 0$; the proof for $p = -1$ and, hence, for $p < 0$ is very similar. From (3.12), it follows that $\{\alpha, \beta^p\} = \{\alpha, \beta\}v$, where $v := 1 + \beta + \dots + \beta^{p-1}$. On the other hand, $\beta^p - 1 = (\beta - 1)v$, and there remains to apply the element $v \in \Lambda$ to the relation

$$am + bl = 0 \pmod{(\beta - 1)},$$

cf. (3.13), computing the value $\kappa_r(\beta) = -b/a$. Here, l and m are the elements of $A_n \otimes_{\Lambda} \mathbb{C}(\xi_r)$ given by (3.14) and (3.15) (with $\omega = \xi_r$), respectively; this determines the action of v . One should also observe that l is \mathbb{B}_n -invariant, as so is $\rho := [\partial D^\circ] = \alpha_1 \dots \alpha_n$, whence $lv = pl$.

For Statement (2), recall that the action of Δ^2 on $\pi_1(D^\circ)$ is the conjugation by ρ^{-1} and, since $\deg \rho = n$, this action is identical on $A_n \otimes_{\Lambda} \mathbb{C}(\omega)$. Then, since l is the image of ρ , we have $\{\alpha_1, \Delta^2\} = \rho^{-1} \alpha_1 \rho \alpha_1^{-1} \mapsto (1 - \omega)l$ and, by (3.15) and (3.12), the new meridian is $m - l$.

For Statement (3), just recall that $\Delta^2 = (\sigma_1 \dots \sigma_{n-1})^n$ and use (1) and (2). \square

3.4. Rationality and Conway functions. In this section, we discuss the rationality of the slope as function of ω and its relation to the Conway function. We illustrate the results with a number of examples, postponing the proofs till the next two sections.

Throughout the section, $K \cup L \subset \mathbb{S}$ is a fixed $(1, \mu)$ -colored link.

Theorem 3.19 (see §3.5). *Pick a component $\mathcal{A} \subset \mathcal{A}(K/L)$ and let r be the minimal integer such that $\Delta_{L,r}|_{\mathcal{A}} \neq 0$, i.e., $\mathcal{A} \setminus \mathcal{V}_{r+1}(L)$ is dense in \mathcal{A} . Denote by R the coordinate ring of \mathcal{A} and fix a normalization of $\Delta_{L,r}$. Then, either*

(1) *there exists a unique polynomial $\Delta_{\mathcal{A}} \in R$ such that*

$$(K/L)(\omega) = \frac{\Delta_{\mathcal{A}}(\omega)}{\Delta_{L,r}(\omega)}$$

holds for each character $\omega \in \mathcal{A}^\circ \setminus \mathcal{V}_{r+1}(L)$, or

(2) *the slope $(K/L)(\omega) = \infty$ is well defined and infinite at each character ω in a certain dense Zariski open subset of \mathcal{A} .*

Case (2) cannot occur if $r = 0$, i.e., if $\Delta_L|_{\mathcal{A}} \neq 0$, cf. [Theorem 3.21](#) below.

Remark 3.20. The slope $(K/L)(\omega)$ at a character $\omega \in \mathcal{A} \cap \mathcal{V}_{r+1}(L)$ does not need to be given by the rational function in [Theorem 3.19\(1\)](#), even if the latter admits an analytic continuation through ω .

In the next theorem, as well as in several other statements below, we need to evaluate the Conway potential at the radical $\sqrt{\omega} := (\sqrt{\omega_1}, \dots, \sqrt{\omega_\mu})$, which is not quite well defined. We use the convention that one of the values of each radical is chosen and used consistently *throughout the whole formula*; then, the nature of the formula guarantees that the result is independent of the initial choice.

Theorem 3.21 (see [§3.6](#)). *For a $(1, \mu)$ -colored link $K \cup L \subset \mathbb{S}$, denote $\nabla' := \frac{\partial}{\partial t} \nabla_{K \cup L}$. Then, for a character $\omega \in \mathcal{A}(K/L)$, one has*

$$(K/L)(\omega) = -\frac{\nabla'(1, \sqrt{\omega})}{2\nabla_L(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

provided that the expression in the right hand side makes sense, i.e., $\nabla'(1, \sqrt{\omega})$ and $\nabla_L(\sqrt{\omega})$ do not vanish simultaneously. In particular, the slope is well defined in this case.

[Theorem 3.21](#) is inconclusive if $\nabla_L(\sqrt{\omega}) = \nabla'(1, \sqrt{\omega}) = 0$: just as in the freshman calculus, the indeterminate form $0/0$ should be resolved by other means. Note also that, even in the case of $(1, 1)$ -coloring (univariate polynomials), l'Hôpital's rule does *not* apply! We illustrate this phenomenon in [Examples 3.28](#) and [3.29](#) below; in a sense, cf. also [Example 3.4](#).

Remark 3.22. The mysterious polynomial $\frac{1}{2}\nabla'$ in the statement can be understood as follows: if ω is admissible, then $\nabla_{K \cup L}(1, \sqrt{\omega}) = 0$, i.e., $\nabla_{K \cup L}(t, \sqrt{\omega}) = (t - t^{-1})R(t)$ for a certain Laurent polynomial $R \in \mathbb{C}[t^{\pm 1}]$, and we substitute $t = 1$ to the residual factor R .

Remark 3.23. [Theorem 3.21](#) can almost be restated in terms of the Alexander rather than Conway polynomials, thus avoiding the radicals: the slope is the only monomial multiple of the ratio

$$\pm \frac{\Delta'_{K \cup L}(1, \omega)}{\Delta_L(\omega)}$$

satisfying [Proposition 3.5](#), i.e., such that $(K/L)(\omega^*) = (K/L)(\omega)$. Unfortunately, this simple description misses one vital bit of information —the sign!

As an indirect consequence of [Theorem 3.21](#), we have that the slope is a multivariate generalization of the Kojima-Yamaski η -function [[KY79](#)].

Corollary 3.24. *Let $K \cup L$ be a two component $(1, 1)$ -colored link such that $\ell k(L, K) = 0$. Then, for any $\omega \in S^1 \setminus 1$ such that $\Delta_L(\omega) \neq 0$, the η -function and the slope coincide at ω .*

[Corollary 3.24](#) follows from the formula in [[Jin88](#)] (first suggested in [[KY79](#), Theorem 1]), computing the η -function in terms of the Alexander polynomials of $K \cup L$ and L . The main theorem in [[Jin88](#)] is stated with the sign ambiguity (cf. our [Remark 3.23](#)), and the reader is told that the sign can be determined *via* Bailey's presentation matrix of the first homology group of

the universal abelian cover of $S^3 \setminus L$. It is worth noticing that the slope is defined in the more general context of links with non-zero linking number. In the restricted case of two component links, it is defined at each root of unity of order $lk(K, L)$, see [Example 3.2](#). Besides, it may contain certain extra information at the roots of Δ_L . (Neither [\[KY79\]](#) nor [\[Jin88\]](#) suggest any clue on the value of the η -function at the zeroes of the denominator.)

Example 3.25 (the Whitehead link). Let $K \cup L$ be the $(1, 1)$ -colored Whitehead link. We have $\nabla_{K \cup L}(t, t_1) = (t - t^{-1})(t_1 - t_1^{-1})$ and $\nabla_L(t_1) = 1/(t_1 - t_1^{-1})$. Hence, for any $\omega \in \mathbb{C}^\times$,

$$(K/L)(\omega_1) = -(\sqrt{\omega_1} - \sqrt{\omega_1^{-1}})^2 = (1 - \omega_1)(1 - \omega_1^{-1}),$$

which agrees with [Example 3.11](#). This example illustrates also the independence of the ratio in [Theorem 3.21](#) of the choice of $\sqrt{\omega}$.

Example 3.26 (equal Alexander polynomials). Let $K \cup L_1 \cup L_2$ and $K' \cup L'_1 \cup L'_2$ be the links [L11n353](#) and [L11n384](#) (see [\[KAT\]](#)), respectively. Both have 11 crossings and 3 components, and their Alexander polynomials are equal:

$$\Delta_{K \cup L} = \Delta_{K' \cup L'} = (t_2 - 1)(t - 1)^3(t_1 - 1), \quad \Delta_L = \Delta_{L'} = 0, \quad \Delta_{L,1} = \Delta_{L',1} = 1,$$

so that [Theorem 3.21](#) is inconclusive. Since $\bar{\ell}k(K, L) = \bar{\ell}k(K', L') = (0, 0)$, one has $\mathcal{A}^\circ(K/L) = \mathcal{A}^\circ(K'/L') = (S^1 \setminus 1)^2$, and a direct computation using the link diagrams (*cf.* [§3.2](#)) gives us, for any $\omega := (\omega_1, \omega_2) \in (\mathbb{C}^\times)^2$, up to the common normalizing factor $-(\omega_1 \omega_2)^{-1}$,

$$(K/L)(\omega) = (\omega_1 \omega_2^2 + \omega_1^2 - 4\omega_1 \omega_2 + \omega_2^2 + \omega_1), \quad (K'/L')(\omega) = (\omega_1 - 1)(\omega_1 \omega_2^2 - 1).$$

Thus, the slope can distinguish links with equal Alexander polynomials. (Here and in the next example, since the first nonvanishing order $\Delta_{L,1}$ is identically 1, the slope is given by a Laurent polynomial on the whole torus $(\mathbb{C}^\times)^2$, see [Theorem 3.19](#).)

Example 3.27 (vanishing Alexander polynomial). Let $K \cup L_1 \cup L_2$ be the link [L11n396](#) in [\[KAT\]](#) with 11 crossings and 3 components. Both polynomials $\Delta_{K \cup L}$ and Δ_L vanish identically, and hence [Theorem 3.21](#) fails. One has $\bar{\ell}k(K, L) = (0, 0)$ and $\mathcal{A}^\circ(K/L) = (\mathbb{C}^\times)^2$. A direct computation using the link diagrams (*cf.* [§3.2](#)) gives us, for any $\omega := (\omega_1, \omega_2) \in (\mathbb{C}^\times)^2$,

$$(K/L)(\omega) = -(\omega_1 \omega_2 - 1)^2 / \omega_1 \omega_2.$$

Furthermore, the first non-trivial orders are $\Delta_{K \cup L,1} = (t - 1)(t^2 - t + 1)$ and $\Delta_{L,1} = 1$. These last two examples suggest that the slope is independent of the higher order Fitting ideals.

Example 3.28 (l'Hôpital's rule). Consider the family of two component algebraically split links $K \cup L$ described in the left diagram of [Figure 1](#). The knot $L \subset S^3$ depends on two parameters: a stands for an odd number of half crossings between the two bands, while c stands for an arbitrary number of full crossings between the two strands. The sign of these two parameters determines whether the crossings are positive or negative. In the right diagram of [Figure 1](#) the case $a = 3$ and $c = 6$ is shown. Setting $b := (a - 1)/2$, we obtain that the Seifert matrix for the Seifert surface F in [Figure 1](#) is given by

$$\Theta = \begin{bmatrix} 0 & b + 1 \\ b & c \end{bmatrix},$$

and the roots of Δ_L are $\omega_\pm := (1 + 1/b)^{\pm 1}$. The component K in [Figure 1](#) is the unknot, and the parameters x and y on the left diagram stand for the linking numbers between K and the fixed generators for $H_1(F)$, depicted in the figure to the right. In [Figure 1](#), $x \leq 0$ and $y \geq 0$; changing the direction of the twisting of K around the bands, one can obtain the other signs. We denote $\lambda_K := (x, y) \in \mathbb{Z}^2$. (In fact, K does not need to be an unknot: the computation *via*

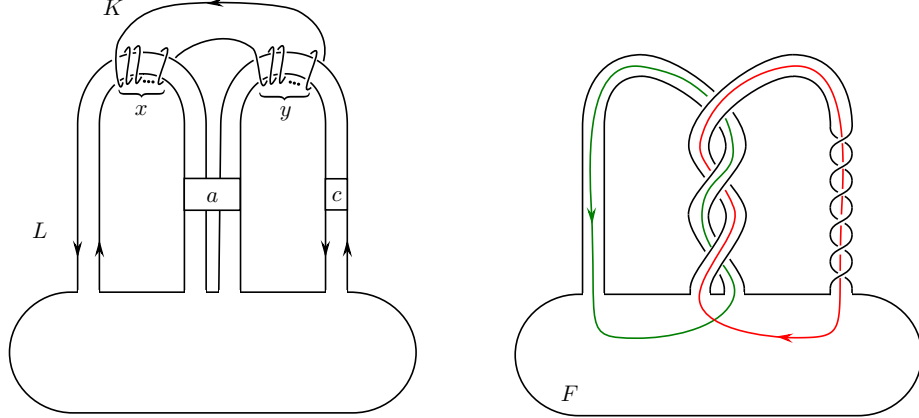


FIGURE 1. The diagram to the left shows the oriented two component algebraically split link $K \cup L$ of [Example 3.28](#). The one to the right shows the knot L with parameters $a = 3$ and $c = 6$.

Seifert surfaces, which will be explained in [\[DFL\]](#), only makes use of the linking homomorphism $H_1(F) \rightarrow \mathbb{Z}$, $\alpha \mapsto \ell k(\alpha, K)$, *i.e.*, of λ_K .

Computing the slope (see [\[DFL\]](#)), we obtain

$$(K/L)(\omega) = -\frac{\omega x(2by + y - cx)}{(\omega b - b - 1)(\omega b + \omega - b)}$$

for $\omega \neq 1, \omega_{\pm}$. Let $\lambda_+ := (2b + 1, c)$ and $\lambda_- := (0, 1)$. Further analysis using [\[DFL\]](#) shows that

- (1) if $\lambda_K \neq 0, \lambda_{\pm}$, then $(K/L)(\omega_{\pm}) = \infty = \lim_{\omega \rightarrow \omega_{\pm}} (K/L)(\omega)$;
- (2) if $\lambda_K = 0$, then $(K/L)(\omega) = 0$ for all $\omega \in \mathbb{C}^{\times} \setminus 1$;
- (3) if $\lambda_K = \lambda_{\pm}$, then $(K/L)(\omega) = 0$ for all $\omega \in \mathbb{C}^{\times} \setminus \{1, \omega_{\pm}\}$. However, in this case we have $\dim \mathcal{Z}(\omega_{\pm}) = 2$ and $\dim \mathcal{Z}(\omega_{\mp}) = 0$, *i.e.*, the slopes $(K/L)(\omega_{\pm})$ are not defined, even though $\lim_{\omega \rightarrow \omega_{\pm}} (K/L)(\omega) = 0$, (*cf.* [Remark 3.20](#)).

Now, we can consider a connected sum L' of two copies of L , so that the new Seifert matrix is $\Theta \oplus \Theta$, and choose K' so that the linking homomorphism is represented by $\lambda_{K'} := \lambda_+ \oplus \lambda_-$. Then, $(K'/L')(\omega) = 0$ on $\omega \in \mathbb{C}^{\times} \setminus \{1, \omega_{\pm}\}$ and $(K'/L')(\omega_{\pm}) = \infty$, whereas $\lim_{\omega \rightarrow \omega_{\pm}} (K'/L')(\omega) = 0$.

In the previous example, the “special” characters ω_{\pm} are not unitary. The next one is more advanced, although slightly less explicit.

Example 3.29 (l’Hôpital’s rule again). Consider the two-component link [L10a39](#) (see [\[KAT\]](#)). Taking for K the unknotted component, we have

$$\nabla_{K \cup L} = -(t - t^{-1})(t_1 - t_1^{-1})(t_1^2 - 1 + t_1^{-2})^2, \quad \nabla_L = (t_1^2 - 1 + t_1^{-2})^2 / (t_1 - t_1^{-1}),$$

so that

$$(K/L)(\omega) = (\sqrt{\omega} - \sqrt{\omega^{-1}})^2 = -(1 - \omega)(1 - \omega^{-1})$$

(*cf.* [Example 3.25](#)) unless $\Delta_L(\omega) = 0$, *i.e.*, $\omega = \omega_{\pm} := (1 \pm i\sqrt{3})/2$ is a primitive 6-th root of unity. A separate computation using [§3.2](#) shows that $(K/L)(\omega_{\pm}) = \infty$ instead of the “predicted” -1 . Similarly, for [L10n36](#) one has $K/L \equiv 0$ except $(K/L)(\omega_{\pm}) = \infty$.

Example 3.30 (non-vanishing linking numbers). Consider the link [L10n85](#) (see [\[KAT\]](#)) with three (trivial) components $C_1 \cup C_2 \cup C_3$ ordered and oriented so that

$$\ell k(C_1, C_2) = 1, \quad \ell k(C_1, C_3) = 0, \quad \ell k(C_2, C_3) = 2.$$

We have

$$\nabla = (t_2^{-1} - t_2)(t_1^2 t_2^{-1} t_3 + t_1^{-2} t_2 t_3^{-1} - t_1^2 t_2^{-1} t_3^{-1} - t_1^{-2} t_2 t_3 - 2t_2^{-1} t_3 - 2t_2 t_3^{-1} + t_2^{-1} t_3^{-1} + t_2 t_3)$$

for the Conway potential of $C_1 \cup C_2 \cup C_3$. There are three possible choices for the distinguished component K (where we keep the colors 1, 2, 3, making one of them distinguished and showing the corresponding value $t_i = \mathbf{1}$ in bold):

- (1) If $K = C_1$ and $L = C_2 \cup C_3$, then $\bar{\ell}k(K, L) = (1, 0)$ and $\mathcal{A}(K/L) = \{\mathbf{1}\} \times \{1\} \times \mathbb{C}^\times$. In view of [Remark 2.28](#), the slope (K/L) is defined by patching the component C_2 , and we have $(K/L)(1, \omega) = (C_1/C_3)(\omega) = 0$ since $C_1 \cup C_3$ is the trivial link.
- (2) If $K = C_2$ and $L = C_1 \cup C_3$, then $\bar{\ell}k(K, L) = (1, 2)$ and

$$\mathcal{A}(K/L) = \{(\omega^2, \mathbf{1}, \omega^{-1}) \mid \omega \in \mathbb{C}^\times\}.$$

We have $\Delta_L = 0$ and $\Delta_{L,1} = 1$. Moreover, $\nabla_2(u, \mathbf{1}, u^{-1/2}) = 2u^{-5/2}(u+1)(u^2 - u + 1)^2$, whose roots are the 3-rd roots of -1 , *i.e.*, -1 and $-\xi_3^{\pm 1}$. At $u = 1$, the character stops being nonvanishing: C_1 needs to be patched, and the slope of the resulting link [L4a1](#) at $\omega = -1$ is given by [Proposition 3.18](#). The slopes at $u = -\xi_3^{\pm 1}$ are computed directly, as in [§3.2](#). We obtain

$$(K/L)(\omega^2, \omega^{-1}) = \begin{cases} 0, & \text{if } \omega = 1, \\ -2, & \text{if } \omega^3 = -1, \\ \infty, & \text{otherwise.} \end{cases}$$

- (3) If $K = C_3$ and $L = C_1 \cup C_2$, then $\bar{\ell}k(K, L) = (0, 2)$ and $\mathcal{A}(K/L) = \mathbb{C}^\times \times \{-1, 1\} \times \{\mathbf{1}\}$ consists of two components. For all $\omega \in \mathbb{C}^\times$ we have $(K/L)(\omega, 1) = (C_3/C_1)(\omega) = 0$ as in case (1), whereas $(K/L)(\omega, -1) = 2(\omega - 3 + \omega^{-1})$ is given by [Theorem 3.21](#), since L is the positive Hopf link and $\nabla_L = 1$ (*cf.* also case (2) for the value at $(1, -1)$).

A number of other examples are given by the univariate specialization of the slopes of many table links with three or more components. We have observed all sorts of behaviour of K/L at unitary roots of Δ_L : continuity, infinite value *vs.* finite limit, finite value *vs.* infinite limit, *etc.*

3.5. Proof of [Theorem 3.19](#). After a sequence of elementary collapses preserving $\partial_K \bar{X}$, we can assume that $\bar{X} = \mathbb{S} \setminus (K \cup L)$ is a 2-complex. Furthermore, the *CW*-decomposition of the torus $\partial_K \bar{X}$ can be chosen standard, with a single 0-cell e^0 , two 1-cells m, l representing the meridian and longitude, respectively, and a 2-cell k^2 . Consider the complexes

$$\bar{S}_* := C_*(\bar{X}; R) \subset S_* := C_*(X; R).$$

Since they are generated by lifts of the cells to the corresponding coverings, we retain the same orientation and notation as for the original cells of \bar{X} and X . By the construction of \mathcal{A} and R , both m and l are cycles in \bar{S}_* . Furthermore, $S_1 = \bar{S}_1$ and the image $\partial_1 S_2$ differs from $\partial_1 \bar{S}_2$ by an extra generator $m = \partial_1 e^2$ (for some extra 2-cell e^2). Thus, we have the class

$$[l] \in H_1(X; R) \subset H_1(X, x_0; R) = \bar{S}_1 / (\partial_1 \bar{S}_2 + Rm),$$

where $x_0 \in X$ is the basepoint. The R -module $H_1(X, x_0; R)$ gives rise to a coherent sheaf \mathcal{H} on \mathcal{A} , so that $H_1(X, x_0; R) = \Gamma(\mathcal{A}; \mathcal{H})$, which restricts to a locally free sheaf (*aka* vector bundle) over $\mathcal{A} \setminus \mathcal{V}_{r+1}(L)$: the fiber over ω is $H_1(X, x_0; \mathbb{C}(\omega))$, which has constant rank $(r+1)$. Hence, a section s of \mathcal{H} is in the torsion submodule $\text{Tors}_R H_1(X, x_0; R)$ if and only if the support of s is contained in $\mathcal{A} \cap \mathcal{V}_{r+1}(L)$.

We consider separately two cases.

Case (1): $[l] \in \text{Tors}_R H_1(X, x_0; R)$. For any character $\omega \in \mathcal{A}^\circ \setminus \mathcal{V}_{r+1}(L)$, there is a polynomial $p \in R$ such that $p(\omega) \neq 0$ and $p[l] = 0$ in $H_1(X, x_0; R)$. This means that $pl = \tilde{p}m \bmod \partial_1 \bar{S}_2$ for

some $\tilde{p} \in R$ and, specializing at ω , we obtain a nontrivial relation $p(\omega)l = \tilde{p}(\omega)m$ in $H_1(\bar{X}; \mathbb{C}(\omega))$; in particular, $\dim \mathcal{Z}(\omega) \geq 1$. Repeating this argument for ω^* (recall that $\mathcal{V}_{r+1}(L)$ is symmetric, at least away from the divisors $\omega_i = 1$), we conclude that the slope at ω is well defined and the relation above is the only one, yielding

$$(K/L)(\omega) = \frac{\tilde{p}(\omega)}{p(\omega)}.$$

This identity holds on the dense set $\{\omega \in \mathcal{A}^\circ \mid p(\omega) \neq 0\}$; in particular, $\tilde{p} \in R$ is uniquely determined by p . Any two rational functions obtained in this way coincide on a dense set; hence, their reduced forms are equal, and altogether these functions give rise to a rational function on $\mathcal{A}^\circ \setminus \mathcal{V}_{r+1}(L)$. The denominator of this common fraction must divide the order $\Delta_{L,r} = \text{g.c.d. } E_{r+1}(H_1(X, x_0; \mathbb{Z}H))$.

Case (2): $[l] \notin \text{Tors}_R H_1(X, x_0; R)$. In this case, the zero set $Z \subset \mathcal{A} \setminus \mathcal{V}_{r+1}(L)$ of $[l]$ regarded as a section of the restricted vector bundle \mathcal{H} is a proper Zariski closed set; by [Proposition 3.7](#), the dense Zariski open subset as in the statement is $\mathcal{A}^\circ \setminus (\mathcal{V}_{r+1}(L) \cup Z \cup Z^*)$, where we let $Z^* := \{\omega^* \mid \omega \in Z\}$.

The last statement of the theorem follows from [Corollary 3.8](#). \square

3.6. Proof of [Theorem 3.21](#). As in the proof of [Theorem 3.19](#), after elementary collapses preserving $\partial_K \bar{X}$, we assume that \bar{X} is a 2-complex and that the CW-decomposition of $\partial_K \bar{X}$ has a single 0-cell e^0 , two 1-cells m, l representing the meridian and longitude, respectively, and a 2-cell k^2 . We also assume that e^0 is the only 0-cell of \bar{X} and that the meridian m_C of one fixed component $C \subset L$ is represented by a 1-cell m_i . Denoting by r the number of 2-cells, the number of 1-cells is $r + 1$. (Recall that $\chi(\bar{X}) = 0$.) We require that the lift of each 1-cell e^1 starts at the chosen lift of e^0 , and we order the resulting bases of $\bar{C}_* := C_*(\bar{X}; \mathbb{Z}\bar{H})$ as follows:

$$(3.31) \quad \{e^0\} \subset \bar{C}_0, \quad \{m, l, \dots, m_i\} \subset \bar{C}_1, \quad \{k^2, \dots\} \subset \bar{C}_2.$$

The same vectors form a basis for $\bar{S}_* := \bar{C}_* \otimes_{\mathbb{Z}\bar{H}} \mathbb{Q}(\bar{H})$.

Up to the same simple homotopy equivalence, the space X is obtained from \bar{X} by adjoining an extra 2-cell e^2 bounded by m and an extra 3-cell e^3 filling the torus T_K . Besides, the passage from $\mathbb{Z}\bar{H}$ to $\mathbb{Z}H$ is the specialization of the coefficients at $t = 1$. To respect the order, the generator e^2 is to be inserted right after k^2 ; thus, the distinguished bases for $C_* := C_*(X; \mathbb{Z}H)$ are

$$(3.32) \quad \{e^0\} \subset C_0, \quad \{m, l, \dots, m_i\} \subset C_1, \quad \{k^2, e^2, \dots\} \subset C_2, \quad \{e^3\} \subset C_3.$$

We orient and lift e^2 and e^3 so that $\partial_2 e^3 = k^2$ and $\partial_1 e^2 = m$.

First, assume that $\omega \in \mathcal{A}^\circ(K/L)$. Let $b_2 := \{k^2, \dots\}$ be the chosen basis for \bar{S}_2 , see [\(3.31\)](#). With appropriate orientation and lift of k^2 , the matrix of ∂_1 has the form

$$(3.33) \quad \begin{bmatrix} 1 - \bar{\varphi}(l) & t - 1 & \mathbf{0} & 0 \\ \mathbf{a} & \mathbf{b} & M & \mathbf{c} \end{bmatrix},$$

where M is a certain $(r-1) \times (r-2)$ -matrix, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are certain column vectors of dimension $r-1$, and $\mathbf{0}$ is the trivial row vector of dimension $(r-2)$. The vector $\partial_0(m_i) = (t_i - 1)e^0 \neq 0$ forms a basis for \bar{S}_0 . It follows that the complex \bar{S}_* is acyclic if and only if $\partial_1(b_2)m_i$ is a basis for \bar{S}_1 , i.e., if the determinant D_1 of the matrix

$$(3.34) \quad T_1 := \begin{bmatrix} 1 - \bar{\varphi}(l) & t - 1 & \mathbf{0} & 0 \\ \mathbf{a} & \mathbf{b} & M & \mathbf{c} \\ 0 & 0 & \mathbf{0} & 1 \end{bmatrix}$$

does not vanish. Then, letting $b_0 = \emptyset$ and $b_1 = \{m_i\}$ in §2.3, we obtain

$$\tau_{\bar{\varphi}}(\bar{X})(t, t_1, \dots, t_\mu) = \frac{-(t-1) \det[\mathbf{a} | M] - (\bar{\varphi}(l) - 1) \det[\mathbf{b} | M]}{t_i - 1}.$$

Specializing this at (t, ω) (and thus letting $\bar{\varphi}(l) = 1$), we arrive at

$$\tau_{\bar{\varphi}}(\bar{X})(t, \omega) = -\frac{(t-1) \det[\mathbf{a} | M](t, \omega)}{\omega_i - 1}, \quad \tau'_{\bar{\varphi}}(\bar{X})(1, \omega) = -\frac{\det[\mathbf{a} | M](1, \omega)}{\omega_i - 1}.$$

A similar computation, using the bases (3.32) and matrices

$$\partial_1: \begin{bmatrix} 0 & 0 & \mathbf{0} & 0 \\ 1 & 0 & \mathbf{0} & 0 \\ \mathbf{a} & \mathbf{b} & M & \mathbf{c} \end{bmatrix}_{t=1}, \quad T_1 = \begin{bmatrix} 1 & 0 & \mathbf{0} & 0 \\ \mathbf{a} & \mathbf{b} & M & \mathbf{c} \\ 0 & 0 & \mathbf{0} & 1 \end{bmatrix}_{t=1}$$

instead of (3.33) and (3.34) gives us

$$\tau_{\varphi}(X)(\omega) = \frac{\det[\mathbf{b} | M](1, \omega)}{\omega_i - 1},$$

no matter whether $C_*(X; \mathbb{Q}(H))$ is acyclic or not.

Now, in order to compute the slope, we consider the complex $C_*(\bar{X}; \mathbb{C}(\omega))$, which is merely $C_*(\bar{X}; \mathbb{Z}\bar{H})$ specialized at $(1, \omega)$; we assume this specialization till the end of the computation. Clearly, a linear combination $am + bl$ represents a class in $\mathcal{Z}(\omega)$ if and only if it is in the image of ∂_1 , which is given by (3.33), *i.e.*, essentially, by $[\mathbf{a} | \mathbf{b} | M | \mathbf{c}]$, as the first row vanishes under the specialization. For a finite slope κ , we need $-\kappa m + l \in \text{Im } \partial_1$; clearly, $\kappa = -\mathbf{x} \cdot \mathbf{a}$, where \mathbf{x} is a solution to the overdetermined linear system

$$\mathbf{x} \cdot [\mathbf{b} | M | \mathbf{c}] = [1 | \mathbf{0} | 0].$$

If $\tau_{\varphi}(X)(\omega) \neq 0$, then $\det[\mathbf{b} | M] \neq 0$ and, disregarding the last column \mathbf{c} , we see that $\mathcal{Z}(\omega)$ contains at most one vector as above, with κ given by Cramer's rule

$$(3.35) \quad \kappa = -[1 | \mathbf{0}] \cdot [\mathbf{b} | M]^{-1} \cdot \mathbf{a} = -\frac{\det[\mathbf{a} | M]}{\det[\mathbf{b} | M]} = \frac{\tau'_{\bar{\varphi}}(\bar{X})(1, \omega)}{\tau_{\varphi}(X)(\omega)},$$

in particular, $\dim \mathcal{Z}(\omega) \leq 1$. Replacing ω with the dual character ω^* and repeating the same argument, we conclude that also $\dim \mathcal{Z}(\omega^*) \leq 1$; hence, both slopes are well defined and $\kappa = (K/L)(\omega)$ is given by (3.35).

If $\tau_{\varphi}(X)(\omega) = 0$ but $\tau'_{\bar{\varphi}}(\bar{X})(1, \omega) \neq 0$, *i.e.*, $\det[\mathbf{b} | M] = 0$ and $\det[\mathbf{a} | M] \neq 0$, then, arguing as above and searching for vectors $m - \kappa l \in \mathcal{Z}(\omega)$, we conclude that $\kappa = 0$, *i.e.*, the slope is well defined and equals ∞ .

To complete the proof, we need to take into account the ambiguity of torsion: (3.35) was obtained by computing both torsions in compatible distinguished bases.

First, compare the signs (assuming both torsions nonvanishing). We will use the notation introduced in §2.3 for $C_*(X; \mathbb{R})$ and its barred counterparts for $C_*(\bar{X}; \mathbb{R}) \subset C_*(X; \mathbb{R})$. Let n be the number of components of L and r , as above, the number of 2-cells in \bar{X} . Clearly,

$$b_0 = \bar{b}_0 = b_1 = \bar{b}_1 = \emptyset, \quad b_3 = \{e^3\} = c_3, \quad h_0 = \bar{h}_0 = \{e^0\} = c_0 = \bar{c}_0,$$

so that all transition matrices except T_i, \bar{T}_i , $i = 1, 2$, are the identities. We can assume that \bar{h}_1 is obtained from h_1 by prepending $m = \partial_1 e^2$. Similarly, we can select $h_2 \subset \bar{C}_2$ and assume that \bar{h}_2 is obtained by prepending $k^2 = \partial_2 e^3$ to h_2 , whereas b_2 is obtained by prepending e^2 to \bar{b}_2 . Then

$$\partial_1(b_2)h_1b_1 = (m)\partial_1(\bar{b}_2)h_1, \quad \partial_1(\bar{b}_2)\bar{h}_1\bar{b}_1 = \partial_1(\bar{b}_2)(m)h_1$$

and, thus, $\det T_1 / \det \bar{T}_1 = (-1)^{|\bar{b}_2|} = (-1)^{r-n}$, whereas in dimension 2 we have

$$\partial_2(b_3)h_2b_2 = (k^2)h_2(e^2)\bar{b}_2, \quad \partial_2(\bar{b}_3)\bar{h}_2\bar{b}_2 = (k^2)h_2\bar{b}_2;$$

since, on the other hand, c_2 is obtained from \bar{c}_2 by inserting e^2 as the second vector, it follows that $\det T_2 / \det \bar{T}_2 = (-1)^{|h_2|} = (-1)^{n-1}$. Since $|C_*(X; \mathbb{R})| = 1 + nr - r$ and $|C_*(\bar{X}; \mathbb{R})| = 1 + nr$, we conclude that $\tau_0(X) = -\tau_0(\bar{X})$, *i.e.*, one should reverse the sign in (3.35) when switching to the sign-determined torsions.

After passing to the sign-determined torsions, still the quotient in the right hand side of (3.35) is only well defined up to multiplicative units H , and there is at most one renormalization of this quotient invariant under the involution $(t_1, \dots, t_\mu) \mapsto (t_1^{-1}, \dots, t_\mu^{-1})$, see Proposition 3.5. The passage to the Conway functions, through (2.25), is a way to obtain such a renormalization (*cf.* also Remark 3.23).

Finally, if $\omega \notin \mathcal{A}^\circ(K/L)$, we patch the sublinks L_i corresponding to the vanishing components $\omega_i = 1$ and express the slope in terms of the Conway functions of the two smaller links obtained. The passage to the Conway functions of the original links is immediate *via* the classical Torres relations. (The original article [Tor53] deals with the Alexander polynomials, and the translation to the Conway function case is found in [Cim04, Proposition 7].) Note that, in this last passage, information may be lost, as we may have to multiply both functions by 0. \square

4. MULTIVARIATE SIGNATURE OF COLORED LINKS

Classically, the 4-dimensional approach to the multivariate signature of a colored link uses branched covers and the G -signature theorem (see, among others, [CF08, DFL17]). Viro [Vir09] suggested an alternative construction, *via* regular coverings of the complement of the branching surfaces and cobordisms arguments. This view point (*cf.* also [CNT17]) allows one to extend the signature from rational characters to the whole character torus $(S^1 \setminus 1)^\mu$. In this section, we further extend Viro's construction to links in integral homology spheres. At the end, we also deal with the subtleties of vanishing characters, studying the literal extension of the signature (*cf.* Remark 2.28) in some special cases. We advise the reader that most results of this section apply to *unitary* characters only.

4.1. Spanning pairs. Let N be a compact smooth oriented 4-manifold with boundary ∂N . Recall that a compact smooth oriented surface $F \subset N$ is said to be *properly embedded* if $\partial F = F \cap \partial N \neq \emptyset$ and F is transversal to ∂N along ∂F . We define a *properly immersed* surface $F \subset N$ as a finite union $\bigcup_i F_i$ of connected properly embedded surfaces $F_i \subset N$ such that all pairwise intersections of the components F_i are transversal, at double points, and away from the boundary ∂N . By a *tubular neighborhood* of F in N we mean an *open* regular neighborhood $T := T_F \supset F$ which is a union of tubular neighborhoods $T_i \supset F_i$.

Lemma 4.1. *Let $F = \bigcup_i F_i \subset N$ be a properly immersed surface. Fix a tubular neighborhood $T \supset F$ and let $W := W_F = N \setminus T$. Then, the following three statements are equivalent:*

- $[F_i, \partial F_i] = 0 \in H_2(N, \partial N; \mathbb{Q})$ for each index i ;
- the inclusion homomorphism $H_2(W; \mathbb{Q}) \rightarrow H_2(N; \mathbb{Q})$ is an epimorphism;
- the meridians m_i of all F_i are linearly independent in $H_1(W)$.

Furthermore, the group $H_1(W)$ is generated by the meridians m_i if and only if $H_1(N) = 0$.

Proof. Let B be the union of all pairwise intersections $T_i \cap T_j$, $i \neq j$: it is the union of small balls about the points of intersection of the components of F . Consider $\bar{N} := N \setminus B$ and let $\bar{F} := F \cap \bar{N}$ and $\bar{T} := T \cap \bar{N}$. Note that $\bar{F} \subset \bar{N}$ is a properly embedded surface and $\bar{T} \supset \bar{F}$ is a tubular neighborhood in the usual sense. Furthermore, up to homotopy equivalence, \bar{N} is obtained from N by removing a finite set of points, *i.e.*, a subset of codimension 4; therefore,

the inclusion homomorphisms $H_n(\bar{N}) \rightarrow H_n(N)$ are isomorphisms for $n = 0, 1, 2$, and so are the homomorphisms $H_2(\bar{N}, \partial\bar{N}) \rightarrow H_2(N, B \cup \partial N) \leftarrow H_2(N, \partial N)$.

Consider the exact sequence of the pair (\bar{N}, W) :

$$(4.2) \quad \longrightarrow H_2(W) \longrightarrow H_2(\bar{N}) \xrightarrow{\text{rel}} H_2(\bar{N}, W) \xrightarrow{\partial} H_1(W) \longrightarrow H_1(\bar{N}) \longrightarrow H_1(\bar{N}, W) \longrightarrow .$$

By the excision and Thom isomorphism, we have

$$H_n(\bar{N}, W) = H_n(\bar{T}, \bar{S}) = H_{n-2}(\bar{F})$$

(where \bar{S} is the S^1 -bundle associated with the disk bundle $\bar{T} \rightarrow \bar{F}$). In particular, the last term in (4.2) vanishes, whereas the group $H_2(\bar{N}, W)$ is generated by the classes d_i of fibers of the disk bundles $\bar{T}_i \rightarrow \bar{F}_i$, so that $\partial d_i = m_i$. This completes the proof of the last statement, and the second statement follows immediately.

For the first statement, there remains to observe that the homomorphism $\text{rel} \otimes \mathbb{Q}$ is given by $x \mapsto \sum_i (x \circ [\bar{F}_i, \partial\bar{F}_i])d_i$ and, by Poincaré–Lefschetz duality, $\text{rel} \otimes \mathbb{Q} = 0$ if and only if all classes $[\bar{F}_i, \partial\bar{F}_i]$ vanish in $H_2(\bar{N}, \partial\bar{N}; \mathbb{Q}) = H_2(N, \partial N; \mathbb{Q})$. \square

Definition 4.3. Let L be a μ -colored link in an integral homology sphere \mathbb{S} . A *spanning pair* for (\mathbb{S}, L) is a pair (N, F) , where N is a compact smooth oriented 4-manifold such that $\partial N = \mathbb{S}$ and $F = F_1 \cup \dots \cup F_\mu \subset N$ is a properly immersed surface such that $\partial F_i = F_i \cap \partial N = L_i$ for all $i = 1, \dots, \mu$. We require in addition that $H_1(N) = 0$ and $[F_i, \partial F_i] = 0 \in H_2(N, \partial N)$ for each index i ; equivalently, we require that the group $H_1(N \setminus F)$ should be freely generated by the meridians of the components of F .

The existence of a spanning pair for a colored link is given by [DFL17, Proposition 3.4].

Proposition 4.4. Fix a μ -colored link $L \subset \mathbb{S}$, consider a spanning pair (N, F) , and let $T := T_F$ be a tubular neighborhood of F in N . Then:

- (1) each character on $\mathbb{S} \setminus L$ extends to a unique character on $N \setminus T$;
- (2) for each index i , the Seifert framing of L_i extends to a framing of F_i ;
- (3) for each pair $i \neq j$, the algebraic intersection $F_i \circ F_j$ equals $\ell k(L_i, L_j)$.

Proof. Statement (1) is given by Lemma 4.1, since a character, both on L and on $N \setminus T$, is uniquely determined by its values on the meridians. Statements (2) and (3) follow from the assumption that the classes $[F_i, \partial F_i]$ vanish in $H_2(N, \partial N)$. \square

4.2. Invariance of the signature. Fix a μ -colored link $L \subset \mathbb{S}$. Given a spanning pair (N, F) , we fix an open tubular neighborhood T_F of F and let $W_F := N \setminus T_F$.

Recall that, according to Proposition 4.4, any character ω on $\mathbb{S} \setminus L$ extends to a unique character on W_F ; for this reason, we retain the same notation ω for the extension. In this section, we consider unitary characters only.

Definition 4.5. The *signature* of a μ -colored link $L \subset \mathbb{S}$ is the map

$$\sigma_L: (S^1 \setminus 1)^\mu \rightarrow \mathbb{Z}, \quad \omega \mapsto \text{sign}^\omega(W_F) - \text{sign}(W_F).$$

Following Remark 2.28, we extend the signature function to arbitrary characters ω by patching the components of the link on which ω vanishes. Occasionally (most notably, in the proof of Theorem 5.3), we need to use the literal extension $\tilde{\sigma}_L$ of the signature, which is not very well defined; we discuss these subtleties in Lemma 4.9 below.

In view of this definition, in the rest of the paper we mainly confine ourselves to nonvanishing characters. Furthermore, we usually use the following alternative definition:

$$(4.6) \quad \sigma_L(\omega) = \text{sign}^\omega(W_F) - \text{sign}(N);$$

indeed, by [Lemma 4.1](#), the isometry $H_2(W_F; \mathbb{Q}) \rightarrow H_2(N; \mathbb{Q})$ is surjective and the two forms have the same signature.

Theorem 4.7. *The signature σ_L is independent of the choice of a spanning pair (N, F) .*

Proof. The proof is essentially that of [[Vir09](#), Theorem 2.A]; we merely fill in a few details.

Given two spanning pairs (N', F') , (N'', F'') , consider the closed manifold $N := N' \cup_{\mathbb{S}} -N''$ and closed surface $F := F' \cup -F'' \subset N$. The character ω on N' , N'' defines a character, also denoted by ω , on $W := N \setminus T_F$. We can assume that the tubular neighborhoods $T_{F'} \subset N'$ and $T_{F''} \subset N''$ cut the same tubular neighborhood $T_L \subset \mathbb{S}$ of L . Then, $W = W' \cup_{\mathbb{S} \setminus T_L} -W''$ and, by [Theorem 2.16](#), we have

$$\text{sign}(N) = \text{sign}(N') - \text{sign}(N''), \quad \text{sign}^\omega(W) = \text{sign}^\omega(W') - \text{sign}^\omega(W'').$$

(Indeed, in the former case, the ‘‘corner locus’’ T in [Theorem 2.16](#) is empty, and in the latter case $T = T_L$ is the union of tori, each with nontrivial restriction of ω , so that $H_*(T; \mathbb{C}(\omega)) = 0$ by [Corollary 2.4](#).) Thus, there remains to prove that $\text{sign}^\omega(W) = \text{sign}(N)$.

By the definition of spanning pair and Mayer–Vietoris exact sequence, we have $H_1(N) = 0$ and $[F_i] = 0 \in H_2(N)$ for each component F_i of F . Since $H_2(X) = \Omega_2(X)$ for any CW-complex X , each component F_i is null-cobordant in N . Pick a cobordism, push it off to the cylinder $N \times I$, and smoothen the result to obtain a smooth 3-manifold $D_i \subset N \times I$ transversal to N along the boundary $\partial D_i = F_i$. Do this for each surface F_i and put the results in general position to obtain an immersed 3-manifold $D := \bigcup_i D_i \subset N \times I$. Let $T_D \subset N \times I$ be a tubular neighborhood of D , and consider the 5-manifold $U := (N \times I) \setminus T_D$. It is immediate (cf. [Lemma 4.1](#) and the beginning of this subsection) that $H_1(U)$ is generated by the meridians about the components D_i and, hence, ω extends to a unique character (also denoted by ω) on U ; thus, by [Corollary 2.15](#), we have $\text{sign}^\omega(\partial U) = 0$. On the other hand,

$$\partial U = W \cup_{\partial T_F} \partial T_D \sqcup -N$$

(where $W \subset N \times \{0\}$ and the other copy of N is $N \times \{1\}$) and the manifold ∂T_D is obtained by gluing, along whole components of boundaries, several 4-manifolds fibered into circles. Hence, by [Theorem 2.16](#) and [Corollary 2.3](#), we have $\text{sign}^\omega(\partial T_D) = 0$ and

$$0 = \text{sign}^\omega(\partial U) = \text{sign}^\omega(W) - \text{sign}(N),$$

as stated. (Note that $H_1(N) = 0$ and, hence, $\omega = 1$ on $N \times \{1\}$.) □

4.3. Previous versions of the signature. Classically, the signature was defined only for characters of finite order, via ramified coverings, and its invariance was proved using the G -signature theorem. We recall briefly the constructions; a more detailed exposition can be found in [[DFL17](#)]. Let (N, F) be a spanning pair for (\mathbb{S}, L) and $\omega \in (\mathbb{Q}/\mathbb{Z})^\mu \subset (S^1)^\mu$ a character of finite order. Then, ω defines a normal covering $N^G \rightarrow N$ with finite abelian group $G \cong \text{Im } \omega$ of deck translations. Regarding $H_2(N^G; \mathbb{C})$ as a $\mathbb{C}[G]$ -module, we consider the eigenspace

$$H_2^\omega(N, F) := H_2(N^G; \mathbb{C}) \otimes_{\mathbb{C}[G]} \mathbb{C}$$

and the restricted hermitian intersection form; its signature is denoted by $\text{sign}^\omega(N, F)$. The next lemma asserts that the signature considered in [§4.2](#) extends this definition from the rational points to the whole character torus $(S^1 \setminus 1)^\mu$.

Lemma 4.8. *For any spanning pair (N, F) and $\omega \in (S^1 \setminus 1)^\mu$ of finite order, one has*

$$\sigma_L(\omega) = \text{sign}^\omega(N, F) - \text{sign}(N).$$

Proof. Let T_F be an open tubular neighborhood of F and $W_F := N \setminus T_F$. By (4.6), we only need to prove that $\text{sign}^\omega(W_F) = \text{sign}^\omega(N, F)$. In the notation introduced prior to the statement, we have an isomorphism of $\mathbb{C}[G]$ -complexes

$$C_*(W_F; \mathbb{Z}\pi_1(W_F)) \otimes_{\mathbb{Z}\pi_1(W_F)} \mathbb{C}[G] \cong C_*(W_F^G; \mathbb{C}).$$

Hence there is an isomorphism preserving the intersection form

$$H_2(W_F; \mathbb{C}(\omega)) \cong H_2^\omega(W_F).$$

The character ω induces a branched covering T_F^G of the tubular neighborhood T_F , branched along F . Then N^G is obtained by gluing T_F^G to W_F^G , along parts of ∂T_F^G . The 3-manifold ∂T_F is a plumbing constructed from $F_i \times S^1$, and the fibers $\{\cdot\} \times S^1$ are meridians of the components of L . Since ω is nonvanishing, by Corollary 2.3, T_F^G and W_F^G are glued along bundles which have trivial homology. By Wall's Theorem 2.16, we obtain $\text{sign}^\omega(W_F) = \text{sign}^\omega(N, F)$. \square

4.4. Literal extension of the signature. The following technical lemma, which is used in the proof of Theorem 5.3, illustrates the level of difficulties that one would encounter if the definitions of the nullity and signature were extended to arbitrary characters literally. Still, we consider a very special case of one vanishing component only, and even in this case, the literal extension $\tilde{\sigma}_{K \cup L}(1, \omega)$ is not very well defined unless $\ell k(K, L) \neq 0$.

When dealing with spanning pairs of a $(1, \mu)$ -colored link $K \cup L \subset \mathbb{S}$, we adopt the notation $(N, D \cup F)$, assuming that $K = \partial D$ and $L = \partial F$.

Lemma 4.9. *Let $K \cup L \subset \mathbb{S}$ be a $(1, \mu)$ -colored link. Then, for a nonvanishing unitary character $\omega \in (S^1 \setminus 1)^\mu$, one has*

$$\tilde{\sigma}_{K \cup L}(1, \omega) = \begin{cases} \sigma_L(\omega) + \text{sg}[(K/L)(\omega)] & \text{if } \omega \in \mathcal{A}^\circ(K/L) \text{ and } D \cap F = \emptyset, \\ \sigma_L(\omega) & \text{in all other cases,} \end{cases}$$

$$\tilde{\eta}_{K \cup L}(1, \omega) = \begin{cases} \eta_L(\omega) + 1 & \text{if } \omega \in \mathcal{A}^\circ(K/L) \text{ and } (K/L)(\omega) \neq \infty, \\ \eta_L(\omega) & \text{in all other cases,} \end{cases}$$

where $\tilde{\sigma}_{K \cup L}(1, \omega)$ is computed using a spanning pair $(N, D \cup F)$.

Proof. We can assume that $L \neq \emptyset$, as otherwise both statements become the tautology $0 = 0$.

Fix a spanning pair $(N, D \cup F)$. Consider two transversal tubular neighborhoods $T_F \supset F$ and $T_D \cong B^2 \times D \supset D$ and introduce $W_{D \cup F} := N \setminus (T_D \cup T_F)$, $W_F := N \setminus T_F$ and $W_D := N \setminus T_D$. By (4.6),

$$\tilde{\sigma}_{K \cup L}(1, \omega) = \text{sign}^{1, \omega}(W_{D \cup F}) - \text{sign } N, \quad \sigma_L(\omega) = \text{sign}^\omega(W_F) - \text{sign } N,$$

and to prove the first statement of the lemma we will compare $\text{sign}^{1, \omega}(W_{D \cup F})$ and $\text{sign}^\omega(W_F)$ using Wall's nonadditivity theorem. The surface F meets D transversally in a collection of $m \geq 0$ points, and $F \cap T_D$ is a collection of parallel disks $B_1, \dots, B_m \subset T_D$; we denote by U the link $(B_1 \cup \dots \cup B_m) \cap \partial T_D$ in the 3-manifold ∂T_D , and $T_U \subset \partial T_D$ is its tubular neighborhood $T_{B_1 \cup \dots \cup B_m} \cap \partial T_D$. Let, further, $E_m^2 := D \setminus T_{B_1 \cup \dots \cup B_m}$, so that $T_D \setminus T_{B_1 \cup \dots \cup B_m} \cong B^2 \times E_m^2$. We have

$$W_F = W_{D \cup F} \cup (B^2 \times E_m^2),$$

glued along $S^1 \times E_m^2$, where $S^1 \times \{\cdot\}$ is identified to a meridian of K in $W_{D \cup F}$.

In the rest of the proof, we assume that $\omega \in \mathcal{A}^\circ(K/L)$ is admissible: otherwise, the homology $H_1(\partial T_K; \mathbb{C}(\omega))$ vanishes (see Corollary 2.4) and the proof simplifies.

Following the notation of Theorem 2.16, we have $T = \partial T_K \sqcup \partial T_U$ and

$$X_0 = S^1 \times E_m^2, \quad X_1 = (\mathbb{S} \setminus T_{K \cup L}) \cup \{\partial T_F \cap W_D\}, \quad X_2 = T_K \cup \{\partial T_F \cap T_D\},$$

where shown in braces $\{\cdot\}$ are ω -invisible parts, see the definition prior to [Corollary 2.5](#). Since ω is a nonvanishing character, we have $H_1(\partial T_U; \mathbb{C}(\omega)) = 0$. Since ω is also admissible, we also have $H_1(T; \mathbb{C}(1, \omega)) = H_1(\partial T_K; \mathbb{C}(1)) = \mathbb{C}^2$. Thus, using [Corollary 2.5](#) and ignoring the ω -invisible parts, we can easily compute the subspaces $A_i \subset H_1(X_i; \mathbb{C}(1, \omega))$ in [Theorem 2.16](#):

$$\begin{aligned} H_1(X_0; \mathbb{C}(\omega)) &= H_1(S^1 \times E_m^2; \mathbb{C}(1, \omega)), & A_0 &= \langle m_K \rangle \text{ or } \langle l_K \rangle, \\ H_1(X_1; \mathbb{C}(1, \omega)) &= H_1(\mathbb{S} \setminus T_{K \cup L}; \mathbb{C}(1, \omega)), & A_1 &= \mathcal{Z}_{K \cup L}(\omega) = \langle am_K + bl_K \rangle, \\ H_1(X_2; \mathbb{C}(1, \omega)) &= H_1(T_K; \mathbb{C}(1)), & A_2 &= \langle m_K \rangle, \end{aligned}$$

where $a, b \in \mathbb{C}$ and $-a/b = (K/L)(\omega)$. Here, $A_0 = \langle m_K \rangle$ if $m > 0$ and $A_0 = \langle l_K \rangle$ if $m = 0$; this space is computed exactly as in [Example 3.4](#). (If $m = 0$, we also use the obvious fact that the Seifert framing of K extends to a framing of D , so that l_K bounds a parallel copy of D .)

If $m > 0$, then $A_0 = A_2$, the correction term $\text{sign } f$ in [Theorem 2.16](#) vanishes (just like in the easier case where ω is not admissible, where $A_0 = A_1 = A_2 = 0$), and we obtain, in both cases,

$$\text{sign}^\omega(W_F) = \text{sign}^{1, \omega}(W_{D \cup F}) + \text{sign}^\omega(B^2 \times E_m^2).$$

If $m = 0$, the left hand side of this last expression acquires an extra summand

$$\text{sign } f = \text{sign}(0, \kappa, \infty) = \text{sg } \kappa, \quad \kappa := -a/b = (K/L)(\omega),$$

see [Corollary 2.22](#). There remains to observe that E_m^2 is a surface with nonempty boundary and, hence, we have $H_2(B^2 \times E_m^2; \mathbb{C}(\omega)) = H_2(E_m^2; \mathbb{C}(\omega)) = 0$ and $\text{sign}^\omega(B^2 \times E_m^2) = 0$.

The formula for the nullity follows from the Mayer–Vietoris exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{K} \longrightarrow H_1(\partial T_K; \mathbb{C}(1)) \longrightarrow \\ H_1(\mathbb{S} \setminus (K \cup L); \mathbb{C}(1, \omega)) \oplus H_1(T_K; \mathbb{C}(1)) \longrightarrow H_1(\mathbb{S} \setminus L; \mathbb{C}(\omega)) \longrightarrow 0, \end{aligned}$$

where $\mathcal{K} = \mathcal{Z}_{K \cup L}(\omega) \cap Z_1(T_K, \mathbb{C}(1)) = A_1 \cap A_2$. Hence, we have $\dim \mathcal{K} \leq 1$ and $\dim \mathcal{K} = 1$ if and only if $(K/L)(\omega) = \infty$, implying the statement. (Recall that we assume the character both unitary and admissible and, hence, the slope is well defined.) \square

5. THE SPLICE FORMULA

Recall that the *splice* of two $(1, \mu^*)$ -colored links $K^* \cup L^* \subset \mathbb{S}^*$, $* = \prime$ or $\prime\prime$, is defined as follows. Denote by $T^* \subset \mathbb{S}^*$ a small tubular neighborhood of K^* disjoint from L^* and let $m^*, l^* \subset \partial T^*$ be its meridian and Seifert longitude, respectively. Then, the splice of the two links is the $(\mu' + \mu'')$ -colored link $L := L' \cup L''$ in the integral homology sphere

$$\mathbb{S} := (\mathbb{S}' \setminus T') \cup_\varphi (\mathbb{S}'' \setminus T''),$$

where the gluing homeomorphism $\varphi: \partial T' \rightarrow \partial T''$ takes (m', l') to (l'', m'') , respectively.

5.1. Statement of the splice formula. A formula for the colored signature of the splice of two links was established in [\[DFL17\]](#), under some restrictions on the characters. In order to state the general formula, we first introduce some notation.

Let $L \subset \mathbb{S}$ be the splice of the $(1, \mu^*)$ -colored links $K^* \cup L^* \subset \mathbb{S}^*$, $* = \prime$ or $\prime\prime$. Consider the linking vectors $\lambda^* := \overline{\ell k}(K^*, L^*)$, see [\(2.30\)](#), and, for characters $\omega^* \in (S^1)^{\mu^*}$, denote

$$(5.1) \quad v^* := \omega^*([K^*]) = \prod_{i=1}^{\mu^*} (\omega_i^*)^{\lambda_i^*} = (\omega^*)^{\lambda^*} \in S^1 \subset \mathbb{C}^\times.$$

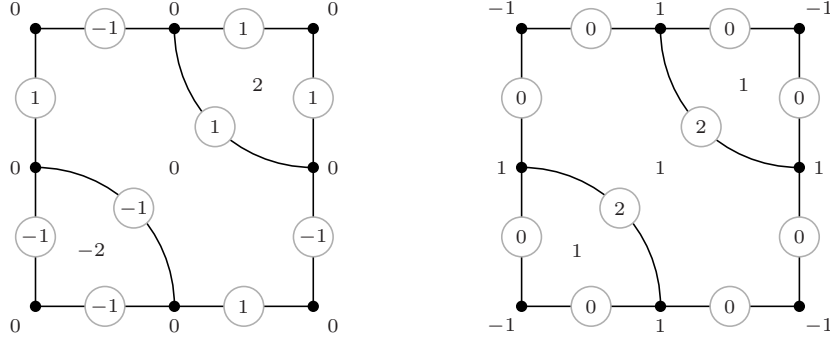


FIGURE 2. The correction terms $\Delta\sigma$ (left) and $\Delta\eta$ (right) in [Theorem 5.3](#). See [Remark 5.4](#) for further details on this figure.

Define the *defect function*

$$\begin{aligned} \delta_\lambda: (S^1)^\mu &\longrightarrow \mathbb{Z} \\ \omega &\longmapsto \text{ind}\left(\sum_{i=1}^\mu \lambda_i \text{Log } \omega_i\right) - \sum_{i=1}^\mu \lambda_i \text{ind}(\text{Log } \omega_i), \end{aligned}$$

where $\lambda \in \mathbb{Z}^\mu$, the *index* of a real number x is defined *via* $\text{ind}(x) := [x] - [-x] \in \mathbb{Z}$, and the *Log-function* $\text{Log}: S^1 \rightarrow [0, 1)$ sends $\exp(2\pi it)$ to $t \in [0, 1)$. Then, the main result in [\[DFL17\]](#) can essentially be stated as follows.

Theorem 5.2 (cf. [\[DFL17\]](#)). *In the notation introduced at the beginning of the section, under the assumption that $(v', v'') \neq (1, 1)$, one has*

$$\begin{aligned} \sigma_L(\omega', \omega'') &= \sigma_{K' \cup L'}(v'', \omega') + \sigma_{K'' \cup L''}(v', \omega'') + \delta_{\lambda'}(\omega') \delta_{\lambda''}(\omega''), \\ \eta_L(\omega', \omega'') &= \eta_{K' \cup L'}(v'', \omega') + \eta_{K'' \cup L''}(v', \omega''). \end{aligned}$$

The assumption that $(v', v'') \neq (1, 1)$ is crucial, as [\[DFL17, Example 2.5\]](#) shows.

Strictly speaking, the signature formula is proved in [\[DFL17\]](#) only for rational characters ω^* , *i.e.*, such that $\text{Log } \omega^* \in \mathbb{Q}^{\mu^*}$. However, once the signature is defined, the extension of the formula to the whole character torus $(S^1 \setminus 1)^{\mu^*}$ is immediate, as Wall's non-additivity formula still works. (Alternatively, one can follow the proof found in [§5.2](#) below, omitting all slope computations, as the homology groups of all tori involved vanish.) The nullity formula, not stated explicitly in [\[DFL17\]](#), follows from [Corollary 2.4](#) and the Mayer–Vietoris exact sequence (again, cf. the more involved case treated in [§5.2](#)).

Our goal is extending [Theorem 5.2](#) to the special case $v' = v'' = 1$. By definition, $v^* = 1$ if and only if ω^* is an admissible character, *i.e.*, there is a well defined slope $\kappa^* := (K^*/L^*)(\omega^*)$. These slopes give rise to an extra correction term, described in the following statement.

Theorem 5.3 (see [§5.2](#)). *Consider two $(1, \mu^*)$ -colored links $K^* \cup L^* \subset \mathbb{S}^*$ as at the beginning of the section and their splice $L := L' \cup L'' \subset \mathbb{S}$. Let $\omega^* \in \mathcal{A}(K^*/L^*) \cap (S^1)^{\mu^*}$ be two admissible characters (so that $v' = v'' = 1$, see [\(5.1\)](#)), and denote $\kappa^* = (K^*/L^*)(\omega^*)$. Then*

$$\begin{aligned} \sigma_L(\omega', \omega'') &= \sigma_{L'}(\omega') + \sigma_{L''}(\omega'') + \delta_{\lambda'}(\omega') \delta_{\lambda''}(\omega'') + \Delta\sigma(\kappa', \kappa''), \\ \eta_L(\omega', \omega'') &= \eta_{L'}(\omega') + \eta_{L''}(\omega'') + \Delta\eta(\kappa', \kappa''), \end{aligned}$$

where the correction terms $\Delta\sigma, \Delta\eta \in \{0, \pm 1, \pm 2\}$ are as shown in [Figure 2](#) (see [Remark 5.4](#)).

It is worth emphasizing that, in both statements of [Theorem 5.3](#), the knots K^* contribute *only* through the slopes: all other terms depend on the links L^* only.

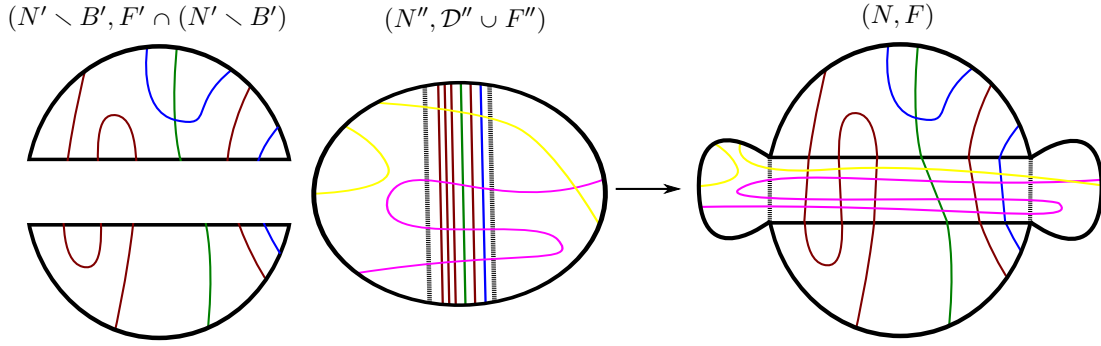


FIGURE 3. The third diagram represents the pair (N, F) used to compute the signature of the splice of $K' \cup L'$ and $K'' \cup L''$. This pair is obtained identifying parts of the boundary of $(N' \setminus B', F' \cap (N' \setminus B'))$ and $(N'', D'' \cup F'')$.

Remark 5.4. Figure 2 shows the correction terms $\Delta\sigma(\kappa', \kappa'')$ (left) and $\Delta\eta(\kappa', \kappa'')$ (right) in Theorem 5.3. The domain is the square $[-\infty, \infty]^2 \ni (\kappa', \kappa'')$, and the curve in the figures is the hyperbola $\kappa'\kappa'' = 1$. For the term $\Delta\sigma$, we have an explicit formula, which was actually found in the course of the proof:

$$(5.5) \quad \Delta\sigma(\kappa', \kappa'') = \text{sg } \kappa' - \text{sg} \left(\frac{1}{\kappa'} - \kappa'' \right)$$

(see Corollary 2.22 for the conventions on sg); in spite of its appearance, it is symmetric in (κ', κ'') . Note that always

$$\Delta\eta(\kappa', \kappa'') = |\Delta\sigma(\kappa', \kappa'')| \pm 1.$$

Intuitively, this means that the matrices always differ by an extra eigenvector rather than by the eigenvalue of a common eigenvector.

5.2. Proof of Theorem 5.3. In view of our uniform conventions on the signature, nullity, and slope (see Remark 2.28 and the respective definitions), we can start with patching the components of L^* on which ω^* vanish. Thus, from now on, we assume both characters nonvanishing.

If L^* becomes empty ($\omega^* = 1$) or was empty in the first place, we take for L^* a small unknot contained in a ball disjoint from K^* , endowed with its own color and arbitrary nonvanishing unitary character; this change does not affect any of the quantities involved. (Alternatively, one can also repeat the computation below, taking into account the difference between the slopes of $H_{1,0}$ and $H_{1,m}$, $m > 0$, see Example 3.4 and cf. the proof of Lemma 4.9.)

Thus, assume that $L^* \neq \emptyset$ and let $(N^*, D^* \cup F^*)$ be a *special* spanning pair for $(\mathbb{S}^*, K^* \cup L^*)$, i.e., such that D is a disk and $F \cap D \neq \emptyset$. (The existence of special spanning pairs follows from [DFL17, Lemma 4.1] and the obvious fact that, if $F \neq \emptyset$, one can always create an extra pair of intersection points.) We will construct an appropriate spanning pair for (\mathbb{S}, L) by cut and paste on the manifolds $(N^*, D^* \cup F^*)$.

The main idea is as in [DFL17], see [DFL17, Figure 4] (reproduced here as Figure 3 for the reader's convenience): we cut off tubular neighborhoods of D^* (thus, passing to the literal extension $\tilde{\sigma}_{K^* \cup L^*}(1, \omega^*) = \sigma_{L^*}(\omega^*)$, see Lemma 4.9), fill one of the gaps with a “standard” spanning pair for the generalized Hopf link, and attach the second manifold to the result. (If it is Theorem 5.2 that is to be proved, the two tubular neighborhoods have already been removed in the definition and the first step is skipped.) An important difference is the fact that, while in [DFL17] we kept the ramification surfaces (the colored curves in Figure 3) inside the 4-manifolds, here we need to carve them out and work with $N^* \setminus (D^* \cup F^*)$, which makes the

description of the boundary more involved. However, the extra boundary parts acquired are (ω', ω'') -invisible 3-manifolds and this change does not affect the computation of the homology groups, see [Corollary 2.5](#).

Thus, fix neighborhoods T_{F^*} and $T_{D^*} = B^2 \times D^*$ and let

$W_{D^* \cup F^*} = N^* \setminus (T_{F^*} \cup T_{D^*})$. The intersection $B_1^* \cup \dots \cup B_{m^*}^* = F^* \cap T_{D^*}$ is a collection of parallel disks in T_{D^*} , and their boundary is the unlink $U^* \subset S^3 = \partial T_{D^*}$ with $m^* > 0$ components. Consider the generalized Hopf link $H_{m', m''} = U' \cup U''$ and orient its components according to the signs of the intersection points $F^* \cap D^*$. Let $B = B^2 \times B^2$ be a 4-ball, and let $C \subset B$ be two families of parallel disks, transversal to each other, so that (B, C) is a spanning pair for $(S^3, H_{m', m''})$. Then,

$$(N, F) := (W_{D'} \cup B \cup W_{D''}, (F' \setminus (F' \cap T_{D'})) \cup C \cup (F'' \setminus (F'' \cap T_{D''}))),$$

after smoothing the corners, is a spanning pair for (\mathbb{S}, L) (see [Figure 3](#)).

The components of F^* induce a $(\mu' + \mu'')$ -coloring on $H_{m', m''}$, and the characters ω^* induce a unique character $\omega := (\omega', \omega'')$ on $H_{m', m''}$. In the rest of the proof, we switch to ω , as all other characters are essentially its restrictions. To compute $\sigma_L(\omega)$, we consider the manifold

$$W := W_{D' \cup F'} \cup (B \setminus T_C) \cup W_{D'' \cup F''},$$

and apply Wall's theorem to the first two terms in the above decomposition, that is, to the manifold

$$W_1 := W_{D' \cup F'} \cup (B \setminus T_C),$$

for which the gluing takes place along $S^3 \setminus H_{1, m'}$.

In the notation of [Theorem 2.16](#), we have $T = \partial T_{K'} \sqcup \partial T_{U'}$ and

$$X_0 = S^3 \setminus H_{1, m'}, \quad X_1 = (\mathbb{S}' \setminus T_{K' \cup L'}) \cup \{\partial T_{F'} \cap (N' \setminus T_{D'})\}, \quad X_2 = (S^3 \setminus H_{m'', 1}) \cup \{\partial T_C\},$$

where, as usual, embraced are ω -invisible parts. Since ω' is admissible and nonvanishing, we also have $H_1(T; \mathbb{C}(\omega)) = H_1(\partial T_{K'}; \mathbb{C}(1)) = \mathbb{C}^2$, and, using [Corollary 2.5](#) to ignore the ω -invisible parts, we obtain the following expressions for the spaces $A_i \subset H_1(X_i; \mathbb{C}(\omega))$ in [Theorem 2.16](#):

$$\begin{aligned} H_1(X_0; \mathbb{C}(\omega)) &= H_1(S^3 \setminus H_{1, m'}; \mathbb{C}(1, \omega')), & A_0 &= \langle m_{K'} \rangle, \\ H_1(X_1; \mathbb{C}(\omega)) &= H_1(\mathbb{S}' \setminus T_{K' \cup L'}; \mathbb{C}(\omega)), & A_1 &= \langle a' m_{K'} + b' l_{K'} \rangle, \\ H_1(X_2; \mathbb{C}(\omega)) &= H_1(S^3 \setminus H_{m'', 1}; \mathbb{C}(\omega'', 1)), & A_2 &= \langle m_{K''} \rangle = \langle l_{K''} \rangle, \end{aligned}$$

where $-a'/b' = \kappa'$ and A_0 and A_2 are computed using [Example 3.4](#). In accordance with the settings of [§2.5](#), the orientation on $\partial T_{K'}$ is induced from $\mathbb{S}' \setminus K'$; hence, $m_{K'} \circ l_{K'} = -1$ in T and [Corollary 2.22](#) yields

$$\text{sign } f = \text{sign}(\infty, \kappa', 0) = -\text{sg } \kappa'.$$

Applying [Theorem 2.16](#) and subtracting $\text{sign}(N')$, we arrive at

$$\begin{aligned} \text{sign}^\omega(W_1) - \text{sign}(N') &= [\text{sign}^\omega(W_{D' \cup F'}) - \text{sign}(N')] + \text{sign}^\omega(B \setminus T_C) + \text{sg } \kappa' \\ &= \sigma_{L'}(\omega') + \delta_{\lambda'}(\omega') \delta_{\lambda''}(\omega'') + \text{sg } \kappa' \end{aligned}$$

(see [Lemma 4.9](#) for the first term, which is $\tilde{\sigma}_{K' \cup L'}(1, \omega')$, and [[DFL17](#), Lemma 4.2] for the second term, which, up to the summand $-\text{sign}(B) = 0$, is the signature of $H_{m', m''}$.)

Now, we use [Theorem 2.16](#) again, this time for the decomposition $W = W_1 \cup W_{D'' \cup F''}$. Since ω'' is also admissible and nonvanishing, we have $H_1(T; \mathbb{C}(\omega)) = H_1(\partial T_{K''}; \mathbb{C}(1)) = \mathbb{C}^2$ for the

new space T and, arguing as above and ignoring the ω -invisible parts, we obtain

$$\begin{aligned} H_1(X_0; \mathbb{C}(\omega)) &= H_1(S^3 \setminus H_{1,m''}; \mathbb{C}(1, \omega'')), & A_0 &= \langle m_{K''} \rangle = \langle l_{K'} \rangle \\ H_1(X_1; \mathbb{C}(\omega)) &= H_1(S' \setminus T_{K' \cup L'}; \mathbb{C}(\omega)), & A_1 &= \langle a' m_{K'} + b' l_{K'} \rangle = \langle a' l_{K''} + b' m_{K''} \rangle, \\ H_1(X_2; \mathbb{C}(\omega)) &= H_1(S'' \setminus T_{K'' \cup L''}; \mathbb{C}(\omega)), & A_2 &= \langle a'' m_{K''} + b'' l_{K''} \rangle, \end{aligned}$$

where $-a^*/b^* = \kappa^*$. Again, $m_{K'} \circ l_{K'} = -1$ and, hence, $m_{K''} \circ l_{K''} = 1$, contrary to the usual convention of §2.5. Hence, by Corollary 2.22,

$$\text{sign } f = -\text{sign}\left(\infty, \frac{1}{\kappa'}, \kappa''\right) = \text{sg}\left(\frac{1}{\kappa'} - \kappa''\right),$$

and

$$\text{sign}^\omega(W) = \text{sign}^\omega(W_1) + \text{sign}^\omega(W_{D'' \cup F''}) - \text{sg}\left(\frac{1}{\kappa'} - \kappa''\right).$$

To complete the proof of the signature formula, with the correction term $\Delta\sigma$ given by (5.5), there remains to subtract the sum $\text{sign}(N') + \text{sign}(N'')$ and observe that

$$\text{sign}(W_{D'} \cup B \cup W_{D''}) = \text{sign}(N') + \text{sign}(N'').$$

For the latter statement, one can either refer to [DFL17] or directly repeat the computation above for the ordinary signature, when all “slopes” vanish.

For the nullity formula, consider the Mayer–Vietoris exact sequence related to

$$X := \mathbb{S} \setminus T_L = X' \cup X'',$$

where the manifolds $X^* := \mathbb{S}^* \setminus T_{K^* \cup L^*}$ are identified along the common boundary component $\partial_{K'} X' \cong \partial_{K''} X''$. We have

$$\begin{aligned} 0 \longrightarrow \mathcal{K} \longrightarrow H_1(\partial_{K^*} X^*; \mathbb{C}(1)) \longrightarrow \\ H_1(X'; \mathbb{C}(\omega)) \oplus H_1(X''; \mathbb{C}(\omega)) \longrightarrow H_1(X; \mathbb{C}(\omega)) \longrightarrow \mathbb{C} \longrightarrow 0, \end{aligned}$$

where $\mathcal{K} = \mathcal{Z}_{K' \cup L'}(\omega') \cap \mathcal{Z}_{K'' \cup L''}(\omega'')$. Hence,

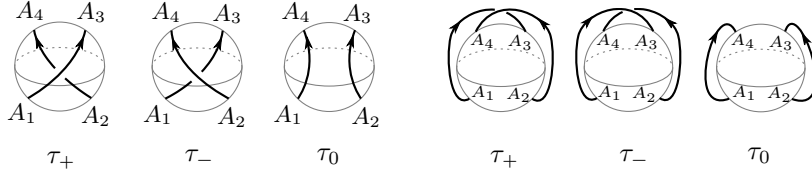
$$\eta_L(\omega', \omega'') = \tilde{\eta}_{K' \cup L'}(1, \omega') + \tilde{\eta}_{K'' \cup L''}(1, \omega'') + \dim \mathcal{K} - 1.$$

Clearly, $\dim \mathcal{K} = 1$ if $\mathcal{Z}_{K' \cup L'}(\omega') = \mathcal{Z}_{K'' \cup L''}(\omega'')$, and $\dim \mathcal{K} = 0$ otherwise. Since $m_{K'} = l_{K''}$ and $l_{K'} = m_{K''}$ in the homology of $\partial_{K'} X' \cong \partial_{K''} X''$, we have $\dim \mathcal{K} = 1$ if and only if $\kappa' = 1/\kappa''$, and there remains to apply Lemma 4.9 to relate $\tilde{\eta}_{K^* \cup L^*}(1, \omega^*)$ and $\eta_{L^*}(\omega^*)$. A case-by-case analysis gives us Figure 2, right, for which we could not find a “nice” formula. \square

6. SKEIN RELATIONS FOR THE SIGNATURE

We conclude the paper with another illustration, *viz.* we develop the concept of slope for tangles with four marked loose ends and analyze its relation to the signature.

For any tangle T with four fixed ends in a homology 3-ball \mathbb{B} and any *generic* character ω in $(\mathbb{C}^\times \setminus 1)^\mu$, we define the slope $\kappa_T(\omega) \in \mathbb{C} \cup \infty$. We show that it can be computed as the quotient of the Conway polynomials of the links obtained by attaching to T certain *elementary tangles*. Then, we define the sum of tangles and prove that the signatures of three pairwise sums of three tangles are related by the sign (as in Corollary 2.22) of their three slopes. Finally, we use these results to derive the conventional skein relations for the signature as in [CF08].

FIGURE 4. Two views of the basic tangles τ_+ , τ_- and τ_0 .

6.1. Preliminaries. Let \mathbb{B} be a homology 3-ball with boundary $S := \partial\mathbb{B} \cong S^2$. We fix an oriented equator $E \subset S$ and four points $A_1, \dots, A_4 \in E$ ordered according to the orientation of E . (We number these points cyclically, so that $A_{i+4} = A_i$. In other words, the index takes values in $\mathbb{Z}/4$.)

Definition 6.1. A *tangle* is a smooth compact oriented submanifold $T \subset \mathbb{B}$ of dimension 1 such that

- T is transversal to the boundary S ;
- $T \cap S = \{A_1, A_2, A_3, A_4\}$, and
- T has incoming branches at A_1, A_2 and outgoing branches at A_3, A_4 .

Similarly to links, a μ -*coloring* on T is a surjective function $\pi_0(T) \rightarrow \{1, \dots, \mu\}$.

Two tangles T and T' are equivalent if there exists an orientation preserving homeomorphism $(\mathbb{B}, T) \rightarrow (\mathbb{B}', T')$ taking E to E' (respecting the orientation) and A_i to A'_i , $i \in \mathbb{Z}/4$.

Given a tangle $T \subset \mathbb{B}$, we let $\mathbb{B}_T = \mathbb{B} \setminus \text{Tub } T$ and $S^\circ := S \setminus \{A_1, A_2, A_3, A_4\}$. Let n be the number of components of T . Then, we have $H_1(\mathbb{B}_T) \cong \mathbb{Z}^n$, and a character $\omega: H_1(\mathbb{B}_T) \rightarrow \mathbb{C}^\times$ is determined by its values $\omega_i \neq 1$ on the meridians m_i about the components $T_i \subset T$. If T is colored, we assume that ω takes equal values on the meridians of the components having the same color. We denote by $\partial\omega$ the restriction of ω to the boundary sphere S° . It is uniquely determined by the restrictions $\omega[i]$ of ω to the meridian about A_i , $i \in \mathbb{Z}/4$. Obviously, $\omega[1]\omega[3] = \omega[2]\omega[4]$. In fact, each of $\omega[1], \omega[2]$ equals one of $\omega[3], \omega[4]$ and *vice versa*.

Example 6.2. The *basic tangles* $\tau_\pm, \tau_0 \subset B^3$ are the three tangles shown in Figure 4 (“inside” and “outside” views). These tangles can be described as the intersection of a link undergoing a skein transformation with a small neighborhood of the relevant crossing of the diagram.

Convention 6.3. We assume that two branches of the tangle connect A_1 to A_3 and A_2 to A_4 . If the tangle is colored, we denote the two colors (that may coincide) assigned to these branches by $-$ and $+$. Therefore, we assume that

$$\omega_- := \omega[2] = \omega[4] \neq 1, \quad \omega_+ := \omega[1] = \omega[3] \neq 1.$$

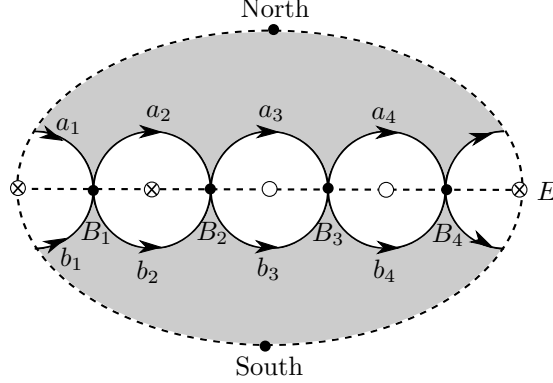
The other convention $\omega[1] = \omega[4] \neq 1$ and $\omega[2] = \omega[3] \neq 1$ can be treated similarly.

Up to homotopy equivalence, the punctured sphere S° has a CW-decomposition shown in Figure 5 (where the sphere is cut along a “geographic” meridian): there are two 2-cells (about each of the poles), eight 1-cells a_i, b_i , and four 0-cells B_i , $i \in \mathbb{Z}/4$. We choose the lifts of the cells so that, for $i \in \mathbb{Z}/4$, with coefficients in $\mathbb{C}(\omega)$,

$$\partial a_i = B_i - B_{i-1}, \quad \partial b_i = \omega[i]^{\epsilon_i} B_i - B_{i-1},$$

where $\epsilon_1 = \epsilon_2 = -1$ and $\epsilon_3 = \epsilon_4 = 1$. We define

$$\tilde{c}_- := b_1 + \omega_+^{-1} a_2 + \omega_+^{-1} b_3 + a_4 \quad \tilde{c}_+ := a_1 + b_2 + \omega_-^{-1} a_3 + \omega_-^{-1} b_4.$$

FIGURE 5. The CW-decomposition of S° .

Lemma 6.4. *For any unitary character ω as in [Convention 6.3](#), the cycles*

$$c_- := (1 - \omega_-^{-1})^{-1} \cdot \tilde{c}_- \quad c_+ := (1 - \omega_+^{-1})^{-1} \cdot \tilde{c}_+$$

form a standard symplectic basis of $H_1(S^\circ; \mathbb{C}(\omega)) = \mathbb{C}^2$, in the sense of [§2.5](#). Furthermore, the chain c_\pm bounds a 2-cell in \mathbb{B}_{τ_\pm} (see [Example 6.2](#) and [Figure 4](#)).

Note that (c_-, c_+) is a basis for $H_1(S^\circ; \mathbb{C}(\omega))$, and will be used as such, for *any* character ω as in [Convention 6.3](#). However, it is a *symplectic* basis only if ω is unitary, as otherwise we do not have a well-defined intersection index.

Proof of [Lemma 6.4](#). We compute the intersections in the maximal abelian covering of S° and specialize the result at ω . Since $\omega_\pm \neq 1$ and the space S° has homotopy type of a wedge of three circles, we have

$$\dim H_1(S^\circ; \mathbb{C}(\omega)) = 2.$$

The oriented loops \tilde{c}_- and \tilde{c}_+ could also be defined as the lifts of the cells of S° :

$$b_1 + a_2 + b_3 + a_4 \quad a_1 + b_2 + a_3 + b_4$$

starting at the chosen lift of B_4 . They are in general position and intersect in four points (with signs according to the orientations):

$$(b_1 + a_2 + b_3 + a_4) \cap (a_1 + b_2 + a_3 + b_4) = \{-B_4, B_1, -B_2, B_3\}.$$

Since $\omega(a_1 - b_1) = \omega_+^{-1}$, $\omega(a_2 - b_2) = \omega_-^{-1}$, and $\omega(a_3 - b_3) = \omega_+$, we obtain

$$\tilde{c}_- \circ \tilde{c}_+ = -1 + \omega_+^{-1} - \omega_+^{-1} \omega_-^{-1} + \omega_-^{-1} = -(1 - \omega_-^{-1})(1 - \omega_+^{-1}).$$

Hence $c_- \circ c_+ = -1$. Since \tilde{c}_- and \tilde{c}_+ are represented in the covering by honest loops, which can be moved off, both classes and their multiples c_-, c_+ are isotropic; thus, the latter constitute a standard symplectic basis of $H_1(S^\circ; \mathbb{C}(\omega)) = \mathbb{C}^2$.

Using the relations $a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4 = 0$, one sees that \tilde{c}_\pm can be defined as the lift of $h_\pm(a_4 + b_4 + a_1 + b_1)$ where h_\pm is a positive or negative (from the south pole) half twist exchanging A_1 and A_2 . This shows that \tilde{c}_\pm bounds a 2-cell in \mathbb{B}_{τ_\pm} . \square

6.2. Slope of a tangle. Fix a tangle $T \subset \mathbb{B}$ and consider a character ω as in [Convention 6.3](#). Then, by [Corollary 2.5](#) and [Lemma 6.4](#), there are canonical isomorphisms

$$H_1(\partial\mathbb{B}_T; \mathbb{C}(\omega)) = H_1(S^\circ; \mathbb{C}(\omega)) = \mathbb{C}^2$$

and, as in [§3.1](#), we can introduce the subspace

$$\mathcal{Z}(\omega) := \mathcal{Z}_T(\omega) := \text{Ker}[\text{inclusion}_* : H_1(S^\circ; \mathbb{C}(\omega)) \rightarrow H_1(\mathbb{B}_T; \mathbb{C}(\omega))].$$

Definition 6.5. Suppose that ω is nonvanishing and $\dim \mathcal{Z}(\omega) = 1$, *i.e.* $\mathcal{Z}(\omega)$ is generated by a vector $u_-c_- + u_+c_+$. Then the *slope* of T at ω is the quotient

$$\kappa_T(\omega) := -\frac{u_-}{u_+} \in \mathbb{C} \cup \infty.$$

As before, we extend κ_T to all characters by patching the components of the tangle on whose meridians ω takes value 1. Recall that we *always* assume that $\omega_\pm \neq 1$ (see [Convention 6.3](#)).

Propositions [3.5](#) and [3.6](#) and their proofs extend to κ_T literally; in particular, if ω is a unitary character, then $\kappa_T(\omega)$ is well-defined and real (possibly infinite).

The *sum* $T' \sqcup T''$ of two tangles $T' \subset \mathbb{B}'$ and $T'' \subset \mathbb{B}''$ is the link $T' \cup -T''$ in the homology sphere $\mathbb{S} := \mathbb{B}' \cup_{\partial} -\mathbb{B}''$, where the attaching homeomorphism $\varphi: S' \rightarrow S''$ restricts to an orientation preserving homeomorphism $E' \rightarrow E''$ taking A'_i to A''_i for each $i = 1, \dots, 4$. If the tangles are colored, we also ask the gluing to respect the colors $-$ and $+$. Then, the result is a colored link in \mathbb{S} , and $-$ and $+$ become ordinary colors.

Example 6.6. Consider a link in the sphere S^3 and denote by L the tangle obtained by removing a small neighborhood of a crossing of its diagrams. Then, the sums $L_* := L \sqcup \tau_*$ with the basic tangles (see [Example 6.2](#)) are the usual links L_\pm and L_0 involved in a skein relation.

Theorem 6.7. Fix a tangle $T \subset \mathbb{B}$ and consider a character ω as in [Convention 6.3](#), so that ω extends to the links $L_\pm := T \sqcup \tau_\pm$ (see [Example 6.6](#)). If the polynomials $\nabla_{L_\pm}(\sqrt{\omega})$ do not vanish simultaneously, then the slope $\kappa_T(\omega)$ is well defined and one has

$$\kappa_T(\omega) = \frac{\nabla_0(\sqrt{\omega_+})}{\nabla_0(\sqrt{\omega_-})} \cdot \frac{\nabla_{L_+}(\sqrt{\omega})}{\nabla_{L_-}(\sqrt{\omega})} \in \mathbb{C} \cup \infty,$$

where $\nabla_0(t) = (t - t^{-1})^{-1}$ is the Conway potential of the unknot.

Similarly to [Theorem 3.21](#), the statement is inconclusive if $\nabla_{L_-}(\sqrt{\omega}) = \nabla_{L_+}(\sqrt{\omega}) = 0$.

Example 6.8. The sums of the basic tangles (see [Example 6.2](#) and [Figure 4](#)) are as follows:

- $\tau_- \sqcup \tau_-$ and $\tau_+ \sqcup \tau_+$ are the Hopf links (up to orientation),
- $\tau_- \sqcup \tau_+ = \tau_+ \sqcup \tau_-$ is the trivial link with two components,
- $\tau_- \sqcup \tau_0 = \tau_+ \sqcup \tau_0$ is the unknot.

Hence, by [Theorem 6.7](#), at any *nonvanishing* character, we have

$$\kappa_{\tau_-} = \infty, \quad \kappa_{\tau_+} = 0, \quad \kappa_{\tau_0} = 1,$$

where the latter slope is defined only on the diagonal $\omega_- = \omega_+$.

Proof of Theorem 6.7. As in [§3.6](#), assume that the character ω nonvanishing. A crucial observation is the fact that, up to homotopy equivalence, the complement of L_\pm is obtained from \mathbb{B}_T by attaching a single 2-cell k_\pm along the cycle \tilde{c}_\pm (see [Lemma 6.4](#)). We proceed exactly as in the proof of [Theorem 3.21](#), computing and comparing the torsions of the two links L_\pm . Let X_\pm be the complement of $\text{Tub } L_\pm$ in $\mathbb{S} = \mathbb{B} \cup B^3$. Up to homotopy equivalence, the *CW*-decomposition of X_\pm is partially given as follows

$$C_0 = \{B_1, B_2, B_3, B_4\}, \quad \{\tilde{c}_-, \tilde{c}_+, a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \dots\} \subset C_1, \quad \{k_\pm, \dots\} \subset C_2.$$

Then, we fix the following bases for $C_*(X_\pm; \mathbb{C}(\omega))$:

$$c_0 := B_1, B_2, B_3, B_4, \quad c_1 := \tilde{c}_-, \tilde{c}_+, \dots, a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4, \quad c_2 := k_\pm, \dots$$

(These bases are not quite as in the definition of the torsion. However, they differ from cellular ones by a transition matrix in C_1 that is common to X_- and X_+ . Thus, our choice would not affect the *ratio* of the two torsions.)

The vectors

$$\partial_0(a_i - b_i) = (1 - \omega[i]^{\epsilon_i})B_i, \quad i = 1, \dots, 4,$$

constitute a basis for C_0 . The complex $C_*(X_\pm; \mathbb{C}(\omega))$ is acyclic if and only if

$$\partial_1(c_2), a_1 - b_1, a_2 - b_2, a_3 - b_3, a_4 - b_4$$

is a basis for C_1 . The homomorphisms ∂_1 in X_- and X_+ are given by certain matrices of the form

$$\begin{bmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{a} & \mathbf{b} & M \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & \mathbf{0} \\ \mathbf{a} & \mathbf{b} & M \end{bmatrix},$$

respectively, where M is a certain matrix and \mathbf{a} , \mathbf{b} , are column vectors, also common for X_- and X_+ . Thus, the two complexes are acyclic if and only if neither of the two determinants vanish, and then we have

$$\frac{\tau_\varphi(X_+)(\omega)}{\tau_\varphi(X_-)(\omega)} = -\frac{\det[\mathbf{a} | M]}{\det[\mathbf{b} | M]}.$$

Now, a linear combination $(1 - \omega_-^{-1})^{-1}u_- \tilde{c}_- + (1 - \omega_+^{-1})^{-1}u_+ \tilde{c}_+$ represents a class in $\mathcal{Z}(\omega)$ if and only if it is in the image of ∂_1 in the complex $C_*(\mathbb{B}_T; \mathbb{C}(\omega))$, which is given by $[\mathbf{a} | \mathbf{b} | M]$. If the slope κ is finite, we are looking for a vector of the form $-\tilde{\kappa} \tilde{c}_- + \tilde{c}_+ \in \text{Im } \partial_1$, where

$$\tilde{\kappa} := \frac{(1 - \omega_+^{-1})}{(1 - \omega_-^{-1})} \kappa = \frac{\nabla_0(\sqrt{\omega_-})}{\nabla_0(\sqrt{\omega_+})} \kappa.$$

Clearly, if such a vector exists, then $\tilde{\kappa} = -\mathbf{x} \cdot \mathbf{a}$, where \mathbf{x} is a solution to the linear system

$$\mathbf{x} \cdot [\mathbf{b} | M] = [1 | \mathbf{0}].$$

If $\tau_\varphi(X_-)(\omega) \neq 0$, then $\det[\mathbf{b} | M] \neq 0$ and $\tilde{\kappa}$ can be computed by Cramer's rule:

$$\tilde{\kappa} = -[1 | \mathbf{0}] \cdot [\mathbf{b} | M]^{-1} \cdot \mathbf{a} = -\frac{\det[\mathbf{a} | M]}{\det[\mathbf{b} | M]} = \frac{\tau_\varphi(X_+)(\omega)}{\tau_\varphi(X_-)(\omega)},$$

so that there is at most one solution. As in §3.6, replacing ω with ω^* , we conclude that the slope is well defined and given by the expression above. If $\tau_\varphi(X_-)(\omega) = 0$, but $\tau_\varphi(X_+)(\omega) \neq 0$, the same argument with X_\pm interchanged shows that the slope is well defined and equal to ∞ .

The passage from τ_φ to ∇ and verification of the sign is immediate (cf. §3.6). \square

6.3. The skein relations. Let T' and T'' be a pair of colored tangles such that the sum $T' \sqcup T''$ is well-defined. Clearly, any pair of characters ω', ω'' on T', T'' such that $\partial\omega' = \partial\omega''$ gives rise to a character on $T' \sqcup T''$; we denote the latter by $\omega' \sqcup \omega''$.

Theorem 6.9. *Consider three tangles $T^i \subset \mathbb{B}^i$, $i \in \mathbb{Z}/3$, and three characters ω^i on $\mathbb{B}^i \setminus T^i$ as in Convention 6.3 such that $\partial\omega^i = \text{const}(i)$ (i.e., all three characters have the same restriction to the common boundary sphere S). Then, denoting $\kappa^i := \kappa_{T^i}(\omega^i)$, one has*

$$\sum_{i \in \mathbb{Z}/3} \sigma_{T^{i+1} \sqcup T^i}(\omega^{i+1} \sqcup \omega^i) = \text{sign}(\kappa^0, \kappa^1, \kappa^2)$$

(see Corollary 2.22 for the definition of sign).

[Theorem 6.9](#) could be derived directly from [[CC](#), Theorem 1.1]. The short proof given below, although using essentially the same argument, fits better the framework of our paper.

Remark 6.10. Changing the indices $(0, 1, 2)$ to $(\text{nothing}, \iota, \iota')$, one can rewrite the conclusion of [Theorem 6.9](#) in the following, less symmetric, form:

$$\sigma_{T \sqcup T'}(\omega \sqcup \omega') + \sigma_{T' \sqcup T''}(\omega' \sqcup \omega'') = \sigma_{T \sqcup T''}(\omega \sqcup \omega'') - \text{sign}(\kappa, \kappa', \kappa'').$$

It is, essentially, this identity that is actually proved below.

Proof of [Theorem 6.9](#). Since the orientation of T'' is reversed in the definition, the operation $T' \sqcup T''$ is skew-symmetric, *i.e.*, for any colored tangles T', T'' and character ω', ω'' such that the operation is well-defined, we have

$$\sigma_{T' \sqcup T''}(\omega' \sqcup \omega'') = -\sigma_{T'' \sqcup T'}(\omega'' \sqcup \omega').$$

This observation justifies [Remark 6.10](#) and, in the latter form, the identity is an immediate consequence of Wall's [Theorem 2.16](#). Indeed, if (N_1, F_1) is a spanning pair for $T' \sqcup T$ and (N_2, F_2) is a spanning pair for $T \sqcup T''$, then a spanning pair for $T' \sqcup T''$ is $(N, F) := (N_1, F_1) \cup (N_2, F_2)$. The three characters $\omega, \omega', \omega''$ define a common character $\tilde{\omega}$ on $\mathbb{B} \cup \mathbb{B}' \cup \mathbb{B}''$, which extends uniquely to $N_1 \setminus F_1$ and $N_2 \setminus F_2$, and there remains to apply [Theorem 2.16](#). The few technical details related to removing $\text{Tub } F_i$, $i = 1, 2$, are filled in as in [§5.2](#), using [Corollary 2.5](#); we leave this exercise to the reader. \square

The following corollary generalizes and refines [[CF08](#), Theorem 5.1].

Corollary 6.11. *Let $L_{\pm}, L_0 \subset S^3$ be colored links involved into the skein relation at a crossing of the diagram, and denote by $L \subset B^3$ the tangle obtained by removing a small tubular neighborhood of the crossing (see [Example 6.6](#)). Pick a unitary character $\omega \in (S^1 \setminus 1)^\mu$ on L_{\pm} and, contrary to the usual convention, fix $\sqrt{\omega}$ so that $\text{Im } \sqrt{\omega_{\pm}} > 0$. Then,*

$$\sigma_{L_+}(\omega) - \sigma_{L_-}(\omega) = \text{sg } \kappa_L(\omega) = \text{sg} \left(\frac{\nabla_{L_+}(\sqrt{\omega})}{\nabla_{L_-}(\sqrt{\omega})} \right).$$

If $\omega_- = \omega_+$, then also

$$\sigma_{L_{\pm}}(\omega) - \sigma_{L_0}(\omega) = \text{sg}(\kappa_L(\omega)^{\mp 1} - 1) = \pm \text{sg} \left(i \cdot \frac{\nabla_{L_{\pm}}(\sqrt{\omega})}{\nabla_{L_0}(\sqrt{\omega})} \right).$$

In both case, the second expression makes sense if at least one of the two Conway potentials does not vanish; in this case, we assert, in particular, that the argument of sg is real.

Proof. Let $T = L$, $T'' = \tau_+$, and $T' = \tau_-$ or τ_0 in [Theorem 6.9](#). Since $\tau_+ \sqcup \tau_-$ and $\tau_{\pm} \sqcup \tau_0$ are trivial links/knots (see [Example 6.2](#)), their signature is 0 and, hence, letting $\kappa := \kappa_T(\omega)$, from [Theorem 6.9](#) and [Example 6.8](#) we have

$$\begin{aligned} \sigma_{L_+}(\omega) - \sigma_{L_-}(\omega) &= \text{sign}(\kappa, \infty, 0) &&= \text{sg } \kappa, \\ \sigma_{L_+}(\omega) - \sigma_{L_0}(\omega) &= \text{sign}(\kappa, 1, 0) = \text{sg}(\kappa(1 - \kappa)) = \text{sg}(\kappa^{-1} - 1). \end{aligned}$$

To relate these expressions to the Conway potentials, we use [Theorem 6.7](#) and the following simple observation: if $\xi \in S^1$, the difference $\xi - \xi^{-1} = 2i \text{Im } \xi$ makes a predictable contribution to the sgn . In view of our choice of $\sqrt{\omega_{\pm}}$, this completes the proof of the first formula. For the second one, we employ the classical skein relation (letting $\xi := \sqrt{\omega_+} = \sqrt{\omega_-}$)

$$\nabla_{L_+}(\sqrt{\omega}) - \nabla_{L_-}(\sqrt{\omega}) = (\xi - \xi^{-1}) \nabla_{L_0}(\sqrt{\omega}),$$

which implies

$$\kappa^{-1} - 1 = \frac{\nabla_{L_-}(\sqrt{\omega})}{\nabla_{L_+}(\sqrt{\omega})} - 1 = \frac{\nabla_{L_-}(\sqrt{\omega}) - \nabla_{L_+}(\sqrt{\omega})}{\nabla_{L_+}(\sqrt{\omega})} = (\xi^{-1} - \xi) \frac{\nabla_{L_0}(\sqrt{\omega})}{\nabla_{L_+}(\sqrt{\omega})} \in \mathbb{R} \cup \infty$$

and, since $\text{Im } \xi > 0$,

$$\text{sg}(\kappa^{-1} - 1) = \text{sg} \left(-i \frac{\nabla_{L_0}(\sqrt{\omega})}{\nabla_{L_+}(\sqrt{\omega})} \right) = \text{sg} \left(i \frac{\nabla_{L_+}(\sqrt{\omega})}{\nabla_{L_0}(\sqrt{\omega})} \right).$$

The computation for $\sigma_{L_-}(\omega) - \sigma_{L_0}(\omega)$ is similar. \square

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