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## The optimum inertial amplifier tuned mass dampers for nonlinear dynamic systems

Sudip Chowdhury\*

*Civil Engineering Department, Indian Institute of Technology Delhi, India  
sudip.chowdhury@civil.iitd.ac.in*

Arnab Banerjee

*Civil Engineering Department, Indian Institute of Technology Delhi, India  
abanerjee@iitd.ac.in*

Sondipon Adhikari

*James Watt School of Engineering, The University of Glasgow, Glasgow, UK  
Sondipon.Adhikari@glasgow.ac.uk*

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The optimum inertial amplifier tuned mass dampers (IATMD) for vibration reduction of linear and nonlinear dynamic systems are introduced in this paper.  $H_2$  and  $H_\infty$  optimization methods are applied to derive the exact closed-form expressions for optimal design parameters such as frequency and viscous damping ratios in simplified form mathematically for IATMD. From the parametric study, using these optimal closed-form solutions, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle are recommended to design optimum IATMD to achieve robust dynamic response reduction capacity having moderate viscous damping and lower frequency ratios at an affordable range. The optimum IATMD systems are installed on top of linear and nonlinear single-degree-of-freedom systems to mitigate their dynamic responses. The linear dynamic responses are determined through transfer matrix formations, and nonlinear dynamic responses are derived using the harmonic balance (HB) method.  $H_2$  optimized IATMD is significantly 44.78 % and 48.62 % superior to the  $H_2$  optimized conventional tuned mass damper one (CTMD1) and conventional tuned mass damper two (CTMD2). Furthermore,  $H_\infty$  optimized IATMD is significantly 39.98 % superior to the  $H_\infty$  optimized conventional tuned mass damper (CTMD). According to the nonlinear dynamic analysis,  $H_2$  optimized IATMD systems are significantly 35.33 %, 76.97 %, and 35.33 % superior to the  $H_2$  optimized conventional tuned mass damper (CTMD). Furthermore,  $H_\infty$  optimized IATMD systems are significantly 25.92 %, 73.64 %, and 25.92 % superior to the  $H_\infty$  optimized conventional tuned mass damper (CTMD). The results of this study are mathematically accurate and feasible for practical applications.

*Keywords:* inertial amplifier tuned mass dampers (IATMD), conventional tuned mass damper (CTMD),  $H_2$  optimization, Closed-form expressions, Dynamic response reduction capacity, Nonlinear dynamic analysis.

\*Corresponding author: Sudip Chowdhury, Email id: sudip.chowdhury@civil.iitd.ac.in

## 1. Introduction

In order to mitigate the dynamic responses of the structures from natural calamities like earthquakes and cyclones, passive vibration control devices are implemented. Tuned mass dampers (TMD) are one of these devices that assist in preventing vibrations. In 1909, Frahm was the first to patent the theory of TMD without addressing damping in TMD Frahm [1909]. When the natural frequency of the TMD is close to the excitation frequency, it is highly effective, but when the excitation frequency deviates from the natural frequency, there is no vibration reduction.

Ormondroyd and Den Hartog later addressed this drawback by integrating damped TMD and establishing closed-form expressions for the optimal design parameters Ormondroyd [1928]. This optimization method is referred to as the  $H_\infty$  optimization method, and its name is the fixed-point theory Chen and Hu [2019]; Sun *et al.* [2019]. When the controlled structure is subjected to harmonic excitation, this method is appropriate. Den Hartog authored a book containing a comprehensive illustration of this method Den Hartog [1985]. Ever since that period, comprehensive research has been conducted on TMD, and it has been implemented in a wide range of mechanical and civil applications Zhang and Li [2001]; Lu *et al.* [2020], such as car suspension systems, offshore platforms, buildings, bridges, etc Adhikari and Bhattacharya [2012]; Batou and Adhikari [2019]; Kasinos *et al.* [2021]. When the controlled structure is subjected to white-noise random excitation, another method known as  $H_2$  optimization is also employed to determine the optimal design parameters Palmeri and Lombardo [2011]; Khodaparast *et al.* [2008]; Adhikari *et al.* [2016]. Previous research has shown that the ability of a TMD to stop vibrations grows as its mass increase.

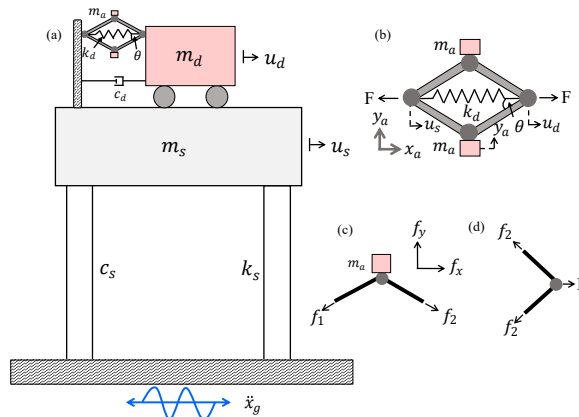
Smith has recently introduced a mechanical system called an inerter Smith [2020], which contradicts the conventional analogy for mitigating the vibration responses of structures. This inerter has been induced within or parallel to traditional passive vibration control devices in order to increase its energy dissipation capacity by amplifying the large effective mass through rotating mass using motion transformers within the system Pietrosanti *et al.* [2017]. A lot of these inerters have been used to improve the performance of machinery and parts in the field of mechanical engineering, especially in the suspension systems of cars and trains. Researchers have used inerter in civil engineering structures after seeing how well it worked in mechanical engineering De Domenico *et al.* [2019]. So far, many researchers have had satisfactory accuracy. Particularly, inerters have been put into the classical tuned mass damper and base isolator to control how buildings, wind turbines, and bridges respond to vibrations. In this paper, TMDs will be the primary topic Petrini *et al.* [2020]; Wagg [2021]. Most inerter-based TMDs, on the other hand, have been made with a flywheel-gear inerter. Apart from the flywheel-based inerter, there are also inertial amplifiers that have already achieved significant mass amplification and a large wide-bandgap at low frequencies. Because of these properties, inertial amplifiers can be used in construction to reduce vibration Banerjee *et al.* [2019, 2021];

Adhikari and Banerjee [2021]. However, the use of inertial amplifiers in finished constructions like buildings and bridges is rather limited. The majority of investigations, according to the present evaluation of the literature, were based on structural members such as beams and columns Chowdhury *et al.* [2021]. The inertial amplifier mechanism-tuned mass damper, whose structural typologies are comparable to inertial amplifiers, was recently explored by Cheng *et al.* However, they did not research any inertial amplifier-based TMDs and did not apply them to the nonlinear dynamic systems for vibration mitigation. Thus, a research scope has been detected.

Therefore, to address these research scopes, the inertial amplifier tuned mass dampers for nonlinear dynamic systems to mitigate their dynamic responses are introduced in this paper. In addition, the exact-closed form expressions for optimal design parameters of IATMD in simplified form have been introduced in this paper using  $H_2$  and  $H_\infty$  optimization methods Chowdhury and Banerjee [2022]; Chowdhury *et al.* [2022, 0]. Applying these newly derived optimal closed-form solutions for the design parameters to IATMD, robust vibration reduction capacity has been achieved. Thus, the optimum IATMD is applied to the linear and nonlinear dynamic systems individually to mathematically determine the exact vibration reduction capacity of the optimum inertial amplifier tuned mass damper (IATMD). The vibration reduction capacity of optimum IATMD has been compared to the vibration reduction capacity of conventional tuned mass dampers (CTMD).

## 2. Structural model and equations of motion

The schematic diagram of a single degree of freedom system equipped with inertial amplifier tuned mass damper (IATMD) has been shown in Figure 1 (a). The schematic diagram of inertial amplifier and the corresponding free-body diagrams are shown in Figure 1 (b), Figure 1 (c), and Figure 1 (d).  $m_d$ ,  $k_d$ , and  $c_d$  refer to



**Figure 1:** The schematic diagram of (a) structure with IATMD, (b) inertial amplifier, and (c-d) free-body diagrams.

the static mass of IATMD.  $m_a$  and  $\theta$  refer to the static mass and inertial angle of the inertial amplifier.  $m_s$ ,  $k_s$ , and  $c_s$  refer to the mass, stiffness, and damping of the primary structure.  $\ddot{x}_g$  refers to the base excitation.  $u_s$  and  $u_d$  refer to the absolute dynamic responses of the primary structure and IATMD.  $x_a$  and  $y_a$  refer to the dynamic responses of inertial amplifier in  $x$  and  $y$ -directions. Hence, the dynamic responses of the amplifier's masses in  $x$  and  $y$ -directions are derived as

$$x_a = \frac{u_d + u_s}{2} \quad \text{and} \quad y_a = \pm \frac{u_d - u_s}{2 \tan \theta} \quad (1)$$

The inertial forces generated through amplifier's mass have been derived as

$$f_x = m_a \ddot{x}_a \quad \text{and} \quad f_y = m_a \ddot{y}_a \quad (2)$$

The forces generated through rigid links are derived as

$$f_1 = \frac{1}{2} \left( \frac{f_y}{\sin \theta} - \frac{f_x}{\cos \theta} \right) \quad \text{and} \quad f_2 = \frac{1}{2} \left( \frac{f_y}{\sin \theta} + \frac{f_x}{\cos \theta} \right) \quad (3)$$

The resultant force develops at the lateral terminals of the inertial amplifier has been derived as

$$\begin{aligned} F &= 2f_2 \cos \theta + k_d(u_d - u_s) \\ &= \underbrace{\frac{0.5m_a}{\tan^2 \theta}}_{c_1} (\ddot{u}_d - \ddot{u}_s) + \underbrace{0.5m_a}_{c_2} (\ddot{u}_s + \ddot{u}_d) + k_{ad}(u_d - u_s) \end{aligned} \quad (4)$$

where  $c_1 = (0.5m_a / \tan^2 \theta)$  and  $c_2 = 0.5m_a$  are added with the static mass of the IATMD  $m_d$ . Therefore, the effective mass of IATMD has been derived as

$$m_{ad} = m_d + 0.5m_a \left( 1 + \frac{1}{\tan^2 \theta} \right) \quad (5)$$

The effective stiffness and damping of IATMD derives as

$$k_{ad} = m_{ad} \omega_d^2 \quad \text{and} \quad c_{ad} = 2\zeta_d m_{ad} \omega_d \quad (6)$$

Newton's second law applies to derive the equations of motion for a single-degree-of-freedom system equipped with inertial amplifier tuned mass dampers (IATMD) after considering all the effective system parameters, such as effective mass stiffness and damping of the IATMD and expressed as.

$$\begin{aligned} m_s \ddot{x}_s + c_s \dot{x}_s + k_s x_s - k_{ad} x_d - c_{ad} \dot{x}_d &= -m_s \ddot{x}_g \\ m_{ad} \ddot{x}_d + m_{ad} \ddot{x}_s + k_{ad} x_d + c_{ad} \dot{x}_d &= -m_{ad} \ddot{x}_g \end{aligned} \quad (7)$$

The controlled dynamic system is subjected to harmonic base excitation. Therefore, the steady state solutions are considered as  $x_s = X_s e^{i\omega t}$ ,  $x_d = X_d e^{i\omega t}$ , and  $\ddot{x}_g = A_g e^{i\omega t}$ . Hence, the transfer function has been derived as

$$\begin{bmatrix} 2q\zeta_d \omega_d \mu_{ad} + q^2 \mu_{ad} + \omega_d^2 \mu_{ad} & q^2 \mu_{ad} \\ -2q\zeta_d \omega_d \mu_{ad} - \omega_d^2 \mu_{ad} & 2\zeta_s \omega_s q + q^2 + \omega_s^2 \end{bmatrix} \begin{Bmatrix} X_d \\ X_s \end{Bmatrix} = - \begin{bmatrix} \mu_{ad} \\ 1 \end{bmatrix} A_g \quad (8)$$

The dynamic response of IATMD has been derived as

$$H_d = \frac{X_d}{A_g} = \frac{-\omega_s (2q\zeta_s + \omega_s)}{\Delta} \quad (9)$$

The dynamic response of primary structure has been derived as

$$H_s = \frac{X_s}{A_g} = \frac{-2q\zeta_d\omega_d\mu_{ad} - 2\zeta_d\omega_dq - \omega_d^2\mu_{ad} - q^2 - \omega_d^2}{\Delta} \quad (10)$$

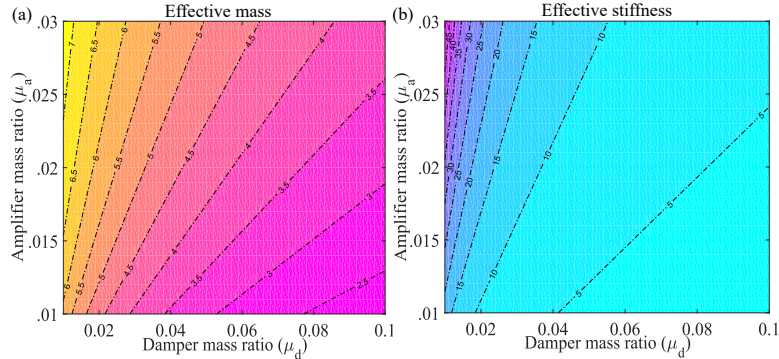
$\Delta$  has been derived as

$$\begin{aligned} \Delta = & q^4 + (2\zeta_d\omega_d\mu_{ad} + 2\zeta_d\omega_d + 2\zeta_s\omega_s)q^3 \\ & + (4\zeta_d\zeta_s\omega_s\omega_d + \omega_d^2\mu_{ad} + \omega_s^2 + \omega_d^2)q^2 \\ & + (2\zeta_d\omega_s^2\omega_d + 2\zeta_s\omega_s\omega_d^2)q + \omega_s^2\omega_d^2 \end{aligned} \quad (11)$$

The effective mass ratio for IATMD has been derived as

$$\mu_e = \frac{\mu_{ad}}{\mu_d + 2\mu_a} = \frac{\mu_d}{\mu_d + 2\mu_a} + 0.5 \frac{\mu_a}{\mu_d + 2\mu_a} \left(1 + \frac{1}{\tan^2 \theta}\right) \quad (12)$$

$\mu_{ad} = m_{ad}/m_s$ ,  $\mu_d = m_d/m_s$ , and  $\mu_a = m_a/m_s$  refer to the effective, damper, and amplifier mass ratios of IATMD. The contour diagram of effective mass ratio as a function of amplifier mass ratio and damper mass ratio of IATMD has been shown in Figure 2 (a). The inertial angle considers as  $\theta = 10^\circ$ . The effective mass ratio increases when the damper mass and amplifier's mass ratio increases. The contour



**Figure 2:** The contour diagrams of (a) effective mass ratio and (b) effective stiffness ratio as a function of amplifier mass ratio and damper mass ratio of IATMD.

diagram of effective stiffness ratio as a function of amplifier mass ratio and damper mass ratio of IATMD has been shown in Figure 2 (b). The effective stiffness ratio increases when the damper mass and amplifier's mass ratio increases. The effective stiffness ratio for IATMD has been derived as

$$\kappa_e = \frac{k_{ad}}{k_d} = \frac{\mu_{ad}\omega_d^2}{\mu_d\omega_d^2} = 1 + 0.5 \frac{\mu_a}{\mu_d} \left(1 + \frac{1}{\tan^2 \theta}\right) \quad (13)$$

### 3. $H_2$ optimization

$H_2$  optimization has been performed to minimize the standard deviation of dynamic response of the controlled structures subjected to random-white noise excitations. The mathematical expressions for deriving standard deviations are determined as

$$\sigma_{x_s,d}^2 = \int_{-\infty}^{\infty} \frac{\varepsilon_n(\omega) d\omega}{\varrho_n(i\omega)\varrho_n^*(i\omega)} = \frac{\pi \det[\mathbf{A}_4]}{u_4 \det[\mathbf{B}_4]} \quad (14)$$

$$A_4 = \begin{bmatrix} v_3 & b_2 & v_1 & v_0 \\ -u_4 & u_2 & -u_0 & 0 \\ 0 & -u_3 & u_1 & 0 \\ 0 & u_4 & -u_2 & u_0 \end{bmatrix} \quad \text{and} \quad B_4 = \begin{bmatrix} u_3 & -u_1 & 0 & 0 \\ -u_4 & u_2 & -u_0 & 0 \\ 0 & -u_3 & u_1 & 0 \\ 0 & u_4 & -u_2 & u_0 \end{bmatrix} \quad (15)$$

Hence, the standard deviation of dynamic response of primary structure has been derived as

$$\sigma_{x_s}^2 = \frac{S_0\pi \left( \begin{array}{l} 4\omega_s^2\omega_d^2\mu_{ad}^3\zeta_d^2 + 12\omega_s^2\omega_d^2\mu_{ad}^2\zeta_d^2 + \mu_{ad}^4\omega_d^4 \\ + 12\omega_s^2\omega_d^2\mu_{ad}\zeta_d^2 + \omega_s^2\omega_d^2\mu_{ad}^3 + 4\mu_{ad}^3\omega_d^4 \\ + 4\omega_s^2\omega_d^2\zeta_d^2 + 6\mu_{ad}^2\omega_d^4 - 3\omega_s^2\omega_d^2\mu_{ad} \\ + 4\mu_{ad}\omega_d^4 + \omega_s^4 - 2\omega_s^2\omega_d^2 + \omega_d^4 \end{array} \right)}{2\zeta_d\omega_d\omega_s^6\mu_{ad}} \quad (16)$$

The mathematical formulations for derivations of optimal design parameters are listed below.

$$\frac{\partial\sigma_{x_s}^2}{\partial\zeta_d} = 0 \quad \text{and} \quad \frac{\partial\sigma_{x_s}^2}{\partial\omega_d} = 0 \quad (17)$$

Equation (16) inserted into the first equation of Eq. (17). Therefore, the viscous damping ratio of IATMD has been derived as

$$\zeta_d = \frac{1}{2} \sqrt{\frac{(\mu_{ad} + 1)^4 \omega_d^4 + \omega_s^2 (\mu_{ad} - 2) (\mu_{ad} + 1)^2 \omega_d^2 + \omega_s^4}{\omega_s^2 \omega_d^2 (\mu_{ad} + 1)^3}} \quad (18)$$

Equation (18) substituted into Eq. (16) and the modified SD of primary structure derived as

$$\sigma_{x_s}^2 = \frac{2S_0\pi \left( (\mu_{ad} + 1)^4 \omega_d^4 + \omega_s^2 (\mu_{ad} - 2) (\mu_{ad} + 1)^2 \omega_d^2 + \omega_s^4 \right)}{\sqrt{\frac{(\mu_{ad} + 1)^4 \omega_d^4 + \omega_s^2 (\mu_{ad} - 2) (\mu_{ad} + 1)^2 \omega_d^2 + \omega_s^4}{\omega_s^2 \omega_d^2 (\mu_{ad} + 1)^3}} \omega_d \omega_s^6 \mu_{ad}} \quad (19)$$

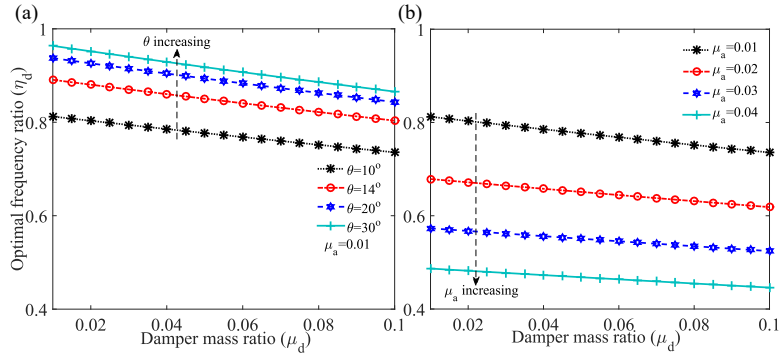
Equation (19) inserted in the second equation of Eq. (17). Hence, the optimal frequency for IATMD has been derived as

$$(\omega_d)_{opt} = \frac{\sqrt{4 - 2\mu_{ad}\omega_s}}{2(\mu_{ad} + 1)} \quad \text{and} \quad (\eta_d)_{opt} = \frac{\sqrt{4 - 2\mu_{ad}}}{2(\mu_{ad} + 1)} \quad (20)$$

Equation (20) substituted into Eq. (18). Hence, the optimal viscous damping ratio for IATMD has been derived as

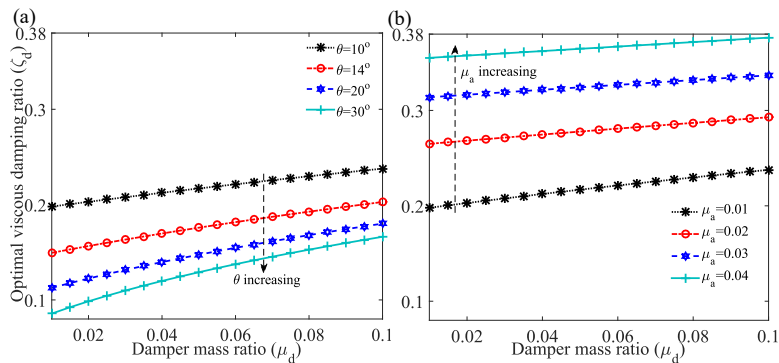
$$(\zeta_d)_{opt} = \frac{\sqrt{2} \sqrt{\frac{(\mu_{ad} - 4)\mu_{ad}}{(\mu_{ad} + 1)(\mu_{ad} - 2)}}}{4} \quad (21)$$

The variations of optimal frequency ratio versus damper mass ratio for different values of inertial angle of IATMD have been shown in Figure 3 (a). The optimal frequency ratio decreases as the damper mass ratio increases and increases as the inertial angle increases. The variations of optimal frequency ratio versus damper



**Figure 3:** The variations of optimal frequency ratio versus damper mass ratio for different values of (a) inertial angle and (b) amplifier’s mass ratio of IATMD.

mass ratio for different values of amplifier mass ratio of IATMD have been shown in Figure 3 (b). The optimal frequency ratio decreases as the amplifier mass ratio increases. Therefore, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle achieve optimum IATMD with a lower frequency ratio. The variations of optimal viscous damping ratio versus damper mass ratio for different values of inertial angle of IATMD have been shown in Figure 4 (a). The optimal viscous damping ratio increases as the damper mass ratio increases and decreases as the inertial angle increases. The variations of optimal viscous damping ratio versus



**Figure 4:** The variations of optimal viscous damping ratio versus damper mass ratio for different values of (a) inertial angle and (b) amplifier’s mass ratio of IATMD.

damper mass ratio for different values of amplifier mass ratio of IATMD have been shown in Figure 4 (b). The optimal viscous damping ratio increases as the amplifier



mass ratio increases. For tuned mass dampers, a moderate viscous damping ratios with lower frequencies are recommended for achieving optimum vibration reduction capacity. Therefore, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle achieve optimum IATMD with a moderate viscous damping ratio.

#### 4. $H_\infty$ optimization

$H_\infty$  optimization method has been applied to minimize the maximum dynamic responses of the controlled structures subjected to harmonic excitations. Hence, to perform that Eq. (8) has been re-written as

$$\begin{bmatrix} \mu_{ad} (2i\eta\zeta_d\eta_d - \eta^2 + \eta_d^2) & -\eta^2\mu_{ad} \\ -\eta_d\mu_{ad} (2i\eta\zeta_d + \eta_d) & -\eta^2 + 1 + 2i\zeta_s\eta \end{bmatrix} \begin{Bmatrix} X_d \\ X_s \end{Bmatrix} = - \begin{bmatrix} \mu_{ad} \\ 1 \end{bmatrix} \frac{A_g}{\omega_s^2} \quad (22)$$

The dynamic response of IATMD has been derived as

$$H_d = \frac{X_d \omega_s^2}{A_g} = \frac{2i\zeta_s\eta + 1}{\Delta} \quad (23)$$

The dynamic response of primary structure has been derived as

$$H_s = \frac{X_s \omega_s^2}{A_g} = \frac{\eta_d^2 \mu_{ad} - \eta^2 + \eta_d^2 + 2i\eta\zeta_d\eta_d (\mu_{ad} + 1)}{\Delta} \quad (24)$$

$\Delta$  has been derived as

$$\Delta = \begin{aligned} & 4\eta^2\zeta_d\zeta_s\eta_d + \eta^2\eta_d^2\mu_{ad} - \eta^4 + \eta^2\eta_d^2 + \eta^2 - \eta_d^2 \\ & + i(2\eta^3\zeta_d\eta_d\mu_{ad} + 2\eta^3\zeta_d\eta_d + 2\eta^3\zeta_s - 2\eta\zeta_s\eta_d^2 - 2\eta\zeta_d\eta_d) \end{aligned} \quad (25)$$

The resultant of  $H_s$  has been written as

$$|H_s| = \sqrt{\frac{x_1^2 + \zeta_d^2 x_2^2}{x_3^2 + \zeta_d^2 x_4^2}} = \frac{x_2}{x_4} \sqrt{\frac{\frac{x_1^2}{x_2^2} + \zeta_d^2}{\frac{x_3^2}{x_4^2} + \zeta_d^2}} \quad (26)$$

From Eq. (26), the first constraint Den Hartog [1985] has been derived as

$$\left| \frac{x_1}{x_2} \right| = \left| \frac{x_3}{x_4} \right| \quad (27)$$

A equation has been derived from Eq. (27) and expressed as

$$\begin{aligned} & (2\mu_{ad} + 2)\eta^4 + 2\eta_d^2\mu_{ad} + 2\eta_d^2 \\ & + (-2\eta_d^2\mu_{ad}^2 - 4\eta_d^2\mu_{ad} - 2\eta_d^2 - \mu_{ad} - 2)\eta^2 = 0 \end{aligned} \quad (28)$$

$$\eta_1^2 + \eta_2^2 = \frac{(\eta_d^2\mu_{ad}^2 + 2\eta_d^2\mu_{ad} + \eta_d^2 + (\mu_{ad}/2) + 1)}{(\mu_{ad} + 1)} \quad (29)$$

From Eq. (26), the second constraint Den Hartog [1985] has been derived as

$$(H_s)_{\eta_1, \eta_2} = \left| \frac{x_2}{x_4} \right| \quad \text{and} \quad (H_s)_{\eta_1, \eta_2} = \frac{1 + \mu_{ad}}{|1 - \eta_{1,2}^2(1 + \mu_{ad})|} \quad (30)$$

$$\eta_1^2 + \eta_2^2 = \frac{2}{1 + \mu_{ad}} \quad (31)$$

Equation (29) and Eq. (31) are equating to derive the closed-form expression for optimal frequency ratio of IATMD and expressed as

$$(\eta_d)_{opt} = \frac{\sqrt{1 - 0.5\mu_{ad}}}{1 + \mu_{ad}} \quad (32)$$

The optimum  $\eta_{1,2}$ ,  $H_s$ , and  $\zeta_d$  has also been derived in similar manner and expressed as

$$(\eta_{1,2})_{opt} = \frac{\sqrt{1 \pm \sqrt{0.5\mu_{ad}}}}{1 + \mu_{ad}} \quad (33)$$

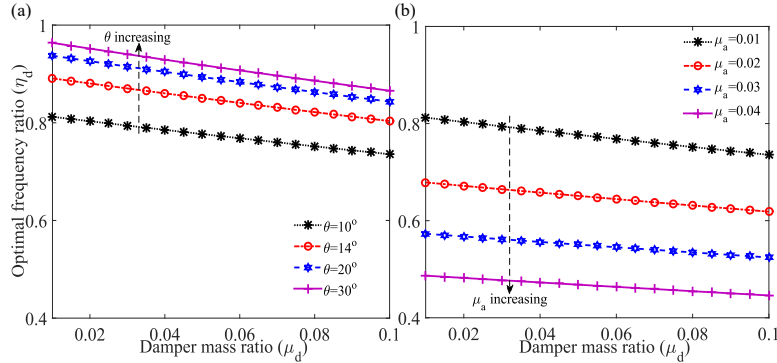
The optimal dynamic response of IATMD has been derived as

$$(H_s)_{opt} = \frac{1 + \mu_{ad}}{\sqrt{0.5\mu_{ad}}} \quad (34)$$

The exact closed-form expression for optimal viscous damping ratio of IATMD has been derived as

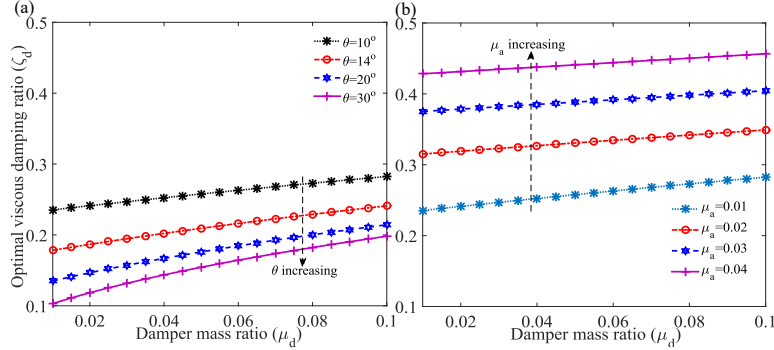
$$(\zeta_d)_{opt} = \sqrt{\frac{\mu_{ad}(3 - \sqrt{0.5\mu_{ad}})}{8(1 + \mu_{ad})(1 - 0.5\mu_{ad})}} \quad (35)$$

The variations of optimal frequency ratio versus damper mass ratio for different values of inertial angle of IATMD have been shown in Figure 5 (a). The optimal frequency ratio decreases as the damper mass ratio increases and increases as the inertial angle increases. The variations of optimal frequency ratio versus damper



**Figure 5:** The variations of optimal frequency ratio versus damper mass ratio for different values of (a) inertial angle and (b) amplifier's mass ratio of IATMD.

mass ratio for different values of amplifier mass ratio of IATMD have been shown in Figure 5 (b). The optimal frequency ratio decreases as the amplifier mass ratio increases. Therefore, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle achieve optimum IATMD with a lower frequency ratio. The variations of optimal viscous damping ratio versus damper mass ratio for different values of inertial angle of IATMD have been shown in Figure 6 (a). The optimal



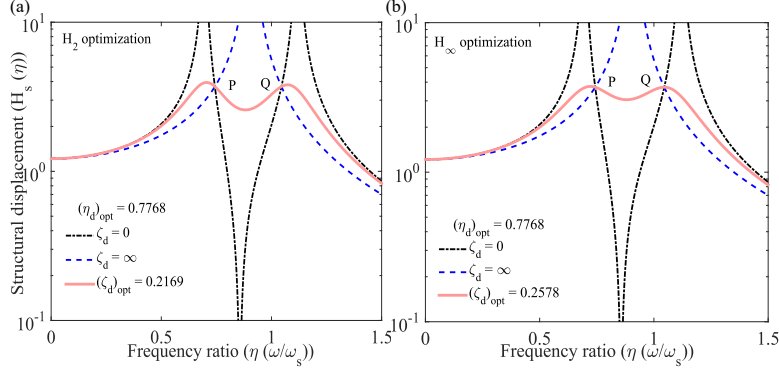
**Figure 6:** The variations of optimal viscous damping ratio versus damper mass ratio for different values of (a) inertial angle and (b) amplifier's mass ratio of IATMD.

viscous damping ratio increases as the damper mass ratio increases and decreases as the inertial angle increases. The variations of optimal viscous damping ratio versus damper mass ratio for different values of amplifier mass ratio of IATMD have been shown in Figure 6 (b). The optimal viscous damping ratio increases as the amplifier mass ratio increases. For tuned mass dampers, a moderate viscous damping ratios with lower frequencies are recommended for achieving optimum vibration reduction capacity. Therefore, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle achieve optimum IATMD with a moderate viscous damping ratio.

## 5. Robustness of $H_2$ and $H_\infty$ optimized IATMD

The variations of optimal dynamic responses of primary structures controlled by  $H_2$  optimized IATMD versus frequency ratio for different values of viscous damping ratio have been shown in Figure 7 (a). The primary structure's viscous damping is considered  $\zeta_s = 0.00$ . For  $\zeta_d = 0$ , the controlled structures are vibrating at their Eigen frequencies, i.e.,  $\eta = 0.6943, 1.119$ . The anti-resonance frequency point located at  $\eta = 0.8566$ . For  $\zeta_d > 0$ , the controlled structures are vibrating at their resonating frequencies, i.e.,  $\eta = 0.6983, 1.081$ . The minima frequency point located at  $\eta = 0.8763$ . At  $\zeta_d = \infty$ , the maximum peaks of controlled structures are merged into one, i.e., SDOF and frequency peak located as  $\eta = 0.9072$ . The variations of optimal dynamic responses of primary structures controlled by  $H_\infty$  optimized IATMD versus frequency ratio for different values of viscous damping ratio have been shown in Figure 7 (b). The primary structure's viscous damping is considered  $\zeta_s = 0.00$ . For  $\zeta_d = 0$ , the controlled structures are vibrating at their Eigen frequencies, i.e.,  $\eta = 0.6944, 1.119$ . The anti-resonance frequency point located at  $\eta = 0.8566$ . For  $\zeta_d > 0$ , the controlled structures are vibrating at their resonating frequencies, i.e.,  $\eta = 0.7223, 1.046$ . The minima frequency point located at  $\eta = 0.8803$ . At  $\zeta_d = \infty$ , the maximum peaks of controlled structures are merged into one, i.e., SDOF and frequency peak located as  $\eta = 0.9069$ .

The variations of optimal dynamic responses of primary structures controlled by



**Figure 7:** The variations of optimal dynamic responses of primary structures controlled by (a)  $H_2$  and (b)  $H_\infty$  optimized IATMD versus frequency ratio for different values of viscous damping ratio.

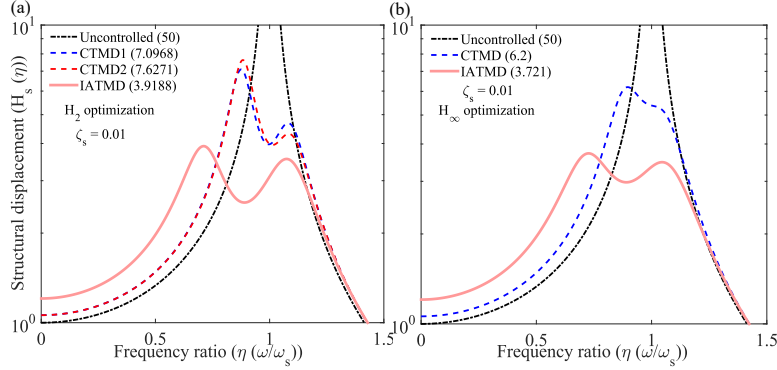
$H_2$  optimized IATMD and conventional tuned mass dampers (CTMD) versus frequency ratio have been shown in Figure 8 (a). The details of design parameters for these graphs are listed in Table 1. The maximum dynamic response of the uncontrolled structure have been determined as 50. The maximum dynamic response of the structure controlled by conventional tuned mass damper 1 (CTMD1) Warburton [1982]; Zilletti *et al.* [2012] and conventional tuned mass damper 2 (CTMD2) Iwata [1982] have been determined as 7.0968 and 7.6271. The maximum dynamic response

**Table 1:** The optimal design parameters of uncontrolled and controlled structures. Equation (20) and Eq. (21) are applied for  $H_2$  optimized mass dampers.

Symbols			$H_2$ optimization		
CTMD1	CTMD2	IATMD	CTMD1	CTMD2	IATMD
$\zeta_s$	$\zeta_s$	$\zeta_s$	0.01	0.01	0.01
$\zeta_d$	$\zeta_d$	$\zeta_d$	0.1198	0.1225	0.2124
$\eta_d$	$\eta_d$	$\eta_d$	0.9574	0.9713	0.7855
$\mu_d$	$\mu_d$	$\mu_d + 2\mu_a$	0.06	0.06	0.06
$\mu_d$	$\mu_d$	$\mu_d$	0.06	0.06	0.04
...	...	$\mu_a$	...	...	0.01
...	...	$\theta$	...	...	$10^\circ$

Note: CTMD1 = Warburton [1982]; Zilletti *et al.* [2012]  
 CTMD2 = Iwata [1982]

of the structure controlled by IATMD has been determined as 3.9188. Therefore, the dynamic response capacity of  $H_2$  optimized IATMD is significantly 44.78 % and 48.62 % superior to the  $H_2$  optimized CTMD1 and CTMD2. The variations of optimal dynamic responses of primary structures controlled by  $H_\infty$  optimized IATMD and conventional tuned mass dampers (CTMD) versus frequency ratio have been shown in Figure 8 (b). The details of design parameters for these graphs are listed in Table 2. The maximum dynamic response of the uncontrolled struc-



**Figure 8:** The variations of optimal dynamic responses of primary structures controlled by (a)  $H_2$  and (b)  $H_\infty$  optimized IATMD and conventional tuned mass damper (CTMD) versus frequency ratio.

ture have been determined as 50. The maximum dynamic response of the structure controlled by conventional tuned mass damper (CTMD) Krenk [2005]; Den Hartog and Ormondroyd [1928]; Nishihara and Asami [2002] has been determined as 6.2. The maximum dynamic response of the structure controlled by IATMD has been

**Table 2:** The optimal design parameters of uncontrolled and controlled structures. Equation (32) and Eq. (35) are applied for  $H_\infty$  optimized mass dampers.

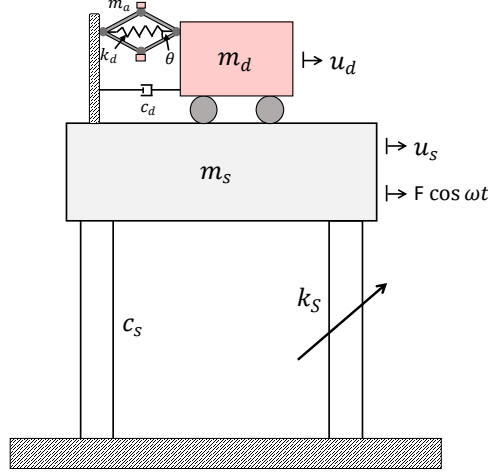
Symbols		$H_\infty$ optimization	
CTMD	IATMD	CTMD	IATMD
$\zeta_s$	$\zeta_s$	0.01	0.01
$\zeta_d$	$\zeta_d$	0.1682	0.2524
$\eta_d$	$\eta_d$	0.9434	0.7855
$\mu_d$	$\mu_d + 2\mu_a$	0.06	0.06
$\mu_d$	$\mu_d$	0.06	0.04
...	$\mu_a$	...	0.01
...	$\theta$	...	$10^\circ$

*Note:* CTMD = Krenk [2005]; Den Hartog and Ormondroyd [1928]; Nishihara and Asami [2002]

determined as 3.721. Therefore, the dynamic response capacity of  $H_\infty$  optimized IATMD is significantly 39.98 % superior to the  $H_\infty$  optimized CTMD.

## 6. Nonlinear dynamic analysis

IATMD has been installed on the nonlinear dynamic system, and the corresponding schematic diagram has been displayed in Figure 9. Newton's second law applies to derive the equations of motion for a nonlinear dynamic system equipped with inertial amplifier tuned mass dampers (IATMD) after considering all the effective system parameters, such as effective mass stiffness and damping of the IATMD and



**Figure 9:** The schematic diagram of IATMD equipped with nonlinear dynamic system.

expressed as

$$\begin{aligned} m_s \ddot{u}_s + c_s \dot{u}_s + k_{s1} u_s + k_{s2} u_s^3 - k_{ad} x_d - c_{ad} \dot{x}_d &= F \cos \omega t \\ m_{ad} \ddot{x}_d + m_{ad} \ddot{u}_s + k_{ad} x_d + c_{ad} \dot{x}_d &= 0 \end{aligned} \quad (36)$$

The controlled structure is subjected to harmonic excitation. Therefore, the harmonic balance (HB) method has been applied to derive the dynamic responses of the controlled structures analytically. The steady state solutions are considered as

$$x_d = X_d \cos(\omega t + \beta) \quad \text{and} \quad u_s = U_s \cos(\omega t + \alpha) \quad (37)$$

Equation (37) has been substituted in the second equation of Eq. (36). Therefore, Eq. (36) has been modified as

$$-\mu_{ad} \left( \begin{aligned} &X_d (\omega^2 - \omega_d^2) \cos(\omega t + \beta) \\ &+ 2 \sin(\omega t + \beta) X_d \omega_d \omega \zeta_d + U_s \omega^2 \cos(\omega t + \alpha) \end{aligned} \right) = 0 \quad (38)$$

The trigonometric function has been derived as

$$\begin{aligned} \cos(\omega t + \alpha) &= \cos((\omega t + \beta) + (\alpha - \beta)) \\ &= \cos(\omega t + \beta) \cos(\alpha - \beta) - \sin(\omega t + \beta) \sin(\alpha - \beta) \end{aligned} \quad (39)$$

Equation (38) has been re-written as

$$\begin{aligned} X_d (\omega^2 - \omega_d^2) \cos(\omega t + \beta) + 2 \sin(\omega t + \beta) X_d \omega_d \omega \zeta_d \\ + U_s \omega^2 \cos(\omega t + \beta) \cos(\alpha - \beta) - U_s \omega^2 \sin(\omega t + \beta) \sin(\alpha - \beta) \end{aligned} = 0 \quad (40)$$

After applying the harmonic balance, the trigonometric functions are derived as

$$\begin{aligned} \cos(\alpha - \beta) &= -\frac{X_d (\omega^2 - \omega_d^2)}{\omega^2 U_s} \quad \text{and} \quad \sin(\alpha - \beta) = \frac{2 \zeta_d X_d \omega_d}{\omega U_s} \\ \tan(\alpha - \beta) &= \frac{2 \omega \zeta_d \omega_d}{\omega_d^2 - \omega^2} = \frac{2 \eta \zeta_d \eta_d}{\eta_d^2 - \eta^2} \end{aligned} \quad (41)$$

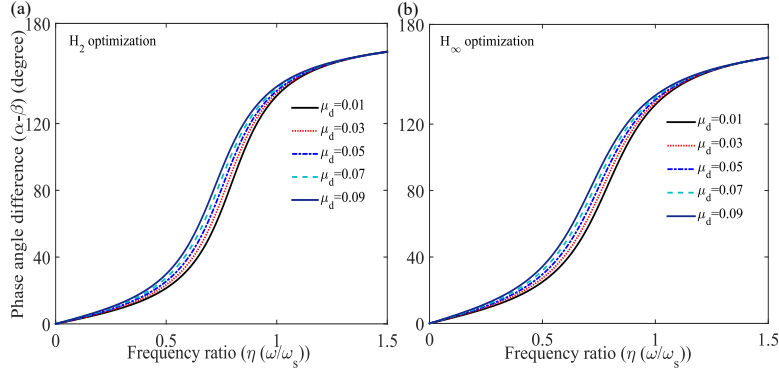
The closed-form expression for  $X_d$  has been derived using Eq. (41). Hence, the derivations are listed below.

$$\sin^2(\alpha - \beta) + \cos^2(\alpha - \beta) = \left( \frac{X_d(\omega^2 - \omega_d^2)}{\omega^2 U_s} \right)^2 + \left( \frac{2\zeta_d X_d \omega_d}{\omega U_s} \right)^2 \quad (42)$$

The closed-form expression for dynamic response of IATMD has been derived as

$$X_d = \frac{U_s \omega^2}{\sqrt{4\omega^2 \zeta_d^2 \omega_d^2 + \omega^4 - 2\omega^2 \omega_d^2 + \omega_d^4}} \quad (43)$$

The variations of phase angle differences versus frequency ratio of controlled structures for  $H_2$  optimized IATMD have been shown in Figure 10 (a). The variations of



**Figure 10:** The variations of phase angle differences versus frequency ratio of controlled structures for (a)  $H_2$  and (b)  $H_\infty$  optimized IATMD.

phase angle differences versus frequency ratio of controlled structures for  $H_\infty$  optimized IATMD have been shown in Figure 10 (b). From both figures, it has been observed that the controlled structures are in phase condition at  $\eta = 0.813$  and out of phase conditions are located at  $\eta > 0.813$ . Equation (37) has been substituted in the first equation of Eq. (36). Therefore, the equation has been modified as

$$\begin{aligned} & \frac{1}{4} \gamma \omega_s^2 U_s^3 \cos(3\omega t + 3\alpha) \\ + \frac{1}{4} & \left( \begin{array}{l} 3\gamma \omega_s^2 U_s^3 + (-4\omega^2 + 4\omega_s^2) U_s \\ -8B\omega \zeta_d X_d \mu_{ad} \omega_d - 4AX_d \mu_{ad} \omega_d^2 - 4Fw_1 \end{array} \right) \cos(\omega t + \alpha) = 0 \quad (44) \\ + 2 \sin(\omega t + \alpha) & \left( \begin{array}{l} A\omega \zeta_d X_d \mu_{ad} \omega_d - 1/2 BX_d \mu_{ad} \omega_d^2 \\ -\zeta_s \omega_s U_s \omega - 1/2 Fw_2 \end{array} \right) \end{aligned}$$

where  $A = \cos(\alpha - \beta)$ ,  $B = \sin(\alpha - \beta)$ ,  $w_1 = \cos \alpha$ ,  $w_2 = \sin \alpha$ . The HB method has been applied and the trigonometric functions are derived as

$$\cos \alpha = \frac{1}{F} \left( 3/4 \gamma \omega_s^2 U_s^3 - (\omega^2 - \omega_s^2) U_s - \frac{\mu_{ad} \omega_d^2 X_d^2 (4\omega^2 \zeta_d^2 - \omega^2 + \omega_d^2)}{\omega^2 U_s} \right) \quad (45)$$

$$\sin \alpha = -\frac{2\omega \left( \zeta_d X_d^2 \mu_{ad} \omega_d + \zeta_s \omega_s U_s^2 \right)}{U_s F} \quad (46)$$

Equation (45) and Eq. (46) have been written as

$$\begin{aligned} \sin^2 \alpha + \cos^2 \alpha &= \left( \frac{2\omega \left( \zeta_d X_d^2 \mu_{ad} \omega_d + \zeta_s \omega_s U_s^2 \right)}{U_s F} \right)^2 \\ &+ \left( \frac{1}{F} \left( 3/4 \gamma \omega_s^2 U_s^3 - (\omega^2 - \omega_s^2) U_s - \frac{\mu_{ad} \omega_d^2 X_d^2 (4\omega^2 \zeta_d^2 - \omega^2 + \omega_d^2)}{\omega^2 U_s} \right) \right)^2 \end{aligned} \quad (47)$$

Equation (43) has been substituted in the first equation of Eq. (50). Therefore, the nonlinear dynamic response of the primary structure has been derived as

$$\begin{aligned} u_3 U_s^6 + u_2 U_s^4 + u_1 U_s^2 + u_0 &= 0 \\ p_1 = U_{s1}^2 &= \frac{\sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}}{6 u_3} \end{aligned} \quad (48)$$

$$\begin{aligned} &- \frac{2 \left( 3 u_3 u_1 - u_2^2 \right)}{3 u_3 \sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}} - \frac{u_2}{3 u_3} \\ p_2 = U_{s2}^2 &= -\frac{\sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}}{12 u_3} \\ &+ \frac{3 u_3 u_1 - u_2^2}{3 u_3 \sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}} - \frac{u_2}{3 u_3} \end{aligned} \quad (49)$$

$$\begin{aligned} &+ \frac{1}{2} \left( i \sqrt{3} \left( + \frac{\frac{\sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}}{6 u_3}}{2 \left( 3 u_3 u_1 - u_2^2 \right)} \right) \right) \\ p_3 = U_{s3}^2 &= -\frac{\sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}}{12 u_3} \\ &+ \frac{3 u_3 u_1 - u_2^2}{3 u_3 \sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}} - \frac{u_2}{3 u_3} \\ &- \frac{1}{2} \left( i \sqrt{3} \left( + \frac{\frac{\sqrt[3]{-108 u_0 u_3^2 + 36 u_3 u_2 u_1 + 12 \sqrt{3} V_1 u_3 - 8 u_2^3}}{6 u_3}}{2 \left( 3 u_3 u_1 - u_2^2 \right)} \right) \right) \end{aligned} \quad (50)$$

where

$$V_1 = \sqrt{27 u_3^2 u_0^2 - 18 u_3 u_2 u_1 u_0 + 4 u_3 u_1^3 + 4 u_2^3 u_0 - u_2^2 u_1^2} \quad (51)$$

The closed-form expressions for  $u_3$ ,  $u_2$ ,  $u_1$ , and  $u_0$  are derived as

$$\begin{aligned} u_3 &= 9 \omega_s^4 \gamma^2 \left( \omega_d^4 + \left( 4 \zeta_d^2 - 2 \right) \omega^2 \omega_d^2 + \omega^4 \right)^2 \\ &- 24 \left( \omega_d^4 + \left( 4 \zeta_d^2 - 2 \right) \omega^2 \omega_d^2 + \omega^4 \right) \omega_s^2 \gamma \\ u_2 &= \left( \begin{aligned} &\omega^6 + \left( \left( 4 \mu_{ad} + 4 \right) \zeta_d^2 - \mu_{ad} - 2 \right) \omega_d^2 - \omega_s^2 \right) \omega^4 \\ &+ \left( \left( \mu_{ad} + 1 \right) \omega_d^4 + \left( -4 \zeta_d^2 + 2 \right) \omega_s^2 \omega_d^2 \right) \omega^2 - \omega_s^2 \omega_d^4 \end{aligned} \right) \end{aligned} \quad (52)$$



$$u_1 = \begin{pmatrix} \omega^8 + \left( \begin{array}{l} 16\omega_d^4 + 16(4\zeta_d^2 - 2)\omega^2\omega_d^2 + 16\omega^4 \\ 4(-1/2 + (\mu_{ad} + 1)\zeta_d^2)(\mu_{ad} + 1)\omega_d^2 \\ + 8\zeta_d\zeta_s\omega_s\mu_{ad}\omega_d + (4\zeta_s^2 - 2)\omega_s^2 \end{array} \right) \omega^6 \\ + \left( \begin{array}{l} (\mu_{ad} + 1)^2\omega_d^4 + \omega_s^4 \\ + 16\omega_s^2 \left( \begin{array}{l} (\zeta_s^2 - 1/2\mu_{ad} - 1/2)\zeta_d^2 \\ - 1/2\zeta_s^2 + 1/8\mu_{ad} + 1/4 \end{array} \right) \omega_d^2 \end{array} \right) \omega^4 \\ + 4 \left( \begin{array}{l} (\zeta_s^2 - 1/2\mu_{ad} - 1/2)\omega_d^2 + \omega_s^2(\zeta_d^2 - 1/2) \\ + \omega_s^4\omega_d^4 \end{array} \right) \omega_s^2\omega_d^2\omega^2 \end{pmatrix} \quad (53)$$

$$u_0 = -16F^2 \left( \omega_d^4 + (4\zeta_d^2 - 2)\omega^2\omega_d^2 + \omega^4 \right)^2 \quad (54)$$

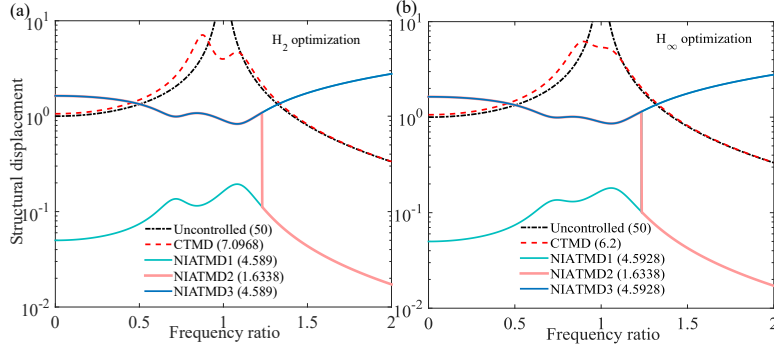
The variations of optimal nonlinear dynamic responses of primary structures controlled by  $H_2$  optimized IATMD and conventional tuned mass dampers (CTMD) versus frequency ratio have been shown in Figure 11 (a). The details of design parameters for these graphs are listed in Table 3. The maximum dynamic response of the uncontrolled structure have been determined as 50. The maximum dynamic response of the structure controlled by conventional tuned mass damper (CTMD) Warburton [1982]; Zilletti *et al.* [2012] has been determined as 7.0968. Three analyt-

**Table 3:** The optimal design parameters of uncontrolled and controlled structures. Equation (20) and Eq. (21) are applied for  $H_2$  optimized mass dampers.

Symbols		$H_2$ optimization	
CTMD	IATMD	CTMD	IATMD
$\zeta_s$	$\zeta_s$	0.01	0.01
$\zeta_d$	$\zeta_d$	0.1198	0.2124
$\eta_d$	$\eta_d$	0.9574	0.7855
$\mu_d$	$\mu_d + 2\mu_a$	0.06	0.06
$\mu_d$	$\mu_d$	0.06	0.04
...	$\mu_a$	...	0.10
...	$\theta$	...	10°

*Note:* CTMD = Warburton [1982]; Zilletti *et al.* [2012]

ical closed-form expressions for nonlinear dynamic responses of the primary structures have been derived. Equation (50) has been utilized to determine the nonlinear dynamic responses for primary structures controlled by  $H_2$  optimized IATMD. The maximum nonlinear dynamic responses of the primary structures controlled by  $H_2$  optimized IATMD have been evaluated as 4.589, 1.6338, 4.589. Therefore,  $H_2$  optimized IATMD systems are significantly 35.33 %, 76.97 %, and 35.33 % superior to the  $H_2$  optimized CITMD. The variations of optimal dynamic responses of primary structures controlled by  $H_\infty$  optimized IATMD and conventional tuned mass dampers (CTMD) versus frequency ratio have been shown in Figure 11 (b). The details of design parameters for these graphs are listed in Table 4. The maximum dynamic response of the uncontrolled structure have been determined as 50. The



**Figure 11:** The variations of optimal nonlinear dynamic responses of primary structures controlled by (a)  $H_2$  and (b)  $H_\infty$  optimized CTMD and IATMD versus frequency ratio.

maximum dynamic response of the structure controlled by conventional tuned mass damper (CTMD) Krenk [2005]; Den Hartog and Ormondroyd [1928]; Nishihara and Asami [2002] has been determined as 6.2. The maximum nonlinear dynamic re-

**Table 4:** The optimal design parameters of uncontrolled and controlled structures. Equation (32) and Eq. (35) are applied for  $H_\infty$  optimized mass dampers.

Symbols		$H_\infty$ optimization	
CTMD	IATMD	CTMD	IATMD
$\zeta_s$	$\zeta_s$	0.01	0.01
$\zeta_d$	$\zeta_d$	0.1682	0.2524
$\eta_d$	$\eta_d$	0.9434	0.7855
$\mu_d$	$\mu_d + 2\mu_a$	0.06	0.06
$\mu_d$	$\mu_d$	0.06	0.04
...	$\mu_a$	...	0.10
...	$\theta$	...	$10^\circ$

*Note:* CTMD = Krenk [2005]; Den Hartog and Ormondroyd [1928]; Nishihara and Asami [2002]

sponses of the primary structures controlled by  $H_\infty$  optimized IATMD have been evaluated as 4.5928, 1.6338, 4.5928. Therefore,  $H_\infty$  optimized IATMD systems are significantly 25.92 %, 73.64 %, and 25.92 % superior to the  $H_\infty$  optimized CITMD.

## 7. Summary and conclusions

The dynamic response reduction capacity of optimum IATMD has been determined in the present study.  $H_2$  and  $H_\infty$  optimization methods are applied to derive the closed-form expressions for optimal design parameters for IATMD. The harmonic balance (HB) method has been applied to derive the nonlinear dynamic responses of the controlled structures. The dynamic response reduction capacity of optimum IATMD has been compared with the dynamic response reduction capacity of CTMD. The significant outcomes of the study are listed below.

18 *References*

- A higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle are recommended to achieve  $H_2$  optimized IATMD with a moderate viscous damping and lower frequency ratios.
- For  $H_\infty$  optimized IATMD, a higher damper mass ratio, a higher amplifier mass ratio, and a lower inertial angle are also recommended.
- The dynamic response capacity of  $H_2$  optimized IATMD is significantly 44.78 % and 48.62 % superior to the  $H_2$  optimized CTMD1 and CTMD2.
- The dynamic response capacity of  $H_\infty$  optimized IATMD is significantly 39.98 % superior to the  $H_\infty$  optimized CTMD.
- According to the nonlinear dynamic analysis,  $H_2$  optimized IATMD systems are significantly 35.33 %, 76.97 %, and 35.33 % superior to the  $H_2$  optimized CITMD.
- Furthermore,  $H_\infty$  optimized IATMD systems are significantly 25.92 %, 73.64 %, and 25.92 % superior to the  $H_\infty$  optimized CITMD.

The closed-form expressions for optimal design parameters of IATMD are one of the significant contributions of the paper. The experimental works considering this novel tuned mass dampers will be the future perspective of the research.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper

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