



Bispectrality of AG_2 Calogero–Moser–Sutherland System

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Abstract

We consider the generalised Calogero–Moser–Sutherland quantum integrable system associated to the configuration of vectors AG_2 , which is a union of the root systems A_2 and G_2 . We establish the existence of and construct a suitably defined Baker–Akhiezer function for the system, and we show that it satisfies bispectrality. We also find two corresponding dual difference operators of rational Macdonald–Ruijsenaars type in an explicit form.

Keywords Bispectrality · Macdonald–Ruijsenaars operators · Baker–Akhiezer functions

Mathematics Subject Classification 37J70

1 Introduction

The Calogero–Moser–Sutherland (CMS) systems form an important class of integrable systems with deep relations to geometry, algebra, combinatorics, and other areas. The original works of Calogero, Sutherland and Moser dealt with Hamiltonians describing a pairwise interaction of systems of particles on a circle or a line [2, 20, 28]. This case was related with the root system A_n by Olshanetsky and Perelomov, who extended CMS Hamiltonians to the case of any root system [21]. In the quantum case, these Hamiltonians are closely related to the radial parts of Laplace–Beltrami operators on symmetric spaces [1, 22].

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Ruijsenaars and Schneider introduced a relativistic version of the CMS system in the classical case in [26]. The quantum Hamiltonian and its integrals, which are difference operators, were studied by Ruijsenaars in [23]. For an arbitrary (reduced) root system, the corresponding commuting operators were introduced by Macdonald [19].

Ruijsenaars established a duality relation between the classical (trigonometric) CMS system and its (rational) relativistic version which essentially swaps action and angle variables of the two systems [24]. He also conjectured a duality relation in the quantum case [25]. In this case, there exists a function ψ of two sets of variables x and z such that

$$\begin{aligned} L\psi &= \lambda\psi, \\ M\psi &= \mu\psi, \end{aligned} \tag{1.1}$$

where $L = L(x, \partial_x)$ is the CMS Hamiltonian or its quantum integral, M is Ruijsenaars' relativistic CMS operator acting in the variables z , and $\lambda = \lambda(z)$, $\mu = \mu(x)$ are the corresponding eigenvalues. The relations (1.1) may be viewed as a higher-dimensional differential-difference version of the one-dimensional differential-differential bispectrality relation studied by Duistermaat and Grünbaum [11].

The bispectrality relation (1.1) for the CMS Hamiltonian L related to any root system and for the corresponding Macdonald–Ruijsenaars operators M was established by Chalykh in [4]. In the case of integer coupling parameters, the function ψ can be taken to be a multi-dimensional Baker–Akhiezer (BA) function [6, 8, 10]. It has the form

$$\psi(z, x) = P e^{\langle x, z \rangle}, \tag{1.2}$$

where P is a polynomial in z which also depends on x , and ψ can be characterised by its properties as a function of the variables z . The case of general coupling parameters is deduced by some analytic continuation in parameters and related arguments [4]. The relation (1.1) for the root system A_n and another function ψ given by integral formulas was established recently by Kharchev and Khoroshkin in [17]. We refer to [16] and references therein for aspects of this and various other dualities in the broad realm of integrable systems.

In general, the eigenfunctions of quantum integrable systems may be complicated. However, the multi-dimensional BA functions turn out to be manageable algebraic functions. An axiomatic definition of BA functions was formulated by Chalykh, Veselov and Styrkas for an arbitrary finite collection of non-collinear vectors with integer multiplicities in [6, 8] (see also [13] for a weaker version of the function; we also refer to [18] for one-dimensional BA functions). The key properties are the *quasi-invariant* conditions of the form

$$\psi(z + s\alpha, x) = \psi(z - s\alpha, x)$$

which are satisfied at $\langle \alpha, z \rangle = 0$ for any vector α from the configuration, and s takes special values.

It was shown that the BA function exists for very special configurations only, which includes positive subsystems of reduced root systems [8]. With a slight modification, one can also formulate an axiomatic definition of the BA function in the case of the (only) non-reduced root system BC_n [4].

The BA function is known to exist for certain generalisations of CMS systems related to special configurations of vectors which are not root systems. The known examples are deformations $A_{n,1}(m)$, $C_n(m, l)$ of the root systems A_n and C_n , respectively, found by Chalykh, Veselov and one of the authors in [9, 10]; and another deformation $A_{n,2}(m)$ of the root system A_{n+1} [7], which satisfies a weaker axiomatics [13]. The BA function for the configuration $A_{n,1}(m)$ was constructed in [4], and for the cases $C_n(m, l)$, $A_{n,2}(m)$ it was constructed in [13]. It was also shown in the papers [4, 13] that the bispectrality relation (1.1) holds for the corresponding generalisations of Macdonald–Ruijsenaars operators. An important property of these operators is the preservation of a space of quasi-invariant analytic functions.

In the process of classification of algebraically integrable two-dimensional generalised CMS operators, the configuration of vectors AG_2 emerged in the work of Fairley and one of the authors [12]. We studied the corresponding generalised CMS operator in [15]. There we found its eigenfunction ψ of the form (1.2) as $\psi = \mathcal{D}\psi_0$, where ψ_0 is the BA function for the root system G_2 and \mathcal{D} is an explicit differential operator of order three.

In this paper, we study the function ψ further. We describe it as a BA function by finding the conditions in the variables z which this function satisfies. We find a pair of explicit difference operators which have ψ as their common eigenfunction. Thus we establish bispectral duality for the generalised CMS system of type AG_2 with integer coupling parameters. We also obtain an expression for the function ψ by iterations of the action of the difference operators, in analogy with formulas from the works [4, 13] for other configurations.

Let us now introduce more precisely the generalised CMS operators and the configuration of vectors AG_2 . Let $\langle \cdot, \cdot \rangle$ denote the standard Euclidean inner product on \mathbb{R}^n . We consider the complexification of the real Euclidean space \mathbb{R}^n and extend $\langle \cdot, \cdot \rangle$ bilinearly. Let $x = (x_1, \dots, x_n) \in \mathbb{C}^n$.

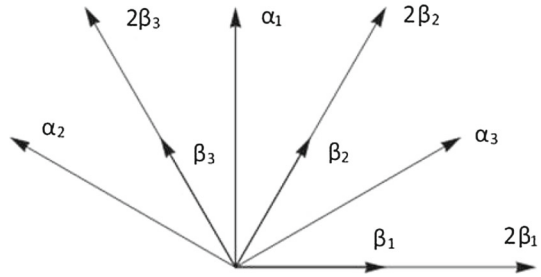
Let $A \subset \mathbb{C}^n$ be a finite collection of non-isotropic vectors. A multiplicity map is a function $m: A \rightarrow \mathbb{C}$, $\alpha \mapsto m_\alpha = m(\alpha)$. One calls m_α the multiplicity of α . One denotes such collections of vectors with multiplicities by $\mathcal{A} = (A, m)$, or just A if the intended multiplicity map is clear from the context. The corresponding generalised CMS operator has the form

$$L = -\Delta + \sum_{\alpha \in A} g_\alpha \sinh^{-2} \langle \alpha, x \rangle, \tag{1.3}$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the Laplacian on \mathbb{C}^n and we have the convention of writing the constants g_α as

$$g_\alpha = m_\alpha(m_\alpha + 2m_{2\alpha} + 1) \langle \alpha, \alpha \rangle \tag{1.4}$$

Fig. 1 A positive half of the configuration AG_2 [15]



with $m_{2\alpha} = 0$ if $2\alpha \notin A$. The convention (1.4) is used in symmetric spaces theory (see e.g. [27]).

The Olshanetsky–Perelomov operators correspond to the case when A is a positive half of a root system and the multiplicity function m is Weyl-invariant. In that case, if $m_\alpha \in \mathbb{N}$ for all α then the operator (1.3) admits additional integrals and is algebraically integrable (see [6, 8]).

The configuration of vectors AG_2 is a non-reduced configuration in \mathbb{C}^2 obtained as a union of the root systems G_2 and A_2 . A positive half $AG_{2,+}$ is shown in Fig. 1, where $G_{2,+} = \{\alpha_i, \beta_i : i = 1, 2, 3\}$, and $A_{2,+} = \{2\beta_i : i = 1, 2, 3\}$. There are in total 18 vectors in the configuration AG_2 . The subscripts on the α_i 's are assigned in such a way that $\langle \alpha_i, \beta_i \rangle = 0$ for all $i = 1, 2, 3$. The ratio of the squared lengths of the long roots α_i relative to the short roots β_i from the root system G_2 is $\alpha_i^2/\beta_i^2 = 3$ for all $i = 1, 2, 3$. The vectors α_i, β_i and $2\beta_i$ are assigned respectively the multiplicities $m, 3m$ and 1, where $m \in \mathbb{C}$ is a parameter.

We adopt a coordinate system where the vectors take the form

$$\alpha_1 = \omega(0, \sqrt{3}), \quad \alpha_2 = \omega\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \quad \alpha_3 = \omega\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$$

and

$$\beta_1 = \omega(1, 0), \quad \beta_2 = \omega\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \beta_3 = \omega\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

for some scaling $\omega \in \mathbb{C}^\times$.

We have $\beta_1 + \beta_3 = \beta_2, \alpha_2 + \alpha_3 = \alpha_1$, and also

$$\begin{aligned} \beta_1 &= 2\beta_2 - \alpha_1 = \alpha_1 - 2\beta_3 = \alpha_3 - \beta_2 = \beta_3 - \alpha_2, \\ \alpha_1 &= \frac{3}{2}\beta_2 + \frac{1}{2}\alpha_2 = \frac{3}{2}\beta_3 + \frac{1}{2}\alpha_3, \text{ and } \beta_1 = \frac{1}{2}\beta_2 - \frac{1}{2}\alpha_2 = -\frac{1}{2}\beta_3 + \frac{1}{2}\alpha_3. \end{aligned}$$

The configuration AG_2 is contained in the lattice $\mathbb{Z}\beta_1 \oplus \mathbb{Z}\alpha_2 \subset \mathbb{C}^2$. It is invariant under the Weyl group of type G_2 , but it is not a crystallographic root system because, for instance, the vectors β_1 and $2\beta_2$ have $2\langle \beta_1, 2\beta_2 \rangle / \langle 2\beta_2, 2\beta_2 \rangle = \frac{1}{2} \notin \mathbb{Z}$.

The corresponding generalised CMS quantum Hamiltonian (1.3) with $A = AG_{2,+}$ has recently been shown to be quantum integrable for any value of the parameter $m \in \mathbb{C}$ [15]. It is moreover algebraically integrable for $m \in \mathbb{N}$ by virtue of AG_2 being

a locus configuration [12], as a consequence of the general results presented in [5] (see also [15]). In this paper, we continue investigating the generalised CMS operator for AG_2 in the special case when $m \in \mathbb{N}$.

The structure of the paper is as follows. In Sect. 2, we recall the notion of a multi-dimensional BA function associated with a configuration \mathcal{A} following [6, 8], and we generalise it to a case when A has proportional vectors. We show that if such a function exists then it is an eigenfunction for the generalised CMS operator (1.3). In Sect. 3, we give a general ansatz for the dual generalised Macdonald–Ruijsenaars difference operators with rational coefficients, and we find sufficient conditions for these operators to preserve a space of quasi-invariant analytic functions. In Sect. 4, we find a difference operator \mathcal{D}_1 related to the configuration AG_2 which satisfies the conditions from Sect. 3. In Sect. 5, we use this operator to show that the BA function for the configuration AG_2 exists, and we express this function by iterated action of the operator. We also show that the BA function is an eigenfunction of the operator \mathcal{D}_1 , thus establishing bispectral duality. In Sect. 6, we present another dual operator \mathcal{D}_2 for the configuration AG_2 , and we establish the corresponding statements for this operator analogous to the ones from Sects. 4 and 5.

In Sect. 7, we consider the operator \mathcal{D}_1 from Sect. 4 at $m = 0$, which gives a Macdonald–Ruijsenaars operator for the root system A_2 with multiplicity 1. We also consider a version of this operator for the root system A_1 and decompose it into a sum of two non-symmetric commuting difference operators. We relate these operators with the standard Macdonald–Ruijsenaars operator for the minuscule weight of the root system A_1 .

2 Baker–Akhiezer Functions

We propose a modified definition of multi-dimensional BA functions so as to extend the definitions from papers [4, 6, 8] to include the configuration AG_2 . We formulate the definition in such a way that it naturally generalises the case of reduced root systems as well as the case of the root system BC_n .

Let $R \subset \mathbb{C}^n$ be a (possibly non-reduced) finite collection of non-isotropic vectors. We assume there is a subset $R_+ \subset R$ such that any collinear vectors in R_+ are of the form $\alpha, 2\alpha$ and $R = R_+ \sqcup (-R_+)$. Let $R^r = \{\alpha \in R : \frac{1}{2}\alpha \notin R\}$ and $R^r_+ = R^r \cap R_+$. Let $m : R \rightarrow \mathbb{Z}_{\geq 0}$ be a multiplicity map with $m(R^r) \subset \mathbb{N}$. We extend it to a map $m : R \cup 2R^r \rightarrow \mathbb{Z}_{\geq 0}$ by putting $m_{2\alpha} = 0$ if $2\alpha \notin R, \alpha \in R^r$.

Definition 2.1 A function $\psi(z, x)$ ($z, x \in \mathbb{C}^n$) is a BA function for (R, m) if

1. $\psi(z, x) = P(z, x) \exp\langle z, x \rangle$ for some polynomial P in z with highest order term $\prod_{\gamma \in R_+} \langle \gamma, z \rangle^{m_\gamma}$;
2. $\psi(z + s\alpha, x) = \psi(z - s\alpha, x) = 0$ for $s = 1, 2, 3, \dots, m_\alpha, m_\alpha + 2, \dots, m_\alpha + 2m_{2\alpha}$ for all $\alpha \in R^r_+$.

The multiplicities of the vectors in AG_2 are $m_{\beta_i} = 3m, m_{2\beta_i} = 1, m_{\alpha_i} = m$, and we put $m_{2\alpha_i} = 0$ for all $i = 1, 2, 3$, where $m \in \mathbb{N}$. When we apply Definition 2.1 to this configuration $R = AG_2$, we get the notion of a BA function for AG_2 .

For $\gamma \in \mathbb{C}^n$, we denote by δ_γ the operator that acts on functions $f(z, x)$ by

$$\delta_\gamma f(z, x) = f(z + \gamma, x) - f(z - \gamma, x).$$

The condition 2 in Definition 2.1 admits the following equivalent characterisation.

Lemma 2.2 *Let $\alpha \in \mathbb{C}^n$ be non-isotropic, $m_\alpha \in \mathbb{N}$, and $m_{2\alpha} \in \mathbb{Z}_{\geq 0}$. A function $\psi(z, x)$ ($z, x \in \mathbb{C}^n$) analytic in z satisfies $\psi(z + s\alpha, x) = \psi(z - s\alpha, x)$ at $\langle z, \alpha \rangle = 0$ for $s = 1, 2, 3, \dots, m_\alpha, m_\alpha + 2, \dots, m_\alpha + 2m_{2\alpha}$ if and only if*

$$\left(\delta_\alpha \circ \frac{1}{\langle z, \alpha \rangle} \right)^{s-1} \delta_\alpha \psi(z, x) = 0 \text{ at } \langle z, \alpha \rangle = 0, s = 1, \dots, m_\alpha \tag{2.1}$$

and

$$\left(\delta_{2\alpha} \circ \frac{1}{\langle z, \alpha \rangle} \right)^t \circ \left(\delta_\alpha \circ \frac{1}{\langle z, \alpha \rangle} \right)^{m_\alpha-1} \delta_\alpha \psi(z, x) = 0 \text{ at } \langle z, \alpha \rangle = 0, t = 1, \dots, m_{2\alpha}. \tag{2.2}$$

The proof will follow from the one-dimensional considerations in Lemma 2.4 below (see also [8] where the corresponding statements in the case $m_{2\alpha} = 0$ were stated). Let δ_r ($r \in \mathbb{N}$) be the difference operator that acts on functions $F(k)$ ($k \in \mathbb{C}$) by $\delta_r F(k) = F(k+r) - F(k-r)$. Write $\delta = \delta_1$ for short. The next Lemma will be used in the proof of Lemma 2.4.

Lemma 2.3 *Suppose $\delta_r F = k\widehat{F}$ for analytic functions $F(k)$ and $\widehat{F}(k)$. Suppose also that $F(s-r) = F(r-s)$ for some s . Then $\widehat{F}(s) = \widehat{F}(-s)$ if and only if $F(s+r) = F(-s-r)$.*

Proof The statement follows by taking $k = s$ and $k = -s$ in the equality $\delta_r F = k\widehat{F}$. □

Lemma 2.4 *The following two properties are equivalent for any analytic function $F(k)$ ($k \in \mathbb{C}$) and any $n \in \mathbb{N}$, $m \in \mathbb{Z}_{\geq 0}$.*

1. For all $s = 1, \dots, n$,

$$\left(\delta \circ \frac{1}{k} \right)^{s-1} \delta F(k) \Big|_{k=0} = 0, \tag{2.3}$$

and for all $t = 1, \dots, m$,

$$\left(\delta_2 \circ \frac{1}{k} \right)^t \circ \left(\delta \circ \frac{1}{k} \right)^{n-1} \delta F(k) \Big|_{k=0} = 0. \tag{2.4}$$

2. $F(s) = F(-s)$ for all $s = 1, 2, 3, \dots, n, n+2, \dots, n+2m$.

Proof Let us firstly prove by induction on n that conditions (2.3) are equivalent to the existence of analytic functions G_1, \dots, G_n such that $\delta G_q(k) = kG_{q+1}(k)$ for $0 \leq q \leq n - 1$, $G_0 = F$, with the conditions $G_q(s) = G_q(-s)$ for $1 \leq s \leq n - q$.

For the base case $n = 1$, the condition (2.3) is equivalent to $F(1) = F(-1)$, which is equivalent to the existence of an analytic function G_1 such that $\delta F = kG_1$. Thus the base of induction holds.

Let us now assume that the induction hypothesis is satisfied for some $n \in \mathbb{N}$. The relation (2.3) for $s = n + 1$ is equivalent to $\delta G_n(k) = 0$ at $k = 0$, that is $G_n(1) = G_n(-1)$, which is also equivalent to the existence of an analytic function G_{n+1} such that $\delta G_n = kG_{n+1}$.

We also have $kG_n = \delta G_{n-1}$ by the induction hypothesis. By applying Lemma 2.3 with $s = 1$ we get $G_{n-1}(2) = G_{n-1}(-2)$. By considering $kG_{n-1} = \delta G_{n-2}$ and using that by induction hypothesis $G_{n-2}(1) = G_{n-2}(-1)$, we obtain by Lemma 2.3 with $s = 2$ that $G_{n-2}(3) = G_{n-2}(-3)$. Similarly we obtain $G_q(n - q + 1) = G_q(-n + q - 1)$ successively for $q = n - 3, \dots, 0$, which completes the proof of the claim by induction.

For any fixed n let us now prove by induction on m that relations (2.3), (2.4) are equivalent to the existence of functions G_1, \dots, G_n as in the first paragraph of the proof with the additional conditions $G_q(s) = G_q(-s)$ for $s = n - q + 2t$ with $1 \leq t \leq m$, $0 \leq q \leq n - 1$, together with the existence of analytic functions H_1, \dots, H_m satisfying $\delta_2 H_q = kH_{q+1}$ for $0 \leq q \leq m - 1$, $H_0 = G_n$, and $H_q(2s) = H_q(-2s)$ for $1 \leq s \leq m - q$.

The base case $m = 0$ is already done. Let us now assume that the induction hypothesis is satisfied for some $m \in \mathbb{Z}_{\geq 0}$. The relation (2.4) for $t = m + 1$ is equivalent to $\delta_2 H_m(k) = 0$ at $k = 0$, that is $H_m(2) = H_m(-2)$, which is also equivalent to the existence of an analytic function H_{m+1} such that $\delta_2 H_m = kH_{m+1}$.

We also have $kH_m = \delta_2 H_{m-1}$ by the induction hypothesis. By applying Lemma 2.3, we obtain $H_q(2(m - q + 1)) = H_q(-2(m - q + 1))$ successively for $q = m - 1, \dots, 0$.

By considering $kH_0 = kG_n = \delta G_{n-1}$ and using that by induction hypothesis $G_{n-1}(2m + 1) = G_{n-1}(-2m - 1)$, we obtain by Lemma 2.3 for $s = 2(m + 1)$ that $G_{n-1}(2m + 3) = G_{n-1}(-2m - 3)$. Similarly we obtain $G_q(n - q + 2m + 2) = G_q(-n + q - 2m - 2)$ successively for $q = n - 2, \dots, 0$, which completes the proof by induction on m .

It follows that condition 1 in Lemma 2.4 implies condition 2.

Let us now show that condition 2 implies condition 1. Firstly, let us prove it in the case $m = 0$ by induction on n . The base case $n = 1$ is clear. Let us assume it for some $n \in \mathbb{N}$. Suppose now that $F(s) = F(-s)$ for $s = 1, \dots, n, n + 1$. By the induction hypothesis and the above analysis in the first part of the proof, we know that there exist functions G_1, \dots, G_n with properties as stated in the first paragraph of the proof. By using $F(n \pm 1) = F(-n \mp 1)$ and $\delta F = kG_1$ and applying Lemma 2.3 with $s = n$, we get $G_1(n) = G_1(-n)$. Similarly we obtain $G_q(n + 1 - q) = G_q(-n - 1 + q)$ successively for $q = 2, \dots, n$. In particular, $G_n(1) = G_n(-1)$, which implies that the relation (2.3) holds for $s = n + 1$, completing the induction on n .

Suppose that condition 2 implies condition 1 for some $m \in \mathbb{Z}_{\geq 0}$. Now suppose F satisfies condition 2 and $F(n + 2m + 2) = F(-n - 2m - 2)$. By the induction hypothesis and the above analysis in the first part of the proof, we have existence of

functions G_1, \dots, G_n and H_1, \dots, H_m with properties as stated in the fifth paragraph of the proof. By using the assumptions on F , the fact that $\delta F = kG_1$ and applying Lemma 2.3 for $s = n + 2m + 1$, we get that $G_1(n + 2m + 1) = G_1(-n - 2m - 1)$. Similarly we obtain $G_q(n + 2m + 2 - q) = G_q(-n - 2m - 2 + q)$ successively for $q = 2, \dots, n$, and $H_q(2m + 2 - 2q) = H_q(-2m - 2 + 2q)$ successively for $q = 0, \dots, m$. In particular, $H_m(2) = H_m(-2)$, which implies that the relation (2.4) holds for $t = m + 1$, which completes the proof. \square

The following Lemma is a generalisation to the present context of [13, Lemma 1] (see also [8, Proposition 1]). In its proof we use that for a polynomial $p(z)$ we have $p(z + \gamma) = p(z) + \text{l.o.t.}$ by the binomial theorem, where ‘‘l.o.t.’’ denotes terms of lower order in z than $\deg p(z)$. This Lemma will be used below to prove uniqueness of the BA function whenever such a function exists.

Lemma 2.5 *Let $\psi(z, x) = P(z, x) \exp\langle z, x \rangle$ ($z, x \in \mathbb{C}^n$), where $P(z, x)$ is a polynomial in z . Suppose that ψ satisfies conditions (2.1) and (2.2) for some non-zero $\alpha \in \mathbb{C}^n$, $m_\alpha \in \mathbb{N}$, $m_{2\alpha} \in \mathbb{Z}_{\geq 0}$. Then $\langle \alpha, z \rangle^{m_\alpha + m_{2\alpha}}$ divides the highest order term $P_0(z, x)$ of $P(z, x)$.*

Proof Firstly, the condition (2.1) with $s = 1$ gives

$$\begin{aligned} 0 = \delta_\alpha \psi(z, x) &= (P_0(z, x)(\exp\langle \alpha, x \rangle - \exp\langle -\alpha, x \rangle) + \text{l.o.t.}) \exp\langle z, x \rangle \\ &= (2 \sinh\langle \alpha, x \rangle P_0(z, x) + \text{l.o.t.}) \exp\langle z, x \rangle \end{aligned}$$

at $\langle z, \alpha \rangle = 0$ for all $x \in \mathbb{C}^n$. This implies that $P_0(z, x)$ and the l.o.t. must be divisible by $\langle z, \alpha \rangle$. Let $P_0(z, x) = \langle z, \alpha \rangle P_0^{(1)}(z, x)$, and let

$$\psi^{(1)}(z, x) = \frac{\delta_\alpha \psi(z, x)}{\langle z, \alpha \rangle} = (2 \sinh\langle \alpha, x \rangle P_0^{(1)}(z, x) + \text{l.o.t.}) \exp\langle z, x \rangle.$$

The condition (2.1) with $s = 2$ gives

$$0 = \delta_\alpha \psi^{(1)}(z, x) = (4 \sinh^2\langle \alpha, x \rangle P_0^{(1)}(z, x) + \text{l.o.t.}) \exp\langle z, x \rangle$$

at $\langle z, \alpha \rangle = 0$ for all $x \in \mathbb{C}^n$. This implies that $\langle z, \alpha \rangle$ divides $P_0^{(1)}(z, x)$. In particular, $P_0(z, x)$ is divisible by $\langle z, \alpha \rangle^2$: $P_0(z, x)\langle z, \alpha \rangle^{-2} = P_0^{(2)}(z, x)$ for some polynomial $P_0^{(2)}$. We let $\psi^{(2)}(z, x) = \langle z, \alpha \rangle^{-1} \delta_\alpha \psi^{(1)}(z, x)$. By iterating for $s = 3, \dots, m_\alpha$, we get that $P_0(z, x)$ is divisible by $\langle z, \alpha \rangle^{m_\alpha}$, and we recursively obtain polynomials $P_0^{(s)}(z, x) = P_0^{(s-1)}(z, x)\langle z, \alpha \rangle^{-1} = P_0(z, x)\langle z, \alpha \rangle^{-s}$ and functions

$$\psi^{(s)}(z, x) = \frac{\delta_\alpha \psi^{(s-1)}(z, x)}{\langle z, \alpha \rangle} = (2^s \sinh^s\langle \alpha, x \rangle P_0^{(s)}(z, x) + \text{l.o.t.}) \exp\langle z, x \rangle.$$

The condition (2.2) with $t = 1$ then gives

$$0 = \delta_{2\alpha} \psi^{m_\alpha}(z, x) = (2^{m_\alpha+1} \sinh^{m_\alpha}\langle \alpha, x \rangle \sinh\langle 2\alpha, x \rangle P_0^{(m_\alpha)}(z, x) + \text{l.o.t.}) \exp\langle z, x \rangle$$

at $\langle z, \alpha \rangle = 0$. This implies that $\langle z, \alpha \rangle$ divides $P_0^{(m_\alpha)}(z, x)$ and the l.o.t., in particular, $P_0(z, x)$ is divisible by $\langle z, \alpha \rangle^{m_\alpha+1}$. Continuing this iteratively for $t = 2, \dots, m_{2\alpha}$ completes the proof. \square

Lemma 2.5 has the following consequence.

Lemma 2.6 *Let $\psi(z, x) = P(z, x) \exp\langle z, x \rangle$ ($z, x \in \mathbb{C}^n$) satisfy condition 2 in Definition 2.1, where $P(z, x)$ is a polynomial in z with highest order term $P_0(z, x)$. Then $\prod_{\gamma \in R_+} \langle \gamma, z \rangle^{m_\gamma}$ divides $P_0(z, x)$.*

Proof Lemma 2.5 gives that $P_0(z, x)$ is divisible by $\langle z, \alpha \rangle^{m_\alpha+m_{2\alpha}}$ for all $\alpha \in R_+^r$. This is a constant multiple of $\langle z, \alpha \rangle^{m_\alpha} \langle z, 2\alpha \rangle^{m_{2\alpha}}$. The statement follows since we are assuming that collinear vectors in R_+ are only of the form $\alpha, 2\alpha$. \square

Lemma 2.6 leads to the following uniqueness statement analogous to [8, Proposition 1] (cf. also [13, Proposition 1]) with a completely analogous proof.

Proposition 2.7 *If a BA function satisfying Definition 2.1 exists, then it is unique. In particular, if a BA function for AG_2 exists then it is unique.*

The next Theorem generalises [8, Theorem 1] to the present context, and it is proved analogously to how that result is proved. It states that if the BA function satisfying Definition 2.1 exists, then it is a joint eigenfunction of a commutative ring of differential operators in the variables x . Let us first define an isomorphic ring of polynomials.

Let \mathcal{R} be the ring of polynomials $p(z) \in \mathbb{C}[z_1, \dots, z_n]$ satisfying

$$p(z + s\alpha) = p(z - s\alpha) \text{ at } \langle z, \alpha \rangle = 0, s = 1, 2, 3, \dots, m_\alpha, m_\alpha + 2, \dots, m_\alpha + 2m_{2\alpha}$$

for all $\alpha \in R_+^r$ (we remark that this is similar to condition 2 in Definition 2.1). We have $z^2 \in \mathcal{R}$. Indeed, for any $\gamma \in \mathbb{C}^n, s \in \mathbb{N}$, we have $(z \pm s\gamma)^2 = z^2 \pm 2s\langle z, \gamma \rangle + s^2\gamma^2 = z^2 + s^2\gamma^2$ at $\langle z, \gamma \rangle = 0$.

For a polynomial $p(z) = p(z_1, \dots, z_n)$, by $p(\partial_x)$ we will mean $p(\partial_{x_1}, \dots, \partial_{x_n})$. For example, if $p(z) = z^2 = z_1^2 + \dots + z_n^2$, then $p(\partial_x) = \Delta$ is the Laplace operator in n dimensions acting in the variables x .

The following statement takes place.

Theorem 2.8 *If the BA function $\psi(z, x)$ satisfying Definition 2.1 exists, then for any $p(z) \in \mathcal{R}$ there is a differential operator $L_p(x, \partial_x)$ with highest order term $p(\partial_x)$ such that*

$$L_p(x, \partial_x)\psi(z, x) = p(z)\psi(z, x).$$

For any $p, q \in \mathcal{R}$, the operators L_p and L_q commute.

The next Lemma will be used below to prove that the differential operator L_{z^2} from Theorem 2.8 corresponding to the element z^2 of \mathcal{R} coincides with the generalised CMS Hamiltonian (1.3) associated to the configuration $A = R_+$. The next Lemma is a generalisation of [13, Lemma 2] (see also [8]).

Lemma 2.9 Suppose $\psi(z, x) = P(z, x) \exp\langle z, x \rangle$ satisfies Definition 2.1. Let $N = \sum_{\gamma \in R_+} m_\gamma$. Write $P(z, x) = \sum_{i=0}^N P_i(z, x)$ where $P_0(z, x) = \prod_{\gamma \in R_+} \langle \gamma, z \rangle^{m_\gamma}$ and P_i are polynomials homogeneous in z with $\deg P_i = N - i$. Then

$$\frac{P_1(z, x)}{P_0(z, x)} = - \sum_{\gamma \in R_+} \frac{m_\gamma(m_\gamma + 2m_{2\gamma} + 1)\gamma^2}{2\langle \gamma, z \rangle} \coth\langle \gamma, x \rangle. \tag{2.5}$$

Proof We introduce the following notations

$$A(z) = P_0(z, x), \quad A_\alpha(z) = \frac{A(z)}{\langle \alpha, z \rangle}, \quad A_{\alpha\beta}(z) = \frac{A(z)}{\langle \alpha, z \rangle \langle \beta, z \rangle} = A_{\beta\alpha}(z), \quad \text{etc.}$$

As a shorthand, we will write A_{α^2} for $A_{\alpha\alpha}$, etc. In this notation, the equality (2.5) can be written as $P_1 = - \sum_{\gamma \in R_+} \frac{1}{2} m_\gamma(m_\gamma + 2m_{2\gamma} + 1)\gamma^2 A_\gamma(z) \coth\langle \gamma, x \rangle$.

The strategy is as follows. Applying condition 2 in Definition 2.1 and equating the homogeneous terms of each degree on the right- and left-hand sides, one gets equations for P_1, \dots, P_N . We are assuming that ψ satisfies the definition, hence a solution exists. We are only interested in P_1 here.

Fix any $\alpha \in R_+^r$. We will omit arguments of functions whenever convenient, and write e^α for $\exp\langle \alpha, x \rangle$. By the binomial theorem $A(z \pm \alpha)$ has the following homogeneous components

$$A(z \pm \alpha) = A(z) \pm \sum_{\beta \in R_+} m_\beta \langle \alpha, \beta \rangle A_\beta(z) + \text{l.o.t.}$$

and similarly $P_1(z \pm \alpha, x) = P_1(z, x) + \text{l.o.t.}$

We will use the fact that for this fixed α condition 2 in Definition 2.1 admits the equivalent characterisation given in Lemma 2.2. We will apply condition (2.1) for $s = 1, \dots, m_\alpha$, and then if $m_{2\alpha} > 0$ also condition (2.2) for $t = 1, \dots, m_{2\alpha}$, in order to see what conditions this places on P_1 . Applying condition (2.1) with $s = 1$ gives $0 = \delta_\alpha \psi(z, x) = (P(z + \alpha, x)e^\alpha - P(z - \alpha, x)e^{-\alpha}) \exp\langle z, x \rangle$ at $\langle \alpha, z \rangle = 0$. This equation can be rearranged (after dividing through by $\exp\langle z, x \rangle$) as

$$0 = A(z)(e^\alpha - e^{-\alpha}) + \sum_{\beta \in R_+} m_\beta \langle \alpha, \beta \rangle A_\beta(z)(e^\alpha + e^{-\alpha}) + P_1(z, x)(e^\alpha - e^{-\alpha}) + \text{l.o.t.} \tag{2.6}$$

Note that for $\beta \neq \alpha$ and $\beta \neq 2\alpha$ we have $A_\beta(z) = 0$ at $\langle \alpha, z \rangle = 0$. By identifying the homogeneous terms of degree $N - 1$ in (2.6), we get at $\langle \alpha, z \rangle = 0$ that

$$\begin{aligned} 0 &= \left(m_\alpha \alpha^2 A_\alpha + 2m_{2\alpha} \alpha^2 A_{2\alpha} \right) (e^\alpha + e^{-\alpha}) + P_1(e^\alpha - e^{-\alpha}) \\ &= (m_\alpha + m_{2\alpha}) \alpha^2 A_\alpha (e^\alpha + e^{-\alpha}) + P_1(e^\alpha - e^{-\alpha}). \end{aligned} \tag{2.7}$$

Notice that this can be rearranged as $P_1 = -(m_\alpha + m_{2\alpha}) \alpha^2 A_\alpha \coth\langle \alpha, x \rangle$ at $\langle \alpha, z \rangle = 0$.

Assume that $m_\alpha > 1$. Then $A_\alpha(z) = 0$ at $\langle \alpha, z \rangle = 0$, which forces $P_1 = 0$ at $\langle \alpha, z \rangle = 0$. This is equivalent to divisibility of P_1 by $\langle \alpha, z \rangle$. We are next going to consider the condition (2.1) from Lemma 2.2 with $s = 2$, and we are now interested in the degree $N - 2$ terms in the polynomial part of $(\delta_\alpha \circ \langle \alpha, z \rangle^{-1})\delta_\alpha \psi(z, x)$. These terms arise in two ways: firstly they come from the degree $N - 2$ terms in the polynomial part of $\langle \alpha, z \rangle^{-1}\delta_\alpha \psi$ and further action of δ_α on $\exp\langle \alpha, z \rangle$; secondly they come from application of δ_α onto the terms in $\langle \alpha, z \rangle^{-1}\delta_\alpha \psi$ whose polynomial part has degree $N - 1$. Let us examine these two possibilities individually. From (2.6) we see that the terms of degree $N - 2$ in $\exp\langle -z, x \rangle \langle \alpha, z \rangle^{-1}\delta_\alpha \psi$ are

$$\begin{aligned} & \sum_{\beta \in R_+} m_\beta \langle \alpha, \beta \rangle A_{\alpha\beta}(z)(e^\alpha + e^{-\alpha}) + \frac{P_1(z, x)}{\langle \alpha, z \rangle}(e^\alpha - e^{-\alpha}) \\ &= (m_\alpha + m_{2\alpha})\alpha^2 A_{\alpha^2}(z)(e^\alpha + e^{-\alpha}) + \sum_{\beta \in R_+ \setminus \{\alpha, 2\alpha\}} m_\beta \langle \alpha, \beta \rangle A_{\alpha\beta}(z)(e^\alpha + e^{-\alpha}) \\ & \quad + \frac{P_1(z, x)}{\langle \alpha, z \rangle}(e^\alpha - e^{-\alpha}). \end{aligned} \tag{2.8}$$

And from (2.6) we see that the term of degree $N - 1$ in $\exp\langle -z, x \rangle \langle \alpha, z \rangle^{-1}\delta_\alpha \psi$ is $A_\alpha(z)(e^\alpha - e^{-\alpha})$. Since we need to know how δ_α acts on it, we note here that

$$A_\alpha(z \pm \alpha) = A_\alpha(z) \pm (m_\alpha + m_{2\alpha} - 1)\alpha^2 A_{\alpha^2}(z) \pm \sum_{\beta \in R_+ \setminus \{\alpha, 2\alpha\}} m_\beta \langle \alpha, \beta \rangle A_{\alpha\beta}(z) + \text{l.o.t.} \tag{2.9}$$

Let us now consider condition (2.1) from Lemma 2.2 with $s = 2$, namely $0 = (\delta_\alpha \circ \langle \alpha, z \rangle^{-1})\delta_\alpha \psi(z, x)$ at $\langle \alpha, z \rangle = 0$. By making use of formulas (2.8) and (2.9) this condition can be rearranged as

$$\begin{aligned} 0 &= A_\alpha(z)(e^\alpha - e^{-\alpha})^2 + (m_\alpha + m_{2\alpha} - 1)\alpha^2 A_{\alpha^2}(z)(e^\alpha - e^{-\alpha})(e^\alpha + e^{-\alpha}) \\ & \quad + (m_\alpha + m_{2\alpha})\alpha^2 A_{\alpha^2}(z)(e^\alpha + e^{-\alpha})(e^\alpha - e^{-\alpha}) \\ & \quad + \sum_{\beta \in R_+ \setminus \{\alpha, 2\alpha\}} 2m_\beta \langle \alpha, \beta \rangle A_{\alpha\beta}(z)(e^\alpha + e^{-\alpha})(e^\alpha - e^{-\alpha}) \\ & \quad + \frac{P_1(z, x)}{\langle \alpha, z \rangle}(e^\alpha - e^{-\alpha})^2 + \text{l.o.t.} \end{aligned} \tag{2.10}$$

Note that for $\beta \neq \alpha$ and $\beta \neq 2\alpha$ we have $A_{\alpha\beta}(z) = 0$ at $\langle \alpha, z \rangle = 0$. By identifying the homogeneous terms of degree $N - 2$ in (2.10) we get at $\langle \alpha, z \rangle = 0$ that

$$0 = ((m_\alpha + m_{2\alpha}) + (m_\alpha + m_{2\alpha} - 1))\alpha^2 A_{\alpha^2}(z)(e^\alpha - e^{-\alpha})(e^\alpha + e^{-\alpha}) + \frac{P_1}{\langle \alpha, z \rangle}(e^\alpha - e^{-\alpha})^2. \tag{2.11}$$

If $m_\alpha > 2$, then $A_{\alpha^2}(z) = 0$ at $\langle \alpha, z \rangle = 0$, so (2.11) gives divisibility of $\langle \alpha, z \rangle^{-1}P_1$ by $\langle \alpha, z \rangle$. Continuing iteratively for $s = 3, \dots, m_\alpha$ the process above, after the $s = m_\alpha$

step we get that at $\langle \alpha, z \rangle = 0$

$$\begin{aligned}
 0 &= ((m_\alpha + m_{2\alpha}) + (m_\alpha + m_{2\alpha} - 1) + \dots \\
 &\quad + (1 + m_{2\alpha})) \alpha^2 A_{\alpha^{m_\alpha}}(z) (e^\alpha - e^{-\alpha})^{m_\alpha - 1} (e^\alpha + e^{-\alpha}) \\
 &\quad + \frac{P_1}{\langle \alpha, z \rangle^{m_\alpha - 1}} (e^\alpha - e^{-\alpha})^{m_\alpha}.
 \end{aligned} \tag{2.12}$$

We have

$$\begin{aligned}
 (m_\alpha + m_{2\alpha}) + (m_\alpha + m_{2\alpha} - 1) + \dots + (1 + m_{2\alpha}) &= m_\alpha m_{2\alpha} + (1 + \dots + m_\alpha) \\
 &= \frac{m_\alpha(m_\alpha + 2m_{2\alpha} + 1)}{2}.
 \end{aligned}$$

Therefore (2.12) can be rearranged as

$$\frac{P_1}{\langle \alpha, z \rangle^{m_\alpha - 1}} = -\frac{m_\alpha(m_\alpha + 2m_{2\alpha} + 1)}{2} \alpha^2 A_{\alpha^{m_\alpha}}(z) \coth \langle \alpha, x \rangle \text{ at } \langle \alpha, z \rangle = 0. \tag{2.13}$$

If $m_{2\alpha} = 0$ for all $\alpha \in R_+$ for which $\frac{\alpha}{2} \notin R_+$, then for each such α there are no more conditions in Definition 2.1 to be satisfied, and P_1 given by (2.5) satisfies (2.13). This completes the proof in this case, by uniqueness. Otherwise we continue and consider condition (2.2). Note that the highest order term that we are carrying forward is $A_{\alpha^{m_\alpha - 1}}(z)$.

Assume $m_{2\alpha} > 0$. Then $A_{\alpha^{m_\alpha}}(z) = 0$ at $\langle \alpha, z \rangle = 0$, so (2.13) implies divisibility of $\langle \alpha, z \rangle^{1 - m_\alpha} P_1$ by $\langle \alpha, z \rangle$. Consider the condition (2.2) from Lemma 2.2 with $t = 1$. Note that

$$A_{\alpha^{m_\alpha}}(z \pm 2\alpha) = A_{\alpha^{m_\alpha}}(z) \pm m_{2\alpha} (2\alpha)^2 A_{(2\alpha)\alpha^{m_\alpha}} + f(z)$$

where the term $f(z)$ vanishes at $\langle \alpha, z \rangle = 0$. Thus at this step we obtain

$$\begin{aligned}
 0 &= m_{2\alpha} (2\alpha)^2 A_{(2\alpha)\alpha^{m_\alpha}}(z) (e^{2\alpha} + e^{-2\alpha}) (e^\alpha - e^{-\alpha})^{m_\alpha} \\
 &\quad + \frac{m_\alpha(m_\alpha + 2m_{2\alpha} + 1) \alpha^2}{2} A_{\alpha^{m_\alpha + 1}}(z) (e^{2\alpha} - e^{-2\alpha}) (e^\alpha - e^{-\alpha})^{m_\alpha - 1} (e^\alpha + e^{-\alpha}) \\
 &\quad + \frac{P_1}{\langle \alpha, z \rangle^{m_\alpha}} (e^{2\alpha} - e^{-2\alpha}) (e^\alpha - e^{-\alpha})^{m_\alpha}
 \end{aligned} \tag{2.14}$$

at $\langle \alpha, z \rangle = 0$. This can be rearranged as

$$\begin{aligned}
 \frac{P_1}{\langle \alpha, z \rangle^{m_\alpha}} &= -m_{2\alpha} (2\alpha)^2 A_{(2\alpha)\alpha^{m_\alpha}}(z) \coth \langle 2\alpha, x \rangle \\
 &\quad - \frac{m_\alpha(m_\alpha + 2m_{2\alpha} + 1) \alpha^2}{2} A_{\alpha^{m_\alpha + 1}}(z) \coth \langle \alpha, x \rangle.
 \end{aligned} \tag{2.15}$$

Assume $m_{2\alpha} = 1$, then one can check that P_1 given by (2.5) satisfies (2.15). Otherwise $m_{2\alpha} > 1$ and we continue recursively. For $t = 2$, for example, to treat the highest order term we need to consider

$$A_{\alpha^{m_{\alpha}+1}}(z \pm 2\alpha) = A_{\alpha^{m_{\alpha}+1}}(z) \pm (m_{2\alpha} - 1)(2\alpha)^2 A_{(2\alpha)\alpha^{m_{\alpha}+1}} + g(z)$$

where the term $g(z)$ vanishes at $\langle \alpha, z \rangle = 0$.

We thus iterate for $t = 2, \dots, m_{2\alpha}$. One has to use that $1 + \dots + m_{2\alpha} = \frac{1}{2}m_{2\alpha}(m_{2\alpha} + 1)$, and at the end we get

$$\begin{aligned} \frac{P_1}{\langle \alpha, z \rangle^{m_{\alpha} + m_{2\alpha} - 1}} &= - \frac{m_{2\alpha}(m_{2\alpha} + 1)(2\alpha)^2}{2} A_{(2\alpha)\alpha^{m_{\alpha} + m_{2\alpha} - 1}} \coth \langle 2\alpha, x \rangle \\ &\quad - \frac{m_{\alpha}(m_{\alpha} + 2m_{2\alpha} + 1)\alpha^2}{2} A_{\alpha^{m_{\alpha} + m_{2\alpha}}} \coth \langle \alpha, x \rangle \end{aligned} \tag{2.16}$$

at $\langle \alpha, z \rangle = 0$. It is straightforward to check that P_1 given by (2.5) satisfies (2.16) for all $\alpha \in R_+$ such that $\frac{\alpha}{2} \notin R_+$.

If there existed some other \tilde{P}_1 of degree $N - 1$ satisfying (2.16), then (2.16) would imply that $P_1 - \tilde{P}_1$ is divisible by $\langle \alpha, z \rangle^{m_{\alpha} + m_{2\alpha}}$ for all $\alpha \in R_+$ with $\frac{\alpha}{2} \notin R_+$. But unless $P_1 - \tilde{P}_1 = 0$ this would mean that $P_1 - \tilde{P}_1$ has degree at least N which is not possible. This completes the proof. \square

The next Proposition has a completely analogous proof to [13, Proposition 2], it just uses Lemma 2.9 in place of [13, Lemma 2] (see also [8]).

Proposition 2.10 *With notations and assumptions as in Theorem 2.8, the element $p(z) = z^2$ of \mathcal{R} corresponds to the differential operator*

$$L_{z^2} = \Delta - \sum_{\gamma \in R_+} \frac{m_{\gamma}(m_{\gamma} + 2m_{2\gamma} + 1)\gamma^2}{\sinh^2 \langle \gamma, x \rangle},$$

which coincides (up to sign) with the generalised CMS operator (1.3) for $\mathcal{A} = (R_+, m)$.

This implies integrability of the Hamiltonian and provides a quantum integral of motion L_p for each $p(z) \in \mathcal{R}$. The following statement is one of the main results of this paper. The proof will be presented below.

Theorem 2.11 *There exists a BA function for the configuration AG_2 .*

3 Generalised Macdonald–Ruijsenaars Operators

In Sect. 4, we will utilise a method for explicit construction of BA functions which was proposed by Chalykh [4] (see also [13] for further examples where this method is applied, and [3, 10] for the differential case). The construction uses (generalised) Macdonald–Ruijsenaars difference operators. The key element of the method is the preservation of a space of quasi-invariant analytic functions under the action of the

Macdonald–Ruijsenaars operators. In this Section, we find sufficient conditions for a sufficiently general difference operator to preserve such a ring of quasi-invariants.

Let $W = \langle s_\alpha : \alpha \in R \rangle$, where s_α is the orthogonal reflection about the hyperplane $\langle \alpha, x \rangle = 0$. We assume now that the collection R is W -invariant, that is $w(R) = R$ for all $w \in W$, and that the multiplicity map m is W -invariant, too. Let $u^\vee = 2u/\langle u, u \rangle$ for any $u \in \mathbb{C}^n$ such that $\langle u, u \rangle \neq 0$.

Let \mathcal{R}^a be the ring of analytic functions $p(z)$ such that

$$p(z + t\alpha) = p(z - t\alpha) \text{ at } \langle \alpha, z \rangle = 0 \text{ for } t \in A_\alpha$$

for all $\alpha \in R_+^r$, where $A_\alpha \subset \mathbb{N}$ specifies the axiomatics that one wants to consider. For instance, in the next Sections we will use $A_\alpha = \{1, 2, \dots, m_\alpha, m_\alpha + 2, \dots, m_\alpha + 2m_{2\alpha}\}$. We assume that $A_{|w\alpha|} = A_\alpha$ for all $w \in W, \alpha \in R_+^r$, where $|w\alpha| = w\alpha$ if $w\alpha \in R_+^r$ and $|w\alpha| = -w\alpha$ if $w\alpha \in -R_+^r$. For $\alpha \in R$, we let $\text{sgn } \alpha = 1$ if $\alpha \in R_+$ and $\text{sgn } \alpha = -1$ if $\alpha \in -R_+$.

Let $S \subset \mathbb{C}^n \setminus \{0\}$ be a W -invariant finite collection of vectors. Let $z \in \mathbb{C}^n$ and for any $\gamma \in \mathbb{C}^n$ let T_γ be the shift operator that acts on functions $f(z)$ by $T_\gamma f(z) = f(z + \gamma)$. We are interested in difference operators D of the form

$$D = \sum_{\tau \in S} a_\tau(z)(T_\tau - 1), \tag{3.1}$$

where a_τ are rational functions with the following three properties:

(D_1) $\deg a_\tau = 0$.

(D_2) $a_\tau(z)$ has a simple pole at $\langle \alpha, z \rangle = c\alpha^2$ for $\alpha \in R_+^r, c \in \mathbb{C}$ if and only if $\lambda = s_\alpha(\tau) - 2c\alpha \in S \cup \{0\}$ and

$$\langle \tau + c\alpha, \alpha \rangle / \alpha^2 = c + \langle \tau, (2\alpha)^\vee \rangle \in A_\alpha \cup (-A_\alpha).$$

There are no other singularities in a_τ . Denote the set of all such pairs (α, c) for this τ by S_τ .

(D_3) $wa_\tau = a_{w\tau}$ for all $w \in W$.

The condition (D_2) implies that if a_τ has a singularity $\langle \alpha, z \rangle = c\alpha^2$, then for any such z the vectors $z + \tau$ and $z + \lambda$ are of the form $z + \tau = \tilde{z} + t\alpha$ and $z + \lambda = \tilde{z} - t\alpha$ for some \tilde{z} with $\langle \alpha, \tilde{z} \rangle = 0$ and $t = c + \langle \tau, (2\alpha)^\vee \rangle \in A_\alpha \cup (-A_\alpha)$. We note also that $\lambda \neq \tau$ since $0 \notin A_\alpha$.

Note that if a_τ has a singularity $\langle \alpha, z \rangle = c\alpha^2$ and the corresponding $\lambda \neq 0$, then by condition (D_2) $a_\lambda(z)$ necessarily also has a singularity at $\langle \alpha, z \rangle = c\alpha^2$, since $s_\alpha(\lambda) - 2c\alpha = \tau \in S$ and $\langle \lambda + c\alpha, \alpha \rangle = \langle s_\alpha(\tau + c\alpha), \alpha \rangle = -\langle \tau + c\alpha, \alpha \rangle$. In other words, by condition (D_2) we have $(\alpha, c) \in S_\tau$ if and only if $(\alpha, c) \in S_\lambda$ provided that both $\tau, \lambda \neq 0$. We additionally observe the following.

Lemma 3.1 *For any $w \in W, (\alpha, c) \in S_\tau$ if and only if $(|w\alpha|, \text{sgn}(w\alpha)c) \in S_{w\tau}$.*

Proof Let $\varepsilon = \text{sgn}(w\alpha)$. Since $s_{|w\alpha|} = ws_\alpha w^{-1}$ and $\varepsilon|w\alpha| = w\alpha$, we get that $s_{|w\alpha|}(w\tau) - 2\varepsilon c|w\alpha| = w(s_\alpha(\tau) - 2c\alpha)$ belongs to $S \cup \{0\}$ if and only if $s_\alpha(\tau) -$

$2c\alpha \in S \cup \{0\}$, by W -invariance of S . Furthermore, $\langle w\tau + \varepsilon c|w\alpha|, |w\alpha| \rangle / |w\alpha|^2 = \pm \langle \tau + c\alpha, \alpha \rangle / \alpha^2$, and $A_\alpha = A_{|w\alpha|}$, by assumption. \square

More explicitly, we are looking at operators of the form

$$D = \sum_{\tau \in S} P_\tau(z) \left(\prod_{(\alpha, c) \in S_\tau} (\langle \alpha, z \rangle - c\alpha^2)^{-1} \right) (T_\tau - 1)$$

for some polynomials $P_\tau(z)$ of degree $|S_\tau|$ so that $\deg a_\tau = 0$ and such that the condition (D_3) holds. We want to find some sufficient conditions that would ensure that D preserves the ring \mathcal{R}^a .

Theorem 3.2 *Suppose the operator (3.1) satisfies conditions (D_2) and (D_3) . Then for any $\alpha \in R_+^r$ and arbitrary $p(z) \in \mathcal{R}^a$, we have the following two properties.*

1. $Dp(z)$ is non-singular at $\langle \alpha, z \rangle = 0$. Moreover, for any $c \neq 0$, provided that for all $\tau \in S$ such that $(\alpha, c) \in S_\tau$ and such that $\lambda = s_\alpha(\tau) - 2c\alpha \neq 0$ we have

$$\text{res}_{\langle \alpha, z \rangle = c\alpha^2} (a_\tau + a_\lambda) = 0,$$

then $Dp(z)$ is non-singular at $\langle \alpha, z \rangle = c\alpha^2$.

2. Suppose, in addition to assumptions of part 1, that for all $\tau \in S$ and any $t \in A_\alpha$, the following is satisfied whenever $t + \langle \tau, (2\alpha)^\vee \rangle \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$:

(a) $a_\tau(z + t\alpha) = 0$ at $\langle \alpha, z \rangle = 0$ (equivalently, $P_\tau(z)$ has a factor of $\langle \alpha, z \rangle - t\alpha^2$),

or

(b) $\lambda = s_\alpha(\tau) - 2t\alpha \in S$ and $a_\lambda(z + t\alpha) = a_\tau(z + t\alpha)$ at $\langle \alpha, z \rangle = 0$.

Then $Dp(z + t\alpha) = Dp(z - t\alpha)$ at $\langle \alpha, z \rangle = 0$ for all $t \in A_\alpha$.

Proof 1. Let $c \in \mathbb{C}$. We want to show that the residue at $\langle \alpha, z \rangle = c\alpha^2$ of $Dp(z)$ is zero.

Take any $\tau \in S$ such that $(\alpha, c) \in S_\tau$. Write $\tau + c\alpha = t\alpha + \gamma$, where $\langle \gamma, \alpha \rangle = 0$ and $t = \langle \tau + c\alpha, \alpha \rangle / \alpha^2$. Let $\lambda = s_\alpha(\tau) - 2c\alpha$. Then $\lambda + c\alpha = s_\alpha(\tau + c\alpha) = -t\alpha + \gamma$. At $\langle \alpha, z \rangle = c\alpha^2$ we thus have

$$p(z + \tau) = p((z - c\alpha + \gamma) + t\alpha) = p((z - c\alpha + \gamma) - t\alpha) = p(z + \lambda)$$

as $\langle z - c\alpha + \gamma, \alpha \rangle = 0$ and $t \in A_\alpha \cup (-A_\alpha)$ by assumption (D_2) . So, if $\lambda = 0$ then the simple pole at $\langle \alpha, z \rangle = c\alpha^2$ present in $a_\tau(z)$ is cancelled by $(T_\tau - 1)[p(z)] = p(z + \tau) - p(z)$. And if $\lambda \neq 0$, then the sum

$$a_\tau(z)(p(z + \tau) - p(z)) + a_\lambda(z)(p(z + \lambda) - p(z))$$

contributes zero to the residue provided that the residue of $a_\tau + a_\lambda$ is zero. For $c \neq 0$, the latter is satisfied by assumption. In the case of $c = 0$, we have $\lambda = s_\alpha(\tau)$,

hence $a_\tau(s_\alpha(z)) = a_\lambda(z)$ by the symmetry (D_3) of the operator, so we get

$$\lim_{\langle \alpha, z \rangle \rightarrow 0} \langle \alpha, z \rangle a_\tau(z) = \lim_{\langle \alpha, z \rangle \rightarrow 0} \langle \alpha, s_\alpha(z) \rangle a_\tau(s_\alpha(z)) = - \lim_{\langle \alpha, z \rangle \rightarrow 0} \langle \alpha, z \rangle a_\lambda(z),$$

that is, the residue of $a_\tau(z)$ at $\langle \alpha, z \rangle = 0$ is minus that of $a_\lambda(z)$, as needed.

2. Fix $t \in A_\alpha$. By symmetry (D_3) of the operator, at $\langle \alpha, z \rangle = 0$ we have $a_\mu(z + s\alpha) = a_{s_\alpha(\mu)}(z - s\alpha)$ for all generic $s \in \mathbb{C}$ and generic $z \in \mathbb{C}^n$ with $\langle \alpha, z \rangle = 0$, $\mu \in S$. By using that $s_\alpha(S) = S$, we can thus write $Dp(z + t\alpha) - Dp(z - t\alpha)$ at $\langle \alpha, z \rangle = 0$ as

$$\lim_{s \rightarrow t} \sum_{\mu \in S} a_\mu(z + s\alpha) \left(p(z + s\alpha + \mu) - p(z - s\alpha + s_\alpha(\mu)) - p(z + s\alpha) + p(z - s\alpha) \right). \tag{3.2}$$

Firstly, let us consider any $\tau \in S$ for which $a_\tau(z + t\alpha)$ is non-singular at $\langle \alpha, z \rangle = 0$ (for generic z). Then the corresponding $\mu = \tau$ term in the sum (3.2) can be simplified to

$$a_\tau(z + t\alpha) (p(z + t\alpha + \tau) - p(z - t\alpha + s_\alpha(\tau))) \tag{3.3}$$

as $p(z) \in \mathcal{R}^a$. Let $\tau = b\alpha + \delta$, where $\langle \delta, \alpha \rangle = 0$ and $b = \langle \tau, (2\alpha)^\vee \rangle$. Then $s_\alpha(\tau) = -b\alpha + \delta$. Thus

$$\begin{aligned} & p(z + t\alpha + \tau) - p(z - t\alpha + s_\alpha(\tau)) \\ &= p(z + \delta + (t + b)\alpha) - p(z + \delta - (t + b)\alpha), \end{aligned} \tag{3.4}$$

where $\langle \alpha, z + \delta \rangle = 0$. Hence if $t + b \in A_\alpha \cup (-A_\alpha) \cup \{0\}$, then (3.4) equals zero, and the whole term (3.3) vanishes. Else, we have by assumption two possibilities (cases (a) and (b)). If $a_\tau(z + t\alpha) = 0$ at $\langle \alpha, z \rangle = 0$, then (3.3) vanishes; and if $a_\tau(z + t\alpha) \neq 0$, then $\lambda = s_\alpha(\tau) - 2t\alpha \in S \setminus \{\tau\}$ and $a_\lambda(z + t\alpha) = a_\tau(z + t\alpha)$. (Note that $t + b \notin A_\alpha \cup (-A_\alpha) \cup \{0\}$ implies that $\lambda \neq \tau$, and due to (D_2) also that $a_\lambda(z + t\alpha)$ is well-defined at $\langle \alpha, z \rangle = 0$ for generic z). In the latter case, the term corresponding to $\mu = \lambda$ in the sum (3.2) can be simplified to

$$\begin{aligned} & a_\lambda(z + t\alpha) (p(z + t\alpha + \lambda) - p(z - t\alpha + s_\alpha(\lambda))) \\ &= a_\tau(z + t\alpha) (p(z - t\alpha + s_\alpha(\tau)) - p(z + t\alpha + \tau)), \end{aligned}$$

which is the negative of (3.3), hence the terms corresponding to $\mu = \tau$ and $\mu = \lambda$ in (3.2) cancel out.

Secondly, let us consider any $\tau \in S$ for which $a_\tau(z + t\alpha)$ is singular at $\langle \alpha, z \rangle = 0$. Equivalently, $a_\tau(\tilde{z})$ is singular at $\langle \alpha, \tilde{z} \rangle = t\alpha^2$. Hence $(\alpha, t) \in S_\tau$ by assumption (D_2), in particular, $\lambda \in S \cup \{0\}$ and $t + b \in A_\alpha \cup (-A_\alpha)$. From the latter, it follows that expression (3.4) vanishes. We can restate this as

$$p(z + s\alpha + \tau) - p(z - s\alpha + s_\alpha(\tau)) = (s - t)q(s)$$

for some analytic function $q(s)$ ($s \in \mathbb{C}$). Similarly, the condition $p(z+t\alpha) = p(z-t\alpha)$ at $\langle \alpha, z \rangle = 0$ can be restated as

$$p(z + s\alpha) - p(z - s\alpha) = (s - t)r(s) \tag{3.5}$$

for some analytic function $r(s)$. Moreover, we also have

$$\begin{aligned} p(z + s\alpha + \lambda) - p(z - s\alpha + s_\alpha(\lambda)) &= p(z - (2t - s)\alpha + s_\alpha(\tau)) - p(z + (2t - s)\alpha + \tau) \\ &= (s - t)q(2t - s). \end{aligned} \tag{3.6}$$

Suppose firstly that $\lambda \neq 0$. Then in the sum (3.2) the two terms corresponding to $\mu = \tau$ and $\mu = \lambda$ cancel out. Indeed, they equal

$$\begin{aligned} \lim_{s \rightarrow t} a_\tau(z + s\alpha)(s - t)(q(s) - r(s)) + a_\lambda(z + s\alpha)(s - t)(q(2t - s) - r(s)) \\ = (q(t) - r(t)) \operatorname{res}_{\langle z, \alpha \rangle = t\alpha^2} (a_\tau + a_\lambda) = 0, \end{aligned}$$

because $\operatorname{res}_{\langle z, \alpha \rangle = t\alpha^2} (a_\tau + a_\lambda) = 0$ by assumptions of part 1 with $c = t$. Suppose now that $\lambda = 0$, then $r(s) = q(2t - s)$ by equalities (3.5) and (3.6). Therefore, the term corresponding to $\mu = \tau$ in the sum (3.2) equals $\lim_{s \rightarrow t} a_\tau(z + s\alpha)(s - t)(q(s) - q(2t - s)) = 0$.

It follows that the sum (3.2) vanishes, as required. □

It follows that if conditions (D_2) , (D_3) and the assumptions of both parts 1 and 2 of Theorem 3.2 are satisfied for all $\alpha \in R^r_+$, then D preserves the ring \mathcal{R}^a , that is $Dp(z) \in \mathcal{R}^a$ if $p(z) \in \mathcal{R}^a$.

Additionally, we can use the symmetry assumption (D_3) to reduce the number of conditions that we have to consider in Theorem 3.2. The following statements take place.

Lemma 3.3 *Suppose that condition (D_3) holds. If $a_\tau + a_\lambda$ has zero residue at $\langle \alpha, z \rangle = c\alpha^2$, then $a_{w\tau} + a_{w\lambda}$ has zero residue at $\langle w\alpha, z \rangle = c\alpha^2$ for any $w \in W$.*

Proof By the property (D_3) , we have $a_{w\tau}(z) + a_{w\lambda}(z) = a_\tau(w^{-1}z) + a_\lambda(w^{-1}z)$, therefore

$$\begin{aligned} \operatorname{res}_{\langle w\alpha, z \rangle = c\alpha^2} (a_{w\tau}(z) + a_{w\lambda}(z)) &= \lim_{\langle w\alpha, z \rangle \rightarrow c\alpha^2} (\langle w\alpha, z \rangle - c\alpha^2)(a_{w\tau}(z) + a_{w\lambda}(z)) \\ &= \lim_{\langle \alpha, w^{-1}z \rangle \rightarrow c\alpha^2} (\langle \alpha, w^{-1}z \rangle - c\alpha^2)(a_\tau(w^{-1}z) + a_\lambda(w^{-1}z)) \\ &= \operatorname{res}_{\langle \alpha, \tilde{z} \rangle = c\alpha^2} (a_\tau(\tilde{z}) + a_\lambda(\tilde{z})) = 0. \end{aligned}$$

By combining Lemmas 3.1, 3.3 and Theorem 3.2, we obtain the following. □

Corollary 3.4 *Suppose the operator (3.1) satisfies conditions (D_2) and (D_3) . If the assumptions of part 1 of Theorem 3.2 are satisfied for some $\alpha \in R^r_+$, then $Dp(z)$ is non-singular at $\langle w\alpha, z \rangle = c\alpha^2$ for all $w \in W$ and for all $c \in \mathbb{C}$.*

Proof By Theorem 3.2 part 1, it suffices to check that for any $\tilde{\tau} \in S$ and $c \neq 0$ such that $(|w\alpha|, \text{sgn}(w\alpha)c) \in S_{\tilde{\tau}}$ and such that $\tilde{\lambda} = s_{|w\alpha|}(\tilde{\tau}) - 2cw\alpha \neq 0$, we have that the residue of $a_{\tilde{\tau}} + a_{\tilde{\lambda}}$ at $\langle w\alpha, z \rangle = c\alpha^2$ is zero. Since S is W -invariant, we can write $\tilde{\tau} = w\tau$ for some $\tau \in S$. Lemma 3.1 then gives $(\alpha, c) \in S_{\tau}$. Note that $\tilde{\lambda} = w\lambda$ for $\lambda = s_{\alpha}(\tau) - 2c\alpha$ (in particular, $\lambda \neq 0$ as $\tilde{\lambda} \neq 0$). By assumption, part 1 of Theorem 3.2 holds for these α and c , that is $\text{res}_{\langle \alpha, z \rangle = c\alpha^2} (a_{\tau} + a_{\lambda}) = 0$. Lemma 3.3 now gives what we need. \square

Lemma 3.5 *Suppose the operator (3.1) satisfies conditions (D_2) and (D_3) . If the assumptions of parts 1 and 2 of Theorem 3.2 are satisfied for some $\alpha \in R_+^r$, then the assumptions of part 2 are also satisfied for $w\alpha$ for all $w \in W$ such that $w\alpha \in R_+^r$.*

Proof Note that $A_{w\alpha} = A_{\alpha}$. Thus we need to prove that whenever for some $t \in A_{\alpha}$ and $\tilde{\tau} \in S$ we have $t + \langle \tilde{\tau}, (2w\alpha)^{\vee} \rangle \notin A_{\alpha} \cup (-A_{\alpha}) \cup \{0\}$, then either $a_{\tilde{\tau}}(z + tw\alpha) = 0$ at $\langle w\alpha, z \rangle = 0$, or else $\tilde{\lambda} = s_{w\alpha}(\tilde{\tau}) - 2tw\alpha \in S$ and $a_{\tilde{\lambda}}(z + tw\alpha) = a_{\tilde{\tau}}(z + tw\alpha)$ at $\langle w\alpha, z \rangle = 0$.

Suppose that $t + \langle \tilde{\tau}, (2w\alpha)^{\vee} \rangle \notin A_{\alpha} \cup (-A_{\alpha}) \cup \{0\}$. Since S is invariant, we can write $\tilde{\tau} = w\tau$ for some $\tau \in S$. Note that then $\tilde{\lambda} = w(s_{\alpha}(\tau) - 2t\alpha) = w\lambda$. Note also that $(2w\alpha)^{\vee} = w(2\alpha)^{\vee}$. Therefore $t + \langle \tau, (2\alpha)^{\vee} \rangle = t + \langle \tilde{\tau}, (2w\alpha)^{\vee} \rangle \notin A_{\alpha} \cup (-A_{\alpha}) \cup \{0\}$. By assumption, part 2 of Theorem 3.2 holds for this α . Suppose firstly (case (a)) that $a_{\tau}(\tilde{z} + t\alpha) = 0$ at $\langle \alpha, \tilde{z} \rangle = 0$. By the symmetry (D_3) of the operator, at $\langle w\alpha, z \rangle = 0$ (or, equivalently at $\langle \alpha, w^{-1}z \rangle = 0$) we thus get $a_{\tilde{\tau}}(z + tw\alpha) = a_{\tau}(w^{-1}z + t\alpha) = 0$, as required. Otherwise (case (b)), $\lambda \in S$ hence $\tilde{\lambda} = w\lambda \in S$ by invariance, and at $\langle w\alpha, z \rangle = 0$ we have $a_{\tilde{\lambda}}(z + tw\alpha) - a_{\tilde{\tau}}(z + tw\alpha) = a_{\lambda}(w^{-1}z + t\alpha) - a_{\tau}(w^{-1}z + t\alpha) = 0$, as required. \square

Remark 3.6 Let $\alpha \in R_+^r$. Suppose $w \in W$ satisfies $w\alpha = \alpha$. Then, for any $\tau \in S$, in part 2 of Theorem 3.2 it suffices to check the given conditions for either τ or $w\tau$, as one implies the other. Indeed, we have $t + \langle w\tau, (2\alpha)^{\vee} \rangle = t + \langle \tau, (2\alpha)^{\vee} \rangle$. Also $s_{\alpha}(w\tau) - 2t\alpha = w\lambda$, and at $\langle \alpha, z \rangle = 0$ by the symmetry (D_3) we have $a_{w\tau}(z + t\alpha) = a_{\tau}(w^{-1}z + t\alpha)$ and in case (b) also $a_{w\lambda}(z + t\alpha) = a_{\lambda}(w^{-1}z + t\alpha)$.

4 Bispectral Dual Difference Operator for AG_2

In this Section, we find a difference operator \mathcal{D}_1 satisfying the conditions of Theorem 3.2 for the configuration AG_2 . The corresponding axiomatics is determined by the choice

$$A_{\gamma} = \{1, 2, 3, \dots, m_{\gamma}, m_{\gamma} + 2, m_{\gamma} + 4, \dots, m_{\gamma} + 2m_{2\gamma}\} \tag{4.1}$$

for all $\gamma \in G_{2,+}$. We define a difference operator acting in the variable $z \in \mathbb{C}^2$ of the form

$$\mathcal{D}_1 = \sum_{\tau: \frac{1}{2}\tau \in G_2} a_{\tau}(z)(T_{\tau} - 1). \tag{4.2}$$

Let W be the Weyl group of the root system G_2 . For $\tau = 2\varepsilon\alpha_j, \varepsilon \in \{\pm 1\}, j \in \{1, 2, 3\}$, we define

$$\begin{aligned}
 a_{2\varepsilon\alpha_j}(z) = & \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 3}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z \rangle}\right) \left(1 - \frac{(3m+1)\gamma^2}{\langle \gamma, z \rangle + \gamma^2}\right) \left(1 - \frac{3m\gamma^2}{\langle \gamma, z \rangle + 2\gamma^2}\right) \\
 & \times \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle}\right) \times \left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z \rangle}\right) \left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z \rangle + \alpha_j^2}\right). \tag{4.3}
 \end{aligned}$$

For $\tau = 2\varepsilon\beta_j$, we define

$$\begin{aligned}
 a_{2\varepsilon\beta_j}(z) = & 3 \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\beta_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z \rangle}\right) \left(1 + \frac{3m\gamma^2}{\langle \gamma, z \rangle + 2\gamma^2}\right) \left(1 - \frac{(3m-1)\gamma^2}{\langle \gamma, z \rangle - \gamma^2}\right) \\
 & \times \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\beta_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle}\right) \times \left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z \rangle}\right) \left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + \beta_j^2}\right). \tag{4.4}
 \end{aligned}$$

The following Lemma shows that these functions $a_\tau(z)$ have G_2 symmetry.

Lemma 4.1 *Let $a_\tau(z)$ be defined as in (4.3) and (4.4). Then for all $w \in W$, we have $wa_\tau = a_{w\tau}$.*

Proof For any $w \in W$, we have $w(W\alpha_1) = W\alpha_1, w(W\beta_1) = W\beta_1$, and $\langle w\tau, w\gamma \rangle = \langle \tau, \gamma \rangle$ for all $\gamma, \tau \in \mathbb{C}^2$. The multiplicities are invariant, too. The statement follows. □

Define the ring $\mathcal{R}_{AG_2}^a$ of analytic functions $p(z)$ satisfying conditions

$$\begin{aligned}
 p(z + s\alpha_j) &= p(z - s\alpha_j) \text{ at } \langle \alpha_j, z \rangle = 0, s = 1, 2, \dots, m, \\
 p(z + s\beta_j) &= p(z - s\beta_j) \text{ at } \langle \beta_j, z \rangle = 0, s = 1, 2, \dots, 3m, 3m + 2 \tag{4.5}
 \end{aligned}$$

for all $j = 1, 2, 3$.

Theorem 4.2 *The operator (4.2) preserves the ring $\mathcal{R}_{AG_2}^a$.*

Proof One can check that this operator has the property (D_2) where $S = 2G_2$. Let $p(z) \in \mathcal{R}_{AG_2}^a$ be arbitrary. Without loss of generality, we put $\omega = \sqrt{2}$. We introduce new coordinates (A, B) on \mathbb{C}^2 given by $A = \langle \alpha_1, z \rangle$ and $B = \langle \beta_1, z \rangle$.

If $B = 4$ (equivalently, $\langle \beta_1, z \rangle = 2\beta_1^2$), then $\langle \beta_2, z \rangle = 2 + \frac{1}{2}A, \langle \beta_3, z \rangle = -2 + \frac{1}{2}A, \langle \alpha_2, z \rangle = -6 + \frac{1}{2}A$ and $\langle \alpha_3, z \rangle = 6 + \frac{1}{2}A$. The only terms singular at $B = 4$ are $a_{-2\beta_2}, a_{-2\alpha_3}, a_{2\beta_3}$ and $a_{2\alpha_2}$. Note that $s_{\beta_1}(-2\beta_2) - 4\beta_1 = -2\alpha_3$, and we compute that $\text{res}_{B=4}(a_{-2\beta_2}) = -\text{res}_{B=4}(a_{-2\alpha_3})$ equals

$$- \frac{3m(3m+2)(3m+4)(A-12-12m)(A+6m)(A-4+12m)(A+12m)(A+4+12m)(A+12+12m)^2}{(A-12)(A-4)A^3(A+4)(A+12)}.$$

As $s_{\alpha_1}(-2\beta_2) = 2\beta_3$ and $s_{\alpha_1}(-2\alpha_3) = 2\alpha_2$, by Lemma 3.3 with $w = s_{\alpha_1}$ we get that $a_{2\beta_3} + a_{2\alpha_2}$ has zero residue at $B = 4$, too. By Theorem 3.2 part 1, there is thus no singularity at $B = 4$ in $\mathcal{D}_1 p(z)$.

If $B = 2$ (equivalently, $\langle \beta_1, z \rangle = \beta_1^2$), then $\langle \beta_2, z \rangle = 1 + \frac{1}{2}A$, $\langle \beta_3, z \rangle = -1 + \frac{1}{2}A$, $\langle \alpha_2, z \rangle = -3 + \frac{1}{2}A$ and $\langle \alpha_3, z \rangle = 3 + \frac{1}{2}A$. The only $\tau \in 2G_2$ for which a_τ is singular at $B = 2$ and for which the corresponding $\lambda = s_{\beta_1}(\tau) - 2\beta_1 \neq 0$ are $\tau = 2\beta_2, 2\alpha_2, -2\beta_3, -2\alpha_3$. Note that $s_{\beta_1}(2\beta_2) - 2\beta_1 = 2\alpha_2$, and we compute that $\text{res}_{B=2}(a_{2\beta_2}) = -\text{res}_{B=2}(a_{2\alpha_2})$ equals

$$\frac{6(m+1)(3m-1)(3m+1)(A-10-12m)(A-6-12m)(A-2-12m)(A+6-12m)^2(A-6m)(A+6+12m)}{(A-6)(A-2)A(A+2)(A+6)^3}.$$

As $s_{\alpha_1}(2\beta_2) = -2\beta_3$ and $s_{\alpha_1}(2\alpha_2) = -2\alpha_3$, by Lemma 3.3 we get that $a_{-2\beta_3} + a_{-2\alpha_3}$ has zero residue at $B = 2$, too. By Theorem 3.2 part 1, there is thus no singularity at $B = 2$ in $\mathcal{D}_1 p(z)$, nor at $B = 0$.

It follows from the above analysis and from the form of the coefficient functions (4.3) and (4.4) that there are no singularities in $\mathcal{D}_1 p(z)$ at $B = c$ for all $c \geq 0$. By Corollary 3.4, there is also no singularity in $\mathcal{D}_1 p(z)$ at $\langle \beta_i, z \rangle = c$ for all $i = 1, 2, 3$ and all $c \in \mathbb{C}$.

The only singularity at $A = \text{const} > 0$ present in the coefficients a_τ for some τ is at $A = 6$ (equivalently, $\langle \alpha_1, z \rangle = \alpha_1^2$) when $\tau = -2\alpha_1$. This singularity cancels in $\mathcal{D}_1 p(z)$ by Theorem 3.2 part 1, since the corresponding $\lambda = s_{\alpha_1}(-2\alpha_1) - 2\alpha_1 = 0$. By Corollary 3.4, there is also no singularity in $\mathcal{D}_1 p(z)$ at $\langle \alpha_i, z \rangle = c$ for all $i = 1, 2, 3$ and for all $c \in \mathbb{C}$. This completes the proof that $\mathcal{D}_1 p(z)$ is analytic.

Let us now show $\mathcal{D}_1 p(z)$ satisfies the axiomatics of $\mathcal{R}_{AG_2}^a$. We have $A_{\beta_i} = \{1, 2, 3, \dots, 3m, 3m + 2\}$ and $A_{\alpha_i} = \{1, 2, \dots, m\}$ ($i = 1, 2, 3$). Let us show firstly that $\mathcal{D}_1 p(z + t\beta_1) = \mathcal{D}_1 p(z - t\beta_1)$ at $\langle \beta_1, z \rangle = 0$ for all $t \in A_{\beta_1}$. To do so we will check condition 2 in Theorem 3.2 for all $\tau \in 2G_2$.

Note that $(2\beta_1)^\vee = \frac{1}{2}\beta_1$. Let $\tau = 2\beta_1$. Then $|t + \langle \tau, (2\beta_1)^\vee \rangle| = t + 2$ which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m - 1$ or $t = 3m + 2$. But

$$a_{2\beta_1}(z + (3m + 2)\beta_1) = a_{2\beta_1}(z + (3m - 1)\beta_1) = 0 \text{ at } \langle \beta_1, z \rangle = 0$$

because $a_{2\beta_1}(z)$ contains the factors $\left(1 - \frac{(3m+2)\beta_1^2}{\langle \beta_1, z \rangle}\right) \left(1 - \frac{3m\beta_1^2}{\langle \beta_1, z \rangle + \beta_1^2}\right)$.

Let now $\tau = -2\beta_1$. Then $|t + \langle \tau, (2\beta_1)^\vee \rangle| = |t - 2| \in A_{\beta_1} \cup \{0\}$ for all $t \in A_{\beta_1}$, as needed.

Let now $\tau = 2\beta_2$. Then $|t + \langle \tau, (2\beta_1)^\vee \rangle| = t + 1$ which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m$ or $t = 3m + 2$. But

$$a_{2\beta_2}(z + (3m + 2)\beta_1) = a_{2\beta_2}(z + 3m\beta_1) = 0 \text{ at } \langle \beta_1, z \rangle = 0$$

because $a_{2\beta_2}(z)$ contains the factors $\left(1 - \frac{(3m+2)\beta_1^2}{\langle \beta_1, z \rangle}\right) \left(1 - \frac{(3m-1)\beta_2^2}{\langle \beta_1, z \rangle - \beta_1^2}\right)$.

Let now $\tau = -2\beta_2$. Then $|t + \langle \tau, (2\beta_1)^\vee \rangle| = t - 1$ which does not belong to $A_{\beta_1} \cup \{0\}$ if and only if $t = 3m + 2$. But $a_{-2\beta_2}(z + (3m + 2)\beta_1) = 0$ at $\langle \beta_1, z \rangle = 0$ because $a_{-2\beta_2}$

contains the factor $\left(1 + \frac{3m\beta_1^2}{-\langle\beta_1, z\rangle + 2\beta_1^2}\right)$. Since $s_{\alpha_1}(\beta_1) = \beta_1$, by Remark 3.6 there is nothing to check for $\tau = \pm 2\beta_3 = s_{\alpha_1}(\mp 2\beta_2)$.

For $\tau = \pm 2\alpha_1$, we get $|t + \langle\tau, (2\beta_1)^\vee\rangle| = t \in A_{\beta_1}$, as needed. Similarly for $\tau = 2\alpha_2$, $|t + \langle\tau, (2\beta_1)^\vee\rangle| = |t - 3| \in A_{\beta_1} \cup \{0\}$ for all $t \in A_{\beta_1}$, as needed.

Finally, let $\tau = -2\alpha_2$. Then $|t + \langle\tau, (2\beta_1)^\vee\rangle| = t + 3 \notin A_{\beta_1} \cup \{0\}$ if and only if $t = 3m + 2, t = 3m$ or $t = 3m - 2$, but $a_{-2\alpha_2}(z + t\beta_1) = 0$ at $\langle\beta_1, z\rangle = 0$ for those t because $a_{-2\alpha_2}$ contains the factors $\left(1 - \frac{(3m+2)\beta_1^2}{\langle\beta_1, z\rangle}\right)\left(1 - \frac{(3m+1)\beta_1^2}{\langle\beta_1, z\rangle + \beta_1^2}\right)\left(1 - \frac{3m\beta_1^2}{\langle\beta_1, z\rangle + 2\beta_1^2}\right)$. By Remark 3.6, there is nothing to check for $\tau = \pm 2\alpha_3 = s_{\alpha_1}(\mp 2\alpha_2)$.

Let us show next that $\mathcal{D}_1 p(z + t\alpha_1) = \mathcal{D}_1 p(z - t\alpha_1)$ at $\langle\alpha_1, z\rangle = 0$ for all $t \in A_{\alpha_1}$. By Theorem 3.2 it is sufficient to check its condition 2 for all $\tau \in 2G_2$.

Let $\tau = \pm 2\beta_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t \in A_{\alpha_1}$, as needed.

Let now $\tau = 2\beta_2$. Note that $(2\alpha_1)^\vee = \frac{1}{6}\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 1 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$. But $a_{2\beta_2}(z + m\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ because $a_{2\beta_2}$ contains the factor $\left(1 - \frac{m\alpha_1^2}{\langle\alpha_1, z\rangle}\right)$.

Let now $\tau = -2\beta_2$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t - 1 \in A_{\alpha_1} \cup \{0\}$ for all $t \in A_{\alpha_1}$, as needed. Since $s_{\beta_1}(\alpha_1) = \alpha_1$, by Remark 3.6 there is nothing to check for $\tau = \pm 2\beta_3 = s_{\beta_1}(\pm 2\beta_2)$.

Let now $\tau = 2\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 2 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$ or $t = m - 1$. But $a_{2\alpha_1}(z + t\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ for those t because $a_{2\alpha_1}$ contains the factors $\left(1 - \frac{m\alpha_1^2}{\langle\alpha_1, z\rangle}\right)\left(1 - \frac{m\alpha_1^2}{\langle\alpha_1, z\rangle + \alpha_1^2}\right)$.

Let now $\tau = -2\alpha_1$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = |t - 2| \in A_{\alpha_1} \cup \{0\}$ for all $t \in A_{\alpha_1}$, as needed.

Let now $\tau = 2\alpha_2$. Then $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t + 1 \notin A_{\alpha_1} \cup \{0\}$ if and only if $t = m$. But $a_{2\alpha_2}(z + m\alpha_1) = 0$ at $\langle\alpha_1, z\rangle = 0$ because $a_{2\alpha_2}$ contains the factor $\left(1 - \frac{m\alpha_1^2}{\langle\alpha_1, z\rangle}\right)$.

Finally, for $\tau = -2\alpha_2$ we get $|t + \langle\tau, (2\alpha_1)^\vee\rangle| = t - 1 \in A_{\alpha_1} \cup \{0\}$ for all $t \in A_{\alpha_1}$. And by Remark 3.6 there is nothing to check for $\tau = \pm 2\alpha_3 = s_{\beta_1}(\pm 2\alpha_2)$.

Since all the vectors α_i, β_i are in the W -orbit of $\alpha_1 \cup \beta_1$ the statement follows by Lemma 3.5. □

Let us now look at the expansion of the operator (4.2) as $\omega \rightarrow 0$. It produces the rational CMS operator in the potential-free gauge for the root system of type G_2 with multiplicity m for the long roots and multiplicity $3m + 1$ for the short roots, as the next Proposition shows. Let $\beta_j = \omega^{-1}\beta_j$ and $\tilde{\alpha}_j = \omega^{-1}\alpha_j$ ($j = 1, 2, 3$) with the same multiplicities as β_j and α_j , respectively, and let $m_2\tilde{\beta}_j = m_2\beta_j = 1, m_2\tilde{\alpha}_j = m_2\alpha_j = 0$.

Proposition 4.3 *We have*

$$\lim_{\omega \rightarrow 0} \frac{\mathcal{D}_1}{72\omega^2} = \Delta - \sum_{\gamma \in \{\tilde{\beta}_i, \tilde{\alpha}_i : i=1,2,3\}} \frac{2(m_\gamma + m_{2\gamma})}{\langle\gamma, z\rangle} \partial_\gamma,$$

where $\Delta = \partial_{z_1}^2 + \partial_{z_2}^2$ and $\partial_\gamma = \langle\gamma, \partial\rangle$ is the derivative in the direction γ .

Proof We have $T_{\pm 2\beta_j} - 1 = \pm\omega\partial_{2\tilde{\beta}_j} + \frac{1}{2}\omega^2\partial_{2\tilde{\beta}_j}^2 + \dots$, and similarly for the other shifts. The terms at ω in the expansion $\omega \rightarrow 0$ of the operator \mathcal{D}_1 vanish. The terms that are second order in derivatives in the coefficient at ω^2 in the expansion $\omega \rightarrow 0$ of the operator \mathcal{D}_1 are

$$3 \sum_{j=1}^3 \partial_{2\tilde{\beta}_j}^2 + \sum_{j=1}^3 \partial_{2\tilde{\alpha}_j}^2 = 72\Delta.$$

Let us now consider the terms that are first order in derivatives in the coefficient at ω^2 . It is easy to see that such terms containing $\langle \tilde{\beta}_1, z \rangle^{-1}$ are

$$-12(3m + 1) \left(2\partial_{2\tilde{\beta}_1} + \partial_{2\tilde{\beta}_2} - \partial_{2\tilde{\beta}_3} + \partial_{2\tilde{\alpha}_3} - \partial_{2\tilde{\alpha}_2} \right) = -144(m_{\beta_1} + m_{2\beta_1})\partial_{\tilde{\beta}_1}.$$

Altogether, the term at ω^2 in the expansion of the operator \mathcal{D}_1 is as required. □

5 Construction of the Baker–Akhiezer Function for AG_2

In this Section, we employ the method from [4] to give a construction of the BA function for the configuration AG_2 . The BA function will be an eigenfunction for the difference operator from Sect. 4, which establishes bispectrality of the CMS AG_2 Hamiltonian for integer coupling parameters.

The following Lemma gives a useful way of expanding the functions a_τ in the operator (4.2).

Lemma 5.1 *Let $a_\tau(z)$ be defined as in (4.3) and (4.4). Then*

$$a_\tau(z) = \kappa_\tau - \kappa_\tau \sum_{\gamma \in G_{2,+}} \frac{\langle \tau, \gamma \rangle (m_\gamma + m_{2\gamma})}{\langle \gamma, z \rangle} + R_\tau(z), \tag{5.1}$$

where $\kappa_{2\epsilon\beta_j} = 3$ and $\kappa_{2\epsilon\alpha_j} = 1$, and $R_\tau(z)$ is a rational function with $\deg R_\tau \leq -2$.

Proof For the factors in a_τ with shifted singularities at $\langle \gamma, z \rangle + c = 0$ for $c \neq 0$, we can use that

$$\frac{1}{\langle \gamma, z \rangle + c} = \frac{1}{\langle \gamma, z \rangle} - \frac{c}{(\langle \gamma, z \rangle + c)\langle \gamma, z \rangle},$$

which differs from $\langle \gamma, z \rangle^{-1}$ only by a rational function of degree -2 which cannot affect the coefficient at $\langle \gamma, z \rangle^{-1}$. The relation (5.1) is then obtained by multiplying out the factors in each of the a_τ . □

The next Lemma is proved by a direct computation that uses Lemma 5.1. We will apply it in the proof of Theorem 5.3 below.

Lemma 5.2 For $\gamma \in G_{2,+}$ let $n_\gamma \in \mathbb{N}$ be arbitrary. Let $N = \sum_{\gamma \in G_{2,+}} n_\gamma$. Let

$$\mu(x) = \sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau (\exp \langle x, \tau \rangle - 1), \tag{5.2}$$

where κ_τ are as in Lemma 5.1. Let $A(z) = \prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{n_\gamma}$. Write $(\mathcal{D}_1 - \mu(x))[A(z) \exp \langle x, z \rangle] = R(x, z) \exp \langle x, z \rangle$ for some rational function $R(x, z)$ in z , which has degree less than or equal to N . Then

$$R(x, z) = \sum_{\gamma \in G_{2,+}} (n_\gamma - (m_\gamma + m_{2\gamma})) \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle \exp \langle x, \tau \rangle \right) A(z) \langle \gamma, z \rangle^{-1} + S(x, z)$$

for some rational function $S(x, z)$ in z with degree less than or equal to $N - 2$.

Proof By making use of the expression for $a_\tau(z)$ given in Lemma 5.1 we get

$$\begin{aligned} \mathcal{D}_1[A(z) \exp \langle x, z \rangle] &= \sum_{\tau: \frac{1}{2}\tau \in G_2} a_\tau(z) (T_\tau - 1)[A(z) \exp \langle x, z \rangle] \\ &= \exp \langle x, z \rangle \sum_{\tau: \frac{1}{2}\tau \in G_2} a_\tau(z) \left(\exp \langle x, \tau \rangle \prod_{\gamma \in G_{2,+}} (\langle \gamma, z \rangle + \langle \tau, \gamma \rangle)^{n_\gamma} - A(z) \right) \\ &= A(z) \exp \langle x, z \rangle \sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau \left(1 - \sum_{\gamma \in G_{2,+}} \langle \tau, \gamma \rangle (m_\gamma + m_{2\gamma}) \langle \gamma, z \rangle^{-1} + \text{l.o.t.} \right) \\ &\quad \times \left(\exp \langle x, \tau \rangle \left(1 + \sum_{\gamma \in G_{2,+}} n_\gamma \langle \tau, \gamma \rangle \langle \gamma, z \rangle^{-1} + \text{l.o.t.} \right) - 1 \right) \\ &= A(z) \exp \langle x, z \rangle \left(\mu(x) + \sum_{\gamma \in G_{2,+}} (n_\gamma - (m_\gamma + m_{2\gamma})) \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle \exp \langle x, \tau \rangle \right) \right. \\ &\quad \left. \langle \gamma, z \rangle^{-1} + \text{l.o.t.} \right), \end{aligned}$$

where l.o.t. denotes lower degree terms in z , and where we used that $\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle = 0$ for all $\gamma \in G_{2,+}$, since if $\frac{1}{2}\tau \in G_2$ then also $-\frac{1}{2}\tau \in G_2$ and $\kappa_\tau = \kappa_{-\tau}$. \square

We are ready to give the main result of this Section.

Theorem 5.3 Let $M = \sum_{\gamma \in AG_{2,+}} m_\gamma = 12m + 3$. Let

$$c(x) = \frac{M!}{8} \prod_{\gamma \in G_{2,+}} \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau \langle \tau, \gamma \rangle \exp \langle x, \tau \rangle \right)^{m_\gamma + m_{2\gamma}} \tag{5.3}$$

for $x \in \mathbb{C}^2$. Define the polynomial

$$Q(z) = \prod_{\substack{\gamma \in G_{2,+} \\ s \in A_\gamma}} (\langle \gamma, z \rangle^2 - s^2 \gamma^2), \tag{5.4}$$

$z \in \mathbb{C}^2$, where A_γ is given by (4.1). Then the function

$$\psi(z, x) = c^{-1}(x)(\mathcal{D}_1 - \mu(x))^M [Q(z) \exp\langle z, x \rangle], \tag{5.5}$$

where $\mu(x)$ is given by (5.2), is the BA function for $R = AG_2$. Moreover, ψ is also an eigenfunction of the operator \mathcal{D}_1 with $\mathcal{D}_1 \psi = \mu(x)\psi$, thus bispectrality holds.

Proof The operator \mathcal{D}_1 preserves the ring $\mathcal{R}_{AG_2}^a$ by Theorem 4.2. The function $Q(z) \exp\langle z, x \rangle$ belongs to $\mathcal{R}_{AG_2}^a$ since it is analytic and satisfies conditions (4.5) given that $Q(z + s\gamma) = Q(z - s\gamma) = 0$ at $\langle \gamma, z \rangle = 0, s \in A_\gamma$, for all $\gamma \in G_{2,+}$. Since \mathcal{D}_1 preserves $\mathcal{R}_{AG_2}^a$, so does $\mathcal{D}_1 - \mu(x)$, hence $\psi(z, x)$ given by (5.5) belongs to $\mathcal{R}_{AG_2}^a$. Its analyticity and the form of the functions a_τ imply that it equals $c^{-1}(x)P(z, x) \exp\langle z, x \rangle$ for some polynomial $P(z, x)$ in z . To prove that $\psi(z, x)$ satisfies the definition of the BA function, it thus suffices to calculate the highest degree term in $P(z, x)$.

The highest degree term in $Q(z)$ is $Q_0(z) = \prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{2(m_\gamma + m_{2\gamma})}$ and $\deg Q_0 = 2M$. For all $k \in \mathbb{N}$ with $k \leq M$, an analogous argument as above gives that $(\mathcal{D}_1 - \mu(x))^k [Q(z) \exp\langle z, x \rangle]$ belongs to $\mathcal{R}_{AG_2}^a$ and is of the form $Q^{(k)}(z, x) \exp\langle z, x \rangle$ for some polynomial $Q^{(k)}(z, x)$ in z . Let its highest-degree homogeneous component be $Q_0^{(k)}(z, x)$. Lemma 5.2 allows to compute $Q_0^{(k)}(z, x)$.

Lemma 5.2 gives that after the first application of $\mathcal{D}_1 - \mu(x)$ onto $Q(z) \exp\langle z, x \rangle$ we get

$$Q_0^{(1)} = \sum_{\gamma \in G_{2,+}} (m_\gamma + m_{2\gamma}) \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau(\tau, \gamma) \exp\langle x, \tau \rangle \right) \langle \gamma, z \rangle^{-1} Q_0(z).$$

The second application gives

$$\begin{aligned} Q_0^{(2)} &= \sum_{\gamma \in G_{2,+}} (m_\gamma + m_{2\gamma})(m_\gamma + m_{2\gamma} - 1) \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau(\tau, \gamma) \exp\langle x, \tau \rangle \right)^2 \langle \gamma, z \rangle^{-2} Q_0(z) \\ &+ \sum_{\substack{\gamma \neq \delta \\ \gamma, \delta \in G_{2,+}}} (m_\gamma + m_{2\gamma})(m_\delta + m_{2\delta}) \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau(\tau, \gamma) \exp\langle x, \tau \rangle \right) \\ &\times \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_\tau(\tau, \delta) \exp\langle x, \delta \rangle \right) \langle \gamma, z \rangle^{-1} \langle \delta, z \rangle^{-1} Q_0(z). \end{aligned}$$

By repeatedly applying Lemma 5.2 we get

$$Q_0^{(k)} = \sum_{\mathbf{n}} f_{\mathbf{n}}(x) Q_0(z) \prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{-n_{\gamma}}$$

where $\mathbf{n} = (n_{\gamma})_{\gamma \in G_{2,+}}$ for $n_{\gamma} \in \mathbb{Z}_{\geq 0}$ such that n_{γ} add up to k and where $f_{\mathbf{n}}(x)$ is non-zero only if $n_{\gamma} \leq m_{\gamma} + m_{2\gamma}$ for all γ . It follows that $\deg P \leq M$ and that the highest degree term of $P(z, x)$ is

$$d(x) \prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{m_{\gamma} + m_{2\gamma}} = \frac{1}{8} d(x) \prod_{\gamma \in AG_{2,+}} \langle \gamma, z \rangle^{m_{\gamma}}$$

for some function $d(x)$. It also implies that the polynomial part of $(\mathcal{D}_1 - \mu(x))^{M+1} [Q(z) \exp \langle z, x \rangle]$ has degree less than M hence vanishes as a consequence of Lemma 2.6, giving $\mathcal{D}_1 \psi = \mu(x) \psi$. So to complete the proof we just need to verify that $c(x)$ given by (5.3) equals $\frac{1}{8} d(x)$.

To arrive at $\prod_{\gamma \in G_{2,+}} \langle \gamma, z \rangle^{m_{\gamma} + m_{2\gamma}}$ starting from $Q_0(z)$ we overall need to reduce the power on each of the factors $\langle \gamma, z \rangle$ by $m_{\gamma} + m_{2\gamma}$ and we do this by reducing the power of one of them by one at each step. The total number of possible orderings of doing that corresponds to the number of words of length M in the alphabet $G_{2,+}$ such that γ appears in the word $m_{\gamma} + m_{2\gamma}$ times for each $\gamma \in G_{2,+}$. This gives

$$\frac{M!}{\prod_{\gamma \in G_{2,+}} (m_{\gamma} + m_{2\gamma})!}$$

possibilities, and for each of them the total proportionality factor picked up equals by Lemma 5.2

$$\prod_{\gamma \in G_{2,+}} (m_{\gamma} + m_{2\gamma})! \left(\sum_{\tau: \frac{1}{2}\tau \in G_2} \kappa_{\tau}(\tau, \gamma) \exp \langle x, \tau \rangle \right)^{m_{\gamma} + m_{2\gamma}}.$$

It follows that $c(x)$ has the required form. □

Remark 5.4 We expect bispectrality of system AG_2 to hold for any value of the parameter m by adopting arguments from [4] for non-integer parameters.

6 Another Dual Operator

In this Section, we present another difference operator for the configuration AG_2 which preserves the quasi-invariants. We also give the corresponding second construction of the BA function.

We define a difference operator acting in the variable $z \in \mathbb{C}^2$ of the form

$$\mathcal{D}_2 = \sum_{\tau: \frac{1}{2}\tau \in AG_2} a_\tau(z)(T_\tau - 1). \tag{6.1}$$

We now proceed to specify the functions $a_\tau(z)$. Let $\lambda_\tau = g_{\tau/2}$ where g is defined in terms of the multiplicity map of AG_2 in accordance with the convention (1.4) for couplings. That is, we set

$$\lambda_\tau = \frac{1}{4}m_{\frac{1}{2}\tau}(m_{\frac{1}{2}\tau} + 2m_\tau + 1)\tau^2.$$

That means $\lambda_{4\varepsilon\beta_j} = 8\beta_j^2$, $\lambda_{2\varepsilon\beta_j} = 9m(m+1)\beta_j^2$ and $\lambda_{2\varepsilon\alpha_j} = m(m+1)\alpha_j^2$ ($j = 1, 2, 3$, $\varepsilon \in \{\pm 1\}$, $m \in \mathbb{N}$). For $\tau = 2\varepsilon\alpha_j$ we define

$$\begin{aligned} a_{2\varepsilon\alpha_j}(z) &= \lambda_{2\varepsilon\alpha_j} \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 3}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z \rangle}\right) \\ &\times \left(1 - \frac{(3m+1)\gamma^2}{\langle \gamma, z \rangle + \gamma^2}\right) \left(1 - \frac{3m\gamma^2}{\langle \gamma, z \rangle + 2\gamma^2}\right) \\ &\times \prod_{\substack{\gamma \in W\beta_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 0}} \left(1 - \frac{6\gamma^2}{\langle \gamma, z \rangle - \gamma^2}\right) \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 2\varepsilon\alpha_j, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle}\right) \\ &\times \left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z \rangle}\right) \left(1 - \frac{m\alpha_j^2}{\langle \varepsilon\alpha_j, z \rangle + \alpha_j^2}\right). \end{aligned} \tag{6.2}$$

For $\tau = 4\varepsilon\beta_j$ we define

$$\begin{aligned} a_{4\varepsilon\beta_j}(z) &= \lambda_{4\varepsilon\beta_j} \prod_{\substack{\gamma \in W\alpha_1 \\ \langle 4\varepsilon\beta_j, (2\gamma)^\vee \rangle = 2}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle}\right) \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle + \gamma^2}\right) \\ &\times \prod_{\substack{\gamma \in W\beta_1 \\ \langle 4\varepsilon\beta_j, (2\gamma)^\vee \rangle = 2}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z \rangle}\right) \left(1 - \frac{3m\gamma^2}{\langle \gamma, z \rangle + \gamma^2}\right) \\ &\times \left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z \rangle}\right) \left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + \beta_j^2}\right) \\ &\times \left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + 2\beta_j^2}\right) \left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + 3\beta_j^2}\right). \end{aligned} \tag{6.3}$$

For $\tau = 2\varepsilon\beta_j$ we define

$$\begin{aligned}
 a_{2\varepsilon\beta_j}(z) &= \lambda_{2\varepsilon\beta_j} \prod_{\substack{\gamma \in W\alpha_1 \\ (2\varepsilon\beta_j, 2\gamma)^\vee = 0}} \left(1 - \frac{\frac{2}{3}\gamma^2}{\langle \gamma, z \rangle - \gamma^2}\right) \prod_{\substack{\gamma \in W\alpha_1 \\ (2\varepsilon\beta_j, 2\gamma)^\vee = 1}} \left(1 - \frac{m\gamma^2}{\langle \gamma, z \rangle}\right) \\
 &\times \prod_{\substack{\gamma \in W\beta_1 \\ (2\varepsilon\beta_j, 2\gamma)^\vee = 1}} \left(1 - \frac{(3m+2)\gamma^2}{\langle \gamma, z \rangle}\right) \left(1 + \frac{3m\gamma^2}{\langle \gamma, z \rangle + 2\gamma^2}\right) \left(1 - \frac{(3m-1)\gamma^2}{\langle \gamma, z \rangle - \gamma^2}\right) \\
 &\quad \times \left(1 - \frac{(3m+2)\beta_j^2}{\langle \varepsilon\beta_j, z \rangle}\right) \left(1 - \frac{3m\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + \beta_j^2}\right) \\
 &\quad \times \left(1 + \frac{4\beta_j^2}{\langle \varepsilon\beta_j, z \rangle + 3\beta_j^2}\right) \left(1 - \frac{4\beta_j^2}{\langle \varepsilon\beta_j, z \rangle - \beta_j^2}\right). \tag{6.4}
 \end{aligned}$$

The next Lemma shows that the functions $a_\tau(z)$ have G_2 symmetry.

Lemma 6.1 *Let $a_\tau(z)$ be defined as above. Then for all $w \in W$, we have $wa_\tau = a_{w\tau}$.*

Proof For any $w \in W$, $\lambda_{w\tau} = \lambda_\tau$ for any τ such that $\frac{1}{2}\tau \in AG_2$. The statement follows as in the proof of Lemma 4.1. □

Theorem 6.2 *The operator (6.1) preserves the ring $\mathcal{R}_{AG_2}^a$.*

Proof One can check that the operator satisfies condition (D_2) . Let $p(z) \in \mathcal{R}_{AG_2}^a$ be arbitrary. Without loss of generality, we put $\omega = \sqrt{2}$. We introduce new coordinates (A, B) on \mathbb{C}^2 given by $A = \langle \alpha_1, z \rangle$ and $B = \langle \beta_1, z \rangle$.

It follows from the form of the coefficient functions (6.2), (6.3) and (6.4) and Theorem 3.2 that there are no singularities in $\mathcal{D}_2 p(z)$ at $B = c$ for $c \geq 0$ unless $B = 2, 4, 6$. Let us consider these cases.

If $B = 6$ (equivalently, $\langle \beta_1, z \rangle = 3\beta_1^2$), then $\langle \beta_2, z \rangle = 3 + \frac{1}{2}A$, $\langle \beta_3, z \rangle = -3 + \frac{1}{2}A$, $\langle \alpha_2, z \rangle = -9 + \frac{1}{2}A$ and $\langle \alpha_3, z \rangle = 9 + \frac{1}{2}A$. The only terms singular at $B = 6$ are $a_{-4\beta_1}$ and $a_{-2\beta_1}$. We note that $s_{\beta_1}(-4\beta_1) - 6\beta_1 = -2\beta_1$, and we compute that $\text{res}_{B=6}(a_{-4\beta_1}) = -\text{res}_{B=6}(a_{-2\beta_1})$ equals

$$\frac{48m(m+1)(3m+2)(3m+5)(A-2-12m)(A-6-12m)(A-14-12m) \times (A-18-12m)(A+2+12m)(A+6+12m)(A+14+12m)(A+18+12m)}{(A-18)(A-6)^2(A-2)(A+2)(A+6)^2(A+18)}.$$

Therefore, by Theorem 3.2 part 1, there is no singularity at $B = 6$ in $\mathcal{D}_2 p(z)$.

If $B = 4$ (equivalently, $\langle \beta_1, z \rangle = 2\beta_1^2$), then $\langle \beta_2, z \rangle = 2 + \frac{1}{2}A$, $\langle \beta_3, z \rangle = -2 + \frac{1}{2}A$, $\langle \alpha_2, z \rangle = -6 + \frac{1}{2}A$ and $\langle \alpha_3, z \rangle = 6 + \frac{1}{2}A$. The only $\tau \in AG_2$ for which a_τ is singular at $B = 4$ and for which the corresponding $\lambda = s_{\beta_1}(\tau) - 4\beta_1 \neq 0$ are $\tau = -2\beta_2, -2\alpha_3, 2\beta_3$ and $2\alpha_2$. We note that $s_{\beta_1}(-2\beta_2) - 4\beta_1 = -2\alpha_3$, and we compute that $\text{res}_{B=4}(a_{-2\beta_2}) = -\text{res}_{B=4}(a_{-2\alpha_3})$ equals

$$\frac{18m^2(m+1)(3m+2)(3m+4)(A-32)(A+24)(A-12-12m)(A+6m) \times (A-4+12m)(A+12m)(A+4+12m)(A+12+12m)^2}{(A-12)(A-8)(A-4)A^4(A+4)(A+12)}.$$

Since $s_{\alpha_1}(-2\beta_2) = 2\beta_3$ and $s_{\alpha_1}(-2\alpha_3) = 2\alpha_2$, by Lemma 3.3 the residue of $a_{2\beta_3} + a_{2\alpha_2}$ at $B = 4$ is also zero. Thus, by Theorem 3.2 part 1, there is no singularity at $B = 4$ in $\mathcal{D}_2p(z)$.

If $B = 2$ (equivalently, $\langle \beta_1, z \rangle = \beta_1^2$), then $\langle \beta_2, z \rangle = 1 + \frac{1}{2}A$, $\langle \beta_3, z \rangle = -1 + \frac{1}{2}A$, $\langle \alpha_2, z \rangle = -3 + \frac{1}{2}A$ and $\langle \alpha_3, z \rangle = 3 + \frac{1}{2}A$. The only $\tau \in AG_2$ for which a_τ is singular at $B = 2$ and for which the corresponding $\lambda = s_{\beta_1}(\tau) - 2\beta_1 \neq 0$ are $\tau = -4\beta_1, 2\beta_1, -4\beta_2, 4\beta_3, \pm 2\alpha_1, 2\beta_2, 2\alpha_2, -2\beta_3$ and $-2\alpha_3$. We note that $s_{\beta_1}(-4\beta_1) - 2\beta_1 = 2\beta_1$, and we compute that $\text{res}_{B=2}(a_{-4\beta_1}) = -\text{res}_{B=2}(a_{2\beta_1})$ equals

$$\frac{144m(m+1)(3m-2)(3m+1)(A+6-12m)(A+2-12m)(A-6-12m) \times (A-10-12m)(A-6+12m)(A-2+12m)(A+6+12m)(A+10+12m)}{(A-6)^2(A-2)^2(A+2)^2(A+6)^2}.$$

Similarly, we note that $s_{\beta_1}(-4\beta_2) - 2\beta_1 = -2\alpha_1$, and we compute that $\text{res}_{B=2}(a_{-4\beta_2}) = -\text{res}_{B=2}(a_{-2\alpha_1})$ equals

$$\frac{288m(m+1)(A-6+6m)(A+6m)(A-10+12m)(A-6+12m)^2 \times (A-2+12m)(A+2+12m)(A+6+12m)^2(A+10+12m)}{(A-10)(A-6)^4(A-2)^2A(A+2)(A+6)}.$$

Since $s_{\alpha_1}(-4\beta_2) = 4\beta_3$ and $s_{\alpha_1}(-2\alpha_1) = 2\alpha_1$, it follows by Lemma 3.3 that the residue of $a_{4\beta_3} + a_{2\alpha_1}$ at $B = 2$ is also zero. Next we note that $s_{\beta_1}(2\beta_2) - 2\beta_1 = 2\alpha_2$, and we compute that $\text{res}_{B=2}(a_{2\beta_2}) = -\text{res}_{B=2}(a_{2\alpha_2})$ equals

$$\frac{36m(m+1)^2(3m-1)(3m+1)(A-26)(A+30)(A-10-12m)(A-6-12m) \times (A-2-12m)(A+6-12m)^2(A-6m)(A+6+12m)}{(A-6)(A-2)^2A(A+2)(A+6)^4}.$$

Since $s_{\alpha_1}(2\beta_2) = -2\beta_3$ and $s_{\alpha_1}(2\alpha_2) = -2\alpha_3$, it follows by Lemma 3.3 that the residue of $a_{-2\beta_3} + a_{-2\alpha_3}$ at $B = 2$ is also zero. Thus, by Theorem 3.2 part 1 there is no singularity at $B = 2$ in $\mathcal{D}_2p(z)$.

Let us now consider possible singularities in $\mathcal{D}_2p(z)$ at $A = c \geq 0$. By Theorem 3.2 part 1 and the form of the coefficients (6.2)–(6.4) it is sufficient to consider the case $A = 6$ (equivalently, $\langle \alpha_1, z \rangle = \alpha_1^2$). In this case $\langle \beta_2, z \rangle = \frac{1}{2}B + 3$, $\langle \beta_3, z \rangle = -\frac{1}{2}B + 3$, $\langle \alpha_2, z \rangle = -\frac{3}{2}B + 3$ and $\langle \alpha_3, z \rangle = \frac{3}{2}B + 3$. The only $\tau \in AG_2$ for which a_τ is singular at $A = 6$ and for which the corresponding $\lambda = s_{\alpha_1}(\tau) - 2\alpha_1 \neq 0$ are $\tau = -4\beta_2, -4\beta_3$ and $\pm 2\beta_1$. We note that $s_{\alpha_1}(-4\beta_2) - 2\alpha_1 = -2\beta_1$, and we compute that $\text{res}_{A=6}(a_{-4\beta_2}) = -\text{res}_{A=6}(a_{-2\beta_1})$ equals

$$\frac{96m(m+1)(B-14-12m)(B-2-12m)(B-2+4m)(B+2+4m)(B-2+6m) \times (B+4+6m)(B-6+12m)(B+2+12m)(B+6+12m)(B+14+12m)}{(B-6)^2(B-2)^4B(B+2)^2(B+6)}.$$

Since $s_{\beta_1}(-4\beta_2) = -4\beta_3$ and $s_{\beta_1}(-2\beta_1) = 2\beta_1$, it follows by Lemma 3.3 that the residue of $a_{-4\beta_3} + a_{2\beta_1}$ at $A = 6$ is also zero. By Theorem 3.2 part 1 there is thus no singularity at $A = 6$ in $\mathcal{D}_2p(z)$.

By Corollary 3.4 it follows that $\mathcal{D}_2 p(z)$ has no singularities. The proof that $\mathcal{D}_2 p(z)$ belongs to $\mathcal{R}_{AG_2}^a$ can be completed in an analogous way to how it was done for the operator (4.2) in the proof of Theorem 4.2. \square

We now give a second construction of the Baker–Akhiezer Function for AG_2 .

Theorem 6.3 *Let $M = \sum_{\gamma \in AG_{2,+}} m_\gamma = 12m + 3$. Let*

$$\mu(x) = \sum_{\tau: \frac{1}{2}\tau \in AG_2} \lambda_\tau(\exp\langle x, \tau \rangle - 1),$$

where $x \in \mathbb{C}^2$, and let

$$c(x) = \frac{M!}{8} \prod_{\gamma \in G_{2,+}} \left(\sum_{\tau: \frac{1}{2}\tau \in AG_2} \lambda_\tau(\tau, \gamma) \exp\langle x, \tau \rangle \right)^{m_\gamma + m_{2\gamma}}. \tag{6.5}$$

Then the function

$$\psi(z, x) = c^{-1}(x)(\mathcal{D}_2 - \mu(x))^M [Q(z) \exp\langle z, x \rangle], \tag{6.6}$$

where polynomial $Q(z)$ is given by (5.4), is the BA function for $R = AG_2$. Moreover, ψ is also an eigenfunction of the operator \mathcal{D}_2 with $\mathcal{D}_2 \psi = \mu(x)\psi$, thus bispectrality holds.

The proof is similar to the proof of Theorem 5.3 and it can be found in [30].

Remark 6.4 One can show that operators (4.2)–(4.4), (6.1)–(6.4) commute: $[\mathcal{D}_1, \mathcal{D}_2] = 0$. This can be proven by taking the rational limit of the more general trigonometric versions of these operators, which also commute [14] Another approach is to use the relation $[\mathcal{D}_1, \mathcal{D}_2]\psi = 0$.

7 Relation with A_2 and A_1 Macdonald–Ruijsenaars Systems

In the case when $m = 0$, the configuration AG_2 reduces to the root system $\{\pm 2\beta_i : i = 1, 2, 3\}$ of type A_2 with multiplicity 1 for all vectors. In this limit, the operator (6.1) reduces to the quasiminuscule operator for (twice) this root system. Let us now consider the $m = 0$ limit of the operator (4.2). After a rescaling, this gives an operator of the form

$$D_0 = \sum_{\tau \in G_2} a_{\tau,0}(z)(T_\tau - 1) = -24 + \sum_{\tau \in G_2} a_{\tau,0}(z)T_\tau, \tag{7.1}$$

where for $\tau = \varepsilon\beta_j, \varepsilon \in \{\pm 1\}, j = 1, 2, 3$, we have

$$a_{\tau,0}(z) = 3 \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = \frac{1}{2}}} \left(1 - \frac{\frac{1}{2}\gamma^2}{\langle \gamma, z \rangle - \frac{1}{2}\gamma^2} \right) \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = 1}} \left(1 - \frac{\gamma^2}{\langle \gamma, z \rangle} \right),$$

and for $\tau = \varepsilon\alpha_j$ we have

$$a_{\tau,0}(z) = \prod_{\substack{\gamma \in W\beta_1 \\ \langle \tau, (2\gamma)^\vee \rangle = \frac{3}{2}}} \left(1 - \frac{\frac{3}{2}\gamma^2}{\langle \gamma, z \rangle + \frac{1}{2}\gamma^2} \right).$$

Proposition 7.1 *The operator (7.1) preserves the ring of analytic functions $p(z)$ satisfying $p(z + \beta_i) = p(z - \beta_i)$ at $\langle \beta_i, z \rangle = 0$ for all $i = 1, 2, 3$.*

The proof is parallel to the proof of Theorem 4.2. In this case, though, the condition 2(b) of Theorem 3.2 is needed while it does not play a role in the proofs of Theorems 4.2 and 6.2.

Let us rewrite the operator D_0 for the more standard realisation of the root system A_2 given by $A_2 = \{e_i - e_j : 1 \leq i \neq j \leq 3\} \subset \mathbb{R}^3$, where e_i are the standard basis vectors.

Proposition 7.2 *Define the set $S = S_1 \cup S_2$, where*

$$S_1 = \{3e_i : i = 1, 2, 3\} \cup \{2e_i + 2e_j - e_k : 1 \leq i < j \neq k \leq 3, i \neq k\} \text{ and} \\ S_2 = \{2e_i + e_j : 1 \leq i \neq j \leq 3\}.$$

Then the operator acting in the variable $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ given by

$$\begin{aligned} \tilde{D}_0 = & 3 \sum_{\tau \in S_2} \left(\prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 1}} \left(1 - \frac{1}{z_i - z_j - 1} \right) \prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 2}} \left(1 - \frac{2}{z_i - z_j} \right) \right) T_\tau \\ & + \sum_{\tau \in S_1} \left(\prod_{\substack{i \neq j \\ \langle \tau, e_i - e_j \rangle = 3}} \left(1 - \frac{3}{z_i - z_j + 1} \right) \right) T_\tau \end{aligned} \tag{7.2}$$

preserves the ring of analytic functions $p(z)$ satisfying $p(z + e_i - e_j) = p(z - e_i + e_j)$ at $z_i = z_j$ for all $i, j = 1, 2, 3$.

Note that $S_1 = \{\tau \in S : |\langle \tau, e_i - e_j \rangle| \in \{0, 3\} \text{ for all } i, j = 1, 2, 3\}$ and $S_2 = \{\tau \in S : |\langle \tau, e_i - e_j \rangle| \in \{0, 1, 2\} \text{ for all } i, j = 1, 2, 3\}$.

Let us now consider a version of the operator (7.2) for the root system A_1 . Let \sim denote equality of operators when acting on functions constant along the direction normal to the hyperplane $z_1 + z_2 = 0$.

Proposition 7.3 *Let $S'_1 = \{3e_1, 3e_2\}$ and $S'_2 = \{2e_1 + e_2, e_1 + 2e_2\}$. Then formula (7.2) after replacement of S_i with S'_i , $i = 1, 2$, gives an operator \widehat{D}_0 acting in the variable $z = (z_1, z_2) \in \mathbb{C}^2$ that preserves the ring $\mathcal{R}_{A_1}^a$ of analytic functions $p(z)$ satisfying $p(z + e_1 - e_2) = p(z - e_1 + e_2)$ at $z_1 = z_2$. Moreover, if we split the operator $\widehat{D}_0 = D_1 + D_2$, where*

$$D_1 = 3 \left(1 - \frac{1}{z_2 - z_1 - 1} \right) T_{e_1 + 2e_2} + \left(1 - \frac{3}{z_1 - z_2 + 1} \right) T_{3e_1}$$

and

$$D_2 = 3 \left(1 - \frac{1}{z_1 - z_2 - 1} \right) T_{2e_1 + e_2} + \left(1 - \frac{3}{z_2 - z_1 + 1} \right) T_{3e_2},$$

then $D_i(\mathcal{R}_{A_1}^a) \subseteq \mathcal{R}_{A_1}^a$ for $i = 1, 2$. The operators D_i satisfy commutativity relations

$$[D_1, D_2] = [D_1, D^{msl}] = [D_2, D^{msl}] = 0,$$

where D^{msl} is the operator for the minuscule weight $2e_1$ of the root system $2A_1$ with multiplicity 1 given by

$$D^{msl} = \left(1 - \frac{2}{z_1 - z_2} \right) T_{2e_1} + \left(1 - \frac{2}{z_2 - z_1} \right) T_{2e_2}.$$

We also have $\widehat{D}_0^2 \sim (D^{msl} + 2)^3$, and $D_1 D_2 \sim 3D^{qm} + 16 \sim 3(D^{msl})^2 + 4$, where

$$D^{qm} = \left(1 - \frac{2}{z_1 - z_2} \right) \left(1 - \frac{2}{z_1 - z_2 + 2} \right) (T_{4e_1} - 1) + \left(1 - \frac{2}{z_2 - z_1} \right) \left(1 - \frac{2}{z_2 - z_1 + 2} \right) (T_{4e_2} - 1)$$

is the operator for the quasiminuscule weight $4e_1$.

We note that the operators D_1 and D_2 are not symmetric under the swap of the variables z_1, z_2 . The operator D^{msl} is symmetric and all three operators commute.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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