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Cohomology and the Brauer groups of diagonal surfaces

Damián Gvirtz and Alexei N. Skorobogatov

June 1, 2021

in memory of Sir Peter Swinnerton-Dyer

Abstract

We present a method for calculating the Brauer group of a surface given by a diagonal equation in projective space. For diagonal quartic surfaces with coefficients in \mathbb{Q} we determine the Brauer groups over \mathbb{Q} and $\mathbb{Q}(i)$.

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Introduction

Diagonal equations occupy a special place in number theory: due to separation of variables many questions about such equations can be reduced to questions about equations in one variable. This provided André Weil with a key example for the Weil conjectures [Wei49, Wei52]. Diagonal hypersurfaces are a particular case of varieties dominated by products of curves (DPC). The Tate conjecture is known for DPC varieties defined over fields finitely generated over \mathbb{Q} , see [Tat94, (5.5)]. Using the results of [SZ14] (based on the work of Faltings) it is easy to show that for DPC varieties defined over such fields the Galois-invariant subgroup of the geometric Brauer group is finite.

In this paper we use diagonal surfaces to test new techniques for computing the Brauer group, a task motivated by the Brauer–Manin obstruction to the local-to-global principle for rational points over number fields (see [CTS21] for a general introduction). To put this into perspective, in characteristic zero, the Brauer group of any quadric X over any field k is trivial, that is, equal to the image $\text{Br}_0(X)$ of the canonical map $\text{Br}(k) \rightarrow \text{Br}(X)$. The Brauer group of a cubic surface X can be non-trivial, but it is equal to the *algebraic Brauer group*, that is, to the kernel $\text{Br}_1(X)$ of the canonical map $\text{Br}(X) \rightarrow \text{Br}(\bar{X})$, where $\bar{X} = X \times_k \bar{k}$ for an algebraic closure \bar{k} of k . This simplifies matters considerably, because $\text{Br}_1(X)/\text{Br}_0(X)$ is a subgroup of $H^1(k, \text{Pic}(\bar{X}))$ and is equal to this group if k is a number field. For diagonal cubic surfaces over \mathbb{Q} all elements of $\text{Br}(X)$ modulo $\text{Br}_0(X)$ and the ensuing Brauer–Manin conditions were determined in [CKS87].

The arithmetic of diagonal quartic surfaces is a recurrent theme in the work of Peter Swinnerton-Dyer, from his first paper [SD43], revisited in [SD68, PSD91, SD00, SD14], until his last published paper [SD16]. The algebraic Brauer groups $\text{Br}_1(X)$ of diagonal quartic surfaces over arbitrary fields of characteristic zero were classified by Bright in his thesis [Bri02, Bri06], though an explicit form of the elements of this group is not known in general, cf. [Bri11]. Ieronymou [Ier10, Thm. 3.1] constructed generators of $\text{Br}(\bar{X})[2]$ and showed [Ier10, Prop. 4.9] that they come from 4-torsion elements of the Brauer group of the quartic Fermat surface over $\mathbb{Q}(i, \sqrt[4]{2})$ (in fact, they come from 2-torsion elements, see Corollary 3.5 below). He then obtained sufficient conditions on the diagonal quartic surface X over \mathbb{Q} under which the 2-torsion subgroup of $\text{Br}(X)/\text{Br}_1(X)$ is trivial [Ier10, Thm. 5.2]. This, together with Mizukami’s isomorphism between the quartic Fermat surface over $\mathbb{Q}(i, \sqrt{2})$ and a certain Kummer surface, was used in [ISZ11] to deduce an explicit upper bound for the size of $\text{Br}(X)/\text{Br}_0(X)$. Finally, the odd order torsion subgroup of $\text{Br}(X)$

was determined in [IS15] using a different method; the elements of this group have an explicit description, enabling one to compute the associated Brauer–Manin conditions. The complete determination of the 2-primary torsion subgroup of $\text{Br}(X)$ could not be done by previously known methods. We now describe a new approach that allowed us to achieve this.

Let k be a number field and let $\Gamma = \text{Gal}(\bar{k}/k)$. For a smooth and projective *surface* X over k such that $\text{Pic}(\bar{X})$ is torsion-free and $\text{Br}(\bar{X})^\Gamma$ is finite, the determination of $\text{Br}(X)/\text{Br}_0(X)$ involves the following steps. (These conditions hold, for example, when X is a K3 surface [SZ08] or a diagonal surface in \mathbb{P}_k^3 , see [SGA7, XI.1.8] and Proposition 1.7 below.)

(a) Determine the action of Γ on the *geometric Brauer group* $\text{Br}(\bar{X})$, hence compute $\text{Br}(\bar{X})^\Gamma$. For a K3 surface with complex multiplication this involves identifying the Grössencharakter which describes the action of Γ on the Tate module of $\text{Br}(\bar{X})$. For diagonal quartic surfaces, this has essentially been done by Pinch and Swinnerton-Dyer in [PSD91] using work of Weil [Wei52].

(b) Determine the image of $\text{Br}(X) \rightarrow \text{Br}(\bar{X})^\Gamma$, called the *transcendental Brauer group*. The crucial fact is that when $\text{Pic}(\bar{X})$ is torsion-free, the 2-term complex of Γ -modules

$$\text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z}) \longrightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \quad (1)$$

is quasi-isomorphic to the truncated complex $\tau_{[1,2]} \mathbf{R}p_* \mathbb{G}_{m,X}$, where $p: X \rightarrow \text{Spec}(k)$ is the structure morphism. Here $T(X_{\mathbb{C}})$ is the *transcendental lattice*, and the map in (1) comes from the cup-product bilinear form on $H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$.

(c) The group $\text{Br}_1(X)/\text{Br}_0(X) \cong H^1(k, \text{Pic}(\bar{X}))$ is finite; it can be determined if we know an integral basis of $\text{Pic}(\bar{X})$ and the action of Γ on it. The group $\text{Br}(X)/\text{Br}_0(X)$, an extension of finite abelian groups $\text{Br}(X)/\text{Br}_1(X)$ by $\text{Br}_1(X)/\text{Br}_0(X)$, is computed as the first Galois hypercohomology group of the complex (1).

The methods in (b) and (c) are used for the first time in this paper. We apply them to advance this program for diagonal surfaces of arbitrary degree d in \mathbb{P}_k^3 , using the description of integral homology of the Fermat hypersurface F of degree d over \mathbb{C} obtained by Pham [Pha65], see also [Lo10, ABB13, DS16].

Pham considers a natural “vanishing cycle” in the complement $U_{\mathbb{C}}$ to a coordinate hyperplane section of $F_{\mathbb{C}}$ and shows that its orbit under the action of the group of diagonal automorphisms $G = (\mu_d)^3$ generates the primitive subgroup of $H_2(F_{\mathbb{C}}, \mathbb{Z})$ (the kernel of the intersection product with the hyperplane section). He also computes the intersection product in the resulting integral basis. The group $H^2(F_{\mathbb{C}}, \mathbb{C})$ decomposes into a direct sum of 1-dimensional eigenspaces of G , compatibly with the Hodge decomposition (which was determined by Shioda [Sh79] using work of Griffiths). The Galois representation in the étale cohomology of F goes back to Weil [Wei49, Wei52], and was studied by Katz, Shioda, and Ulmer, among others, see [Ulm02]. Combining these data, the computation of the Brauer group of diagonal surfaces of any degree d over the cyclotomic field $\mathbb{Q}(\mu_d)$ can in principle be done

by a computer. We work out precise details in the case of diagonal quartic surfaces with coefficients in \mathbb{Q} over the ground field $\mathbb{Q}(i)$, which are then used to deal with the case of the ground field \mathbb{Q} .

To state our results, we call two diagonal quartic forms $\sum_{i=0}^3 a_i x_i^4$, where $a_i \in K^\times$ for $i = 0, 1, 2, 3$, *equivalent* if one is obtained from another by permuting the variables x_0, x_1, x_2, x_3 , multiplying the coefficients a_i by fourth powers in K^\times , and multiplying all four coefficients by a common multiple in K^\times . A diagonal quartic surface is the surface given by a diagonal quartic form in \mathbb{P}_K^3 ; surfaces given by equivalent forms will be called equivalent. It is clear that equivalent surfaces are isomorphic.

Main theorem. *Let a_1, a_2, a_3 be non-zero rational numbers. Let $X \subset \mathbb{P}_{\mathbb{Q}}^3$ be the surface*

$$x_0^4 + a_1 x_1^4 + a_2 x_2^4 + a_3 x_3^4 = 0.$$

(1) *Let $k = \mathbb{Q}(i)$ and let $X_k = X \times_{\mathbb{Q}} k$. If X_k is equivalent to the diagonal quartic surface with coefficients $(a_1, a_2, a_3) = (1, 2, -2)$, then the 2-primary torsion subgroup of $\text{Br}(X_k)/\text{Br}_1(X_k)$ is $\mathbb{Z}/2$; otherwise this subgroup is 0.*

(2) *If X is equivalent to the diagonal quartic surface with coefficients $(1, 2, -2)$ or $(1, 8, -8)$, then the 2-primary torsion subgroup of $\text{Br}(X)/\text{Br}_1(X)$ is $\mathbb{Z}/2$; otherwise this subgroup is 0.*

The two exceptional surfaces in (2) are non-equivalent over \mathbb{Q} but equivalent, hence isomorphic over $\mathbb{Q}(i)$. We do not know whether they are isomorphic over \mathbb{Q} . Both have rational points over \mathbb{Q} . In Proposition 4.7 we show that multiplying two of the coefficients by -4 gives rise to surfaces with isomorphic Galois representations in the second étale cohomology group with coefficients \mathbb{Q}_2 and \mathbb{Z}_ℓ for any odd prime ℓ . By analogy with the Tate module of abelian varieties one may conjecture that these surfaces are related by an isogeny whose degree is a power of 2.

By the main theorem, only finitely many diagonal quartic surfaces with coefficients in \mathbb{Q} , considered over \mathbb{Q} or $\mathbb{Q}(i)$, up to isomorphism, have non-zero 2-torsion in the transcendental Brauer group. In contrast, there are infinitely many non-equivalent diagonal quartic surfaces over \mathbb{Q} with odd order torsion in the transcendental Brauer group. Indeed, by [IS15, Thm. 1.1] the *odd* torsion subgroup of $\text{Br}(X)/\text{Br}_1(X)$ is 0, unless $-3a_1 a_2 a_3$ is a fourth power or -4 times a fourth power in \mathbb{Q} when this subgroup is $\mathbb{Z}/3$, or $125a_1 a_2 a_3$ is a fourth power or -4 times a fourth power in \mathbb{Q} when this subgroup is $\mathbb{Z}/5$. (When \mathbb{Q} is replaced by $\mathbb{Q}(i)$ we have $(\mathbb{Z}/3)^2$ and $(\mathbb{Z}/5)^2$ instead of $\mathbb{Z}/3$ and $\mathbb{Z}/5$, respectively.) Thus we now have a complete classification of the transcendental Brauer groups of diagonal quartic surfaces with coefficients in \mathbb{Q} over \mathbb{Q} and $\mathbb{Q}(i)$. Note that the method of our paper works for all primes ℓ , so our Proposition 3.2 directly implies [IS15, Thm. 1.1], which was proved by a different method that works only for odd ℓ .

As a consequence, we show non-existence of a 2-primary Brauer–Manin obstruction to the Hasse principle alluded to in [ISZ11, p. 660].

Corollary *Let X be a diagonal quartic surface over \mathbb{Q} with coefficients a_1, a_2, a_3 such that $a_1 a_2 a_3$ is a square in \mathbb{Q} . Then $\mathrm{Br}(X) = \mathrm{Br}_1(X)$.*

We can be even more precise: carrying out calculations of step (c) we describe the structure of the extension

$$0 \longrightarrow \mathrm{Br}_1(X)/\mathrm{Br}_0(X) \longrightarrow \mathrm{Br}(X)/\mathrm{Br}_0(X) \longrightarrow \mathrm{Br}(X)/\mathrm{Br}_1(X) \longrightarrow 0. \quad (2)$$

Supplement to the main theorem (1) *Let X_k be the diagonal quartic surface with coefficients $(1, 2, -2)$ over $k = \mathbb{Q}(i)$. Then the exact sequence (2) for X_k over k is the extension*

$$0 \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/4 \longrightarrow (\mathbb{Z}/4)^2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

(2) *Let X be the diagonal quartic surface with coefficients $(1, 2, -2)$ or $(1, 8, -8)$ over \mathbb{Q} . Then the exact sequence (2) for X over \mathbb{Q} is the extension*

$$0 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/8 \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Combined with Bright's classification of algebraic Brauer groups of diagonal quartic surfaces [Bri02], this completes the classification of the Brauer groups of diagonal quartic surfaces with coefficients in \mathbb{Q} over the ground fields \mathbb{Q} and $\mathbb{Q}(i)$.

The supplement illustrates a difficulty in lifting Galois-invariant elements of $\mathrm{Br}(\overline{X})[2]$ to $\mathrm{Br}(X)$ already apparent in work of Ieronymou [Ier10]: an element in the image of $\mathrm{Br}(X) \rightarrow \mathrm{Br}(\overline{X})$ may not lift to an element of $\mathrm{Br}(X)$ of the same order. This difficulty does not arise for odd order elements because $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ is always a 2-group, as Γ acts on $\mathrm{Pic}(\overline{X})$ via a finite 2-group.

The 2-primary torsion subgroup of $\mathrm{Br}(\overline{X})$ contains a unique non-zero element invariant under the group of diagonal automorphisms of \overline{X} . This element is Galois-invariant (Proposition 3.6, Corollary 3.11). Our calculations in step (b) show that the map $\mathrm{Br}(X) \rightarrow \mathrm{Br}(\overline{X})$ is almost always zero on the 2-primary torsion subgroups, but when $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is non-zero in the exceptional cases above, it is generated by this element. This answers Question 1 raised at the end of [Sko17].

The first named author has successfully applied the methods developed in this paper to diagonal surfaces of degrees 5 and 6. These methods also work for diagonal surfaces in weighted projective spaces, see [GLN, Gvi19].

Let us outline the structure of the paper. In the preliminary Section 1 we develop necessary cohomological machinery and prove a finiteness result for the Brauer group of diagonal surfaces over fields finitely generated over \mathbb{Q} . In Section 2 we recall the description of the integral primitive cohomology of the complex Fermat surface of degree d , including the cup-product and Hodge decomposition, and explain how to recover the full cohomology lattice from the primitive cohomology sublattice. We also recall a known description of the associated representation of the absolute

Galois group of the cyclotomic field $\mathbb{Q}(\mu_d)$. In Section 3 we specialise to the case $d = 4$ and compute the transcendental lattice of the quartic Fermat surface F and the representation of the absolute Galois group of \mathbb{Q} in the ℓ -adic Tate module of $\text{Br}(\overline{F})$. This allows us to carry out step (a). Although our general methods do not rely on the assumption that the geometric Picard group of the Fermat surface is integrally generated by lines (which, by a theorem of Degtyarev [Deg15], holds if and only if $d \leq 4$ or d is coprime to 6; for the case $d = 4$ see [PSD91]), our calculations in step (b) become simpler if we use this fact. We prove the main theorem in Section 4 (see Theorems 4.2 and 4.6); details of computations are collected in Appendix A. The computational proof of the supplement can be found in Appendix B.

This paper owes its existence to many happy discussions of the second named author with Peter Swinnerton-Dyer, and to Peter's lifelong interest in diagonal quartics. We dedicate this paper to his memory.

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1 Surfaces with torsion-free Picard group

1.1 Cohomological tools

We start by recalling some basic homological algebra from [CTS13, §3.1]. Let \mathcal{A}, \mathcal{B} be abelian categories and let $G: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{3}$$

be an exact sequence in \mathcal{A} . Using injective resolutions one constructs a 2-term complex $\mathcal{E} = [E^0 \rightarrow E^1]$ in \mathcal{B} together with maps of complexes

$$(\tau_{[1,2]}G(B^\bullet)) [1] \longleftarrow \mathcal{E} \longrightarrow [(R^1G)C \rightarrow (R^2G)A], \tag{4}$$

where B^\bullet is an injective resolution of B and $\tau_{[1,2]}$ is the natural truncation. (See the diagram in [CTS13], the proof of Lemma 3.2.) It satisfies the following properties.

(1) The leftward arrow induces an isomorphism of cohomology groups in degree 0. In degree 1 it induces the natural map from $\text{Coker}[(R^1G)C \rightarrow (R^2G)A]$ to $(R^2G)B$.

(2) The rightward arrow induces an isomorphism in degree 1. In degree 0 it induces the natural map from $(R^1G)B$ to $\text{Ker}[(R^1G)C \rightarrow (R^2G)A]$.

Let k be a field with separable closure \bar{k} and Galois group $\Gamma = \text{Gal}(\bar{k}/k)$. Let $p: X \rightarrow \text{Spec}(k)$ be a variety over k . We write $\overline{X} = X \times_k \bar{k}$. The structure morphism

p defines the derived functor $\mathbf{R}p_*$ from the derived category $\mathcal{D}(X)$ of bounded below complexes of étale sheaves of abelian groups on X to the derived category $\mathcal{D}(k)$ of bounded below complexes of continuous discrete Γ -modules.

Lemma 1.1 *Let $p: X \rightarrow \mathrm{Spec}(k)$ be a proper, geometrically reduced and geometrically connected variety over a field k . There is a natural map*

$$\mathrm{Br}(X)/\mathrm{Br}_0(X) \longrightarrow \mathbb{H}^2(k, \tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X}),$$

which is an isomorphism when k is a number field.

Proof. The assumption on X implies $\mathrm{H}^0(\overline{X}, \mathcal{O}_X) \cong \overline{k}$, hence $\mathrm{H}^0(\overline{X}, \mathbb{G}_m) \cong \overline{k}^*$. The derived functor of $\mathrm{H}^0(k, \cdot)$ is a functor from $\mathcal{D}(k)$ to the derived category of bounded below complexes of abelian groups. Applying it to the exact triangle

$$\tau_{[0]}\mathbf{R}p_*\mathbb{G}_{m,X} \longrightarrow \tau_{[0,2]}\mathbf{R}p_*\mathbb{G}_{m,X} \longrightarrow \tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X}$$

gives an exact sequence

$$\mathrm{Br}(k) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathbb{H}^2(k, \tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X}) \longrightarrow \mathrm{H}^3(k, \overline{k}^\times).$$

If k is a number field, then $\mathrm{H}^3(k, \overline{k}^\times) = 0$. \square

We now assume that k has characteristic zero and that X is a smooth, proper and geometrically integral variety over k such that $\mathrm{Pic}(\overline{X})$ is torsion-free. Let (3) be the Kummer exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1,$$

where $n \geq 1$. This sequence shows that $\mathrm{H}^1(\overline{X}, \mu_n) = 0$ for any $n \geq 1$, thus in this case the rightward arrow in (4) is a quasi-isomorphism. In the derived category $\mathcal{D}(k)$ we obtain a morphism

$$[\mathrm{Pic}(\overline{X}) \rightarrow \mathrm{H}^2(\overline{X}, \mu_n)] \longrightarrow (\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X})[1]$$

inducing the isomorphism $n\mathrm{Pic}(\overline{X}) \xrightarrow{\sim} \mathrm{Pic}(\overline{X})$, $nx \mapsto x$, in degree 0 and the natural inclusion $\mathrm{Br}(\overline{X})[n] \rightarrow \mathrm{Br}(\overline{X})$ in degree 1. If ℓ is a prime and m is a positive integer, then setting $n = \ell^m$ and passing to the inductive limit as $m \rightarrow \infty$ we obtain a morphism

$$[\mathrm{Pic}(\overline{X}) \otimes \mathbb{Z}[\ell^{-1}] \rightarrow \mathrm{H}^2(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))] \longrightarrow (\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X})[1].$$

In degree 0 it induces the identity on $\mathrm{Pic}(\overline{X})$ and in degree 1 the natural inclusion $\mathrm{Br}(\overline{X})\{\ell\} \hookrightarrow \mathrm{Br}(\overline{X})$, where $\mathrm{Br}(\overline{X})\{\ell\}$ is the ℓ -primary torsion subgroup, see [Gro68,

II, Thm. 3.1]. Since X is smooth, $\text{Br}(\overline{X})$ is a torsion group [Gro68, II, Prop. 1.4]. Thus summing over all primes ℓ we obtain a quasi-isomorphism

$$[\text{Pic}(\overline{X}) \otimes \mathbb{Q} \rightarrow \text{H}^2(\overline{X}, \mathbb{Q}/\mathbb{Z}(1))] \xrightarrow{\sim} (\tau_{[1,2]} \mathbf{R}p_* \mathbb{G}_{m,X})[1]. \quad (5)$$

Let X be a smooth, projective, geometrically integral *surface* over a field $k \subset \mathbb{C}$ such that $\text{Pic}(X_{\mathbb{C}})$ is torsion-free. Then $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ is a torsion-free abelian group and the first Chern class identifies $\text{Pic}(X_{\mathbb{C}})$ with a saturated subgroup of $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ (that is, the quotient is torsion-free). The cup-product defines a unimodular symmetric bilinear pairing

$$\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \times \text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \longrightarrow \text{H}^4(X_{\mathbb{C}}, \mathbb{Z}(2)) = \mathbb{Z}.$$

Since $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ is torsion-free, the cup-product induces an isomorphism

$$\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \xrightarrow{\sim} \text{Hom}(\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)), \mathbb{Z}).$$

The Hodge index theorem implies that the restriction of the cup-product to $\text{Pic}(X_{\mathbb{C}})$, which coincides with the intersection pairing on $\text{Pic}(X_{\mathbb{C}})$, has non-zero discriminant. Define the *transcendental lattice* $T(X_{\mathbb{C}})$ as the orthogonal complement to $\text{Pic}(X_{\mathbb{C}})$ in $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ with respect to the cup-product. Thus $\text{Pic}(X_{\mathbb{C}}) \cap T(X_{\mathbb{C}}) = 0$, and $\text{Pic}(X_{\mathbb{C}}) \oplus T(X_{\mathbb{C}})$ has finite index in $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$. Since $\text{Pic}(X_{\mathbb{C}})$ is a saturated subgroup of $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$, it is the orthogonal complement to $T(X_{\mathbb{C}})$. Using the unimodularity of the cup-product on $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ we obtain canonical isomorphisms of finite abelian groups

$$\frac{\text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z})}{\text{Pic}(X_{\mathbb{C}})} \cong \frac{\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1))}{\text{Pic}(X_{\mathbb{C}}) \oplus T(X_{\mathbb{C}})} \cong \frac{\text{Hom}(T(X_{\mathbb{C}}), \mathbb{Z})}{T(X_{\mathbb{C}})}.$$

Let us call this group Δ_X , or simply Δ , if there is no confusion. We write $\Delta = \bigoplus_{\ell} \Delta_{\ell}$, where the order of Δ_{ℓ} is a power of a prime number ℓ . We deduce an exact sequence of abelian groups

$$0 \longrightarrow \Delta_{\ell} \longrightarrow (\text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}) \oplus (T(X_{\mathbb{C}}) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}) \longrightarrow \text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} \longrightarrow 0.$$

The group of connected components of the Picard scheme of $X \times_k K$, where K is an algebraically closed field extension of k , does not depend on the choice of K . Thus, under the assumption that $\text{Pic}(X_{\mathbb{C}})$ is torsion-free, the groups $\text{Pic}(\overline{X})$ and $\text{Pic}(X_{\mathbb{C}})$ are canonically isomorphic. By the comparison theorem between classical and étale cohomology we have an isomorphism $\text{H}^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Z}_{\ell} \cong \text{H}^2(\overline{X}, \mathbb{Z}_{\ell}(1))$, compatible with the cycle class map and the cup-product. Thus $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_{\ell} \cong T(\overline{X})_{\ell}$, where $T(\overline{X})_{\ell} \subset \text{H}^2(\overline{X}, \mathbb{Z}_{\ell}(1))$ is the orthogonal complement to the (injective) image of $\text{Pic}(\overline{X})$ under the class map, and the previous exact sequence is canonically identified with the exact sequence of Γ -modules

$$0 \longrightarrow \Delta_{\ell} \longrightarrow (\text{Pic}(\overline{X}) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}) \oplus (T(\overline{X})_{\ell} \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}) \longrightarrow \text{H}^2(\overline{X}, \mathbb{Z}_{\ell}(1)) \otimes \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell} \longrightarrow 0.$$

This gives rise to the commutative diagram of Γ -modules

$$\begin{array}{ccc} \Delta_\ell \hookrightarrow & \longrightarrow & T(\overline{X})_\ell \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \\ \downarrow & & \downarrow [-1] \\ \text{Pic}(\overline{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \hookrightarrow & \longrightarrow & H^2(\overline{X}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \end{array}$$

Summing over all primes ℓ we obtain a commutative diagram of Γ -modules

$$\begin{array}{ccc} \Delta \hookrightarrow & \longrightarrow & T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow [-1] \\ \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow & \longrightarrow & H^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Q}/\mathbb{Z} \end{array}$$

Using the non-degeneracy of the intersection pairing on $\text{Pic}(\overline{X})$ we identify the Γ -modules $\text{Pic}(\overline{X}) \otimes \mathbb{Q}$ and $\text{Hom}(\text{Pic}(\overline{X}), \mathbb{Q})$. This gives rise to the commutative diagram of Γ -modules

$$\begin{array}{ccc} \text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z}) & \longrightarrow & \Delta \\ \downarrow & & \downarrow \\ \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q} & \longrightarrow & \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \end{array}$$

Finally, we compose horizontal arrows in the two last diagrams to obtain a commutative diagram of Γ -modules

$$\begin{array}{ccc} \text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z}) & \longrightarrow & T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \\ \downarrow & & \downarrow [-1] \\ \text{Pic}(X_{\mathbb{C}}) \otimes \mathbb{Q} & \longrightarrow & H^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Q}/\mathbb{Z} \end{array} \quad (6)$$

Since X is a surface, by Poincaré duality $H^2(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}} = 0$ implies $H^3(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}} = 0$, for all ℓ . By the Kummer sequence this implies that $\text{Br}(\overline{X}) \cong \text{Br}(X_{\mathbb{C}})$ is divisible and the ℓ -primary torsion subgroup $\text{Br}(\overline{X})\{\ell\}$ is the quotient of $H^2(\overline{X}, \mathbb{Z}_\ell(1)) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$ by $\text{Pic}(\overline{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell$, see [CTS13], diagram (26), p. 157. Taking the direct sum over all primes ℓ we obtain a canonical isomorphism of Γ -modules

$$\text{Hom}_{\mathbb{Z}}(T(X_{\mathbb{C}}), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \text{Br}(\overline{X}). \quad (7)$$

For future use we point out that (7) gives rise to an isomorphism of Γ -modules

$$\text{Hom}_{\mathbb{Z}}(T(X_{\mathbb{C}}), \mathbb{Z}/\ell^m) \xrightarrow{\sim} \text{Br}(\overline{X})[\ell^m], \quad (8)$$

for any prime ℓ and any positive integer m .

Proposition 1.2 *Let X be a smooth, projective, geometrically integral surface over a subfield $k \subset \mathbb{C}$ such that $\text{Pic}(X_{\mathbb{C}})$ is torsion-free. Then the map of horizontal 2-term complexes of Γ -modules in (6) is a quasi-isomorphism. Each of these complexes concentrated in degrees 0 and 1 represents $(\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X})[1]$ in $\mathcal{D}(k)$.*

Proof. The first statement follows from the construction of (6). Indeed, the induced map in degree 0 is the identity on $\text{Pic}(X_{\mathbb{C}}) = \text{Pic}(\overline{X})$. The induced map in degree 1 is the canonical isomorphism (7).

Since $H^3(X_{\mathbb{C}}, \mathbb{Z})_{\text{tors}} = 0$, the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

gives a natural isomorphism

$$H^2(X_{\mathbb{C}}, \mathbb{Z}(1)) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\sim} H^2(X_{\mathbb{C}}, \mathbb{Q}/\mathbb{Z}(1)).$$

Now the quasi-isomorphism (5) gives that the bottom complex of (6) is quasi-isomorphic to $(\tau_{[1,2]}\mathbf{R}p_*\mathbb{G}_{m,X})[1]$. \square

Remark 1.3 For completeness we note that $\tau_{[0,1]}\mathbf{R}p_*\mathbb{G}_{m,X}$ is represented by the complex of Γ -modules $\text{div}: \bar{k}(X)^{\times} \rightarrow \text{Div}(\overline{X})$, where div sends a function to its divisor, see [BvH09, Lemma 2.3].

Corollary 1.4 *Let X be a smooth, projective, geometrically integral surface over a number field k such that $\text{Pic}(\overline{X})$ is torsion-free. Then the group $\text{Br}(X)/\text{Br}_0(X)$ is canonically isomorphic to the Galois hypercohomology group*

$$\mathbb{H}^1(k, [\text{Hom}(\text{Pic}(\overline{X}), \mathbb{Z}) \rightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z}]).$$

Proof. This follows from Proposition 1.2 in view of Lemma 1.1. \square

Recall that we have exact sequences of Γ -modules

$$0 \longrightarrow \Delta \longrightarrow T(X_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \longrightarrow \text{Br}(\overline{X}) \longrightarrow 0 \quad (9)$$

and

$$0 \longrightarrow \text{Pic}(\overline{X}) \longrightarrow \text{Hom}_{\mathbb{Z}}(\text{Pic}(\overline{X}), \mathbb{Z}) \longrightarrow \Delta \longrightarrow 0. \quad (10)$$

Corollary 1.5 *Let X be a smooth, projective, geometrically integral surface over a number field k such that $\text{Pic}(\overline{X})$ is torsion-free. Then the image of the natural map $\text{Br}(X) \rightarrow \text{Br}(\overline{X})$ is equal to the kernel of the composition*

$$\text{Br}(\overline{X})^{\Gamma} \xrightarrow{\partial_1} H^1(k, \Delta) \xrightarrow{\partial_2} H^2(k, \text{Pic}(\overline{X})),$$

where ∂_1 is defined by (9) and ∂_2 is defined by (10).

Proof. This follows from Proposition 1.2 using that $H^3(k, \bar{k}^\times) = 0$ for a number field k . See also [CTS13, Prop. 4.1]. \square

Corollary 1.6 *Let X be a smooth, projective, geometrically integral surface over a number field k such that $\text{Pic}(\bar{X})$ is torsion-free. Suppose that the image of the natural map $\text{Br}(X) \rightarrow \text{Br}(\bar{X})^\Gamma$ is contained in $\text{Br}(\bar{X})[n]$ for some $n \geq 1$. Let $T_n \subset T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z}$ be the inverse image of $\text{Br}(\bar{X})[n]$ in (9). Let Γ_K be the kernel of the natural map*

$$\Gamma \longrightarrow \text{Aut}(\text{Pic}(\bar{X})) \times \text{Aut}(T_n),$$

and let $K = (\bar{k})^{\Gamma_K}$, $G = \text{Gal}(K/k)$. Then we have the following statements.

(a) *The image of the natural map $\text{Br}(X) \rightarrow \text{Br}(\bar{X})^\Gamma$ is equal to the kernel of the composition*

$$\text{Br}(\bar{X})[n]^G \xrightarrow{\partial_1} H^1(G, \Delta) \xrightarrow{\partial_2} H^2(G, \text{Pic}(\bar{X})).$$

(b) *The group $\text{Br}(X)/\text{Br}_0(X)$ is canonically isomorphic to the hypercohomology group*

$$\mathbb{H}^1(G, [\text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z}) \rightarrow T_n]). \quad (11)$$

Proof. (a) It is clear that the action of Γ on the terms of (10) and on the terms of

$$0 \longrightarrow \Delta \longrightarrow T_n \longrightarrow \text{Br}(\bar{X})[n] \longrightarrow 0$$

factors through G . We have $\text{Pic}(\bar{X})^{\Gamma_K} = \text{Pic}(\bar{X})$, which is a finitely generated free abelian group, hence $H^1(K, \text{Pic}(\bar{X})) = 0$. From the Hochschild–Serre spectral sequence we deduce the injectivity of the inflation map

$$H^2(G, \text{Pic}(\bar{X})) \longrightarrow H^2(k, \text{Pic}(\bar{X})).$$

Now (a) follows from Corollary 1.5.

(b) Let \mathcal{K} be the complex $\text{Hom}(\text{Pic}(\bar{X}), \mathbb{Z}) \rightarrow T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z}$ and let \mathcal{K}_n be the complex $\text{Hom}(\text{Pic}(X_{\mathbb{C}}), \mathbb{Z}) \rightarrow T_n$. Using Proposition 1.2 we obtain a commutative diagram, whose lower part is constructed using inflation maps:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(k, \text{Pic}(\bar{X})) & \longrightarrow & \mathbb{H}^1(k, \mathcal{K}) & \longrightarrow & \text{Br}(\bar{X})^\Gamma & \longrightarrow & H^2(k, \text{Pic}(\bar{X})) \\ & & \cong \uparrow & & \uparrow & & \downarrow & & \cong \uparrow \\ 0 & \longrightarrow & H^1(k, \text{Pic}(\bar{X})) & \longrightarrow & \mathbb{H}^1(k, \mathcal{K}_n) & \longrightarrow & \text{Br}(\bar{X})[n]^\Gamma & \longrightarrow & H^2(k, \text{Pic}(\bar{X})) \\ & & \cong \uparrow & & \uparrow & & \cong \uparrow & & \downarrow \\ 0 & \longrightarrow & H^1(G, \text{Pic}(\bar{X})) & \longrightarrow & \mathbb{H}^1(G, \mathcal{K}_n) & \longrightarrow & \text{Br}(\bar{X})[n]^G & \longrightarrow & H^2(G, \text{Pic}(\bar{X})) \end{array}$$

A diagram chase gives canonical isomorphisms

$$\mathbb{H}^1(G, \mathcal{K}_n) \xrightarrow{\sim} \mathbb{H}^1(k, \mathcal{K}_n) \xrightarrow{\sim} \mathbb{H}^1(k, \mathcal{K}) \xrightarrow{\sim} \text{Br}(X)/\text{Br}_0(X),$$

the last one given by Corollary 1.4. \square

1.2 Brauer group of varieties dominated by products of curves

A smooth, projective and geometrically integral variety X over a field k is called a variety *dominated by a product of curves* if there is a dominant rational map from a product of geometrically integral \bar{k} -curves to \bar{X} .

Proposition 1.7 *Let k be a field finitely generated over \mathbb{Q} . If X is a smooth, projective and geometrically integral variety over k dominated by a product of curves, then $\mathrm{Br}(\bar{X})^\Gamma$ is finite.*

Proof. If V and W are smooth, projective and geometrically integral varieties over a field k which is finitely generated over \mathbb{Q} , then the cokernel of the natural map $\mathrm{Br}(\bar{V})^\Gamma \oplus \mathrm{Br}(\bar{W})^\Gamma \rightarrow \mathrm{Br}(\bar{V} \times \bar{W})^\Gamma$ is finite by [SZ14, Thm. A]. The Brauer group of a curve over an algebraically closed field is zero [Gro68, III, Cor. 1.2], [CTS21, Thm. 5.6.1 (ii)]. Thus if Z is a product of smooth, projective and geometrically integral curves over k , then $\mathrm{Br}(\bar{Z})^\Gamma$ is finite.

Replacing k by a finite field extension we obtain geometrically integral k -curves C_1, \dots, C_n such that there is a dominant rational map f from $S = \prod_{i=1}^n C_i$ to X defined over k . Let $d = \dim(X)$. There is a dense open subset $U \subset S$ such that the restriction of f to U is a smooth morphism $U \rightarrow X$. After another finite extension of k we can find a k -point $P \in U(k)$. The induced map of tangent spaces $f_*: T_{S,P} \rightarrow T_{X,f(P)}$ is surjective, so we can choose a d -element subset $I \subset \{1, \dots, n\}$ such that $f_*(T_{S_I,P}) = T_{X,f(P)}$, where $S_I \cong \prod_{i \in I} C_i$ is the k -fibre of the projection $S \rightarrow \prod_{i \notin I} C_i$ which contains P . Thus f restricts to a dominant, generically finite rational map from a product of d geometrically integral k -curves to X .

Since $\mathrm{char}(k) = 0$, we can assume that there is a smooth, projective and geometrically integral variety Y over k , a birational morphism $Y \rightarrow Z$, where Z is a product of smooth, projective and geometrically integral curves, and a dominant, generically finite morphism $f: Y \rightarrow X$. By the birational invariance of the Brauer group [Gro68, III, §7], [CTS21, Cor. 6.2.11] we have $\mathrm{Br}(\bar{Y})^\Gamma \cong \mathrm{Br}(\bar{Z})^\Gamma$. A general theorem of Grothendieck says that the natural map $\mathrm{Br}(\bar{X}) \hookrightarrow \mathrm{Br}(\bar{k}(X))$ is injective, see [Gro68, III, Cor. 6.2], [CTS21, Thm. 3.5.5]. The standard restriction-corestriction argument then gives that the kernel of $f^*: \mathrm{Br}(\bar{X}) \rightarrow \mathrm{Br}(\bar{Y})$ is killed by the degree $[\bar{k}(Y) : \bar{k}(X)]$. Since $\mathrm{Br}(\bar{X})$ is a torsion group of cofinite type [Gro68, II, Cor. 3.4], [CTS21, Prop. 5.2.9], this kernel is finite. Hence $\mathrm{Br}(\bar{X})^\Gamma$ is finite. \square

Corollary 1.8 *Let k be a field finitely generated over \mathbb{Q} . Let $f(t)$ and $g(t)$ be separable polynomials of degree $d \geq 2$. Let $F(x, y)$ and $G(x, y)$ be homogeneous forms of degree d such that $f(t) = F(t, 1)$ and $g(t) = G(t, 1)$. Let $X \subset \mathbb{P}_k^3$ be the surface with equation $F(x, y) = G(z, w)$, for example, a diagonal surface. Then the Brauer group $\mathrm{Br}(X)$ is finite modulo $\mathrm{Br}(k)$.*

Proof. Consider the smooth plane curves C and D given by $z^d = F(x, y)$ and $u^d = G(v, w)$, respectively. Passing to the quotient by the action of μ_d on $C \times D$, which multiplies z and u by the same root of unity, gives a dominant rational map from $C \times D$ to X . Thus X is a DPC surface. Since $\text{Pic}(\overline{X})$ is torsion-free [SGA7, XI.1.8], the group $\text{Br}_1(X)/\text{Br}_0(X)$ is finite. The finiteness of $\text{Br}(\overline{X})^\Gamma$ follows from Proposition 1.7. \square

2 Cohomology of Fermat surfaces

2.1 Topological background

Let X be a smooth irreducible subvariety of $\mathbb{P}_{\mathbb{C}}^{n+1}$ of dimension n . Let $Z \subset X$ be a smooth hyperplane section and let $U = X \setminus Z$. The long exact sequence of relative homology groups and the long exact sequence of cohomology groups with compact support form the following commutative diagram, where vertical arrows are the Poincaré duality isomorphisms (see, e.g., the diagram in the proof of [Hat02, Prop. 3.46]):

$$\begin{array}{ccccccc}
 \mathrm{H}_{n+1}(X, U, \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(U, \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(X, \mathbb{Z}) & \longrightarrow & \mathrm{H}_n(X, U, \mathbb{Z}) \\
 \cong \uparrow & & \cong \uparrow & & \cong \uparrow & & \cong \uparrow \\
 \mathrm{H}^{n-1}(Z, \mathbb{Z}) & \longrightarrow & \mathrm{H}_c^n(U, \mathbb{Z}) & \longrightarrow & \mathrm{H}^n(X, \mathbb{Z}) & \longrightarrow & \mathrm{H}^n(Z, \mathbb{Z})
 \end{array} \tag{12}$$

Next, we have a commutative diagram, where the maps h are defined by evaluating cochains on chains and the isomorphisms are dual to the Poincaré duality isomorphisms:

$$\begin{array}{ccccc}
 \mathrm{H}^n(U, \mathbb{Z}) & \xrightarrow{h} & \mathrm{Hom}(\mathrm{H}_n(U, \mathbb{Z}), \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{H}_c^n(U, \mathbb{Z}), \mathbb{Z}) \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathrm{H}^n(X, \mathbb{Z}) & \xrightarrow{h} & \mathrm{Hom}(\mathrm{H}_n(X, \mathbb{Z}), \mathbb{Z}) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{H}^n(X, \mathbb{Z}), \mathbb{Z})
 \end{array}$$

The commutativity of this diagram implies compatibility of the cup-product pairings

$$\begin{array}{ccc}
 \mathrm{H}^n(U, \mathbb{Z}) \times \mathrm{H}_c^n(U, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\
 \uparrow & & \downarrow \quad \parallel \\
 \mathrm{H}^n(X, \mathbb{Z}) \times \mathrm{H}^n(X, \mathbb{Z}) & \longrightarrow & \mathbb{Z}
 \end{array}$$

Modulo torsion subgroups, these pairings are perfect dualities, cf. [Hat02, Prop. 3.38]. The natural map $\mathrm{H}_c^n(U, \mathbb{Z}) \rightarrow \mathrm{H}^n(U, \mathbb{Z})$ gives rise to a bilinear form on $\mathrm{H}_c^n(U, \mathbb{Z})$. The Poincaré duality isomorphism $\mathrm{H}_c^n(U, \mathbb{Z}) \cong \mathrm{H}_n(U, \mathbb{Z})$ identifies this form with the intersection products of homology classes, see [GH78, pp. 43–60].

2.2 Primitive cohomology and the cup-product

Let $F \subset \mathbb{P}_{\mathbb{Q}}^3$ be the Fermat surface

$$x_0^d + x_1^d + x_2^d + x_3^d = 0.$$

The abelian group $H = H^2(F_{\mathbb{C}}, \mathbb{Z}(1))$ is finitely generated and free, so the cup-product defines a unimodular symmetric bilinear form $\langle x, y \rangle$ on H . Let $L \in H$ be the hyperplane section class. Let $P = P^2(F_{\mathbb{C}}, \mathbb{Z}(1)) \subset H$ be the orthogonal complement to L with respect to the cup-product pairing. Thus P is the kernel of the map $H \rightarrow \mathbb{Z}$ sending $x \in H$ to $\langle x, L \rangle$, so P is a saturated subgroup of H . Since $\langle L, L \rangle = d > 0$, the restriction of the cup-product pairing $H \times H \rightarrow \mathbb{Z}$ to P is non-degenerate.

Let $\Lambda \in H$ be the class of a line contained in $F_{\mathbb{C}}$. Since $\langle L, \Lambda \rangle = 1$, we see that L is a primitive element of H . (This holds for arbitrary smooth complete intersection surfaces in a projective space, see [SGA7, XI.1.8].) Our map $H \rightarrow \mathbb{Z}$ sends Λ to 1, so we have a direct sum decomposition of abelian groups $H = \mathbb{Z}\Lambda \oplus P$.

Let G be the group \mathbb{Q} -scheme $(\mu_d)^3$. We write G for $G(\overline{\mathbb{Q}})$. The choice of a primitive d -th root ε of 1 defines an isomorphism of groups $G \simeq (\mathbb{Z}/d)^3$ so that the elements of G are monomials $u_1^a u_2^b u_3^c$, where

$$u_1 = (\varepsilon, 1, 1), \quad u_2 = (1, \varepsilon, 1), \quad u_3 = (1, 1, \varepsilon),$$

and $(a, b, c) \in (\mathbb{Z}/d)^3$. The group G acts on $F_{\mathbb{C}}$ by automorphisms so that u_i multiplies x_i by ε and leaves the other coordinates unaltered.

Let $U \subset F$ be the complement to the smooth hyperplane section $Z \subset F$ given by $x_0 = 0$. We are in the situation of the previous section, with $X = F_{\mathbb{C}}$. It is clear that the group $G \simeq (\mathbb{Z}/d)^3$ acts on all the terms of diagram (12) so that the rows are exact sequences of G -modules. The Poincaré duality isomorphisms are defined by taking the cap-product with the fundamental class in $H_4(F_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$, given by the orientation of $F_{\mathbb{C}}$. The fundamental class is fixed by the action of G , hence the Poincaré duality isomorphisms are isomorphisms of $\mathbb{Z}[G]$ -modules.

From (12) we thus obtain an exact sequence of G -modules

$$0 \longrightarrow H_1(Z_{\mathbb{C}}, \mathbb{Z}) \longrightarrow H_2(U_{\mathbb{C}}, \mathbb{Z}) \longrightarrow H_2(F_{\mathbb{C}}, \mathbb{Z}) \longrightarrow H_0(Z_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}, \quad (13)$$

where we used that $H_1(F_{\mathbb{C}}, \mathbb{Z}) = 0$. If we ignore the twists, then the last arrow is the map $H \rightarrow \mathbb{Z}$ sending $x \in H$ to $\langle x, L \rangle$, so its kernel is P . Thus (13) gives rise to an exact sequence of G -modules

$$0 \longrightarrow H_1(Z_{\mathbb{C}}, \mathbb{Z}) \longrightarrow H_2(U_{\mathbb{C}}, \mathbb{Z}) \longrightarrow P \longrightarrow 0, \quad (14)$$

where we ignore the twists. (The twist of P plays no role in the description of the cup-product pairing and the action of G on P , but will be essential for the determination of the action of complex conjugation, see Remark 2.3.)

Since the cup-product pairing on P is non-degenerate, the kernel of the intersection pairing on $H_2(U_{\mathbb{C}}, \mathbb{Z})$ is the image of $H_1(Z_{\mathbb{C}}, \mathbb{Z})$.

Let us recall a description of the G -module $H_2(U_{\mathbb{C}}, \mathbb{Z})$ together with the intersection pairing on it, obtained by Pham in [Pha65, Thm. 1]. The affine Fermat surface $U \subset \mathbb{A}_k^3$ is given by

$$x_1^d + x_2^d + x_3^d = -1.$$

Let $S \subset \mathbb{R}^3$ be the standard 2-dimensional simplex $z_1 + z_2 + z_3 = 1$, $z_i \geq 0$, for $i = 1, 2, 3$. Consider the map $\mathbf{e} : S \rightarrow U_{\mathbb{C}}$ given by

$$\mathbf{e}(z_1, z_2, z_3) = \left(\zeta_{2d} z_1^{1/d}, \zeta_{2d} z_2^{1/d}, \zeta_{2d} z_3^{1/d} \right),$$

where ζ_{2d} is a primitive root of unity of degree $2d$ such that $\zeta_{2d}^2 = \varepsilon$. Then

$$e = (1 - u_1^{-1})(1 - u_2^{-1})(1 - u_3^{-1})\mathbf{e} \in H_2(U_{\mathbb{C}}, \mathbb{Z})$$

is a cycle. Pham shows that the topological space of the simplicial complex generated by $g(\mathbf{e})$, for $g \in G$, is a G -equivariant deformation retract of $U_{\mathbb{C}}$. The resulting chain complex gives isomorphisms of $\mathbb{Z}[G]$ -modules

$$H_2(U_{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}[G]e \cong \mathbb{Z}[u_1, u_1, u_3]/(\phi(u_1), \phi(u_2), \phi(u_3)), \quad (15)$$

where $\phi(x)$ is defined as follows:

$$\phi(x) = 1 + x + x^2 + \cdots + x^{d-1} \in \mathbb{Z}[x], \quad \rho(x, y) = \sum_{0 \leq i \leq j \leq d-2} y^i x^j \in \mathbb{Z}[x, y]. \quad (16)$$

Consider the ring $R_x := \mathbb{Z}[x]/(\phi(x)) \simeq \mathbb{Z}^{d-1}$. It is immediate to check that in R_x we have $(1-x)\rho(1, x) = d$, so $d(1-x)^{-1} \in R_x$. In this notation, Pham's isomorphism (15) is $H_2(U_{\mathbb{C}}, \mathbb{Z}) \cong R_{u_1} \otimes R_{u_2} \otimes R_{u_3}$.

Lemma 2.1 *We have an isomorphism of $\mathbb{Z}[G]$ -modules*

$$P \cong \mathbb{Z}[u_1, u_1, u_3]/I, \quad \text{where } I = (\phi(u_1), \phi(u_2), \phi(u_3), \phi(u_1 u_2 u_3)).$$

Proof. In [Pha65, p. 337] Pham defines a $\mathbb{Z}[G]$ -valued form $(x|y)$ on $H_2(U_{\mathbb{C}}, \mathbb{Z})$, as follows:

$$(x|y) = \sum_{g \in G} \langle x, gy \rangle g^{-1}.$$

This form is $\mathbb{Z}[G]$ -linear in y and $\mathbb{Z}[G]$ -antilinear in x , with respect to the involution of the ring $\mathbb{Z}[G]$ that sends g to g^{-1} . Then he computes [Pha65, p. 340]

$$(e|e) = -(1 - u_1)(1 - u_2)(1 - u_3)(1 - (u_1 u_2 u_3)^{-1}). \quad (17)$$

(Note that the right-hand side of (17) is invariant under the involution of $\mathbb{Z}[G]$.) Since x is invertible in R_x , and $1 - x$ becomes invertible in $R_x \otimes \mathbb{Q}$, we deduce that the kernel of the intersection product on $H_2(U_{\mathbb{C}}, \mathbb{Z})$ is the annihilator of $1 - u_1 u_2 u_3$ in the ring $R_{u_1} \otimes R_{u_2} \otimes R_{u_3}$. We need to prove that this ideal is generated by $\phi(u_1 u_2 u_3)$.

This can be rephrased in terms of group cohomology of the cyclic group C_d . Namely, we have an exact sequence of C_d -modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[C_d] \longrightarrow R_x \longrightarrow 0,$$

where \mathbb{Z} is a trivial C_d -module. Multiplication by $u_1 u_2 u_3$ on $R_{u_1} \otimes R_{u_2} \otimes R_{u_3}$ makes it a C_d -module isomorphic to $(\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes 3}$. The annihilator ideal of $1 - u_1 u_2 u_3$ is the C_d -invariant subgroup of $(\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes 3}$, whereas $\phi(u_1 u_2 u_3)$ is the norm element of the group ring $\mathbb{Z}[C_d]$. Thus we need to show that the 0-th Tate cohomology group $\widehat{H}^0(C_d, (\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes 3})$ is trivial.

Taking the triple self-product of the exact complex $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}[C_d]$ we obtain the following exact sequence of C_d -modules:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}[C_d]^{\oplus 3} \longrightarrow (\mathbb{Z}[C_d]^{\otimes 2})^{\oplus 3} \longrightarrow \mathbb{Z}[C_d]^{\otimes 3} \longrightarrow (\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes 3} \longrightarrow 0.$$

The three terms in the middle are free $\mathbb{Z}[C_d]$ -modules, hence all their Tate cohomology groups are trivial. We obtain

$$\widehat{H}^0(C_d, (\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes 3}) \cong H^3(C_d, \mathbb{Z}) \cong \text{Hom}(C_d, \mathbb{Z}) = 0.$$

This finishes the proof. \square

Remark 2.2 This result is stated in [Lo10, Cor. 2.2]. The argument in the above proof gives that $H^0(C_d, (\mathbb{Z}[C_d]/\mathbb{Z})^{\otimes(n+1)})$ is zero when n is even, and is isomorphic to \mathbb{Z}/d when n is odd. This second case is missed in [Lo10, Cor. 2.2] so the statement of *loc. cit.* for odd n needs to be modified.

Remark 2.3 The complex conjugation is an involution of the real manifold $F(\mathbb{C})$. Denote by τ the induced involutions of the homology and cohomology groups of $F_{\mathbb{C}}$ and $U_{\mathbb{C}}$ with coefficients \mathbb{Z} . Then $\tau g = g^{-1} \tau$ for any $g \in G$. From $\tau(\mathbf{e}) = (u_1 u_2 u_3)^{-1} \mathbf{e}$ we deduce that $\tau(e) = -e$.

Let $e' \in H_2(F_{\mathbb{C}}, \mathbb{Z})$ be the image of $e \in H_2(U_{\mathbb{C}}, \mathbb{Z})$. Since the maps in (13) respect the action of τ , we have $\tau(e') = -e'$. The complex conjugation τ preserves the orientation of $F_{\mathbb{C}}$, because $F_{\mathbb{C}}$ has complex dimension 2, hence preserves the fundamental class, so the Poincaré duality isomorphisms respect the action of τ .

Now let us consider the twisted cohomology group $H = H^2(F_{\mathbb{C}}, \mathbb{Z}(1))$. By definition, there is an isomorphism

$$H^2(F_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{\sim} H^2(F_{\mathbb{C}}, \mathbb{Z}(1))$$

sending x to $2\pi ix$. Thus H with its natural (twisted) action of τ is a subgroup of the complex vector space $H^2(F_{\mathbb{C}}, \mathbb{C})$ where the complex conjugation is the product of τ , which is induced by the conjugation acting on $F(\mathbb{C})$, and the complex conjugation in \mathbb{C} . Let us denote by $\tau(1)$ this twisted action on H . In particular, for $2\pi ie' \in H$ we have

$$\tau(1)(2\pi ie') = 2\pi ie'. \quad (18)$$

As $2\pi ie'$ generates the $\mathbb{Z}[G]$ -module P , the isomorphism of Lemma 2.1 identifies $2\pi ie' \in P$ with the image of $1 \in \mathbb{Z}[G]$, so this isomorphism is compatible with the action of $\tau(1)$ on $\mathbb{Z}[G]$ which fixes 1 and sends each $g \in G$ to g^{-1} . (According to [Lo10, Cor. 2.2], permutations of $\{1, 2, 3\}$ act on e' by the sign character.)

The cup-product symmetric bilinear form on P has the following description. Define $\varphi: \mu_d \rightarrow \mathbb{Z}$ by $\varphi(1) = 1$, $\varphi(\varepsilon) = -1$, and $\varphi(\varepsilon^i) = 0$ for $i = 2, \dots, d-1$.

Proposition 2.4 *Under the isomorphism of G -modules $P \cong \mathbb{Z}[G]/I$, the cup-product is the unique symmetric G -invariant bilinear form $\langle x, y \rangle: P \times P \rightarrow \mathbb{Z}$ such that*

$$\langle u_1^a u_2^b u_3^c, 1 \rangle = -\varphi(\varepsilon^a)\varphi(\varepsilon^b)\varphi(\varepsilon^c) - \varphi(\varepsilon^{-a})\varphi(\varepsilon^{-b})\varphi(\varepsilon^{-c}).$$

For any $r \in \mathbb{Z}[G]$ we have $\langle rx, y \rangle = \langle x, \bar{r}y \rangle$, where $r \mapsto \bar{r}$ is the automorphism of the abelian group $\mathbb{Z}[G]$ that sends g to g^{-1} for any $g \in G$.

Proof. This is obtain by a straightforward calculation from (17). The last statement follows from G -invariance of the form. \square

We write $E = \mathbb{Q}(\mu_d)$. It is well known that $\text{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/d)^\times$.

Let $\widehat{G} = \text{Hom}(G, \mathbb{C}^\times) \cong (\mathbb{Z}/d)^3$ be the group of characters. Write $\chi(u_1^a u_2^b u_3^c) = \varepsilon^{la+mb+nc}$, where l, m, n are elements of $\{0, \dots, d-1\}$. Then

$$E[G] = E[u_1, u_2, u_3]/(u_i^d - 1) = \bigoplus_{\chi \in \widehat{G}} V_\chi$$

is a direct sum of 1-dimensional eigenspaces V_χ generated by the orthogonal idempotents

$$\alpha_\chi = \alpha_l(u_1)\alpha_m(u_2)\alpha_n(u_3) \in E[G], \quad \text{where} \quad \alpha_i(u) = \frac{1}{d} \sum_{j=0}^{d-1} \varepsilon^{-ij} u^j.$$

Indeed, one readily checks that α_χ is an eigenvector with eigenvalue χ :

$$[g]\alpha_\chi = \chi(g)\alpha_\chi \quad \text{for any } g \in G, \quad (19)$$

where we write $[g]$ for the natural action of G on $E[G]$. Hence $\alpha_\chi \alpha_\phi$ is a multiple of $\delta_{\chi, \phi}$, where δ is the Kronecker delta. Our normalisation is such that $\alpha_\chi^2 = \alpha_\chi$.

Corollary 2.5 For any $\chi, \phi \in \widehat{G}$ we have $\langle \alpha_\chi, \alpha_\phi \rangle = 0$ if $\phi \neq \chi^{-1}$, and

$$\langle \alpha_\chi, \alpha_{\chi^{-1}} \rangle = \langle \alpha_\chi^2, 1 \rangle = \langle \alpha_\chi, 1 \rangle = -\frac{2}{d^3} \operatorname{Re} \left((1 - \varepsilon^l)(1 - \varepsilon^m)(1 - \varepsilon^n) \right). \quad (20)$$

Proof. This follows directly from Proposition 2.4. \square

We note that $I_E = I \otimes E$ is the direct sum of the spaces V_χ such that χ restricts trivially to any of the three coordinates $\mu_d \subset G$ (that is, $l = 0$ or $m = 0$ or $n = 0$) or to the diagonal $\mu_d \subset G$ (that is, $l + m + n$ is divisible by d). Let us denote by S the complement to this set in \widehat{G} . In other words, S is the set of triples $(l, m, n) \in \{1, \dots, d-1\}^3$ such that $l + m + n$ is not divisible by d . Then

$$P \otimes E = \bigoplus_{\chi \in S} V_\chi,$$

so that the rank of P is $|S|$. For $d = 2, 3, 4$ we have $|S| = 1, 6, 21$, respectively.

The group $\operatorname{Gal}(E/\mathbb{Q}) \cong (\mathbb{Z}/d)^\times$ acts on $\widehat{G} = \operatorname{Hom}(G, E^\times)$ via its natural action on E , which raises ε to powers coprime to d . Note that this action preserves $S \subset \widehat{G}$. For $\sigma \in \operatorname{Gal}(E/\mathbb{Q})$ we have $\langle \alpha_\chi, \alpha_\phi \rangle = \langle \alpha_{\sigma(\chi)}, \alpha_{\sigma(\phi)} \rangle$.

2.3 Hodge decomposition

The Hodge decomposition $P \otimes \mathbb{C} = P^{1,-1} \oplus P^{0,0} \oplus P^{-1,1}$ can be described as follows. Let q be the function $S \rightarrow \{-1, 0, 1\}$ given by $q(\chi) = \lfloor \frac{l+m+n}{d} \rfloor - 1$. Note that q depends on the choice of ε .

Lemma 2.6 The Hodge subspace $P^{p,q} \subset P \otimes \mathbb{C} = P^2(F_{\mathbb{C}}, \mathbb{C}(1))$ is the direct sum of $V_\chi \otimes_E \mathbb{C}$ for $\chi \in S$ such that $q(\chi) = q$.

Proof. This is [Sh79, Thm. I]. For a proof see [ABB13, Prop. 2.7] based on [Lo10]. \square

In particular, the \mathbb{C} -vector space $P^{p,q}$ comes from the E -vector space $\bigoplus_{q(\chi)=q} V_\chi$.

Let $S_\# \subset S$ be the union of the $\operatorname{Gal}(E/\mathbb{Q})$ -orbits in S that contain an element χ with $q(\chi) \neq 0$. Since $\operatorname{Gal}(E/\mathbb{Q})$ contains complex conjugation, this condition is equivalent to the existence of χ such that $q(\chi) = 1$.

The transcendental lattice $T(F_{\mathbb{C}})$ is the smallest saturated sublattice of P such that $P^{1,-1} \subset T(F_{\mathbb{C}}) \otimes \mathbb{C}$. Thus $T(F_{\mathbb{C}}) = P \cap (T(F_{\mathbb{C}}) \otimes \mathbb{Q})$, where $T(F_{\mathbb{C}}) \otimes \mathbb{Q}$ is the smallest \mathbb{Q} -vector space V such that $\bigoplus_{q(\chi)=1} V_\chi \subset V \otimes E$. It is clear that $T(F_{\mathbb{C}}) \otimes E$ is the direct sum of V_χ for $\chi \in S_\#$.

Lemma 2.7 The action of $\tau(1)$ on the E -vector space $P \otimes E$ is anti-linear and preserves the Hodge decomposition. We have $\tau(1)(\alpha_\chi) = \alpha_\chi$ for every $\chi \in \widehat{G}$.

Proof. We have seen that the action of $\tau(1)$ on $H^2(F_{\mathbb{C}}, \mathbb{C}(1))$ is anti-linear, hence the action on $P \otimes E \subset H^2(F_{\mathbb{C}}, \mathbb{C}(1))$ is also anti-linear. The generator $2\pi i e'$ of P is fixed by $\tau(1)$, see (18). We have $\tau(1)[g] = [g^{-1}]\tau(1)$ for $g \in G$ and $\tau(1)\varepsilon = \varepsilon^{-1}\tau(1)$, thus $\tau(\alpha_\chi) = \alpha_\chi$. \square

That $\tau(1)$ is anti-linear and preserves the Hodge decomposition is a general fact. In Deligne's notation $\tau = F_\infty$, but $F_\infty(H^{p,q}) = H^{q,p}$ by [Del79, 0.2.5]. Our $\tau(1)$ is the composition of F_∞ with the complex conjugation of the coefficients \mathbb{C} , hence $\tau(1)(H^{p,q}) = H^{p,q}$, see [Del79, Prop. 1.4, Cor. 1.6].

2.4 Recovering full cohomology from primitive cohomology

As was noted in Section 2.2, we have $H = \mathbb{Z}\Lambda \oplus P$, where $\Lambda \in H$ is the class of a line in $F_{\mathbb{C}}$. Let $c = L - d\Lambda \in P$, where L is the hyperplane section class. Then H is the subgroup of $\mathbb{Q}L \oplus (P \otimes \mathbb{Q})$ generated by $\mathbb{Z}L \oplus P$ and $\frac{1}{d}(L - c)$. We have $\langle L, L \rangle = d$, and Proposition 2.4 explicitly describes the restriction of the cup-product form to P . Thus, in order to recover the full cohomology group $H = H^2(F_{\mathbb{C}}, \mathbb{Z}(1))$ as a G -module together with its unimodular G -invariant cup-product bilinear form from the G -module P and the restriction of the cup-product form to P , it is enough to find a formula for c . This can be done using calculations of intersection indices from the paper of A. Degtyarev and I. Shimada [DS16]. Recall that $\phi(x)$ and $\rho(x, y)$ were introduced in (16). It is immediate to check that in the ring $\mathbb{Z}[x, y]/(x^d - 1, y^d - 1)$ we have the identity [DS16, Lemma 4.5]

$$(1 - y)\phi(xy) = (1 - x)(1 - y)\rho(x, y).$$

Since $1 - x$ is a unit in $\mathbb{Q}[x]/(\phi(x))$, in the ring $R_x \otimes R_y$ we have $\phi(xy) = (1 - x)\rho(x, y)$.

Proposition 2.8 *Write $u_0 = (u_1 u_2 u_3)^{-1}$ so that $u_0 u_1 u_2 u_3 = 1$. Define*

$$c = (1 - u_0)^{-1} \phi(u_0 u_1) \cdot (1 - u_2)^{-1} \phi(u_2 u_3) = \rho(u_0, u_1) \rho(u_2, u_3) \in P.$$

Then $\frac{1}{d}(L - c)$ is the class of the line in $F_{\mathbb{C}}$ given by $x_1 = \zeta_{2d} x_0$, $x_3 = \zeta_{2d} x_2$. The group $H \subset \mathbb{Q}L \oplus (P \otimes \mathbb{Q})$ is generated by $\mathbb{Z}L \oplus P$ and $\frac{1}{d}(L - c)$.

Proof. Let $\Lambda \in H$ be the class of this line. The lattice H is non-degenerate, so it suffices to show that the images of $\frac{1}{d}(L - c)$ and Λ in $H^* \subset (\mathbb{Z}L)^* \oplus P^*$ coincide. The images of these elements in $(\mathbb{Z}L)^*$ are equal, so it remains to show that they are equal in P^* .

The G -module isomorphism $P \cong \mathbb{Z}[G]/I$ gives rise to the map which evaluates the linear function given by the cup-product with $x \in H$ on the elements of G :

$$\text{ev}: H \longrightarrow P^* \longrightarrow \mathbb{Z}[G], \quad \text{ev}(x) = \sum_{g \in G} \langle x, g \rangle g$$

By [DS16, p. 989], we have $\text{ev}(\Lambda) = \psi$, where $\psi = (1 - u_1)(1 - u_3)\phi(u_2u_3)$. It is clear that $\text{ev}(L) = 0$, so it remains to show that $\text{ev}(c) = -d\psi$. For $h \in G$, we find using the G -invariance of the cup-product pairing that

$$\text{ev}(h) = \sum_{g \in G} \langle h, g \rangle g = \sum_{g \in G} \langle 1, h^{-1}g \rangle g = \sum_{g \in G} \langle 1, g \rangle gh = \text{ev}(1)h.$$

We have $\text{ev}(1) = (e|e) = -(1 - u_0)(1 - u_1)(1 - u_2)(1 - u_3)$, by (17). Hence, we get

$$\begin{aligned} -\text{ev}(c) &= (1 - u_0)(1 - u_1)(1 - u_2)(1 - u_3)\rho(u_0, u_1)\rho(u_2, u_3) \\ &= (1 - u_1)(1 - u_3)\phi(u_0u_1)\phi(u_2u_3) \\ &= (1 - u_1)(1 - u_3)\phi(u_2u_3)^2 \\ &= (1 - u_1)(1 - u_3)d\phi(u_2u_3) = d\psi. \quad \square \end{aligned}$$

2.5 Galois representation over the cyclotomic field

Let ℓ be a prime number. Using the comparison theorem of classical and ℓ -adic étale cohomology, we obtain a G -module isomorphism

$$P_\ell := P \otimes \mathbb{Z}_\ell = P^2(\overline{F}, \mathbb{Z}_\ell(1)) \cong \mathbb{Z}_\ell[G]/(I \otimes \mathbb{Z}_\ell),$$

where $P^2(\overline{F}, \mathbb{Z}_\ell(1)) \subset H_{\text{ét}}^2(\overline{F}, \mathbb{Z}_\ell(1))$ is the orthogonal complement to the hyperplane section class. Since $T_\ell := T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is the orthogonal complement to the subgroup generated by the classes of divisors in P_ℓ , the Galois group Γ acts on T_ℓ .

Let us assume that the ground field k contains $E = \mathbb{Q}(\mu_d)$. Then the action of $\Gamma = \text{Gal}(\overline{k}/k)$ on $P^2(\overline{F}, \mathbb{Z}_\ell(1))$ commutes with the action of G .

Let λ be a prime of E above ℓ . Let O_λ be the ring of integers of E_λ . The E_λ -vector space $P \otimes E_\lambda$ is the direct sum of the 1-dimensional G -eigenspaces $V_\chi \otimes_E E_\lambda$, where $\chi \in S$, which are therefore preserved by the Galois group Γ . We have

$$T(F_{\mathbb{C}}) \otimes E_\lambda = \bigoplus_{\chi \in S_\sharp} V_\chi \otimes_E E_\lambda.$$

Let O_k be the ring of integers of k . Let $\mathfrak{p} \subset O_k$ be a prime ideal coprime to $d\ell$, and let $\mathbb{F}_\mathfrak{p} = O_k/\mathfrak{p}$. Let $p = \text{char}(\mathbb{F}_\mathfrak{p})$. For $x \in \mathbb{F}_\mathfrak{p}^\times$ let $\psi(x)$ be the unique d -th root of unity in $E \subset k$ whose reduction in $\mathbb{F}_\mathfrak{p}$ equals $x^{(N(\mathfrak{p})-1)/d}$, where $N(\mathfrak{p}) = |\mathbb{F}_\mathfrak{p}|$. Thus $\psi: \mathbb{F}_\mathfrak{p}^\times \rightarrow \mu_d$ is a multiplicative character of order d . The Gauss sum $g(r) \in \mathbb{Q}(\varepsilon, \zeta)$ is

$$g(r) = \sum_{x \in \mathbb{F}_\mathfrak{p}^\times} \psi(x)^r \zeta^{\text{Tr}_{\mathbb{F}_\mathfrak{p}/\mathbb{F}_p}(x)}, \quad r = 0, \dots, d-1,$$

where ζ is a primitive p -th root of unity in \mathbb{C} . We have $g(0) = 0$, $\overline{g(r)} = \psi(-1)g(-r)$ and $|g(r)|^2 = N(\mathfrak{p})$. Note that $\psi(-1) = 1$ if d is odd or both d and $(N(\mathfrak{p}) - 1)/d$

are even; if d is even and $(N(\mathfrak{p}) - 1)/d$ is odd, then $\psi(-1) = -1$, see [LN97, §5.2] or [IR90, §10.3].

The Jacobi sum $J(\chi)$ is an element of the ring of integers $O_E = \mathbb{Z}[\varepsilon]$ defined by

$$J(\chi) = \sum_{x_1+x_2+x_3=1} \psi(x_1)^l \psi(x_2)^m \psi(x_3)^n = \frac{g(l)g(m)g(n)}{g(l+m+n)} = \psi(-1) \frac{g(l)g(m)g(n)g(r)}{N(\mathfrak{p})},$$

where $x_1, x_2, x_3 \in \mathbb{F}_{\mathfrak{p}}$. The integer r satisfies $l + m + n + r = \lceil \frac{l+m+n}{d} \rceil d$. The second equality here is [LN97, Thm. 5.21], and the third one follows from the properties of Gauss sums discussed above, cf. [IR90, Ch. 8, Thm. 3].

The following proposition can be found in D. Ulmer's paper [Ulm02], though it goes back to N. Katz and T. Shioda. We give a simpler proof using the Fourier transform on the finite abelian group G .

Proposition 2.9 *Let λ be a prime of $E = \mathbb{Q}(\mu_d)$ above ℓ . Let \mathfrak{p} be a prime of k not dividing $d\ell$. Then $\text{Frob}_{\mathfrak{p}}$ acts on $V_{\chi} \otimes_E E_{\lambda}$, where $\chi \in S$, as multiplication by*

$$\psi(-1)N(\mathfrak{p})^{-1}J(\chi) \in N(\mathfrak{p})^{-1}O_E \subset E.$$

Before giving a proof we explain how the Galois representation attached to an arbitrary diagonal surface can be obtained from the Galois representation attached to the Fermat surface.

Remark 2.10 Let K be a subfield of \mathbb{C} . For $a = (a_1, a_2, a_3) \in (K^{\times})^3$ we write $X = X_a$ for the surface in \mathbb{P}_K^3 given by

$$x_0^d + a_1x_1^d + a_2x_2^d + a_3x_3^d = 0.$$

In particular, the Fermat surface $F = X_{(1,1,1)}$. The surface X_a is obtained by twisting F over K by a 1-cocycle of $\text{Gal}(\overline{K}/K)$ with values in the group K -scheme $G = (\mu_d)^3$, whose class in $H^1(K, G) \cong (K^{\times}/K^{\times d})^3$ is the image of $a = (a_1, a_2, a_3)$ under the natural coordinate-wise map $(K^{\times})^3 \rightarrow (K^{\times}/K^{\times d})^3$. The $\text{Gal}(\overline{K}/K)$ -modules $\text{Pic}(\overline{X}_a)$ and $T_{\ell}(X_{a,\mathbb{C}})$ are obtained from the $\text{Gal}(\overline{K}/K)$ -modules $\text{Pic}(\overline{F})$ and $T_{\ell}(F_{\mathbb{C}})$, respectively, by twisting them by this cocycle with respect to the relevant induced actions of G . If $E \subset K$, then the action of $\text{Gal}(\overline{K}/K)$ commutes with the action of G , so $\text{Gal}(\overline{K}/K)$ preserves the eigenspaces $V_{\chi} \otimes_E E_{\lambda}$. In the case of X_a , the action of $\text{Gal}(\overline{K}/K)$ on $V_{\chi} \otimes_E E_{\lambda}$ is the same as the action in the case of F followed by multiplication by the inverse of the d -power character $\text{Gal}(\overline{K}/K) \rightarrow \mu_d$ associated to $a_1^l a_2^m a_3^n$, where $\chi = (l, m, n)$.

Proof of Proposition 2.9. For $\chi = (l, m, n) \in S$ let $h(\chi) \in E_{\lambda}$ be the eigenvalue of $\text{Frob}_{\mathfrak{p}}$ on $V_{\chi} \otimes E_{\lambda}$. If $\chi \in \widehat{G}$ is not in S , then we set $h(\chi) = 0$.

Let $a = (a_1, a_2, a_3) \in G$. The multiplicative character ψ defines an isomorphism $\mathbb{F}_p^\times / \mathbb{F}_p^{\times d} \xrightarrow{\sim} \mu_d$. Pick any lifting $\tilde{a}_i \in \mathbb{F}_p^\times$ of a_i , for $i = 1, 2, 3$, and consider the surface in $X_a \subset \mathbb{P}_{\mathbb{F}_p}^3$ with equation

$$x^d + \tilde{a}_1 x_1^d + \tilde{a}_2 x_2^d + \tilde{a}_3 x_3^d = 0.$$

Since G acts on V_χ via χ , the twisted action of Frob_p on $V_\chi \otimes E_\lambda$, i.e. the action defined by the action of the Galois group Γ on $P^2(\overline{X}_a, E_\lambda(1))$, is via multiplication by $h(\chi) / \psi(\tilde{a}_1)^l \psi(\tilde{a}_2)^m \psi(\tilde{a}_3)^n$. Hence, by the Lefschetz trace formula, we have $|X_a(\mathbb{F}_p)| = |\mathbb{P}^2(\mathbb{F}_p)| + f(a)N(\mathfrak{p})$, where

$$f(a) = \sum_{\chi \in \hat{G}} \psi(\tilde{a}_1^l \tilde{a}_2^m \tilde{a}_3^n)^{-1} h(\chi) = \sum_{\chi \in \hat{G}} \chi(-a) h(\chi).$$

Weil's classical calculation with exponential sums [Wei49], [LN97] gives that this also holds when $h(\chi)$ is replaced by $g(\chi) = \psi(-1)J(\chi)N(\mathfrak{p})^{-1}$ for $\chi \in S$ and by $g(\chi) = 0$ for $\chi \notin S$. Thus f is the Fourier transform of either h or g . The inverse Fourier transform now gives

$$h(\chi) = d^{-3} \sum_{a \in G} \chi(a) f(a) = g(\chi). \quad \square$$

3 The Fermat quartic surface

From now on let $d = 4$. In this section we let $k = E = \mathbb{Q}(i)$ and $\mathcal{O} = \mathbb{Z}[i]$. For a number field $K \subset \overline{\mathbb{Q}}$ we write $\Gamma_K = \text{Gal}(\overline{\mathbb{Q}}/K)$.

3.1 Explicit transcendental lattice

By definition, the transcendental lattice $T(F_{\mathbb{C}})$ is the smallest saturated \mathbb{Z} -submodule of P whose complexification contains $P^{1,-1}$. Thus we have

$$T(F_{\mathbb{C}}) = P \cap (V_{(1,1,1)} \oplus V_{(3,3,3)}) \subset P \otimes k.$$

Lemma 3.1 *We have $T(F_{\mathbb{C}}) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$, where*

$$w_1 = 8(\alpha_{(1,1,1)} + \alpha_{(3,3,3)}), \quad w_2 = 8i(\alpha_{(1,1,1)} - \alpha_{(3,3,3)}).$$

We have

$$\langle w_1, w_1 \rangle = \langle w_2, w_2 \rangle = 8, \quad \langle w_1, w_2 \rangle = 0. \quad (21)$$

Proof. It is clear that w_1, w_2 is a basis of $T(F_{\mathbb{C}}) \otimes \mathbb{Q}$. One immediately obtains (21) from (20). Recall that $2^6 \alpha_{(1,1,1)}$ is the sum of $i^{-(a+b+c)} u_1^a u_2^b u_3^c$, where $a, b, c \in \{0, 1, 2, 3\}$. We calculate

$$\langle 2^6 \alpha_{(1,1,1)}, u_1^\alpha u_2^\beta u_3^\gamma \rangle = \sum_{a,b,c=0}^3 i^{-(a+b+c)} \langle u_1^{a-\alpha} u_2^{b-\beta} u_3^{c-\gamma}, 1 \rangle = 4i^{-(\alpha+\beta+\gamma)},$$

hence $\langle w_1, u_1^\alpha u_2^\beta u_3^\gamma \rangle = \operatorname{Re}(i^{-(\alpha+\beta+\gamma)})$ and $\langle w_2, u_1^\alpha u_2^\beta u_3^\gamma \rangle = -\operatorname{Im}(i^{-(\alpha+\beta+\gamma)})$. This implies that if $sw_1 + tw_2 \in H$ for some $s, t \in \mathbb{Q}$, then $s, t \in \mathbb{Z}$. This also implies $\langle w_i, P \rangle = \mathbb{Z}$. Recall that $H = P \oplus \mathbb{Z}\Lambda$, where Λ is the class of a line in $F_{\mathbb{C}}$. Since $w_i \in T(F_{\mathbb{C}}) \otimes \mathbb{Q}$, we have $\langle w_i, \Lambda \rangle = 0$ for $i = 1, 2$, hence $\langle w_i, H \rangle = \mathbb{Z}$. This implies $w_i \in H^* = H$ by the unimodularity of H . This finishes the proof that $T(F_{\mathbb{C}}) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$. \square

This lemma implies that $\Delta \simeq (\mathbb{Z}/8)^2$, as was already discovered in [PS71].

Let μ_4 be one of the factors of $G = (\mu_4)^3$. In other words, choose one of x_1, x_2, x_3 , say x_3 , and multiply x_3 by fourth roots of unity. Let us write the induced action of $u \in \mu_4$ on P as $[u]$. Extending the coefficients, this gives rise to a k -linear action of μ_4 on $P \otimes k$. By (19) we then have $[i]\alpha_{(l,m,n)} = i^n \alpha_{(l,m,n)}$. In particular, $[i]\alpha_{(1,1,1)} = i\alpha_{(1,1,1)}$ and $[i]\alpha_{(3,3,3)} = -i\alpha_{(3,3,3)}$. (Note that these formulae will be the same if we consider the action of μ_4 on x_1 or x_2 instead of x_3 .) Hence we obtain that $[i]$ acts on $T(F_{\mathbb{C}})$ by the rule $[i]w_1 = w_2$, $[i]w_2 = -w_1$. This gives an isomorphism of \mathcal{O} -modules, where $\mathcal{O} = \mathbb{Z}[i]$:

$$T(F_{\mathbb{C}}) \cong \mathcal{O}. \quad (22)$$

The isomorphism is unique up to the multiplication by an element of μ_4 ; we fix the isomorphism by identifying $1 \in \mathcal{O}$ with w_1 . The cup-product form on $T(F_{\mathbb{C}})$ then becomes $\langle x, y \rangle = 4\operatorname{Tr}_{k/\mathbb{Q}}(x\bar{y})$.

The normalised cup-product form $\frac{1}{8}\langle x, y \rangle$ induces an isomorphism $T(F_{\mathbb{C}}) \xrightarrow{\sim} T(F_{\mathbb{C}})^*$. By the compatibility between classical and ℓ -adic étale cohomology, for any prime ℓ the normalised cup-product in ℓ -adic étale cohomology $\frac{1}{8}\langle x, y \rangle_{\ell}$ gives an isomorphism of Γ -modules

$$T(\overline{X})_{\ell} \cong T(F_{\mathbb{C}}) \otimes \mathbb{Z}_{\ell} \xrightarrow{\sim} T(F_{\mathbb{C}})^* \otimes \mathbb{Z}_{\ell},$$

cf. Section 1.1. Thus from (7) and (8) we obtain isomorphisms of Γ -modules

$$T(X_{\mathbb{C}}) \otimes \mathbb{Q}/\mathbb{Z} \cong \operatorname{Br}(\overline{X}) \quad \text{and} \quad T(X_{\mathbb{C}})/\ell^m \cong \operatorname{Br}(\overline{X})[\ell^m], \quad (23)$$

for any prime ℓ and any positive integer m . Since $T(F_{\mathbb{C}})^* = \frac{1}{8}T(F_{\mathbb{C}})$, the exact sequence (9) becomes the exact sequence of Γ -modules

$$0 \longrightarrow \operatorname{Br}(\overline{F})[8] \longrightarrow \operatorname{Br}(\overline{F}) \xrightarrow{[8]} \operatorname{Br}(\overline{F}) \longrightarrow 0. \quad (24)$$

3.2 Explicit Galois representation

Let ℓ be a prime number and let λ be a prime of $k = \mathbb{Q}(i)$ above ℓ . Let \mathfrak{p} be a prime of k not dividing 2ℓ . The principal ideal $\mathfrak{p} \subset \mathcal{O}$ has a unique generator π which is a *primary prime* $\pi \equiv 1 \pmod{(1+i)^3}$. (Note that this works both in the split and the inert cases.)

Pinch and Swinnerton-Dyer [PSD91], using Weil [Wei49, Wei52] who relied on Stickelberger, computed the eigenvalues of $\operatorname{Frob}_{\mathfrak{p}}$ acting on $P^2(\overline{F}, k_{\lambda}(1)) = P \otimes k_{\lambda}$,

where F is the quartic Fermat surface. The Galois group Γ_k commutes with G , hence the representation of Γ_k in $P \otimes k_\lambda$ attached to F is a direct sum of 1-dimensional representations $V_\chi \otimes k_\lambda$ described in Proposition 2.9. For $d = 4$ the multiplicative character $\psi(x)$ is the biquadratic residue $\left(\frac{x}{\pi}\right)_4$.

Below, for symmetry reasons, we represent characters $\chi \in \widehat{G}$ by 4-tuples of integers $(l, m, n, r) \in \{1, 2, 3\}^4$ with sum divisible by 4.

- $\chi = (1, 1, 1, 1)$. By Lemma 2.6 we have $P^{1,-1} = V_{(1,1,1,1)} \otimes_k \mathbb{C}$. By Proposition 2.9 and the quartic Jacobi sum calculation [IR90, Prop. 9.9.5], the representation of Γ_k in $V_{(1,1,1,1)} \otimes_k k_\lambda$ sends $\text{Frob}_\mathfrak{p}$ to $\pi/\overline{\pi}$. (In the inert case this equals 1.)
- $\chi = (3, 3, 3, 3)$. We have $P^{-1,1} = \overline{P^{1,-1}} = V_{(3,3,3,3)} \otimes_k \mathbb{C}$, so the representation of Γ_k in $V_{(3,3,3,3)} \otimes_k k_\lambda$ is conjugate to the representation of Γ_k in $V_{(1,1,1,1)} \otimes_k k_\lambda$.
- The spaces V_χ for all other characters belong to $P^{0,0}$. The space $V_{(2,2,2,2)}$ is defined over \mathbb{Q} . There are six spaces like $V_{(1,1,3,3)}$ (obtained by permuting the coordinates of χ) that come in conjugate pairs. The action of Γ_k on all of these spaces is trivial. Finally, there are twelve spaces like $V_{(1,2,2,3)}$; the action of $\text{Frob}_\mathfrak{p}$ on them is via multiplication by $\psi(-1)$. In particular, we recover the known fact that $\Gamma_{\mathbb{Q}(\mu_8)}$ acts trivially on $\text{Pic}(\overline{F})$.

It is more delicate to make explicit the Galois representation over \mathbb{Q} . In the case $d = 4$ we use the fact that $\Gamma_{\mathbb{Q}}$ is the semi-direct product of Γ_k and $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \text{Gal}(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R})) \cong \mathbb{Z}/2$.

For odd primes ℓ the following statement was obtained in [IS15] by a different method (see the proof of [IS15, Lemma 4.2]).

Proposition 3.2 *The representation of $\Gamma_{\mathbb{Q}} = \Gamma_k \rtimes \text{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}} \cap \mathbb{R})$ in $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell \cong \mathcal{O} \otimes \mathbb{Z}_\ell$ is as follows:*

if \mathfrak{p} is coprime to 2ℓ , then $\text{Frob}_\mathfrak{p} \in \Gamma_k$ acts as multiplication by $\pi/\overline{\pi}$ where π is a primary generator of \mathfrak{p} ;

the generator of $\text{Gal}(\overline{\mathbb{Q}}/\overline{\mathbb{Q}} \cap \mathbb{R})$ acts on $T(F_{\mathbb{C}})$ as $\tau(1)$, namely, $\tau(1)(w_1) = w_1$, $\tau(1)(w_2) = -w_2$, i.e., as the usual conjugation of \mathcal{O} .

Proof. Recall that $T(F_{\mathbb{C}}) = \mathbb{Z}w_1 \oplus \mathbb{Z}w_2$ was made into an \mathcal{O} -module using the action of $[i]$ which sends w_1 to w_2 and w_2 to $-w_1$. We know that on $(\mathcal{O} \otimes \mathbb{Z}_\ell)\alpha_{(1,1,1,1)}$ the Frobenius element $\text{Frob}_\mathfrak{p}$, where \mathfrak{p} is coprime to 2ℓ , acts as multiplication by $\pi/\overline{\pi} \in (\mathcal{O} \otimes \mathbb{Z}_\ell)^*$. One then immediately checks that $\text{Frob}_\mathfrak{p}$ acts on $T(X_{\mathbb{C}}) \otimes \mathbb{Z}_\ell = \mathbb{Z}_\ell w_1 \oplus \mathbb{Z}_\ell w_2$ as the (2×2) -matrix given by multiplication by $\pi/\overline{\pi}$ on $\mathcal{O} \otimes \mathbb{Z}_\ell$.

Let us prove the second statement. We have comparison isomorphisms

$$\mathrm{H}^2(F(\mathbb{C}), \mathbb{Z}/\ell^n) \xrightarrow{\sim} \mathrm{H}_{\text{ét}}^2(F_{\mathbb{C}}, \mathbb{Z}/\ell^n) \xrightarrow{\sim} \mathrm{H}_{\text{ét}}^2(\overline{F}, \mathbb{Z}/\ell^n)$$

which are functorial with respect to the action of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \text{Gal}(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R}))$. Thus the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $H^2(F(\mathbb{C}), \mathbb{Z})$ induced by the complex conjugation of $F(\mathbb{C})$, that is, the action sending the generator of $\text{Gal}(\mathbb{C}/\mathbb{R})$ to τ , is compatible with the representation of $\text{Gal}(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R}))$ in $H_{\text{ét}}^2(\overline{F}, \mathbb{Z}_\ell)$, cf. [Del79, 0.3]. It follows that the representation of $\text{Gal}(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R}))$ in $H_{\text{ét}}^2(\overline{F}, \mathbb{Z}_\ell(1))$ sends the generator to $\tau(1)$. The representation of $\text{Gal}(\overline{\mathbb{Q}}/(\overline{\mathbb{Q}} \cap \mathbb{R}))$ in $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell \subset H_{\text{ét}}^2(\overline{F}, \mathbb{Z}_\ell(1))$ extends to $T(F_{\mathbb{C}}) \otimes k_\lambda \subset H_{\text{ét}}^2(\overline{F}, k_\lambda(1))$. The k -vector space $T(F_{\mathbb{C}}) \otimes k$ has basis $\alpha_{(1,1,1)}$ and $\alpha_{(3,3,3)}$, which are fixed by $\tau(1)$, see Lemma 2.7. From this we obtain the action of $\tau(1)$ on w_1 and w_2 . \square

Proposition 3.3 (i) *The Γ_k -invariant subgroup of $\text{Br}(\overline{F})\{2\}$ is $\text{Br}(\overline{F})[4]$.*

(ii) *The $\Gamma_{k(\sqrt{2})}$ -invariant subgroup of $\text{Br}(\overline{F})\{2\}$ is $\text{Br}(\overline{F})[8]$.*

(iii) *We have $k(\sqrt[4]{2}) = k(\sqrt[4]{-2})$. The $\Gamma_{k(\sqrt[4]{-2})}$ -invariant subgroup of $\text{Br}(\overline{F})\{2\}$ is $\text{Br}(\overline{F})[16]$.*

(iv) *The $\Gamma_{k(\sqrt[8]{-2})}$ -invariant subgroup of $\text{Br}(\overline{F})\{2\}$ is $\text{Br}(\overline{F})[32]$.*

Proof. (i) By (22) and (23) there is an isomorphism of Γ_k -modules $\text{Br}(\overline{F})[2^m] = \mathcal{O}/2^m$ for each $m \geq 1$. We have $\pi = a + bi$, where a is odd and b is even. Hence $\pi \equiv \overline{\pi} \pmod{4}$. From the explicit description of the Galois action in Proposition 3.2 we see that Γ_k acts trivially on $\text{Br}(\overline{F})[4]$. To finish the proof of (i) it is enough to show that each element of $\mathcal{O}/8$ fixed by Γ_k is divisible by 2. One easily checks that if $x + iy \in \mathcal{O}$ is fixed modulo 8 by Frob_π , where $\pi = -1 + 2i$, then x and y are even.

(ii) A supplement to the quadratic reciprocity law states that 2 is a square modulo a prime $p \equiv 1 \pmod{4}$ precisely when $p \equiv 1 \pmod{8}$. Write $p = N(\pi)$, where π is a primary prime in \mathcal{O} . Since $\mathbb{Z}/p \xrightarrow{\sim} \mathcal{O}/\pi$, we see that 2 is a square modulo π if and only if $p \equiv 1 \pmod{8}$, that is, if and only if 4 divides b , that is, if and only if $\pi \equiv \overline{\pi} \pmod{8}$. This shows that $\Gamma_{k(\sqrt{2})}$ acts trivially on $\text{Br}(\overline{F})[8]$. The same calculation as in (i) shows that if $x + iy \in \mathcal{O}$ is fixed modulo 16 by Frob_π , where $\pi = 1 + 4i$, then x and y are even.

(iii) A supplement to the biquadratic reciprocity law [Cox89, Thm. 4.23], [Lem00, Thm. 6.9] states that 2 is the fourth power modulo π if and only if 8 divides b , that is, if and only if $\pi \equiv \overline{\pi} \pmod{16}$. The same calculation as in (i) shows that if $x + iy \in \mathcal{O}$ is fixed modulo 32 by Frob_π , where $\pi = 5 + 8i$, then x and y are even.

(iv) A supplement to the octic reciprocity law [Wes11, II.12] states that for $\overline{\pi}\pi = p \equiv 1 \pmod{8}$ with $8 \mid b$, 2 is an eighth power modulo π if and only if $b \equiv p - 1 \equiv 0 \pmod{16}$ or $b \equiv p - 1 \equiv 8 \pmod{16}$ and -1 is an eighth power modulo π if and only if $p - 1 \equiv 0 \pmod{16}$. Thus -2 is an eighth power modulo π if and only if $b \equiv 0 \pmod{16}$, that is $\pi \equiv \overline{\pi} \pmod{32}$. The same calculation as in (i) shows that if $x + iy \in \mathcal{O}$ is fixed modulo 64 by Frob_π , where $\pi = 1 + 16i$, then x and y are even. \square

Remark 3.4 The invariant subfield of the kernel of the Galois action on $\text{Br}(\overline{F})[2^n]$ is the ring class field of $k = \mathbb{Q}(i)$ of conductor 2^{n-1} . This follows from the definition:

K_c/K is the abelian extension of K corresponding to the class group $I_K(c)/P_{K,\mathbb{Z}}$; in other words, precisely the primes congruent to an element of \mathbb{Z} modulo c split in K_c . Indeed, Frob_π acts on $\mathcal{O}/2^n$ trivially if and only if $\pi \equiv \bar{\pi} \pmod{2^n}$, which is equivalent to the divisibility of b by 2^{n-1} . See [Cox89, p. 179]. For the calculation $\mathbb{Q}(i)_8 = \mathbb{Q}(i, \sqrt[4]{2})$, see [Cox89, Prop. 9.5].

We now reprove a result of Ieronymou [Ier10, Thm. 4.2]. Recall that if $K_1 \subset K_2$ are algebraically closed fields of characteristic zero, and X is a smooth and proper variety over K_1 , then the induced map $\text{Br}(X) \rightarrow \text{Br}(X \times_{K_1} K_2)$ is an isomorphism. Thus we can use the notation $\text{Br}(\bar{F})$ without specifying the underlying algebraically closed field.

Since F is defined over \mathbb{Q} , from the exact sequence (24) we obtain the exact sequence of $\Gamma_{\mathbb{Q}}$ -modules

$$0 \longrightarrow \mathcal{O}/8 \xrightarrow{[2]} \mathcal{O}/16 \longrightarrow \mathcal{O}/2 = \text{Br}(\bar{F})[2] \longrightarrow 0. \quad (25)$$

The following corollary should be compared to Ieronymou's results [Ier10, Thm. 3.1, Prop. 4.9].

Corollary 3.5 *Let F be the Fermat quartic surface. Let K be a field extension of $\mathbb{Q}(i, \sqrt[4]{2})$. Then the natural map $\text{Br}(F_K)[2] \rightarrow \text{Br}(\bar{F})[2]$ is surjective.*

Proof. It is enough to prove this for $K = \mathbb{Q}(i, \sqrt[4]{2})$. The Fermat quartic surface has obvious rational points over $\mathbb{Q}(\mu_8)$; pick one of them and call it P . We can write $\text{Br}(F_K) = \text{Br}(K) \oplus \text{Br}(F_K)_P$, where $\text{Br}(F_K)_P \subset \text{Br}(F_K)$ consists of the classes vanishing at P . Since $\Gamma_K = \text{Gal}(\bar{\mathbb{Q}}/K)$ acts trivially on $\text{Pic}(\bar{F}) \simeq \mathbb{Z}^{20}$, we have $H^1(K, \text{Pic}(\bar{F})) = 0$. The standard spectral sequence then shows that the canonical map $\text{Br}(F_K) \rightarrow \text{Br}(\bar{F})$ restricted to $\text{Br}(F_K)_P$ is an injective map $\text{Br}(F_K)_P \hookrightarrow \text{Br}(\bar{F})$. It follows that an element of $\text{Br}(\bar{F})[2]$ which can be lifted to an element of $\text{Br}(F_K)$, can also be lifted to an element of $\text{Br}(F_K)[2]_P \subset \text{Br}(F_K)[2]$.

It remains to show that $\text{Br}(\bar{F})[2]$ is contained in the image of $\text{Br}(F_K)$ in $\text{Br}(\bar{F})$. By Proposition 3.3 (iii) the abelian groups in (25) are trivial Γ_K -modules. It follows that $\partial_1(\text{Br}(\bar{F})[2]) = 0$, so we can conclude by applying Corollary 1.5. \square

3.3 Diagonal quartic surfaces

We start with a general observation about arbitrary diagonal quartic surfaces (proved in [Sko17, Prop. 4.8] by a different method).

Proposition 3.6 *Let $X = X_a$ be a diagonal quartic surface over a field K of characteristic zero. We have $\Delta_X[2]^{\text{Gal}(\bar{K}/K)} \neq 0$, hence $\text{Br}(\bar{X})[2]^{\text{Gal}(\bar{K}/K)} \neq 0$.*

Proof. Without loss of generality we assume that $K = \mathbb{Q}(a_1, a_2, a_3)$; in particular, K can be embedded into \mathbb{C} . Then we have a canonical isomorphism $\text{Pic}(\overline{X}) \cong \text{Pic}(X_{\mathbb{C}})$.

We construct an explicit $\text{Gal}(\overline{K}/K)$ -invariant 2-torsion element in Δ_X using the canonical isomorphism of $\text{Gal}(\overline{K}/K)$ -modules $\Delta_X \cong \text{Pic}(\overline{X})^*/\text{Pic}(\overline{X})$. The snake lemma shows that the $\text{Gal}(\overline{K}/K)$ -module $\Delta_X[2]$ is canonically isomorphic to the kernel of the map $\text{Pic}(\overline{X})/2 \rightarrow \text{Pic}(\overline{X})^*/2$ given by the intersection pairing. Thus it is enough to exhibit a divisor class $D \in \text{Pic}(\overline{X})$ with the following properties:

- (a) $\langle D, x \rangle \in 2\mathbb{Z}$ for every $x \in \text{Pic}(\overline{X})$;
- (b) the class of D in $\text{Pic}(\overline{X})/2$ is $\text{Gal}(\overline{K}/K)$ -invariant;
- (c) the class of D in $\text{Pic}(\overline{X})/2$ is not zero.

Consider the surjective morphism $s : X \rightarrow Q$, where Q is the diagonal quadric

$$y_0^2 + a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 = 0,$$

and $s((x_0 : x_1 : x_2 : x_3)) = (x_0^2 : x_1^2 : x_2^2 : x_3^2)$. There are two families of lines on Q defined over \overline{K} . Let $f_{\pm} \in \text{Pic}(\overline{Q})$ be the classes of these families, and let $D_{\pm} = s^*(f_{\pm})$. It is clear that the set $\{D_+, D_-\} \subset \text{Pic}(\overline{X})$ is $\text{Gal}(\overline{K}/K)$ -invariant.

Let us show that we can take D to be either of D_+ and D_- . Indeed, the inverse images of the two families of lines in Q are pencils of genus 1 curves on F such that $D_+ + D_- = 2L$, where L is the hyperplane section class. It is well known and easy to check that each of the 48 lines on \overline{X} is a component of a degenerate fibre of one of these families. Thus for any line $\ell \subset \overline{X}$ we have either $\langle \ell, D_+ \rangle = 0$ or $\langle \ell, D_+ \rangle = 2$. Since $\text{Pic}(\overline{X})$ is generated by lines, we get $\langle D_+, x \rangle \in 2\mathbb{Z}$ for every $x \in \text{Pic}(\overline{X})$. By the formula $D_+ + D_- = 2L$, the same holds for D_- . This gives (a) and (b).

The class of an integral curve of arithmetic genus 1 on a complex K3 surface $X_{\mathbb{C}}$ is primitive in $\text{Pic}(X_{\mathbb{C}}) = \text{Pic}(\overline{X})$ [Huy16, Ch. 2, Rem. 3.13 (i)]. This gives (c). \square

Definition 3.7 *Let X be a diagonal surface over a field K . We say that X is **split** by a field extension L/K if $X_L \cong F_L$.*

Theorem 3.8 *Let X be a diagonal quartic surface over a field K containing $k = \mathbb{Q}(i)$, which is split by a field extension of K not containing $\sqrt[4]{2}$. Then*

$$\text{Br}(\overline{X})[2] \cap \text{Im}[\text{Br}(X) \rightarrow \text{Br}(\overline{X})] = 0.$$

Proof. To prove that this intersection is zero we can enlarge the field K . So we can assume that K contains $\mathbb{Q}(\mu_8) = k(\sqrt{2})$ but does not contain $\sqrt[4]{2}$, and that $X \simeq F_K$. The group $\text{Gal}(\overline{K}/K)$ acts on the terms of the exact sequence (25) via the map $\text{Gal}(\overline{K}/K) \rightarrow \text{Gal}(\overline{\mathbb{Q}}/k(\sqrt{2})) = \Gamma_{k(\sqrt{2})}$. By Proposition 3.3 (iii), $\Gamma_{k(\sqrt{2})}$ acts via the surjective map $\Gamma_{k(\sqrt{2})} \rightarrow \text{Gal}(k(\sqrt[4]{2})/k(\sqrt{2}))$. Since K and $k(\sqrt[4]{2})$ are linearly disjoint over $k(\sqrt{2})$, the following map is surjective:

$$\text{Gal}(\overline{K}/K) \rightarrow \text{Gal}(K(\sqrt[4]{2})/K) \cong \text{Gal}(k(\sqrt[4]{2})/k(\sqrt{2})) \cong \mathbb{Z}/2.$$

Proposition 3.3 (ii) says that the $\Gamma_{k(\sqrt{2})}$ -invariant subgroup of $\mathcal{O}/16$ is $2\mathcal{O}/16$, which is exactly the kernel of $\mathcal{O}/16 \rightarrow \mathcal{O}/2$. Hence this is also the $\text{Gal}(\overline{K}/K)$ -invariant subgroup of $\mathcal{O}/16$. This implies that ∂_1 induces an injective map

$$\partial_1: \text{Br}(\overline{F})[2]^{\Gamma_K} = \mathcal{O}/2 \longrightarrow H^1(K, \Delta_F) = H^1(K, \mathcal{O}/8).$$

We claim that in our situation $\partial_2: H^1(K, \Delta_F) \rightarrow H^2(K, \text{Pic}(\overline{F}))$ is injective. Indeed, since $k(\sqrt{2}) = \mathbb{Q}(\mu_8)$ is contained in K , and $\Gamma_{\mathbb{Q}(\mu_8)}$ acts trivially on $\text{Pic}(\overline{F})$, the group $\text{Gal}(\overline{K}/K)$ acts trivially on $\text{Pic}(\overline{F})$, and hence also on $\text{Pic}(\overline{F})^*$. Thus $H^1(K, \text{Pic}(\overline{F})^*) = 0$, so that ∂_2 is indeed injective. We have proved that $\partial_2\partial_1$ is injective on $\text{Br}(\overline{F})[2]$, so by Corollary 1.5 the intersection of $\text{Br}(\overline{F})[2]$ and the image of $\text{Br}(F_K)$ in $\text{Br}(\overline{F})$ is zero. \square

The following result shows that for any diagonal quartic surface over \mathbb{Q} or $k = \mathbb{Q}(i)$ the Galois-invariant subgroup of $\text{Br}(\overline{X})$ is annihilated by 4.

Proposition 3.9 *Let $X = X_a$ be a diagonal quartic surface over \mathbb{Q} .*

- (i) *If $a_1a_2a_3 \in k^{\times 4}$, then the Γ_k -invariant subgroup of $\text{Br}(\overline{X})\{2\}$ is $\text{Br}(\overline{X})[4]$.*
- (ii) *If $a_1a_2a_3 \in \mathbb{Q}^{\times 4}$, then the $\Gamma_{\mathbb{Q}}$ -invariant subgroup of $\text{Br}(\overline{X})\{2\}$ is $\frac{1}{4}(\mathbb{Z} + 2i\mathbb{Z})/\mathcal{O}$. If $a_1a_2a_3 \in -4\mathbb{Q}^{\times 4}$, then the $\Gamma_{\mathbb{Q}}$ -invariant subgroup of $\text{Br}(\overline{X})\{2\}$ is $\frac{1}{4}(1 - i)\mathbb{Z}/\mathbb{Z}$.*
- (iii) *If $a_1a_2a_3 \in k^{\times 2}$ but $a_1a_2a_3 \notin k^{\times 4}$, then the Γ_k -invariant subgroup of $\text{Br}(\overline{X})\{2\}$ is $\text{Br}(\overline{X})[2]$. The same holds if k is replaced by \mathbb{Q} .*
- (iv) *If $a_1a_2a_3 \notin k^{\times 2}$, then the Γ_k -invariant subgroup of $\text{Br}(\overline{X})\{2\}$ is $\mathbb{Z}/2$ generated by the class of $1 + i$ in $\frac{1}{2}\mathcal{O}/\mathcal{O}$.*

Remark 3.10 Let us recall the description of the twisted Galois representation from Remark 2.10 in the case $d = 4$. The group scheme $G = (\mu_4)^3$ acts on $\alpha_{(1,1,1)}$ via the product character $(\mu_4)^3 \rightarrow \mu_4$. If K is a subfield of \mathbb{C} and $a = (a_1, a_2, a_3) \in (K^\times)^3$, then the action of $\text{Gal}(\overline{K}/K)$ on $T(X_{a,\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is obtained by twisting the action of $\text{Gal}(\overline{K}/K)$ on $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$ by the 1-cocycle $\text{Gal}(\overline{K}/K) \rightarrow \mu_4$ with class $a_1a_2a_3 \in K^\times/K^{\times 4} = H^1(K, \mu_4)$. For example, if $a_1a_2a_3 \in K^{\times 4}$, then the action of $\text{Gal}(\overline{K}/K)$ on $T(X_{a,\mathbb{C}}) \otimes \mathbb{Z}_\ell$ is the same as the action of $\text{Gal}(\overline{K}/K)$ on $T(F_{\mathbb{C}}) \otimes \mathbb{Z}_\ell$. In general, if K contains k , the twisted action is obtained by composing the untwisted action with the inverse of the quartic character given by $a_1a_2a_3$. The value of this character on Frob_p is $\left(\frac{a_1a_2a_3}{\pi}\right)_4^{-1}$.

Proof of Proposition 3.9. (i) By Remark 3.10, in this case the action of Γ_k is the same as for the Fermat surface over k , so we read off the result from Proposition 3.3 (i).

(ii) By the same remark, the action $\Gamma_{\mathbb{Q}}$ is the same as for the Fermat surface, so compared to (a) we only need to take into account the action of $\tau(1)$. As explained in Proposition 3.2, it acts on \mathcal{O} as the usual conjugation, hence the first formula. If $a_1a_2a_3 \in -4\mathbb{Q}^{\times 4}$, then $\tau(1)$ acts on \mathcal{O} as the usual complex conjugation followed by

multiplication by $(1+i)/(1-i)$, which the value on the complex conjugation of the 1-cocycle given by -4 . Thus the twisted action of $\tau(1)$ sends $a+bi$ to $b+ai$, hence the result.

(iii) Recall that $\text{Frob}_{\mathfrak{p}} \in \Gamma_k$ acts on $\text{Br}(\overline{X})[2^m] = \mathcal{O}/2^m$, $m \geq 1$, as multiplication by $(\frac{a_1 a_2 a_3}{\mathfrak{p}})_4^{-1} \frac{\pi}{\pi}$. Here $(\frac{a_1 a_2 a_3}{\mathfrak{p}})_4$ is 1 if $a_1 a_2 a_3$ is a fourth power modulo \mathfrak{p} , and -1 otherwise. One easily checks that, whatever the sign, the Frob_{-1+2i} -invariant subgroup of $\mathcal{O}/2^m$ is killed by 4. But $\text{Frob}_{\mathfrak{p}}$, for any \mathfrak{p} coprime to 2, acts on $\mathcal{O}/4$ as multiplication by $(\frac{a_1 a_2 a_3}{\mathfrak{p}})_4^{-1}$. Since $a_1 a_2 a_3$ is not a fourth power in k , by the Chebotarev density theorem we can find a prime ideal \mathfrak{p} such that $\text{Frob}_{\mathfrak{p}}$ acts as -1 . This proves (iii) for k .

The kernel of the natural map $\mathbb{Q}^\times/\mathbb{Q}^{\times 4} \rightarrow k^\times/k^{\times 4}$ is the group of order 2 generated by -4 . Hence a square in \mathbb{Q} which is not a fourth power in \mathbb{Q} cannot become a fourth power in k . The action of $\tau(1)$ on $\mathcal{O}/2 = \text{Br}(\overline{F})[2]$ is trivial, so the action of $\tau(1)$ on $\mathcal{O}/2 = \text{Br}(\overline{X})[2]$ is by multiplication by the inverse of the value of the 1-cocycle given by $a_1 a_2 a_3$ evaluated at the complex conjugation. This is an element of $\{\pm 1\}$, so it acts trivially on $\mathcal{O}/2$. Thus the statement (iii) for \mathbb{Q} follows from the same statement for k .

(iv) In this case Frob_{-1+2i} acts on $\mathcal{O}/2^m$ as $\frac{-1+2i}{-1-2i} = 1+4i$ multiplied by ± 1 or $\pm i$. One checks that in all cases the Frob_{-1+2i} -invariant subgroup of $\mathcal{O}/2^m$ is killed by 4. By the same argument as in (iii), for any odd prime ideal $\mathfrak{p} \subset \mathcal{O}$, the action of $\text{Frob}_{\mathfrak{p}}$ on $\mathcal{O}/4$ is by multiplication by ± 1 or $\pm i$. Since $a_1 a_2 a_3$ is not a square in k , by the Chebotarev density theorem we can find a prime ideal \mathfrak{p} such that $\text{Frob}_{\mathfrak{p}}$ acts on $\mathcal{O}/4$ as multiplication by $\pm i$. Then the $\text{Frob}_{\mathfrak{p}}$ -invariant subgroup of $\mathcal{O}/4$ is $\mathbb{Z}/2$ generated by $2(1+i)$. This subgroup is Γ_k -invariant, which proves (iv). \square

The following statement answers Question 2 raised at the end of [Sko17] and gives a different proof of Proposition 3.6.

Corollary 3.11 *Let K be a field of characteristic zero, $a = (a_1, a_2, a_3) \in (K^\times)^3$, and let $X = X_a$. Then the action of $\text{Gal}(\overline{K}/K)$ on $\text{Br}(\overline{X})[2]$ fixes the unique non-zero μ_4 -invariant element; the other two non-zero 2-torsion elements are permuted by $\text{Gal}(K(\sqrt{a_1 a_2 a_3})/K)$.*

Proof. The Galois group $\text{Gal}(\overline{K}/K)$ acts on $\text{Br}(\overline{F})$ via $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, but Γ acts trivially on $\text{Br}(\overline{F})[2]$ by the proof of (ii) above. Thus $\text{Gal}(\overline{K}/K)$ acts on $\text{Br}(\overline{X})[2]$ via the inverse of the quartic character associated to $a_1 a_2 a_3$. The action of μ_4 on $\mathcal{O}/2\mathcal{O}$ fixes the class of $1+i$ and permutes the classes of 1 and i , hence the result. \square

Another result of Ieronymou [Ier10, Thm. 5.2] is a corollary of Theorem 3.8.

Proposition 3.12 (Ieronymou) *Let $X = X_a$ be a diagonal quartic surface over \mathbb{Q} such that 2 does not belong to the subgroup of \mathbb{Q}^\times generated by $-4, a_1, a_2, a_3$ and $\mathbb{Q}^{\times 4}$. Then $\text{Br}(\overline{X})[2] \cap \text{Im}[\text{Br}(X) \rightarrow \text{Br}(\overline{X})] = 0$, so that $\text{Br}(X)[2] \subset \text{Br}_1(X)$.*

Proof. The surface X is split by the field $K = k(\sqrt[4]{a_1}, \sqrt[4]{a_2}, \sqrt[4]{a_3})$, where $k = \mathbb{Q}(i)$. The kernel of $\mathbb{Q}^\times/\mathbb{Q}^{\times 4} \rightarrow k^\times/k^{\times 4}$ is generated by the class of -4 , so the kernel of $\mathbb{Q}^\times/\mathbb{Q}^{\times 4} \rightarrow K^\times/K^{\times 4}$ is generated by the classes of $-4, a_1, a_2, a_3$. Hence $2 \notin K^{\times 4}$, so the statement follows from Theorem 3.8. \square

Proposition 3.13 *Let $X = X_a$ be a diagonal quartic surface over a field K containing $k = \mathbb{Q}(i)$. If $a_1 a_2 a_3 \in K^{\times 4}$ and $2 \in K^{\times 4}$, then $\text{Br}(\overline{X})[2] \subset \text{Im}[\text{Br}(X) \rightarrow \text{Br}(\overline{X})]$.*

Proof. By Remark 3.10, the action of $\text{Gal}(\overline{K}/K)$ on $\text{Br}(\overline{X})$ is the same as the action of $\text{Gal}(\overline{K}/K)$ on $\text{Br}(\overline{F})$, so the arguments from the second part of the proof of Corollary 3.5 still work. \square

The proof of Corollary 3.5 shows that in this situation each element of $\text{Br}(\overline{X})[2]$ can be lifted to an element of $\text{Br}(X)/\text{Br}_0(X)$ of order 2.

4 Main theorem

4.1 Reduction to finite group cohomology

Let X_a be a diagonal quartic surface over \mathbb{Q} with coefficients $a_1, a_2, a_3 \in \mathbb{Q}^\times$. To $a = (a_1, a_2, a_3)$ and a field L of characteristic zero we attach a finite Galois extension L_a and its Galois group

$$L_a = L(i, \sqrt[4]{2}, \sqrt[4]{a_1}, \sqrt[4]{a_2}, \sqrt[4]{a_3}), \quad G_{a,L} = \text{Gal}(L_a/L).$$

We shall also consider a larger Galois extension \tilde{L}_a and its Galois group

$$\tilde{L}_a = L(i, \sqrt[8]{-2}, \sqrt[4]{a_1}, \sqrt[4]{a_2}, \sqrt[4]{a_3}), \quad \tilde{G}_{a,L} = \text{Gal}(\tilde{L}_a/L).$$

Note that L_a splits X_a and contains $\mathbb{Q}(\mu_8)$. We see that $\text{Pic}(\overline{X})$, and hence also $\text{Pic}(\overline{X})^*$ and Δ_X , are trivial $\text{Gal}(\overline{\mathbb{Q}}/L_a)$ -modules. By Proposition 3.3 (iii), $\text{Br}(\overline{X})[16]$ is a trivial $\text{Gal}(\overline{\mathbb{Q}}/L_a)$ -module too. Thus $\Gamma_L = \text{Gal}(\overline{\mathbb{Q}}/L)$ acts on these modules via $G_{a,L}$. By Proposition 3.3 (iv), $\text{Br}(\overline{X})[32]$ is a trivial $\text{Gal}(\overline{\mathbb{Q}}/\tilde{L}_a)$ -module, so Γ_L acts on it via $\tilde{G}_{a,L}$.

In view of Proposition 3.9, the main theorem can be proved by applying Corollary 1.6 with $n = 4$ to all possible groups $\tilde{G}_{a,\mathbb{Q}}$ and $\tilde{G}_{a,k}$, where $k = \mathbb{Q}(i)$. This is a finite calculation because each of these groups is a subgroup of the ‘generic’ Galois group \tilde{G} defined by treating a_1, a_2, a_3 as independent variables. However, we have chosen a slightly more elaborate line of proof which has the advantage of reducing the use of computer to a small number of group cohomology calculations.

The exact sequence (9) in our particular case takes the form (24). From it we obtain exact sequences of Γ_L -modules

$$0 \longrightarrow \Delta_X \longrightarrow \text{Br}(\overline{X})[16] \xrightarrow{[8]} \text{Br}(\overline{X})[2] \longrightarrow 0, \quad (26)$$

$$0 \longrightarrow \Delta_X \longrightarrow \mathrm{Br}(\overline{X})[32] \xrightarrow{[8]} \mathrm{Br}(\overline{X})[4] \longrightarrow 0. \quad (27)$$

The sequence (26) is thus an exact sequence of $G_{a,L}$ -modules, and so defines a boundary map $\mathrm{H}^0(G_{a,L}, \mathrm{Br}(\overline{X})[2]) \rightarrow \mathrm{H}^1(G_{a,L}, \Delta_X)$. Similarly, the sequence (27) defines a boundary map $\mathrm{H}^0(\tilde{G}_{a,L}, \mathrm{Br}(\overline{X})[4]) \rightarrow \mathrm{H}^1(\tilde{G}_{a,L}, \Delta_X)$.

Lemma 4.1 *Let $k = \mathbb{Q}(i)$. Suppose that the image of the natural map $G_{a,L} \rightarrow \mathrm{Gal}(k(\sqrt[4]{2})/\mathbb{Q})$ contains the subgroup $\mathrm{Gal}(k(\sqrt[4]{2})/k) \simeq \mathbb{Z}/4$. Then*

(a) *the group $\mathrm{Br}(\overline{X})[2] \cap \mathrm{Im}[\mathrm{Br}(X_L) \rightarrow \mathrm{Br}(\overline{X})]$ is the intersection of images of the following maps:*

$$\mathrm{H}^1(G_{a,L}, \mathrm{Pic}(\overline{X})^*) \longrightarrow \mathrm{H}^1(G_{a,L}, \Delta_X) \longleftarrow \mathrm{H}^0(G_{a,L}, \mathrm{Br}(\overline{X})[2]);$$

(b) *the group $\mathrm{Br}(\overline{X})\{2\} \cap \mathrm{Im}[\mathrm{Br}(X_L) \rightarrow \mathrm{Br}(\overline{X})]$ is the intersection of images of the following maps:*

$$\mathrm{H}^1(\tilde{G}_{a,L}, \mathrm{Pic}(\overline{X})^*) \longrightarrow \mathrm{H}^1(\tilde{G}_{a,L}, \Delta_X) \longleftarrow \mathrm{H}^0(\tilde{G}_{a,L}, \mathrm{Br}(\overline{X})[4]).$$

Proof. Let us show that $\mathrm{Br}(X)\{2\}^{\Gamma_L}$ is killed by 4. For this it is enough to show that $\mathrm{Br}(X)[8]^{\Gamma_L}$ is killed by 4. Let us first assume that $a_1 a_2 a_3$ is a fourth power in L . In this case, the Γ_L -modules $\mathrm{Br}(\overline{X})\{2\}$ and $\mathrm{Br}(\overline{F})\{2\}$ are isomorphic. By Proposition 3.3 (ii), the group Γ_L acts on $\mathrm{Br}(\overline{F})[8]$ via its image $\mathrm{Gal}(L(i, \sqrt{2})/L)$. By assumption, the subgroup $\mathrm{Gal}(L(i, \sqrt{2})/L(i))$ is isomorphic to $\mathrm{Gal}(k(\sqrt{2})/k) \simeq \mathbb{Z}/2$. As in the proof of Proposition 3.3 (i), the non-trivial element of $\mathrm{Gal}(k(\sqrt{2})/k)$ can be taken to be Frob_π , where $\pi = -1 + 2i$. If $x + iy \in \mathcal{O}$ is fixed modulo 8 under the multiplication by $\frac{\pi}{\bar{\pi}}$, then x and y are even. Thus 4 kills $\mathrm{Br}(\overline{X})\{2\}^{\Gamma_L}$ in this case. In general, the Galois representation on $\mathrm{Br}(\overline{X})[8]$ is obtained by twisting the representation on $\mathrm{Br}(\overline{F})[8]$ by the inverse of the quartic character of $a_1 a_2 a_3$, so $\frac{\pi}{\bar{\pi}}$ is multiplied by ± 1 or $\pm i$. An easy check shows that 4 kills $\mathrm{Br}(\overline{X})\{2\}^{\Gamma_L}$ in all cases.

In view of the exact sequence (26) this implies that the boundary maps

$$\mathrm{Br}(\overline{X})[2]^{\Gamma_L} \longrightarrow \mathrm{H}^1(L, \Delta_X), \quad \mathrm{H}^0(G_{a,L}, \mathrm{Br}(\overline{X})[2]) \longrightarrow \mathrm{H}^1(G_{a,L}, \Delta_X)$$

are injective. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{H}^1(L, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & \mathrm{H}^1(L, \Delta_X) & \xleftarrow{\partial_1} & \mathrm{Br}(\overline{X})[2]^{\Gamma_L} \\ \uparrow \cong & & \uparrow & & \uparrow \cong \\ \mathrm{H}^1(G_{a,L}, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & \mathrm{H}^1(G_{a,L}, \Delta_X) & \longleftarrow & \mathrm{H}^0(G_{a,L}, \mathrm{Br}(\overline{X})[2]) \end{array}$$

where the vertical arrows are inflation maps. Claim (a) follows from this diagram.

Similarly, the boundary maps defined by the exact sequence (27)

$$\mathrm{Br}(\overline{X})\{2\}^{\Gamma_L} = \mathrm{Br}(\overline{X})[4]^{\Gamma_L} \longrightarrow \mathrm{H}^1(L, \Delta_X), \quad \mathrm{H}^0(\tilde{G}_{a,L}, \mathrm{Br}(\overline{X})[4]) \longrightarrow \mathrm{H}^1(G_{a,L}, \Delta_X)$$

are injective. We have a similar commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^1(L, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & \mathrm{H}^1(L, \Delta_X) & \xleftarrow{\partial_1} & \mathrm{Br}(\overline{X})[4]^{\Gamma_L} \\
\uparrow \cong & & \uparrow & & \uparrow \cong \\
\mathrm{H}^1(\tilde{G}_{a,L}, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & \mathrm{H}^1(\tilde{G}_{a,L}, \Delta_X) & \longleftarrow & \mathrm{H}^0(\tilde{G}_{a,L}, \mathrm{Br}(\overline{X})[4]).
\end{array}$$

It immediately implies (b). \square

4.2 The case of $\mathbb{Q}(i)$

The main result of this section is the following description of the 2-primary torsion subgroup of the transcendental Brauer group of X over the field $k = \mathbb{Q}(i)$.

Theorem 4.2 *Let $X = X_a$ be a diagonal quartic surface with $a_1, a_2, a_3 \in \mathbb{Q}^\times$. Then $(\mathrm{Br}(X_k)/\mathrm{Br}_1(X_k))\{2\}$ is $\mathbb{Z}/2$ if X_k is equivalent to $X_{1,2,8} \times_{\mathbb{Q}} k$, and 0 otherwise.*

We will apply Lemma 4.1 to a field extension $L = K$ of k not containing $k(\sqrt{2})$ and such that $K_a = K(\sqrt[4]{2})$. Then

$$\begin{aligned}
G_{a,K} &= \mathrm{Gal}(K(\sqrt[4]{2})/K) = \mathrm{Gal}(k(\sqrt[4]{2})/k), \\
\tilde{G}_{a,K} &= \mathrm{Gal}(K(\sqrt[8]{-2})/K) = \mathrm{Gal}(k(\sqrt[8]{-2})/k).
\end{aligned}$$

In particular, the condition of Lemma 4.1 is satisfied.

Lemma 4.3 *Let K be a field extension of k not containing $k(\sqrt{2})$ such that $K_a = K(\sqrt[4]{2})$. Then the cardinality of the image of $\mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})^*) \rightarrow \mathrm{H}^1(G_{a,K}, \Delta_X)$ is equal to the cardinality of the cokernel of $\mathrm{H}^0(G_{a,K}, \mathrm{Pic}(\overline{X})^*) \rightarrow \mathrm{H}^0(G_{a,K}, \Delta_X)$. The same is true if $G_{a,K}$ is replaced by $\tilde{G}_{a,K}$.*

Proof. By the duality theorem for Tate cohomology groups [Bro82, Ch. VI, §7, Exercise 3] we have an isomorphism

$$\mathrm{H}^{-1}(G_{a,K}, \mathrm{Pic}(\overline{X})^*) \cong \mathrm{Hom}(\mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})), \mathbb{Z}/4).$$

Since $G_{a,K}$ is cyclic, we have a canonical isomorphism

$$\mathrm{H}^{-1}(G_{a,K}, \mathrm{Pic}(\overline{X})^*) \cong \mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})^*).$$

Thus $\mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})^*)$ is (non-canonically) isomorphic to $\mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X}))$. The group $\tilde{G}_{a,K}$ is also cyclic, so the same argument applies with $\mathbb{Z}/4$ replaced by $\mathbb{Z}/8$.

We have an exact sequence of abelian groups

$$\mathrm{H}^0(G_{a,K}, \mathrm{Pic}(\overline{X})^*) \rightarrow \mathrm{H}^0(G_{a,K}, \Delta_X) \rightarrow \mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})) \rightarrow \mathrm{H}^1(G_{a,K}, \mathrm{Pic}(\overline{X})^*),$$

and a similar sequence for $\tilde{G}_{a,K}$. The cardinalities of the third and fourth groups are equal, and our statement follows. \square

Lemma 4.4 *Let K be a field extension of k not containing $k(\sqrt{2})$. Let $X = X_a$ be a diagonal quartic surface where (a_1, a_2, a_3) is in the following list:*

$$(1, 1, 2), \quad (1, 1, 4), \quad (1, 1, 8), \quad (1, 2, 2), \quad (1, 2, 4), \quad (1, 4, 4), \quad (1, 4, 8), \quad (2, 4, 8).$$

Then $\text{Br}(\overline{X})[2] \cap \text{Im}[\text{Br}(X_K) \rightarrow \text{Br}(\overline{X})] = 0$.

Proof. In the proof of the statement, we can replace K by a finite extension. In the cases $(1, 1, 4)$ and $(1, 4, 4)$ we go over to $K(\sqrt{2})$ which splits X but does not contain $\sqrt[4]{2}$, so the result follows from Theorem 3.8.

We identify $(1, 1, 2)$ and $(1, 1, 8)$ as case B59, while $(2, 4, 8)$ is case B46 in [Bri02, Table B]. In these cases $H^1(G_{a,K}, \text{Pic}(\overline{X})) = 0$, hence $H^1(G_{a,K}, \text{Pic}(\overline{X})^*) = 0$, so we conclude by Lemma 4.1 (a).

It remains to consider the cases $(1, 2, 2)$, $(1, 2, 4)$, and $(1, 4, 8)$. Using a computer we check that the map $H^0(G_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^0(G_{a,K}, \Delta_X)$ is surjective in these three cases (see Lemma A.1). By Lemma 4.3 it follows that $H^1(G_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^1(G_{a,K}, \Delta_X)$ is the zero map. We conclude by applying Lemma 4.1 (a). \square

Proof of Theorem 4.2. Since -4 is a fourth power in $k = \mathbb{Q}(i)$ we can multiply the coefficients a_1, a_2, a_3 by -4 and fourth powers of their denominators if necessary and so assume that they are positive integers. Write

$$a_1 = 2^\alpha a, \quad a_2 = 2^\beta b, \quad a_3 = 2^\gamma c,$$

where a, b, c are positive odd integers. The triviality of the intersection of $\text{Br}(\overline{X})[2]$ with the image of $\text{Br}(X_k)$ in $\text{Br}(\overline{X})$ can be proved over a finite extension of k . The field $K = k(\sqrt[4]{a}, \sqrt[4]{b}, \sqrt[4]{c})$ satisfies the assumptions of Lemma 4.4, so we conclude the proof by replacing k with K , except in the case when X_K is equivalent to $X_{1,2,8} \times_{\mathbb{Q}} K$ which is not covered by Lemma 4.4.

In the rest of the proof we deal with this exceptional case and assume $a_1 = a$, $a_2 = 2b$, $a_3 = 8c$, where a, b, c are positive odd integers.

The main case. Let us first deal with the case $(a_1, a_2, a_3) = (1, 2, 8)$. We observe that the product of the coefficients is a fourth power in \mathbb{Q} , so the $\Gamma_{\mathbb{Q}}$ -modules $\text{Br}(\overline{X})$ and $\text{Br}(\overline{F})$ are isomorphic. We work over $K = k$ and apply Lemma 4.1 (a). The action of $G_{a,K} = \text{Gal}(k(\sqrt[4]{2})/k)$ on $\mathcal{O}/8$ factors through a group of order 2 whose generator acts as multiplication by $1 + 4i$. By the standard formula for the first cohomology group of a cyclic group we obtain $H^1(G_{a,K}, \Delta_X) \simeq (\mathbb{Z}/2)^2$. The injective map $H^0(G_{a,K}, \text{Br}(\overline{X})[2]) = \mathcal{O}/2 \rightarrow H^1(G_{a,K}, \Delta_X)$ is therefore an isomorphism. By Lemma 4.3 the image of $H^1(G_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^1(G_{a,K}, \Delta_X)$ has the same cardinality as the cokernel of $H^0(G_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^0(G_{a,K}, \Delta_X)$. It can be checked on a computer (cf. Lemma A.2) that this cardinality is 2.

To prove that $(\text{Br}(X_k)/\text{Br}_1(X_k))\{2\} \cong \mathbb{Z}/2$ we show that this group does not have an element of order 4. By Lemma 4.3 the image of the homomorphism

$H^1(\tilde{G}_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^1(\tilde{G}_{a,K}, \Delta_X)$ has the same cardinality as the cokernel of $H^0(\tilde{G}_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^0(\tilde{G}_{a,K}, \Delta_X)$. But this cokernel is the same as the cokernel of $H^0(G_{a,K}, \text{Pic}(\overline{X})^*) \rightarrow H^0(G_{a,K}, \Delta_X)$, because the action of $\tilde{G}_{a,K}$ factors through the action of $G_{a,K}$. By Lemma 4.1 (b) we conclude that the 2-primary torsion subgroup of $\text{Br}(X_k)/\text{Br}_1(X_k)$ is $\mathbb{Z}/2$.

The remaining cases. Having dealt with the main case we need to show that in all other cases $\text{Br}(\overline{X})[2]$ intersects trivially with the image of $\text{Br}(X_k) \rightarrow \text{Br}(\overline{X})$. There is a prime p dividing the positive odd integer abc . Let L be the field obtained by adjoining to k the fourth roots of all the prime factors of abc other than p . Then X_L is equivalent to a diagonal quartic surface of one of the following types.

$(p^n, 2, 8)$, where $n = 1, 2, 3$. Let K be the extension of L obtained by adjoining the fourth root of $p/4, p/2, p/4$, respectively. We note that 2 is not a square in K and $K_a = K(\sqrt[4]{2})$. As X_K is equivalent to $X_{2,4,8} \times_{\mathbb{Q}} K$ considered in Lemma 4.4, we conclude by this lemma.

$(1, 2p^m, 8p^n)$, where $m > 0$ or $n > 0$. Let $K = L(\sqrt[4]{p/2})$. A tedious but easy verification shows that X_K is equivalent to one of the surfaces considered in Lemma 4.4, except in the case $(1, 2p^2, 8p^2)$. In this case we let $M = L(\sqrt[4]{2p^2})$. Note that 2 is a square in M , so $8p^2$ is a fourth power in M , hence M splits X . But 2 is not a fourth power in M , so we conclude by Theorem 3.8.

$(p^2, 2p^m, 8p^n)$. Another tedious but easy verification shows that extending L to $K = L(\sqrt[4]{p/2})$ reduces all cases, except those equivalent over K to $(p^2, 2p^2, 8)$, to the cases covered in Lemma 4.4.

$(p, 2p^m, 8p^n)$. Let $K_0 = L(\pi)$, where $\pi^2 = p$, and let $K = K_0(\sqrt[4]{\pi/2})$. We note that 2 is not a square in K and $K_a = K(\sqrt[4]{2})$. The surface X_K is equivalent to one of the cases treated before, unless it is equivalent to $(\pi^2, 2\pi^2, 8)$.

It remains to show that for the diagonal quartic surface X with coefficients $(r^2, 2r^2, 8)$ over a field $K \supset k$ such that $[K(\sqrt{2}, \sqrt{r}) : K] = 4$ we have $\text{Br}(X)\{2\} = \text{Br}_1(X)\{2\}$. The proof uses the following lemma.

Lemma 4.5 *Let $A \times B$ be a product of two groups. Let M be a B -module. Consider M as an $A \times B$ -module with trivial action of A . Then the inflation-restriction sequence with respect to the subgroup $A \subset A \times B$ is a split exact sequence, so we have a canonical isomorphism*

$$H^1(A \times B, M) \cong H^1(A, M)^B \oplus H^1(B, M),$$

where $H^1(A, M)^B \cong \text{Hom}(A, M^B)$.

Proof. We have $H^1(A, M) \cong \text{Hom}(A, M)$, and since A and B commute, we have $\text{Hom}(A, M)^B = \text{Hom}(A, M^B)$.

The restriction map $H^1(A \times B, M) \rightarrow H^1(A, M)^B = \text{Hom}(A, M^B)$ has a section given by the inflation map $\text{Hom}(A, M^B) \rightarrow H^1(A \times B, M)$. Thus $H^1(A \times B, M)$ is the direct sum of $H^1(A, M)^B$ and $H^1(B, M^A) = H^1(B, M)$. \square

End of proof of Theorem 4.2. The product of the coefficients $(r^2, 2r^2, 8)$ is a fourth power in K , so the $\text{Gal}(\overline{K}/K)$ -modules $\text{Br}(\overline{X})$ and $\text{Br}(\overline{F})$ are isomorphic. We have $K_a = K(\sqrt[4]{2}, \sqrt{r})$ and $G_{a,K} = G \times H$, where

$$G = \text{Gal}(K(\sqrt[4]{2})/K) = \text{Gal}(k(\sqrt[4]{2})/k) \simeq \mathbb{Z}/4, \quad H = \text{Gal}(K(\sqrt{r})/K) \simeq \mathbb{Z}/2,$$

and H acts trivially on $\text{Br}(\overline{X})$ [16]. The condition of Lemma 4.1 is satisfied.

We can apply Lemma 4.5 to $M = \Delta_X$, $A = H$, $B = G$. Note that the inclusion $(\Delta_X)^G \rightarrow \Delta_X$ is the natural map $\mathcal{O}/4 \rightarrow \mathcal{O}/8$; it induces an isomorphism $H^1(H, (\Delta_X)^G) \xrightarrow{\sim} H^1(H, \Delta_X)$ as both groups are $\mathcal{O}/2$. By Lemma 4.5 we obtain

$$H^1(G \times H, \Delta_X) \cong H^1(G, \Delta_X) \oplus \text{Hom}(H, (\Delta_X)^G) \cong H^1(G, \Delta_X) \oplus \mathcal{O}/2.$$

The map $\partial_1: (\mathcal{O}/2)^G \rightarrow H^1(G, \Delta_X)$ was already computed in the previous case $(1, 2, 8)$, and was found to be an isomorphism. The inflation-restriction sequence with respect to the subgroup $H \subset G \times H$ gives rise to the following commutative diagram, where the middle column is a split exact sequence of Lemma 4.5:

$$\begin{array}{ccccc} H^1(H, \text{Pic}(\overline{X})^*) & \longrightarrow & \mathcal{O}/2 & & \\ \text{res} \uparrow & & \text{res} \uparrow & & \\ H^1(G \times H, \text{Pic}(\overline{X})^*) & \longrightarrow & H^1(G, \Delta_X) \oplus \mathcal{O}/2 & \xleftarrow{\partial_1} & (\mathcal{O}/2)^{G \times H} \\ & & \text{inf} \uparrow & & \uparrow = \\ & & H^1(G, \Delta_X) & \xleftarrow{\cong} & (\mathcal{O}/2)^G \end{array}$$

By Lemma 4.1 (a), if the intersection of images of the maps in the middle row is zero, then $\text{Br}(X_K)\{2\} = \text{Br}_1(X_K)\{2\}$. The commutativity of this diagram implies that $\text{Im}(\partial_1)$ coincides with the direct summand $H^1(G, \Delta_X)$ of $H^1(G, \Delta_X) \oplus \mathcal{O}/2$. Thus we need to show that the image of $H^1(G \times H, \text{Pic}(\overline{X})^*)$ intersects $H^1(G, \Delta_X)$ trivially. The cardinality of the image of $H^1(G \times H, \text{Pic}(\overline{X})^*)$ is 4. This can be checked using the exact sequence in the proof of Lemma 4.3 using the fact that each of the groups $H^1(G \times H, \text{Pic}(\overline{X}))$ and $H^1(G \times H, \text{Pic}(\overline{X})^*)$ has 16 elements. Indeed, since $(\Delta_X)^{G \times H} \simeq (\mathbb{Z}/4)^2$, one needs to check that the image of $(\text{Pic}(\overline{X})^*)^{G \times H} \rightarrow (\Delta_X)^{G \times H}$ is 4. Furthermore, the composition of the left upward arrow and the top horizontal arrow is surjective; see Lemma A.3 for all these facts. The commutativity of the diagram now implies the desired result. This finishes the proof of Theorem 4.2. \square

4.3 The case of \mathbb{Q}

The following is the main result of the paper.

Theorem 4.6 *Let $X = X_a$ be a diagonal quartic surface with $a_1, a_2, a_3 \in \mathbb{Q}^\times$. Then $(\mathrm{Br}(X)/\mathrm{Br}_1(X))\{2\}$ is $\mathbb{Z}/2$ if X is equivalent to $X_{1,2,-2}$ or $X_{1,8,-8}$, and 0 otherwise.*

Proof. By Theorem 4.2 we need to deal with the following cases:

$$(1, 2, 8), \quad (1, -2, -8), \quad (2, -2, -4), \quad (-2, 4, 8), \quad (1, 2, -2), \quad (1, 8, -8). \quad (28)$$

Each of these surfaces satisfies $\mathbb{Q}_a = \mathbb{Q}(i, \sqrt[4]{2}) = k(\sqrt[4]{2})$, where $k = \mathbb{Q}(i)$. By Lemma 4.1, in order to conclude that $\mathrm{Br}(X)\{2\} = \mathrm{Br}_1(X)\{2\}$ it is enough to show that the images of $H^0(G_{a,\mathbb{Q}}, \mathrm{Br}(\overline{X})[2])$ and $H^1(G_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*)$ in $H^1(G_{a,\mathbb{Q}}, \Delta_X)$ intersect trivially.

In the first three cases the product of coefficients is in $\mathbb{Q}^{\times 4}$, so the $\Gamma_{\mathbb{Q}}$ -modules $\mathrm{Br}(\overline{X})$ and $\mathrm{Br}(\overline{F})$ are isomorphic, hence $H^1(G_{a,\mathbb{Q}}, \Delta_X) \simeq H^1(G_{a,\mathbb{Q}}, \Delta_F)$. Proposition 3.9 (ii) implies that $\mathrm{Br}(\overline{X})\{2\}^{\Gamma_{\mathbb{Q}}} \simeq \mathbb{Z}/2 \times \mathbb{Z}/4$, hence $H^0(G_{a,\mathbb{Q}}, \mathrm{Br}(\overline{X})[2]) \cong \mathcal{O}/2$. A computer calculation (cf. Lemma A.4) shows that $H^1(G_{a,\mathbb{Q}}, \Delta_X) \simeq (\mathbb{Z}/2)^4$ and in each case the image of $H^1(G_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*)$ has trivial intersection with the image of $H^0(G_{a,\mathbb{Q}}, \mathrm{Br}(\overline{X})[2])$.

In the last three cases, where the product is in $-4\mathbb{Q}^{\times 4}$, we have $\mathrm{Br}(\overline{X})\{2\}^{\Gamma_{\mathbb{Q}}} \simeq \mathbb{Z}/4$, hence $H^0(G_{a,\mathbb{Q}}, \mathrm{Br}(\overline{X})[2]) \cong \mathbb{Z}/2$. One checks that in all three cases the group $H^1(G_{a,\mathbb{Q}}, \Delta_X)$ is $\mathbb{Z}/2$. Then a computer calculation (cf. Lemma A.4) shows that in the case $(-2, 4, 8)$ the image of $H^1(G_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*) \rightarrow H^1(G_{a,\mathbb{Q}}, \Delta_X)$ is 0. In contrast, in the last two cases the map $H^1(G_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*) \rightarrow H^1(G_{a,\mathbb{Q}}, \Delta_X)$ turns out to be the surjective map $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$. Thus $(\mathrm{Br}(X)/\mathrm{Br}_1(X))\{2\} \cong \mathbb{Z}/2$.

It remains to prove that there is no element of order 4 in $(\mathrm{Br}(X)/\mathrm{Br}_1(X))\{2\}$. To do this, consider $\tilde{G}_{a,\mathbb{Q}} = \mathrm{Gal}(k(\sqrt[8]{-2})/\mathbb{Q})$. By Lemma 4.1 it is enough to show that the intersection of the image of $H^0(\tilde{G}_{a,\mathbb{Q}}, \mathrm{Br}(\overline{X})[4])$ with the image of $H^1(\tilde{G}_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*)$ in $H^1(\tilde{G}_{a,\mathbb{Q}}, \Delta_X)$ is killed by 2. We have a commutative diagram

$$\begin{array}{ccc} H^1(\tilde{G}_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & H^1(\tilde{G}_{a,\mathbb{Q}}, \Delta_X) \\ \uparrow \cong & & \uparrow \\ H^1(G_{a,\mathbb{Q}}, \mathrm{Pic}(\overline{X})^*) & \longrightarrow & H^1(G_{a,\mathbb{Q}}, \Delta_X) \end{array}$$

which shows that the image of the upper horizontal arrow is isomorphic to the image of lower horizontal arrow, which is $\mathbb{Z}/2$. \square

Proposition 4.7 *Let X be a diagonal quartic surface over \mathbb{Q} . Let X' be the diagonal quartic surface obtained by multiplying any two coefficients of X by -4 . Then the $\Gamma_{\mathbb{Q}}$ -modules $H_{\mathrm{ét}}^2(\overline{X}, \mathbb{Q}_2)$ and $H_{\mathrm{ét}}^2(\overline{X}', \mathbb{Q}_2)$ are isomorphic, and the $\Gamma_{\mathbb{Q}}$ -modules $H_{\mathrm{ét}}^2(\overline{X}, \mathbb{Z}_\ell)$ and $H_{\mathrm{ét}}^2(\overline{X}', \mathbb{Z}_\ell)$ are isomorphic for all odd primes ℓ . In particular, $|X(\mathbb{F}_q)| = |X'(\mathbb{F}_q)|$ for all prime powers q coprime to $a_1 a_2 a_3$.*

Proof. Twisting both surfaces we can assume that $X = F$ is the Fermat surface and $X' = F' = X_{(1,-4,-4)}$. Since the degree of the polarisation is 4, it is enough to construct an isomorphism between primitive cohomology $\Gamma_{\mathbb{Q}}$ -modules $P_{\ell}(F)$ and $P_{\ell}(F')$, and also between $P(F) \otimes \mathbb{Q}_2$ and $P(F') \otimes \mathbb{Q}_2$. Write $k = \mathbb{Q}(i)$. Since $-4 \in (k^{\times})^4$, we have $F_k \cong F'_k$ and thus the representations of Γ_k are isomorphic. It remains to show that this isomorphism extends to complex conjugation.

Recall that the action of $\tau(1)$ on $P(F)$ fixes the generator: $\tau(1)(2\pi ie') = 2\pi ie'$, see (18). Thus for $P(F')$ we have $\tau(1)(2\pi ie') = u_2 u_3 (2\pi ie')$. In $\mathbb{Z}[x]/(x^3 + x^2 + x + 1)$ we have $4(1-x)^{-1} = x^2 + 2x + 3$. Hence $(1-x)(1-y)$ is an invertible element of $\mathbb{Q}_2[x, y]/(x^3 + x^2 + x + 1, y^3 + y^2 + y + 1)$ and of $\mathbb{Z}_{\ell}[x, y]/(x^3 + x^2 + x + 1, y^3 + y^2 + y + 1)$, where ℓ is odd. It follows that multiplication by $(1-u_2)(1-u_3)$ defines isomorphisms $P(F) \otimes \mathbb{Q}_2 \cong P(F') \otimes \mathbb{Q}_2$ and $P_{\ell}(F) \cong P_{\ell}(F')$ where ℓ is odd. Since Γ_k commutes with $(\mu_4)^3$, these isomorphisms respect the action of Γ_k . As one immediately checks, they also respect the action of $\tau(1)$. \square

A Computations for the main theorem

This appendix collects details of computations used in the proofs of Theorems 4.2 and 4.6. While these are not too challenging to be unachievable by hand, it is more convenient to carry them out with the help of a computer algebra system. Magma [Magma] code to this effect is available at <https://github.com/dgvirtz/diagonalquartics> and from the auxiliary files accompanying this article arXiv:1905.11869 on the arxiv.org preprint server.

Consider the diagonal quartic surface X_t with coefficients t_1, t_2, t_3 over the field $\mathbb{Q}(t_1, t_2, t_3)$, where t_1, t_2, t_3 are independent variables. Let

$$G_{\eta} = \text{Gal}(\mathbb{Q}(i, \sqrt[4]{2}, \sqrt[4]{t_1}, \sqrt[4]{t_2}, \sqrt[4]{t_3})/\mathbb{Q}(t_1, t_2, t_3)).$$

We denote the generators of G_{η} by

$$\tau(i) = -i, \quad \sigma(\sqrt[4]{2}) = i\sqrt[4]{2}, \quad s_1(\sqrt[4]{t_1}) = i\sqrt[4]{t_1}, \quad s_2(\sqrt[4]{t_2}) = i\sqrt[4]{t_2}, \quad s_3(\sqrt[4]{t_3}) = i\sqrt[4]{t_3},$$

with the understanding that τ acts trivially on $\sqrt[4]{2}, \sqrt[4]{t_1}, \sqrt[4]{t_2}, \sqrt[4]{t_3}$, σ acts trivially on $i, \sqrt[4]{t_1}, \sqrt[4]{t_2}, \sqrt[4]{t_3}$, and similarly for s_1, s_2, s_3 . A diagonal quartic surface X_a over a number field K gives rise to a subgroup $G_{a,K} \subset G_{\eta}$ which will be described explicitly in each case.

Fix fourth roots $\sqrt[4]{a_1}, \sqrt[4]{a_2}, \sqrt[4]{a_3}$ and a primitive eighth root of unity ζ . Recall that the abelian group $\text{Pic}(\overline{X})$ is generated by 48 lines of the form

$$\Lambda_{h,i,j}^{m,n} : \quad \zeta^m x_0 = \sqrt[4]{a_h} x_h, \quad \zeta^n \sqrt[4]{a_i} x_i = \sqrt[4]{a_j} x_j,$$

where (h, i, j) is a permutation of $(1, 2, 3)$ and $m, n \in \{1, 3, 5, 7\}$. A basis \mathcal{B} of 20 lines, which Swinnerton-Dyer kindly provided to the authors, is given by

$$\Lambda_{1,2,3}^{1,1}, \Lambda_{1,2,3}^{1,3}, \Lambda_{1,2,3}^{1,5}, \Lambda_{1,2,3}^{1,7}, \Lambda_{1,2,3}^{3,1}, \Lambda_{1,2,3}^{3,3}, \Lambda_{1,2,3}^{3,5}, \Lambda_{1,2,3}^{3,7}, \Lambda_{1,2,3}^{5,1}, \Lambda_{1,2,3}^{5,3}, \Lambda_{1,2,3}^{5,5},$$

$$\begin{aligned} &\Lambda_{2,3,1}^{1,1}, \Lambda_{2,3,1}^{1,3}, \Lambda_{2,3,1}^{1,5}, \Lambda_{2,3,1}^{3,1}, \Lambda_{2,3,1}^{3,3}, \Lambda_{2,3,1}^{3,5}, \\ &\Lambda_{3,1,2}^{1,1}, \Lambda_{3,1,2}^{1,3}, \Lambda_{3,1,2}^{1,5}, \Lambda_{3,1,2}^{3,1}. \end{aligned}$$

(A similar basis of 20 lines appears in [SS].) The description of $\text{Pic}(\overline{X})$ in terms of lines can be matched with the description in the main text by first identifying the line $\Lambda_{1,2,3}^{1,1}$ with $\frac{1}{4}(L-c)$ and then identifying all other lines using the actions of $(\mu_4)^3$ and S_3 on \overline{X} . The images of w_1, w_2 in Δ_X are expressed in the basis \mathcal{B} as

$$\begin{aligned} &\frac{1}{8}(1 \ 1 \ 0 \ 0 \ 1 \ 4 \ 7 \ 0 \ 7 \ 3 \ 6 \ 6 \ 4 \ 2 \ 2 \ 4 \ 0 \ 4 \ 0 \ 4), \\ &\frac{1}{8}(2 \ 7 \ 3 \ 2 \ 1 \ 6 \ 1 \ 1 \ 5 \ 4 \ 4 \ 6 \ 6 \ 4 \ 2 \ 2 \ 4 \ 4 \ 4 \ 4). \end{aligned}$$

We write elements of $T(X_{\mathbb{C}})$ and $\Delta_X \cong T(X_{\mathbb{C}})/8$ in the basis (w_1, w_2) . The 1-cocycles in group cohomology will be explicitly described as maps from $G_{a,K}$ to the coefficient module.

In order to assist an independent verification, we give intermediate steps and details of our computations. Let $k = \mathbb{Q}(i)$ and let $G = \text{Gal}(k(\sqrt[4]{2})/k)$.

Lemma A.1 *Let $a \in \{(1, 2, 2), (1, 2, 4), (1, 4, 8)\}$. Let K be a field extension of k not containing $k(\sqrt{2})$. Then $G \cong G_{a,K}$ and the map $(\text{Pic}(\overline{X})^*)^G \rightarrow (\Delta_X)^G$ is surjective.*

Proof. In the first case, one has $G_{a,K} = \langle \sigma s_2 s_3 \rangle$. A computation of invariants gives $(\text{Pic}(\overline{X})^*)^G \simeq \mathbb{Z}^6$. By Proposition 3.9, one has $(\Delta_X)^G = \Delta_X[2]$; lifting the generators of $\Delta_X[2]$ to $(\text{Pic}(\overline{X})^*)^G$ is easy.

In the last two cases $G_{a,K} = \langle \sigma s_2 s_3^2 \rangle$ and $G_{a,K} = \langle \sigma s_2^2 s_3^3 \rangle$, respectively. A computation of invariants gives $(\text{Pic}(\overline{X})^*)^G \simeq \mathbb{Z}^3$. By Proposition 3.9, the group $(\Delta_X)^G \simeq \mathbb{Z}/2$ is generated by $\frac{1}{2}(w_1 + w_2)$ which lifts to $(\text{Pic}(\overline{X})^*)^G$. \square

Lemma A.2 *Assume $K = k = \mathbb{Q}(i)$ and $a = (1, 2, 8)$. Then $G \cong G_{a,k}$ and the cokernel of the map $(\text{Pic}(\overline{X})^*)^G \rightarrow (\Delta_X)^G$ has cardinality 2.*

Proof. One has $G_{a,k} = \langle \sigma s_2 s_3^3 \rangle$. Proposition 3.9 gives $(\Delta_X)^G = \Delta_X[4]$. A computation of invariants shows that $(\text{Pic}(\overline{X})^*)^G \simeq \mathbb{Z}^6$ and that the image of this group in $\Delta_X[4]$ is generated by the elements $\frac{1}{4}(w_1 \pm w_2)$. \square

Lemma A.3 *Let K be an extension of $k = \mathbb{Q}(i)$ and let $a = (r^2, 2r^2, 8)$, where $r \in K^\times$ is such that $[K(\sqrt{2}, \sqrt{r}) : K] = 4$. We have $G_{a,K} \cong G \times H$, where $H = \text{Gal}(K(\sqrt{r})/K)$. The image of $(\text{Pic}(\overline{X})^*)^{G \times H} \rightarrow (\Delta_X)^{G \times H}$ has cardinality 4,*

$$|\text{H}^1(G \times H, \text{Pic}(\overline{X})^*)| = |\text{H}^1(G \times H, \text{Pic}(\overline{X}))| = 16$$

and the composition of the following maps is surjective:

$$\text{H}^1(G \times H, \text{Pic}(\overline{X})^*) \xrightarrow{\text{res}} \text{H}^1(H, \text{Pic}(\overline{X})^*) \longrightarrow \text{H}^1(H, \Delta_X) = \Delta_X[2].$$

Proof. One has $G_{a,K} = \langle \sigma s_2 s_3^2, s_1^2 s_2^2 \rangle$. A computation gives $(\text{Pic}(\overline{X})^*)^{G \times H} \simeq \mathbb{Z}^4$. The four generators of $(\text{Pic}(\overline{X})^*)^{G \times H}$ lie in $\text{Pic}(\overline{X})$ only after doubling and, furthermore, they are not all in the same equivalence class modulo $\text{Pic}(\overline{X})$. Thus their images span $\Delta_X[2]$.

The first cohomology of $\text{Pic}(\overline{X})$ was already computed in [Bri02, Table B], case B37. Another computation shows that $H^1(G \times H, \text{Pic}(\overline{X})^*)$ is generated by three 1-cocycles c_1, c_2 and c_3 of orders 2, 2 and 4 such that c_2 and c_3 map to generators of $H^1(H, \Delta_X) \cong \Delta_X[2]$. \square

Lemmas A.1–A.3 concern $\text{Pic}(\overline{X}), \text{Pic}(\overline{X})^*$ and Δ_X . In particular, no computational input about the transcendental lattice and its interplay with the algebraic cycles is necessary. Thus the verification of the computational facts up to this point can be achieved using solely the description of $\text{Pic}(\overline{X})$ in terms of lines. This is not so for the next lemma used in the proof of Theorem 4.6.

Lemma A.4 *Assume that $K = \mathbb{Q}$ and a is in the list (28). Then the intersection of images of the following maps*

$$H^1(G_{a,\mathbb{Q}}, \text{Pic}(\overline{X})^*) \xrightarrow{\rho} H^1(G_{a,\mathbb{Q}}, \Delta_X) \xleftarrow{\partial_1} \text{Br}(\overline{X})[2]^{G_{a,\mathbb{Q}}}$$

is trivial in the first four cases and is $\mathbb{Z}/2$ in the last two cases of (28).

Proof. In all cases $G_{a,\mathbb{Q}}$ is generated by two elements g_1, g_2 , which we specify. We describe 1-cocycles by the images of g_1 and g_2 .

Case $a = (1, 2, 8)$. We have $g_1 = \sigma s_2 s_3^3$ and $g_2 = \tau$. A computation gives $H^1(G_{a,\mathbb{Q}}, \text{Pic}(\overline{X})^*) \simeq (\mathbb{Z}/2)^2$. The group $H^1(G_{a,\mathbb{Q}}, \Delta_X) \simeq (\mathbb{Z}/2)^4$ is generated by

$$\begin{array}{cccc} g_1 \mapsto (0, 2) & g_1 \mapsto (-4, 0) & g_1 \mapsto 0 & g_1 \mapsto (-2, 0) \\ g_2 \mapsto 0 & g_2 \mapsto 0 & g_2 \mapsto (-4, 0) & g_2 \mapsto (0, 1). \end{array}$$

In this basis of $H^1(G_{a,\mathbb{Q}}, \Delta_X)$, we have

$$\text{Im}(\partial_1) = \langle (1, 1, 0, 0), (0, 0, 0, 1) \rangle \text{ and } \text{Im}(\rho) = \langle (0, 1, 0, 0) \rangle.$$

Case $a = (1, -2, -8)$. We have $g_1 = \sigma s_2 s_3^3$ and $g_2 = s_2 s_3 \tau$. A computation gives $H^1(G_{a,\mathbb{Q}}, \text{Pic}(\overline{X})^*) \simeq (\mathbb{Z}/2)^2$. The group $H^1(G_{a,\mathbb{Q}}, \Delta_X) \simeq (\mathbb{Z}/2)^4$ is generated by

$$\begin{array}{cccc} g_1 \mapsto (4, 4) & g_1 \mapsto (-2, 0) & g_1 \mapsto 0 & g_1 \mapsto (2, 2) \\ g_2 \mapsto 0 & g_2 \mapsto 0 & g_2 \mapsto (4, 4) & g_2 \mapsto (-1, 0). \end{array}$$

In this basis, $\text{Im}(\partial_1)$ is as in the previous case and $\text{Im}(\rho) = \langle (1, 0, 1, 0) \rangle$.

Case $a = (2, -2, -4)$. We have $g_1 = \sigma s_1 s_2^3$ and $g_2 = s_2 s_3 \tau$. A computation gives an isomorphism $H^1(G_{a,\mathbb{Q}}, \text{Pic}(\overline{X})^*) \simeq (\mathbb{Z}/2)^3$. The group $H^1(G_{a,\mathbb{Q}}, \Delta_X) \simeq (\mathbb{Z}/2)^4$ is

generated by the same cocycles as for $a = (1, -2, -8)$. In this basis, $\text{Im}(\partial_1)$ is as in the previous case and $\text{Im}(\rho) = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$.

Case $a = (-2, 4, 8)$. We have $g_1 = \sigma s_1^3 s_2^2 s_3^3$ and $g_2 = s_1 \tau$. Then $H^1(G_a, \text{Pic}(\overline{X})^*) \simeq (\mathbb{Z}/2)^2$ is generated by two cocycles which map to 0 in $H^1(G_a, \Delta_X)$.

Case $a = (1, 2, -2)$ (respectively, $a = (1, 8, -8)$). We have $g_1 = \sigma s_2 s_3^3$ (respectively, $g_1 = \sigma s_2^3 s_3$) and $g_2 = s_3 \tau$. A computation shows that $H^1(G_a, \text{Pic}(\overline{X})^*) \simeq \mathbb{Z}/4$ maps surjectively onto $H^1(G_a, \Delta_X) \simeq \mathbb{Z}/2$, which is generated by the cocycle sending g_1 to $(-2, 2)$ and g_2 to 0. \square

B Computations for the supplement

Let $a = (1, 2, -2)$ or $(1, 8, -8)$. In view of Theorems 4.2 and 4.6 we can apply Corollary 1.6 (b) with $n = 2$. Thus we need to compute the first hypercohomology group of the finite group $G_{a,K}$ with coefficients in the complex $\text{Pic}(\overline{X})^* \rightarrow T(X_{\mathbb{C}})/16$, where $K = \mathbb{Q}$ and $\mathbb{Q}(i)$.

Hypercohomology functionality is currently not available in Magma and had to be added. Our implementation constructs a part of the Cartan–Eilenberg resolution of the above complex and computes its first total cohomology. More precisely, let $Z^1(M)$ be the group of 1-cocycles of a $G_{a,K}$ -module M . Consider the double complex

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \uparrow & & \\
 & & Z^1(\text{Pic}(\overline{X})^*) & \xrightarrow{h_{01}} & Z^1(T(X_{\mathbb{C}})/16) \\
 & v_{00} \uparrow & & & \uparrow v_{10} \\
 \text{Pic}(\overline{X})^* & \xrightarrow{h_{00}} & T(X_{\mathbb{C}})/16 & \longrightarrow & 0
 \end{array}$$

where h_{01} is the map induced by h_{00} and the vertical maps are the differentials of the standard resolutions sending m to $gm - m$. Details of the calculations are as follows. (Matrices representing linear maps are understood to act on the right.)

Case $K = \mathbb{Q}(i)$. One computes that $Z^1(\text{Pic}(\overline{X})^*)$ is freely generated by 14 cocycles and $Z^1(T(X_{\mathbb{C}})/16)$ is generated by two cocycles which map the generator $\sigma s_2 s_3^3$ of $G_{a,K}$ to $(4, 4)$ and to $(-4, 0)$. Using (w_1, w_2) and the above basis for $Z^1(T(X_{\mathbb{C}})/16)$, one checks that v_{10} is given by

$$\begin{bmatrix} -1 & -3 \\ -2 & 9 \end{bmatrix}.$$

Similarly, one easily finds matrices representing h_{01} and v_{00} . A computation gives that $\text{Ker}(h_{01} - v_{10}) \cong (\mathbb{Z}/4)^2 \oplus \mathbb{Z}^{14}$ and, after choosing generators x_1, x_2 of the torsion subgroup and linearly independent y_1, \dots, y_{14} suitably in this kernel, we find that

$\text{Im}(v_{00} \oplus h_{00})$ is generated by $y_1 + y_2 + 2y_3, 2y_1 + x_1, 2y_2 + x_1, x_2, y_4, \dots, y_{14}$. Thus the first hypercohomology is $(\mathbb{Z}/4)^2$.

Case $K = \mathbb{Q}$. Recall that $G_{a,\mathbb{Q}}$ is generated by $g_1 = \sigma s_2 s_3^3$ (or $\sigma s_2^3 s_3$) and $g_2 = s_3 \tau$. One computes that $Z^1(\text{Pic}(\overline{X})^*)$ is freely generated by 17 cocycles and $Z^1(T(X_{\mathbb{C}})/16)$ is generated by the two cocycles

$$\begin{array}{ll} g_1 \mapsto (4 & 0) & g_1 \mapsto (-4 & 4) \\ g_2 \mapsto (-1 & 1) & g_2 \mapsto 0 \end{array}.$$

Using (w_1, w_2) and the above basis for $Z^1(T(X_{\mathbb{C}})/16)$, one checks that v_{10} is represented by

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Similarly, one easily finds matrices representing h_{01} and v_{00} . A computation gives that $\text{Ker}(h_{01} - v_{10}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}^{17}$ and, after choosing x of order 4 and linearly independent y_1, \dots, y_{17} suitably, $\text{Im}(v_{00} \oplus \phi)$ is generated by $2x, x + 4y_1, y_2, \dots, y_{17}$. Thus the first hypercohomology is $\mathbb{Z}/8$.

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Department of Mathematics, University College London, WC1E 6BT England, U.K.
d.gvirtz@ucl.ac.uk

Department of Mathematics, South Kensington Campus, Imperial College London, SW7 2BZ England, U.K. – and – Institute for the Information Transmission Problems, Russian Academy of Sciences, 19 Bolshoi Karetnyi, Moscow, 127994 Russia
a.skorobogatov@imperial.ac.uk