Received: 18 March 2022 | Revised: 6 October 2022 | Accepted: 25 October 2022

DOI: 10.1112/jlms.12701

RESEARCH ARTICLE

Journal of the London Mathematical Society

Hecke algebras and the Schlichting completion for discrete quantum groups

Adam Skalski[1] | **Roland Vergnioux**[2] | **Christian Voigt**[3]

[1]Institute of Mathematics of the Polish Academy of Sciences, Warszawa, Poland

[2]Normandie Univ, UNICAEN, CNRS, LMNO, Caen, France

[3]School of Mathematics & Statistics, University of Glasgow, University Place, Glasgow, UK

Correspondence

Christian Voigt, School of Mathematics, and Statistics, University of Glasgow, University Place, Glasgow G12 8QQ, UK.
Email: christian.voigt@glasgow.ac.uk

Funding information

NCN, Grant/Award Number: 2020/39/I/ST1/01566; Agence Nationale de la Recherche, Grant/Award Number: ANR-19-CE40-0002; CEFIPRA, Grant/Award Number: 6101-1; EPSRC, Grant/Award Number: EP/T03064X/1

Abstract

We introduce Hecke algebras associated to discrete quantum groups with commensurated quantum subgroups. We study their modular properties and the associated Hecke operators. In order to investigate their analytic properties we adapt the construction of the Schlichting completion to the quantum setting, thus obtaining locally compact quantum groups with compact open quantum subgroups. We study in detail a class of examples arising from quantum HNN extensions.

MSC 2020

46L67 (primary), 16T20, 20C08, 20G42, 46L05, 46L65 (secondary)

1 | INTRODUCTION

Hecke algebras, originally studied in the analysis of Hecke operators for elliptic modular forms, play a prominent rôle in representation theory and harmonic analysis. In applications to number theory one is typically interested in Hecke operators associated with arithmetic groups. Abstractly, the relevant operators can be described starting from a discrete group Γ together with a commensurated subgroup, that is, a subgroup $\Lambda \subset \Gamma$ such that $\Lambda \cap \Lambda^g$ has finite index in Λ for all $g \in \Gamma$, where $\Lambda^g = g\Lambda g^{-1}$. At this level of generality, Hecke operators can be viewed as Γ-equivariant bounded operators on $\ell^2(\Gamma/\Lambda)$ and can be described using the Hecke algebra $\mathcal{H}(\Gamma, \Lambda)$, which is nothing but the space of functions on double cosets $c_c(\Lambda\backslash\Gamma/\Lambda)$ equipped with a suitable convolution product.

In their seminal paper [3], Bost and Connes exhibited an intriguing connection between Hecke algebras, number theory, and non-commutative geometry. The Hecke algebra underlying the Bost–Connes system is part of a quantum statistical mechanical system whose equilibrium states are intimately related to class field theory, and the time evolution of the system can be explicitly described by means of the modular function

$$\nabla : g \mapsto [\Lambda : \Lambda \cap \Lambda^g]/[\Lambda^g : \Lambda \cap \Lambda^g],$$

comparing the number of left and right cosets in a double coset.

Hecke operators and Hecke algebras can also be defined for locally compact groups G together with compact open subgroups $H \subset G$. Moreover, both situations are related by the Schlichting completion construction which associates to each discrete Hecke pair (Γ, Λ) in a canonical way a pair (G, H) consisting of a totally disconnected locally compact group G and a compact open subgroup $H \subset G$ such that $\mathcal{H}(\Gamma, \Lambda) \cong \mathcal{H}(G, H)$ [23]. Analytical properties of the algebra of Hecke operators are often easier to analyze at the level of the Schlichting completion, see [25], [1], and [12].

The aim of this article is to extend some of this theory to the case of locally compact quantum groups, both in the discrete and compact open settings. This exhibits new combinatorial behavior which is invisible in the classical case, related to the ‘relative dimension’ constants naturally associated with quantum subgroups of discrete quantum groups. A major motivation for the passage to the quantum framework, apart from producing interesting examples of von Neumann algebras, is to create new locally compact quantum groups out of known discrete quantum groups. For this purpose we develop a generalization of the Schlichting completion procedure, which provides a slightly new perspective even for classical groups, and leads to a deeper understanding of the quantum quotient spaces $\mathbb{\Gamma}/\mathbb{\Lambda}$ for discrete quantum groups. The Schlichting completion yields algebraic quantum groups in the sense of Van Daele [28], and we obtain concrete examples by considering pairs $(\mathbb{\Gamma}, \mathbb{\Lambda})$ of discrete quantum groups arising from HNN extensions.

Let us describe the main results obtained in the article. After collecting some preliminaries in Section 2, we begin our analysis in the setting of a subgroup $\mathbb{\Lambda}$ in a discrete quantum group $\mathbb{\Gamma}$ in Section 3. We give a detailed description of the non-commutative quotient space $\mathbb{\Gamma}/\mathbb{\Lambda}$ and of the associated module category, see Proposition 3.5 and Theorem 3.10. This allows us to obtain an explicit formula for the quantum analog μ of the counting measure on $\mathbb{\Gamma}/\mathbb{\Lambda}$, in terms of the equivalence relation induced by $\mathbb{\Lambda}$ on irreducible corepresentations of $\mathbb{\Gamma}$, see Definition 3.13. We also give in Proposition 3.17 a categorical interpretation of the constants κ that appear in this formula. These constants are trivial in the classical case.

With this in place it is easy to write down the definition of the convolution product of the Hecke algebra $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$, see Definitions 3.19 and 3.24. We prove in Theorem 3.29 that this algebra is canonically isomorphic to the algebra of Hecke operators, that is, $\mathbb{\Gamma}$-equivariant linear maps on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Moreover, in Theorem 3.32 we give a combinatorial characterization of the boundedness of the Hecke operators on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$, in terms of the constants κ.

Next we investigate the modular properties of the canonical state on $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ and give an explicit formula for the corresponding modular operator in Proposition 3.42 and Theorem 3.36. This formula involves the number of left and right cosets in double cosets, as in the classical case, but in general also the modular structure of the discrete quantum group $\mathbb{\Gamma}$, as well as the constants κ. In Paragraph 3.3 we consider an explicit class of examples arising from HNN extensions.

In Section 4 we switch to the setting of a locally compact quantum group $\mathbb{G}$ with a compact open quantum subgroup $\mathbb{H}$, working in the framework of algebraic quantum groups. In this case it is easier to define the Hecke algebra $\mathcal{H}(\mathbb{G},\mathbb{H})$ since compactly supported functions on $\mathbb{H}\backslash\mathbb{G}/\mathbb{H}$ are also compactly supported on $\mathbb{G}$, and one can directly use the convolution product of the quantum algebra of functions $\mathcal{O}_c(\mathbb{G})$. Again, we establish a canonical isomorphism of $\mathcal{H}(\mathbb{G},\mathbb{H})$ with the algebra of $\mathbb{G}$-equivariant maps on $c_c(\mathbb{G}/\mathbb{H})$ in Proposition 4.3. As an example, we discuss the case of the quantum doubles $\mathbb{G} = \mathbb{H} \bowtie \hat{\mathbb{H}}$ of a compact quantum group $\mathbb{H}$: in this case $\mathcal{H}(\mathbb{G},\mathbb{H})$ identifies with the algebra of characters of $\mathbb{H}$.

In Paragraph 4.3 we associate a pair $(\mathbb{G},\mathbb{H})$ to each discrete Hecke pair (Γ,Λ) by means of a quantum analog of the Schlichting completion. More precisely, we construct $\mathcal{O}_c(\mathbb{G})$ directly as a subalgebra of $\ell^\infty(\Gamma)$ using the Hecke convolution product between $c_c(\Gamma/\Lambda)$ and $c_c(\Lambda\backslash\Gamma)$, see Definition 4.5. This seems to be a new point of view even in the classical case. Moreover we establish in Proposition 4.14 a canonical identification between $\mathcal{H}(\mathbb{G},\mathbb{H})$ and $\mathcal{H}(\Gamma,\Lambda)$. It follows in particular that Hecke operators are bounded on $\ell^2(\mathbb{G}/\mathbb{H})$ as well as on $\ell^2(\Gamma/\Lambda)$, and this yields an analytic proof of the combinatorial property of the constants κ mentioned above, see Definition 3.30 and Theorem 3.32.

Finally, in Paragraph 4.4 we study the notion of reduced pair both in the discrete setting and in the compact open one, with the property of being reduced corresponding to faithfulness of the Γ-action on Γ/Λ, respectively, of the $\mathbb{G}$-action on $\mathbb{G}/\mathbb{H}$. We construct a reduced pair associated to an arbitrary Hecke pair in Propositions 4.19 and 4.23. Moreover we prove that the Schlichting completion $\mathbb{G}$ is non-discrete whenever the Hecke pair (Γ,Λ) is reduced and Λ is infinite, see Lemma 4.21. It follows that the Schlichting completions of the Hecke pairs constructed in Section 3.3 via HNN extensions yield non-discrete locally compact quantum groups, whose modular automorphisms can be computed by the explicit formulas of Paragraph 3.2.

We would like to thank the anonymous referee for their careful reading of our original manuscript and a number of valuable suggestions and comments.

2 | (QUANTUM GROUP) PRELIMINARIES

In this short section we introduce general conventions, fix our notation, and offer a brief review of some definitions and facts from the theory of quantum groups. For more details we refer the reader to the following sources: [14, 16, 21, 28, 31, 32].

Tensor products of algebras, minimal/spatial tensor products of C^*-algebras, and Hilbert space tensor product of spaces/operators will be usually denoted by $\otimes$; if we want to stress that we are dealing with the algebraic tensor product we will use the symbol $\odot$. If φ is a linear form on an algebra A and $a \in A$, we denote $a\varphi$, φa the forms given by $a\varphi(b) = \varphi(ba)$ and $\varphi a(b) = \varphi(ab)$, for all $b \in A$.

2.1 | Algebraic quantum groups

By definition, an algebraic quantum group $\mathbb{G}$ is given by a multiplier Hopf $*$-algebra $\mathcal{O}_c(\mathbb{G})$ together with positive invariant functionals [28]. Recall from [17] that one can associate to $\mathbb{G}$ in a canonical way a locally compact quantum group, that is, a (reduced) Hopf C^*-algebra $C_0(\mathbb{G})$ satisfying the axioms of Kustermans and Vaes [16] and containing $\mathcal{O}_c(\mathbb{G})$ as a dense $*$-subalgebra. We denote by φ, ψ the left and right Haar weights of $\mathbb{G}$, which are defined on $\mathcal{O}_c(\mathbb{G})$. We denote the dual multiplier Hopf algebra by $\mathcal{D}(\mathbb{G})$.

Not all locally compact quantum groups arise in this way. In particular classical locally compact groups which fit into the algebraic quantum group framework are precisely the ones admitting a compact open subgroup [19, Section 3], and this is precisely the class of groups which naturally appears in the study of Hecke algebras.

More specifically, if G is a locally compact group with a compact open subgroup $H \subset G$, then we get an algebraic quantum group by considering

$$\mathcal{O}_c(G) = \operatorname{Span} G \cdot \mathcal{O}(H) \subset C_0(G),$$

where $\mathcal{O}(H)$ is the usual space of representative functions on H, which embeds canonically in $C_0(G)$ via extension by 0, and $(g \cdot f)(x) = f(xg)$ for $g \in G$ and $f \in C_0(G)$ is the action by right translation. The resulting multiplier Hopf $*$-algebra is independent of the choice of H, and in fact uniquely determined [19].

A morphism between algebraic quantum groups from $\mathbb{G}_1$ to $\mathbb{G}_2$ is given by a non-degenerate $*$-homomorphism $\pi : \mathcal{O}_c(\mathbb{G}_2) \to \mathcal{M}(\mathcal{O}_c(\mathbb{G}_1))$, compatible with the comultiplications. Observe that the algebraic multiplier algebra $\mathcal{M}(\mathcal{O}_c(\mathbb{G}_1))$ typically contains operators which are unbounded at the Hilbert space level. However, if $W_{\mathbb{G}_2}$ denotes the multiplicative unitary associated with $\mathbb{G}_2$, then $(\pi \otimes \mathrm{id})(W_{\mathbb{G}_2})$ is a unitary element of $\mathcal{M}(\mathcal{O}_c(\mathbb{G}_1) \odot \mathcal{D}(\mathbb{G}_2))$, and hence it induces a bounded (unitary) operator on $L^2(\mathbb{G}_1) \otimes L^2(\mathbb{G}_2)$. From the properties of the multiplicative unitary at the algebraic level it follows that this yields in fact a bicharacter at the C^*-level, and from [20] we conclude that π extends to a non-degenerate $*$-homomorphism $C_0^u(\mathbb{G}_2) \to M(C_0^u(\mathbb{G}_1))$ between the universal completions constructed in [15]. That is, π determines a morphism between the associated locally compact quantum groups. For simplicity we will abbreviate $C_b(\mathbb{G}_1) = M(C_0^u(\mathbb{G}_1))$.

If $\mathbb{G}$ is an algebraic quantum group, then an algebraic compact open quantum subgroup $\mathbb{H} \subset \mathbb{G}$ is given by a non-zero central projection $p_{\mathbb{H}} \in \mathcal{O}_c(\mathbb{G})$ such that $\Delta(p_{\mathbb{H}})(1 \otimes p_{\mathbb{H}}) = p_{\mathbb{H}} \otimes p_{\mathbb{H}}$, compare [10, Theorem 4.3, Proposition 4.4, Corollary 3.8]. Then $C(\mathbb{H}) = p_{\mathbb{H}} C_0(\mathbb{G})$ is a Woronowicz-C^*-algebra and the canonical morphism from $\mathbb{H}$ to $\mathbb{G}$ corresponds to the Hopf $*$-homomorphism $\pi_{\mathbb{H}} : C_0(\mathbb{G}) \to C(\mathbb{H})$, $f \mapsto p_{\mathbb{H}} f$. We note that it seems unclear whether central projections of $C_0(\mathbb{G})$ satisfying the above condition (which a priori describe all open compact quantum subgroups of $\mathbb{G}$) automatically lie in $\mathcal{O}_c(\mathbb{G})$.

2.2 | Discrete quantum groups

An algebraic quantum group $\mathbb{\Gamma}$ is called discrete if the corresponding algebra $\mathcal{O}_c(\mathbb{\Gamma})$ is a direct sum of matrix algebras. The existence of invariant functionals is then automatic and we have $C_0(\mathbb{\Gamma}) = C_0^u(\mathbb{\Gamma})$. In this case we use a different notation to emphasize the analogy with the classical situation: we put $\mathcal{O}_c(\mathbb{\Gamma}) = c_c(\mathbb{\Gamma})$, $C_0(\mathbb{\Gamma}) = c_0(\mathbb{\Gamma})$, $C_b(\mathbb{\Gamma}) = \ell^\infty(\mathbb{\Gamma})$, $\mathcal{M}(\mathcal{O}_c(\mathbb{\Gamma})) = c(\mathbb{\Gamma})$, $\mathcal{D}(\mathbb{\Gamma}) = \mathbb{C}[\mathbb{\Gamma}]$. The left, respectively, right Haar weights of $\mathbb{\Gamma}$ are denoted by h_L, h_R and their modular groups σ^L, σ^R; the scaling automorphism group will be denoted as τ. We will also use the antipode S, the unitary antipode R, and the co-unit ϵ.

We denote by $\mathrm{Corep}(\mathbb{\Gamma})$ the category of finite-dimensional normal unital $*$-representations $\alpha : \ell^\infty(\mathbb{\Gamma}) \to B(H_\alpha)$, equipped with the tensor structure $\tilde{\otimes}$ coming from the coproduct: $\alpha \tilde{\otimes} \beta := (\alpha \otimes \beta)\Delta$; for simplicity we will simply write $\alpha \otimes \beta$ in what follows. We denote $\dim(\alpha)$, respectively, $\dim_q(\alpha)$ the classical, respectively, quantum dimension of a corepresentation α. The trivial corepresentation is denoted as $1 \in I(\mathbb{\Gamma})$. We denote $\bar{\alpha}$ the conjugate corepresentation of α, which is unique up to equivalence; by definition we have non-zero morphisms $t_\alpha \in \mathrm{Hom}(1, \bar\alpha \otimes \alpha)$ which are uniquely determined up to a scalar when α is irreducible and satisfy the conjugate equation.

We choose a set $I(\mathbb{\Gamma})$ of representatives of irreducible objects up to equivalence. We can then identify $\ell^\infty(\mathbb{\Gamma}) = \ell^\infty\text{-}\bigoplus_{\alpha\in I(\mathbb{\Gamma})} B(H_\alpha)$ and we have $c_c(\mathbb{\Gamma}) = \bigoplus B(H_\alpha)$ and $c(\mathbb{\Gamma}) = \prod B(H_\alpha)$. We denote by $p_\alpha \in c_c(\mathbb{\Gamma})$, $\alpha \in I(\mathbb{\Gamma})$ the minimal central projections and write $a_\alpha = p_\alpha a$ for $a \in c(\mathbb{\Gamma})$. The coproduct is determined by the formula $(p_\beta \otimes p_\gamma)\Delta(a)v = va_\alpha$ for any $v \in \mathrm{Hom}(\alpha, \beta\otimes\gamma)$ and $a \in c_c(\mathbb{\Gamma})$. Let us record the following fact which is certainly well known to experts.

Lemma 2.1. *Let $\alpha, \beta \in I(\mathbb{\Gamma})$. The tensor product corepresentation $\alpha\otimes\beta$ contains at most* $\dim(\beta)^2$ *irreducible subobjects (counted with multiplicities).*

Proof. Decompose $\alpha\otimes\beta = \bigoplus_{i=1}^p \gamma_i$ with $\gamma_i \in I(\mathbb{\Gamma})$. By Frobenius reciprocity we have $\alpha \subset \gamma_i \otimes \bar\beta$, hence $\dim(\gamma_i) \geqslant \dim(\alpha)/\dim(\beta)$. Then we can write

$$\dim(\alpha)\dim(\beta) = \sum \dim(\gamma_i) \geqslant p\,\frac{\dim(\alpha)}{\dim(\beta)},$$

which yields the result. □

A quantum subgroup $\mathbb{\Lambda}$ of $\mathbb{\Gamma}$ is given by a surjective $*$-homomorphism $\pi : \ell^\infty(\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Lambda})$ such that $(\pi\otimes\pi)\Delta = \Delta\pi$. Due to the very special structure of $\ell^\infty(\mathbb{\Gamma})$, one can identify $\ell^\infty(\mathbb{\Lambda})$ with $p_{\mathbb{\Lambda}}\ell^\infty(\mathbb{\Gamma})$ for a uniquely determined central projection $p_{\mathbb{\Lambda}} \in \ell^\infty(\mathbb{\Gamma})$, in such a way that $\pi(a) = p_{\mathbb{\Lambda}}a$. The coproduct of $\ell^\infty(\mathbb{\Lambda})$ is then $\Delta_{\mathbb{\Lambda}}(a) = (p_{\mathbb{\Lambda}}\otimes p_{\mathbb{\Lambda}})\Delta(a) = (p_{\mathbb{\Lambda}}\otimes \mathrm{id})\Delta(a) = (\mathrm{id}\otimes p_{\mathbb{\Lambda}})\Delta(a)$ for $a \in \ell^\infty(\mathbb{\Lambda})$. Note that we have in the same way $c_c(\mathbb{\Lambda}) = p_{\mathbb{\Lambda}}c_c(\mathbb{\Gamma})$, $c(\mathbb{\Lambda}) = p_{\mathbb{\Lambda}}c(\mathbb{\Gamma})$. Using pre-composition with π the category $\mathrm{Corep}(\mathbb{\Lambda})$ is fully and faithfully embedded in $\mathrm{Corep}(\mathbb{\Gamma})$. We take for $I(\mathbb{\Lambda})$ the subset of $I(\mathbb{\Gamma})$ such that $\alpha \in I(\mathbb{\Lambda})$ if and only if $p_{\mathbb{\Lambda}}p_\alpha \neq 0$. We have $1 \in I(\mathbb{\Lambda})$ and if α, $\beta \in I(\mathbb{\Lambda})$, $\gamma \in I(\mathbb{\Gamma})$ and $\gamma \subset \alpha\otimes\beta$, then $\gamma \in I(\mathbb{\Lambda})$ and $\bar\alpha, \bar\beta \in I(\mathbb{\Lambda})$.

3 | DISCRETE QUANTUM HECKE PAIRS

In this section we define the Hecke algebra associated to a discrete quantum group $\mathbb{\Gamma}$ and a commensurated quantum subgroup $\mathbb{\Lambda}$. We start by studying the structure of the quotient spaces $\mathbb{\Gamma}/\mathbb{\Lambda}$, $\mathbb{\Lambda}\backslash\mathbb{\Gamma}$, first at the level of the set of irreducible corepresentations $I(\mathbb{\Gamma})$, and then at the finer level of the quantum algebras of functions $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. We obtain in particular in Theorem 3.10 a description of $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ using the classical quotient space $I(\mathbb{\Gamma})/\mathbb{\Lambda}$.

We can then define the Hecke algebra $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ and its convolution product, and prove that it is represented by Hecke operators on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. We give a combinatorial characterization of the ℓ^2-boundedness of Hecke operators and describe the modular properties of the canonical state of $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$. Finally we investigate examples arising from quantum HNN extensions.

3.1 | Quotient spaces

3.1.1 | Classical cosets

Suppose that $\mathbb{\Gamma}$ is a discrete quantum group with a quantum subgroup $\mathbb{\Lambda}$. Associated to $\mathbb{\Lambda}$ is the equivalence relation $\sim$ on $I(\mathbb{\Gamma})$ such that $\alpha \sim \beta$ if and only if $\alpha \subset \beta\otimes\gamma$ for some $\gamma \in I(\mathbb{\Lambda})$ if and only if $\Delta(p_\alpha)(p_\beta \otimes p_\mathbb{\Lambda}) \neq 0$, see [29, Lemma 2.3]. There is corresponding left version: $\alpha \backsim \beta$ if and only if $\alpha \subset \gamma \otimes \beta$ for some $\gamma \in I(\mathbb{\Lambda})$ if and only if $\Delta(p_\alpha)(p_\mathbb{\Lambda} \otimes p_\beta) \neq 0$. We denote by $I(\mathbb{\Gamma})/\mathbb{\Lambda}$, $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ the corresponding quotient spaces and by $[\alpha]$ the class of $\alpha \in I(\mathbb{\Gamma})$ in $I(\mathbb{\Gamma})/\mathbb{\Lambda}$ or in $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$, with the context determining the choice of left or right cosets. For $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ or $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ we write $p_\sigma = \sum_{\alpha\in\sigma} p_\alpha \in \ell^\infty(\mathbb{\Gamma})$. Note that in both cases $p_{[1]} = p_\mathbb{\Lambda}$, and the projections p_σ are pairwise orthogonal, central and sum up to 1. Note that they need not be minimal central projections in $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, but are finite sums of such, see [6, Section 5] and also the discussion below.

Recall that the quantum quotient spaces are given by the algebras

$$\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \{a \in \ell^\infty(\mathbb{\Gamma}) \mid (1 \otimes p_\mathbb{\Lambda})\Delta(a) = a \otimes p_\mathbb{\Lambda}\},$$

$$\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \{a \in \ell^\infty(\mathbb{\Gamma}) \mid (p_\mathbb{\Lambda} \otimes 1)\Delta(a) = p_\mathbb{\Lambda} \otimes a\}.$$

One can use the same conditions to define $c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ but one has to be a bit more careful for the spaces of finitely supported functions. One can check that $p_\sigma \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$ for any $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ and one defines $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = \mathrm{Span}\{p_\sigma c(\mathbb{\Gamma}/\mathbb{\Lambda}), \sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}\}$ (and similarly for the left versions). One can show that $p_\sigma c(\mathbb{\Gamma}/\mathbb{\Lambda}) = p_\sigma \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ for any $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ [30, Lemma 3.3]. It is easy to check that the coproduct Δ restricts to von Neumann, respectively, algebraic left coactions $\Delta : \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \to \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, respectively, $\Delta : c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \to \mathcal{M}(c_c(\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$, and similarly to right coactions $\Delta : \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \to \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$, respectively, $\Delta : c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \to \mathcal{M}(c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}))$. We have, for example, $(c_c(\mathbb{\Gamma}) \otimes 1)\Delta(c_c(\mathbb{\Gamma}/\mathbb{\Lambda})) = c_c(\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.

The next lemma works in general for open quantum subgroups of locally compact quantum groups, see [10, Lemma 3.1, Corollary 3.9]. The fact that the unitary antipode exchanges the left and right von Neumann algebraic homogeneous spaces for any closed quantum subgroup can be found, for example, in [13]. We include a simple proof for the discrete case.

Lemma 3.1. *Let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$. We have $R(c(\mathbb{\Gamma}/\mathbb{\Lambda})) = c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ and $S(c(\mathbb{\Gamma}/\mathbb{\Lambda})) = c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. The groups τ, σ^L, σ^R stabilize $c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.*

Proof. For any $\alpha \in I(\mathbb{\Gamma})$ and $t \in \mathbb{R}$ we have $S(p_\alpha) = R(p_\alpha) = p_{\bar\alpha}$ and $\tau_t(p_\alpha) = \sigma_t^R(p_\alpha) = \sigma_t^L(p_\alpha) = p_\alpha$ hence $S(p_\mathbb{\Lambda}) = R(p_\mathbb{\Lambda}) = \tau_t(p_\mathbb{\Lambda}) = \sigma_t^R(p_\mathbb{\Lambda}) = \sigma_t^R(p_\mathbb{\Lambda})$. For $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have $(p_\mathbb{\Lambda} \otimes 1)\Delta(S(a)) = \sigma(S \otimes S)[\Delta(a)(1 \otimes S^{-1}(p_\mathbb{\Lambda}))] = \sigma(S \otimes S)[\Delta(a)(1 \otimes p_\mathbb{\Lambda})] = \sigma(S \otimes S)(a \otimes p_\mathbb{\Lambda}) = p_\mathbb{\Lambda} \otimes S(a)$ (with σ denoting the tensor flip), hence $S(a) \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. The same holds for $R(a)$ since we also have $\Delta(R(a)) = \sigma(R \otimes R)\Delta(a)$. On the other hand for any $t \in \mathbb{R}$ we have $(1 \otimes p_\mathbb{\Lambda})\Delta(\tau_t(a)) = (\tau_t \otimes \tau_t)[(1 \otimes p_\mathbb{\Lambda})\Delta(a)] = \tau_t(a) \otimes p_\mathbb{\Lambda}$ hence $\tau_t(a) \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, and similarly on the left. Finally, note that in the discrete case we have $\tau_t = \sigma_t^L = \sigma_{-t}^R$. □

One can proceed similarly for double cosets. More precisely we denote $c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}) = c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. We consider the equivalence relation $\approx$ on $I(\mathbb{\Gamma})$ generated by $\sim$ and $\backsim$, and we have in fact $\alpha \approx \beta$ if and only if there exist $\delta, \gamma \in I(\mathbb{\Lambda})$ such that $\beta \subset \delta \otimes \alpha \otimes \gamma$ — indeed the set of such functions of β is closed under $\sim$ and $\backsim$, and decomposing $\alpha \otimes \gamma$ into α's one sees that $\beta \backsim \alpha' \sim \alpha$ for one of these α's. We denote by $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$ the corresponding quotient space and write $[\![\alpha]\!]$ for the corresponding class of $\alpha \in I(\mathbb{\Gamma})$. For $\sigma \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$ we denote $p_\sigma = \sum_{\alpha\in\sigma} p_\alpha \in \ell^\infty(\mathbb{\Gamma})$, obtaining again a family of central, pairwise orthogonal projections summing up to 1. We clearly have $p_\sigma \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$ and we denote $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}) = \mathrm{Span}\{p_\sigma c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}), \sigma \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}\} \subset c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$.

For $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$ we have $p_\tau = \sum\{p_\sigma, \sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}, \sigma \subset \tau\} = \sum\{p_\sigma, \sigma \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}), \sigma \subset \tau\}$. We denote $R(\tau) = \#\{\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}, \sigma \subset \tau\} \in \mathbb{N} \cup \{+\infty\}$ and $L(\tau) = \#\{\sigma \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}), \sigma \subset \tau\}$. We have

$c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$, and similarly $c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$. Already in the classical case these inclusions can be strict.

Proposition 3.2. *Let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$. The subset $I(\mathbb{\Gamma}') = \{\alpha \in I(\mathbb{\Gamma}) \mid L(\llbracket\alpha\rrbracket), R(\llbracket\alpha\rrbracket) < +\infty\}$ defines an intermediate quantum subgroup $\mathbb{\Lambda} \subset \mathbb{\Gamma}' \subset \mathbb{\Gamma}$ such that $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.*

Proof. It suffices to show that $I(\mathbb{\Lambda}) \subset I(\mathbb{\Gamma}')$, $I(\mathbb{\Gamma}') \otimes I(\mathbb{\Gamma}') \subset \mathbb{Z}I(\mathbb{\Gamma}')$ and $\overline{I(\mathbb{\Gamma}')} \subset I(\mathbb{\Gamma}')$, see, for example, [29, Section 2]. Since $\llbracket 1\rrbracket = I(\mathbb{\Lambda}) = [1]$ we have $R(\llbracket 1\rrbracket) = L(\llbracket 1\rrbracket) = 1$ and $\llbracket 1\rrbracket = I(\mathbb{\Lambda}) \subset I(\mathbb{\Gamma}')$. Since $\alpha \sim \beta \iff \bar\alpha \backsim \bar\beta$, we have $L(\llbracket\alpha\rrbracket) = R(\llbracket\bar\alpha\rrbracket)$ hence $\bar\alpha \in I(\mathbb{\Gamma}')$ if $\alpha \in I(\mathbb{\Gamma}')$. Let $\alpha, \beta \in I(\mathbb{\Gamma}')$ and let $\delta \in I(\mathbb{\Gamma})$, $\delta \subset \alpha \otimes \beta$. Decompose into finite unions of right cosets $\llbracket\alpha\rrbracket = \bigsqcup[\alpha_i]$, $\llbracket\beta\rrbracket = \bigsqcup[\beta_j]$, and consider the finite set of all the elements $\delta_k \in I(\mathbb{\Gamma})$ such that $\delta_k \subset \alpha_i \otimes \beta_j$ for some i, j. Take $\lambda, \mu \in I(\mathbb{\Lambda})$ and $\gamma \subset \lambda \otimes \delta \otimes \mu$. Then we have $\gamma \subset \lambda \otimes \alpha \otimes \beta \otimes \mu$. Decomposing $\lambda \otimes \alpha$ into α's we have by irreducibility $\gamma \subset \alpha' \otimes \beta \otimes \mu$ for some $\alpha' \in I(\mathbb{\Gamma})$ and since $\alpha' \in \llbracket\alpha\rrbracket$ we have $\alpha' \subset \alpha_i \otimes \lambda'$ for some i and some $\lambda' \in I(\mathbb{\Lambda})$. Then $\gamma \subset \alpha_i \otimes \lambda' \otimes \beta \otimes \mu$. Proceeding similarly with $\lambda' \otimes \beta$ we find j and $\lambda'' \in I(\mathbb{\Lambda})$ such that $\gamma \subset \alpha_i \otimes \beta_j \otimes \lambda'' \otimes \mu$. It follows that $\gamma \in [\delta_k]$ for some k. As a result $\llbracket\delta\rrbracket$ is covered by a finite number of right cosets. Applying this to $\bar\delta$ we see that $\delta \in I(\mathbb{\Gamma}')$.

By definition, for $\tau \in \mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda}$ we can write p_τ as a finite sum $p_\tau = \sum p_{\sigma_i}$ with $\sigma_i \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$. It follows that $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda}) \subset c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Similarly $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda}) \subset c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. Conversely, let $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ and consider $\alpha \in I(\mathbb{\Gamma})$ such that $p_\alpha a \neq 0$. Then we claim that $p_\gamma a \neq 0$ for all $\gamma \in \llbracket\alpha\rrbracket$. Indeed we have $p_{[\alpha]} a \neq 0$ and since $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, Lemma 3.3 in [30] shows that $p_\beta a \neq 0$ for all $\beta \subset \alpha \otimes \lambda$, $\lambda \in I(\mathbb{\Lambda})$. Using now the fact that $a \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ and again [30, Lemma 3.3] we have $p_\gamma a \neq 0$ for all $\gamma \subset \mu \otimes \beta$, $\mu \in I(\mathbb{\Lambda})$. In particular we have $p_\sigma a \neq 0$ for all $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, $\sigma \subset \llbracket\alpha\rrbracket$. But since $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, there is at most a finite number of functions of σ such that $p_\sigma a \neq 0$. Hence $R(\llbracket\alpha\rrbracket) < +\infty$. Similarly $L(\llbracket\alpha\rrbracket) < +\infty$. As a result $\alpha \in I(\mathbb{\Gamma}')$ and we can conclude that $a \in c(\mathbb{\Gamma}')$. $\square$

Definition 3.3. Let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$. The quantum subgroup $\mathbb{\Lambda} \subset \mathbb{\Gamma}' \subset \mathbb{\Gamma}$ of the previous proposition is called the *commensurator* of $\mathbb{\Lambda}$ in $\mathbb{\Gamma}$. We say that $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a *Hecke pair* (or that $\mathbb{\Lambda}$ is almost normal in $\mathbb{\Gamma}$) if $R(\tau), L(\tau) < +\infty$ for any $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$, or equivalently, if $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$, that is, $\mathbb{\Gamma}' = \mathbb{\Gamma}$.

3.1.2 | Quantum cosets

We give now a more precise description of the quantum quotient space algebra $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ and of the corresponding Corep($\mathbb{\Gamma}$)-module category.

For every von Neumann subalgebra $M \subset \ell^\infty(\mathbb{\Gamma})$ we have a restriction functor from Corep($\mathbb{\Gamma}$) = Rep($\ell^\infty(\mathbb{\Gamma})$) to the category Rep($M$) of finite-dimensional normal $*$-representations of M which factors the respective forgetful functors to the category of finite-dimensional Hilbert spaces. If M is left invariant, that is, $\Delta(M) \subset \ell^\infty(\mathbb{\Gamma})\bar\otimes M$, then Rep($M$) is naturally equipped with the structure of a left Corep($\mathbb{\Gamma}$)-module-C^*-category, by considering $\alpha \otimes \pi := (\alpha \otimes \pi)\Delta$ and the Hilbert space tensor product of morphisms.

When $M = \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, this module category is equivalent to the one naturally associated with the $\hat{\mathbb{\Gamma}}$-C^*-algebra $C^*_r(\mathbb{\Lambda})$, compare [5, Theorem 6.4]. We write Rep(M) = Corep($\mathbb{\Gamma}/\mathbb{\Lambda}$) in this case and choose a set $I(\mathbb{\Gamma}/\mathbb{\Lambda})$ of representatives of irreducible objects in this category up to equivalence, not to be confused with $I(\mathbb{\Gamma})/\mathbb{\Lambda}$. We denote $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}$ the corresponding morphism spaces, and note that we have a restriction functor from $\mathrm{Corep}(\mathbb{\Gamma})$ to $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$. Moreover, to describe $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$ is suffices to describe its full subcategory with objects from $\mathrm{Corep}(\mathbb{\Gamma})$, that is, the spaces $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(\alpha,\beta)$ for $\alpha,\beta \in I(\mathbb{\Gamma})$ — one can then recover $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$ via idempotent completion.

The next proposition is a variant and improvement of [30, Lemma 3.3] and [6, Theorem 5.2, Theorem 5.6]. It gives a concrete description of $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\mathrm{Corep}(\mathbb{\Gamma}/\mathbb{\Lambda})$. Recall that $\mathrm{Corep}(\mathbb{\Lambda})$ is faithfully and fully embedded in $\mathrm{Corep}(\mathbb{\Gamma})$.

Definition 3.4. For $v \in \mathrm{Corep}(\mathbb{\Gamma}) = \mathrm{Rep}(\ell^\infty(\mathbb{\Gamma}))$ we denote by $v_{\mathbb{\Lambda}}$ the largest subobject of v belonging to $\mathrm{Corep}(\mathbb{\Lambda})$, given by the projection $v(p_{\mathbb{\Lambda}}) \in B(H_v)$.

Let us apply Definition 3.4 to the space $B(H_\alpha, H_\beta)$, viewed as a corepresentation of $\mathbb{\Gamma}$ via the identification with $H_{\bar\alpha} \otimes H_\beta$ given by $S \mapsto (\mathrm{id} \otimes S)t_\alpha$. By Frobenius reciprocity we then have

$$B(H_\alpha, H_\beta)_{\mathbb{\Lambda}} = \mathrm{Span}\{v(1 \otimes \eta) \mid \lambda \in \mathrm{Corep}(\mathbb{\Lambda}), v \in \mathrm{Hom}(\alpha \otimes \lambda, \beta), \eta \in H_\lambda\}.$$

In the next proposition we use the commutant of $B(H_\alpha, H_\alpha)_{\mathbb{\Lambda}} = B(H_\alpha)_{\mathbb{\Lambda}}$ inside $B(H_\alpha)$:

$$\begin{aligned} B(H_\alpha)'_{\mathbb{\Lambda}} &= \{b \in B(H_\alpha) \mid bf = fb \text{ for all } f \in B(H_\alpha, H_\alpha)_{\mathbb{\Lambda}}\} \\ &= \{b \in B(H_\alpha) \mid (b \otimes \mathrm{id})w = wb \text{ for all } \lambda \in I(\mathbb{\Lambda}), w \in \mathrm{Hom}(\alpha, \alpha \otimes \lambda)\}. \end{aligned}$$

Recall that for $a \in \ell^\infty(\mathbb{\Gamma})$ and $\alpha \in I(\mathbb{\Gamma})$ we denote by $a_\alpha \in B(H_\alpha)$ the component $p_\alpha a$ of a.

Proposition 3.5. *Let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$ and α, $\beta \in I(\mathbb{\Gamma})$. The map $(a \mapsto a_\alpha)$ is an injective $*$-homomorphism from $p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ to $B(H_\alpha)$, with image $p_\alpha \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = B(H_\alpha)'_{\mathbb{\Lambda}}$. More generally we have $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(\alpha,\beta) = B(H_\alpha, H_\beta)_{\mathbb{\Lambda}}$.*

Proof. Let $a \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. We have then $(p_\alpha \otimes p_{\mathbb{\Lambda}})\Delta(a) = a_\alpha \otimes p_{\mathbb{\Lambda}}$. Since $(p_\alpha \otimes p_{\mathbb{\Lambda}})\Delta$ is non-zero, hence also injective on any matrix block $p_\beta \ell^\infty(\mathbb{\Gamma})$ with $\beta \subset \alpha \otimes \lambda$, $\lambda \in I(\mathbb{\Lambda})$, we see that $a_\alpha = 0 \Rightarrow p_{[\alpha]}a = 0$. Denote $b = a_\alpha$, and let $v \in \mathrm{Hom}(\alpha, \alpha \otimes \lambda)$ with $\lambda \in I(\mathbb{\Lambda})$. We have $b \otimes p_\lambda = (p_\alpha \otimes p_\lambda)\Delta(a)$ hence $(b \otimes \mathrm{id})v = (p_\alpha \otimes p_\lambda)\Delta(a)v = vp_\alpha a = vb$.

Conversely, start from an element $b \in B(H_\alpha)'_{\mathbb{\Lambda}}$. Then, for any λ, $\mu \in I(\mathbb{\Lambda})$ and any $v \in \mathrm{Hom}(\alpha, \alpha \otimes \lambda \otimes \mu)$ we have $vb = (b \otimes \mathrm{id} \otimes \mathrm{id})v$ — it suffices to decompose $\lambda \otimes \mu$ into irreducibles, which are still in $I(\mathbb{\Lambda})$. Even more, for any $v \in \mathrm{Hom}(\alpha \otimes \lambda, \alpha \otimes \mu)$ we have $v(b \otimes \mathrm{id}) = (b \otimes \mathrm{id})v$ — apply the previous property to $w = (v \otimes \mathrm{id})(\mathrm{id} \otimes t_\lambda) \in \mathrm{Hom}(\alpha, \alpha \otimes \mu \otimes \bar\lambda)$ and the conjugate equation.

Now, for any $\beta \in [\alpha]$, choose $\lambda_\beta \in I(\mathbb{\Lambda})$ and $v_\beta \in \mathrm{Hom}(\beta, \alpha \otimes \lambda_\beta)$ isometric — with $\lambda_\alpha = 1$ and $v_\alpha = \mathrm{id}$. Define $a_\beta \in B(H_\beta)$ by putting $a_\beta = v_\beta^*(b \otimes \mathrm{id})v_\beta$. Take β, $\beta' \in [\alpha]$, $\mu \in I(\mathbb{\Lambda})$ and $w \in \mathrm{Hom}(\beta, \beta' \otimes \mu)$. Then we have, applying the identity $(b \otimes \mathrm{id})u = u(b \otimes \mathrm{id})$ to $u = (v_{\beta'} \otimes \mathrm{id})wv_\beta^* \in \mathrm{Hom}(\alpha \otimes \lambda_\beta, \alpha \otimes \lambda_{\beta'} \otimes \mu)$:

$$\begin{aligned} wa_\beta &= (v_{\beta'}^* \otimes \mathrm{id})(v_{\beta'} \otimes \mathrm{id})wv_\beta^*(b \otimes \mathrm{id})v_\beta \\ &= (v_{\beta'}^* \otimes \mathrm{id})(b \otimes \mathrm{id} \otimes \mathrm{id})(v_{\beta'} \otimes \mathrm{id})wv_\beta^* v_\beta = (a_{\beta'} \otimes \mathrm{id})w. \end{aligned}$$

This shows that $(p_{\beta'} \otimes p_\mu)\Delta(a_\beta) = a_{\beta'} \otimes p_\mu$. Putting all a_β together we obtain $a \in p_{[\alpha]}\ell^\infty(\mathbb{\Gamma})$ such that $(\mathrm{id} \otimes p_{\mathbb{\Lambda}})\Delta(a) = a \otimes p_{\mathbb{\Lambda}}$, that is, $a \in p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. Moreover by our choice of v_α we have $a_\alpha = b$. □

Remark 3.6. In particular $p_{[\alpha]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ is a finite-dimensional C^*-algebra for all $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$; hence, $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ is a von Neumann direct product of matrix algebras. The inclusion $p_\alpha\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \subset B(H_\alpha)$ can be strict, even for all elements α in a given class $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ — for example, we always have $p_{[1]}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \mathbb{C}$, and for the dual of $SO(3)$ seen as a subgroup of the dual of $SU(2)$ we have $p_\sigma\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \mathbb{C}$ for both classes $\sigma \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$. It can also happen, for example, for $\mathbb{\Gamma}_1 \subset \mathbb{\Gamma}_1 \times \mathbb{\Gamma}_2$, that $p_\alpha\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ is a proper, non-trivial sub-C^*-algebra of $B(H_\alpha)$.

Further, recall that [6, Theorem 5.2, Theorem 5.6] show that each $[\alpha]$ corresponds to a certain equivalence class of minimal central projections in $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, determined by the left adjoint action of $\widehat{\mathbb{\Gamma}}$, and $p_{[\alpha]}$ is equal to the sum of the projections in the aforementioned equivalence class. Thus the fact that each $p_\alpha\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ is simple is equivalent to the equivalence relation above being trivial. This need not be the case, as already classical Clifford theory shows.

One can reformulate the Hecke condition of Definition 3.3 using the left action of $\mathbb{\Lambda}$ on $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ given by $\alpha(a) = (p_{\mathbb{\Lambda}} \otimes 1)\Delta(a) \in \ell^\infty(\mathbb{\Lambda})\bar{\otimes}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ for $a \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. Following [4, Section 4], define an equivalence relation on $I(\mathbb{\Gamma}/\mathbb{\Lambda})$ by putting $i \equiv_{\mathbb{\Lambda}} j$ if $\Delta(q_i)(p_{\mathbb{\Lambda}} \otimes q_j) \neq 0$, where q_i, q_j are the minimal central projections in $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ corresponding to i and j, respectively. Equivalently, $i \equiv_{\mathbb{\Lambda}} j$ if and only if $j \subset \lambda \otimes i$ for some $\lambda \in I(\mathbb{\Lambda})$ with respect to the module-category structure mentioned previously. Then the Hecke condition is satisfied if and only if the action of $\mathbb{\Lambda}$ on $\mathbb{\Gamma}/\mathbb{\Lambda}$ 'has finite orbits', as stated precisely in the next proposition.

Proposition 3.7. *The pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a Hecke pair if and only if the equivalence relation $\equiv_{\mathbb{\Lambda}}$ on $I(\mathbb{\Gamma}/\mathbb{\Lambda})$ has finite classes.*

Proof. Denote by $q_i \in \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ the minimal central projection associated with $i \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$. For any $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ we have a finite set $I([\alpha]) \subset I(\mathbb{\Gamma}/\mathbb{\Lambda})$ such that $p_{[\alpha]} = \sum_{i \in I([\alpha])} q_i$. This implies that the action of $\mathbb{\Lambda}$ on $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ has finite orbits if and only if for any $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ there exist only finitely many $[\beta] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ such that $\Delta(p_{[\alpha]})(p_{\mathbb{\Lambda}} \otimes p_{[\beta]}) \neq 0$. This means that we can find $\alpha' \in [\alpha]$, $\beta' \in [\beta]$ such that $\alpha' \backsim \beta'$, in other words $[\![\alpha]\!] = [\![\beta]\!]$. This ends the proof. □

We will now represent invariant functions using $\Delta(p_{\mathbb{\Lambda}})$, see Theorem 3.10, which is essentially a consequence of Proposition 3.5 and 'strong invariance' of the Haar weights ([16, Proposition 5.24]). The quantities κ below will play a fundamental role in the sequel. Recall that for $v \in \mathrm{Corep}(\mathbb{\Gamma})$ we denote by $v_{\mathbb{\Lambda}}$ the sum of all irreducible subobjects of v equivalent to an element of $I(\mathbb{\Lambda})$.

Definition 3.8. For $\alpha, \beta \in I(\mathbb{\Gamma})$ we put $\kappa_{\alpha,\beta} = \dim_q(\alpha \otimes \beta)_{\mathbb{\Lambda}}$ and $\kappa_\alpha = \kappa_{\bar{\alpha},\alpha}$.

We first make the connection between the constants $\kappa_{\alpha,\beta}$ and the Hopf algebra structure of $\ell^\infty(\mathbb{\Gamma})$. Recall that we denote $a\varphi = \varphi(\,\cdot\, a)$, $\varphi a = \varphi(a\,\cdot\,)$ if φ is a linear form on an algebra and a an element of this algebra.

Lemma 3.9. *For every $\alpha, \beta \in I(\mathbb{\Gamma})$ we have*

$$(h_R p_{\mathbb{\Lambda}} \otimes p_\beta)\Delta(p_\alpha) = \frac{\dim_q(\alpha)}{\dim_q(\beta)} \kappa_{\alpha,\bar\beta} p_\beta.$$

In particular $(h_R p_{\mathbb{\Lambda}} \otimes p_\alpha)\Delta(p_\alpha) = \kappa_{\bar\alpha} p_\alpha$.

Proof. We first show that for any $\alpha, \beta, \lambda \in I(\mathbb{\Gamma})$ the linear map $(h_R p_\lambda \otimes p_\beta)\Delta(p_\alpha) \in B(H_\beta)$ is an intertwiner, hence a scalar. Indeed, we have $h_R(a) = \dim_q(\alpha) t_\alpha^*(1 \otimes a)t_\alpha$ for any $a \in p_\alpha c(\mathbb{\Gamma}) = B(H_\alpha)$, where $t_\alpha \in \mathrm{Hom}(1, \bar\alpha \otimes \alpha)$ is such that $\|t_\alpha\|^2 = \dim_q(\alpha)$. Then, if $(v_i)_i$ is an orthonormal basis (ONB) of $\mathrm{Hom}(\alpha, \lambda \otimes \beta)$ and $P_\alpha^{\lambda,\beta} = \sum v_i v_i^* \in B(H_\lambda \otimes H_\beta)$ denotes the orthogonal projection onto the α-isotypic component of $\lambda \otimes \beta$, we have

$$\begin{aligned}(h_R p_\lambda \otimes p_\beta)\Delta(p_\alpha) &= \dim_q(\lambda) \textstyle\sum_i (t_\lambda^* \otimes \mathrm{id})(\mathrm{id} \otimes v_i p_\alpha v_i^*)(t_\lambda \otimes \mathrm{id}) \\ &= \dim_q(\lambda)\, (t_\lambda^* \otimes \mathrm{id})(\mathrm{id} \otimes P_\alpha^{\lambda,\beta})(t_\lambda \otimes \mathrm{id}) \in \mathrm{Hom}(\beta, \beta).\end{aligned}$$

Hence there is a scalar $\kappa^\lambda_{\alpha,\bar\beta} \geqslant 0$ such that $(h_R p_\lambda \otimes p_\beta)\Delta(p_\alpha) = \kappa^\lambda_{\alpha,\bar\beta} p_\beta$. This scalar can be computed by evaluating both sides against h_R. We obtain, after dividing both sides by $\dim_q(\beta)$:

$$\begin{aligned}\dim_q(\beta) \times \kappa^\lambda_{\alpha,\bar\beta} &= \dim_q(\lambda)\, t_\beta^*(\mathrm{id} \otimes t_\lambda^* \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes P_\alpha^{\lambda,\beta})(\mathrm{id} \otimes t_\lambda \otimes \mathrm{id})t_\beta \\ &= \dim_q(\lambda)\, t^*_{\lambda\otimes\beta}(\mathrm{id}_{\bar\beta\otimes\bar\lambda} \otimes P_\alpha^{\lambda,\beta})t_{\lambda\otimes\beta} = \dim_q(\lambda) \textstyle\sum_i t^*_{\lambda\otimes\beta}(\mathrm{id}_{\bar\beta\otimes\bar\lambda} \otimes v_i v_i^*)t_{\lambda\otimes\beta} \\ &= \dim_q(\lambda) c_\alpha^{\lambda,\beta} t_\alpha^* t_\alpha = c_\alpha^{\lambda,\beta} \dim_q(\lambda) \dim_q(\alpha),\end{aligned}$$

where $c_\alpha^{\lambda,\beta} = \dim(\mathrm{Hom}(\alpha, \lambda \otimes \beta))$. Note that an analogous formula appears already in work of Izumi, see the remark before [9, Corollary 3.7].

By Frobenius reciprocity we also have $c_\alpha^{\lambda,\beta} = c_\lambda^{\alpha,\bar\beta}$. Summing over $\lambda \in I(\mathbb{\Lambda})$ we get

$$\begin{aligned}\dim_q(\beta) \times (h_R p_{\mathbb{\Lambda}} \otimes p_\beta)\Delta(p_\alpha) &= \dim_q(\alpha)\Big(\textstyle\sum_{\lambda\in I(\mathbb{\Lambda})} c_\lambda^{\alpha,\bar\beta} \dim_q(\lambda)\Big) p_\beta \\ &= \dim_q(\alpha) \dim_q(\alpha \otimes \bar\beta)_{\mathbb{\Lambda}} p_\beta,\end{aligned}$$

which yields the claim. □

Theorem 3.10 below is a very useful tool for the study of Hecke algebras associated to discrete quantum groups. It is merely a materialization of Proposition 3.5, which says that one can recover $a \in p_{[\alpha]} c(\mathbb{\Gamma}/\mathbb{\Lambda})$ from a_α, and it is a simple consequence of the so-called strong invariance properties of the Haar weights. It is well known in the case when $\mathbb{\Lambda} = \{e\}$ — then $p_{\mathbb{\Lambda}}$ is the support of the co-unit and $\kappa_\alpha = 1$ for every α. It has several corollaries important for what follows.

Theorem 3.10. *For any $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ and any choices of representatives $\alpha \in [\alpha]$, $\beta \in [\beta]$ we have*

$$a = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes \mathrm{id})\Delta(p_\mathbb{\Lambda}) = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(h_R S(a_\alpha) \otimes \mathrm{id})\Delta(p_\mathbb{\Lambda}), \tag{3.1}$$

$$b = \sum_{[\beta]\in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1}(\mathrm{id} \otimes h_L S^{-1}(b_\beta))\Delta(p_\mathbb{\Lambda}) = \sum_{[\beta]\in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1}(\mathrm{id} \otimes S(b_\beta)h_L)\Delta(p_\mathbb{\Lambda}). \tag{3.2}$$

Proof. We start with $a \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, $\alpha \in I(\mathbb{\Gamma})$ and $\lambda \in I(\mathbb{\Lambda})$. By strong right invariance we have $(a_\alpha h_R \otimes \mathrm{id})\Delta(p_\lambda) = (h_R \otimes S)((p_\lambda \otimes \mathrm{id})\Delta(a_\alpha))$ hence $(a_\alpha h_R \otimes p_{\bar\alpha})\Delta(p_\lambda) = (h_R \otimes S)((p_\lambda \otimes p_\alpha)\Delta(a_\alpha))$. By left invariance of a we have $(p_\lambda \otimes p_\alpha)\Delta(a_\alpha) = (1 \otimes a_\alpha)(p_\lambda \otimes p_\alpha)\Delta(p_\alpha)$ so that $(a_\alpha h_R \otimes p_{\bar\alpha})\Delta(p_\lambda) = (h_R \otimes S)((p_\lambda \otimes p_\alpha)\Delta(p_\alpha))S(a_\alpha)$. Summing over $\lambda \in I(\mathbb{\Lambda})$ and applying Lemma 3.9 we obtain

$$(a_\alpha h_R \otimes p_{\bar\alpha})\Delta(p_\mathbb{\Lambda}) = (h_R \otimes S)((p_\mathbb{\Lambda} \otimes p_\alpha)\Delta(p_\alpha))S(a_\alpha) = \kappa_{\bar\alpha}S(a_\alpha).$$

By Lemma 3.1, for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we can apply this formula to $S^{-1}(a_\alpha) \in p_{\bar\alpha}c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, which yields $a_\alpha = \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes p_\alpha)\Delta(p_\mathbb{\Lambda})$. Now we observe that the right-hand side also lies in $p_\alpha c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Indeed we have $\Delta(p_\mathbb{\Lambda})(1 \otimes p_\mathbb{\Lambda}) = p_\mathbb{\Lambda} \otimes p_\mathbb{\Lambda}$ hence $\Delta^2(p_\mathbb{\Lambda})(1 \otimes 1 \otimes p_\mathbb{\Lambda}) = \Delta(p_\mathbb{\Lambda}) \otimes p_\mathbb{\Lambda}$, which implies $\Delta(x)(1 \otimes p_\mathbb{\Lambda}) = x \otimes p_\mathbb{\Lambda}$ for any $x = (\varphi \otimes \mathrm{id})\Delta(p_\mathbb{\Lambda})$, by applying $\varphi \otimes \mathrm{id} \otimes \mathrm{id}$. Consequently we can apply Proposition 3.5 which yields $p_{[\alpha]}a = \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes p_{[\alpha]})\Delta(p_\mathbb{\Lambda})$ and summing over $[\alpha]$ we obtain the first equality in (3.1).

Equation (3.2) follows by applying (3.1) to $a = S^{-1}(b)$: this yields $b = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(h_R b_{\bar\alpha} \otimes S)\Delta(p_\mathbb{\Lambda})$. Since $p_\mathbb{\Lambda} = S^{-1}(p_\mathbb{\Lambda})$ and $\Delta S^{-1} = (S^{-1} \otimes S^{-1})\sigma\Delta$, we also have $b = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(\mathrm{id} \otimes (h_R b_{\bar\alpha})S^{-1})\Delta(p_\mathbb{\Lambda})$. Finally we have $(h_R b_{\bar\alpha})S^{-1} = S(b_{\bar\alpha})h_L$ and we obtain the rightmost side of (3.2) since $[\beta] = [\bar\alpha]$ runs through $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ when $[\alpha]$ runs through $I(\mathbb{\Gamma})/\mathbb{\Lambda}$.

The missing identities in (3.1) and (3.2) follow by replacing a, b with a^*, b^* and taking adjoints on both sides. Alternatively one can use the fact that as we are in the context of discrete quantum groups we have $h_L(ab) = h_L(bS^2(a))$, $h_R(ab) = h_R(bS^{-2}(a))$. □

Corollary 3.11. *Let $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. Then $\kappa_\alpha^{-1}h_L(a_\alpha)$ only depends on the class $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, and $\kappa_{\bar\beta}^{-1}h_R(b_\beta)$ only depends on the class $[\beta] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$.*

Proof. One can assume that $a \in p_{[\alpha]}c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $b = S(a) \in p_{[\beta]}c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ with $\beta = \bar\alpha$. We have $a = \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes \mathrm{id})\Delta(p_\mathbb{\Lambda})$ by (3.1) and $p_{[\beta]} = \kappa_{\bar\beta}^{-1}(\mathrm{id} \otimes h_L S^{-1}(p_\beta))\Delta(p_\mathbb{\Lambda})$ by (3.2). We compute then

$$\begin{aligned}\kappa_{\bar\beta}^{-1}h_R(b_\beta) &= \kappa_{\bar\beta}^{-1}h_L(S^{-1}(p_\beta)a) = \kappa_{\bar\beta}^{-1}\kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes h_L S^{-1}(p_\beta))\Delta(p_\mathbb{\Lambda}) \\ &= \kappa_\alpha^{-1}h_R(p_\beta S^{-1}(a_\alpha)) = \kappa_\alpha^{-1}h_L(a_\alpha S(p_\beta)) = \kappa_\alpha^{-1}h_L(a_\alpha).\end{aligned}$$

Since the left-hand (respectively, right hand) side does not depend on the choice of α in $[\alpha]$ (respectively, $\beta \in [\beta]$), we are done. □

Note in particular that $\kappa_\alpha^{-1}h_L(p_\alpha) = \dim_q(\alpha)^2/\kappa_\alpha$ depends only on $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$. We have in fact the more general result below.

Corollary 3.12. *The quantity $\kappa_{\bar\alpha,\beta}/(\dim_q(\bar\alpha)\dim_q(\beta))$ depends only on the classes $[\alpha]$, $[\beta] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$.*

Proof. We can assume $[\alpha] = [\beta]$ — otherwise $\kappa(\bar\alpha, \beta) = 0$. Since $\kappa_{\bar\alpha,\beta} = \kappa_{\bar\beta,\alpha}$, it suffices to prove the independence on β. We apply (3.1) with $a = p_{[\alpha]}$: this yields $p_{[\alpha]} = \kappa_\alpha^{-1}(h_R p_{\bar\alpha} \otimes \mathrm{id})\Delta(p_{\mathbb{\Lambda}})$. In particular $h_R(p_\beta) = \kappa_\alpha^{-1}(h_R p_{\bar\alpha} \otimes h_R p_\beta)\Delta(p_{\mathbb{\Lambda}})$. Computing as in the proof of Lemma 3.9 and denoting $P^{\bar\alpha,\beta}_{\mathbb{\Lambda}} = \sum_{\lambda\in\mathbb{\Lambda}} P^{\bar\alpha,\beta}_\lambda$ this can be written

$$(\dim_q(\beta))^2 = \kappa_\alpha^{-1}\dim_q(\alpha)\dim_q(\beta)\, t^*_{\bar\alpha\otimes\beta}(\mathrm{id}\otimes\mathrm{id}\otimes P^{\bar\alpha,\beta}_{\mathbb{\Lambda}})t_{\bar\alpha\otimes\beta}$$

$$= \kappa_\alpha^{-1}\dim_q(\alpha)\dim_q(\beta)\dim_q(\bar\alpha\otimes\beta)_{\mathbb{\Lambda}} = \kappa_\alpha^{-1}\dim_q(\alpha)\dim_q(\beta)\,\kappa_{\bar\alpha,\beta}.$$

This shows that $\kappa_{\bar\alpha,\beta}/\dim_q(\beta)$ does not depend on $\beta \in [\alpha]$. □

We remark that Corollary 3.12 also has a purely categorical proof, suggested to us by the referee. Fix $\alpha \in I(\mathbb{\Gamma})$ and consider the right Corep($\mathbb{\Lambda}$)-module category $\mathcal{D} \subset$ Corep($\mathbb{\Gamma}$) generated by α. Consider also the functor $\underline{\alpha} : \mathcal{D} \to \mathrm{Corep}(\mathbb{\Lambda})$, $v \mapsto (\bar\alpha\otimes v)_{\mathbb{\Lambda}}$. This is the internal Hom functor for $\mathcal{D}$, in the sense that $\mathrm{Hom}_{\mathbb{\Gamma}}(\alpha\otimes\lambda, v) \simeq \mathrm{Hom}_{\mathbb{\Lambda}}(\lambda, \underline{\alpha}(v))$ for all $v \in \mathcal{D}$, $\lambda \in \mathrm{Corep}(\mathbb{\Lambda})$. In [22, Lemma A.4] it is shown that the rescaled Frobenius reciprocity isomorphism

$$\mathrm{Hom}_{\mathbb{\Gamma}}(\beta, \beta'\otimes\lambda) \to \mathrm{Hom}_{\mathbb{\Gamma}}(\beta\otimes\bar\lambda, \beta'), \quad T \mapsto \sqrt{\frac{\kappa_{\bar\alpha,\beta'}}{\kappa_{\bar\alpha,\beta}}}\,\tilde T,$$

with $\tilde T = (\mathrm{id}_{\beta'}\otimes \bar R^*_\lambda)(T\otimes\mathrm{id}_{\bar\lambda})$, is unitary with respect to the hermitian structures given by $S^*T = \langle S, T\rangle\mathrm{id}_\beta$, $\tilde S\tilde T^* = \langle\tilde T, \tilde S\rangle\mathrm{id}_{\beta'}$ for $\beta, \beta' \in [\alpha]$. Now choose an isometric embedding $T : \beta \to \beta'\otimes\lambda$ with some $\lambda \in \mathrm{Corep}(\mathbb{\Lambda})$. We have $\|T\|^2 = 1$, and using $\mathrm{qTr}(\tilde T\tilde T^*) = \mathrm{qTr}(TT^*) = \dim_q(\beta)$we obtain $\|\tilde T\|^2 = \dim_q(\beta)/\dim_q(\beta')$. Then unitarity of the map above then shows that $\kappa_{\bar\alpha,\beta'}/\dim_q(\beta') = \kappa_{\bar\alpha,\beta}/\dim_q(\beta)$ as required.

Let us finally introduce the quantum analogs μ, ν of the counting measures on $\mathbb{\Gamma}/\mathbb{\Lambda}$, respectively, $\mathbb{\Lambda}\backslash\mathbb{\Gamma}$. We will see at Proposition 3.21 that μ is the (necessarily unique, up to scalar multiplication) $\mathbb{\Gamma}$-invariant weight on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Observe that, applying (3.1) to a^* we obtain

$$a^* = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(h_R S(a^*_\alpha)\otimes\mathrm{id})\Delta(p_{\mathbb{\Lambda}}).$$

This explains the last identity in the following definition. By Corollary 3.11 we also see that μ, ν and $(a \mid b)$ as defined below do not depend on the choices of $\gamma \in [\gamma]$.

Definition 3.13. Define $\mu \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^*$ by $\mu(a) = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}h_L(a_\alpha)$ where $a_\alpha = p_\alpha a$. We endow $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ with the following positive-definite sesquilinear form, where $a, b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$:

$$(a \mid b) = \mu(a^*b) = \sum_{[\gamma]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\gamma^{-1}h_L(a^*p_\gamma b) \tag{3.3}$$

$$= \sum_{[\alpha],[\beta]} \kappa_\alpha^{-1}\kappa_\beta^{-1}(h_R S(a^*_\alpha)\otimes b_\beta h_L)\Delta(p_{\mathbb{\Lambda}}). \tag{3.4}$$

We similarly define $\nu \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})^*$ by $\nu(b) = \sum_{[\beta]\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1}h_R(b_\beta)$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. Finally for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we write $\|a\|_{\mathbb{\Gamma}/\mathbb{\Lambda}} := (a \mid a)^{1/2}$.

Remark 3.14. This definition agrees via the Fourier transform with the scalar product on the 'dual algebra' $\mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}]$ introduced in [30]. Recall that this algebra is defined as the relative tensor product $\mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}] = \mathbb{C}[\mathbb{\Gamma}] \otimes_{\mathbb{C}[\mathbb{\Lambda}]} \mathbb{C}$ with respect to the canonical embedding $\mathbb{C}[\mathbb{\Lambda}] \subset \mathbb{C}[\mathbb{\Gamma}]$ and to the counit $\epsilon : \mathbb{C}[\mathbb{\Lambda}] \to \mathbb{C}$. The duality between $c_c(\mathbb{\Gamma})$ and $\mathbb{C}[\mathbb{\Gamma}]$ is described by the multiplicative unitary $W = \bigoplus_{\alpha \in I(\mathbb{\Gamma})} u_\alpha \in \mathcal{M}(c_c(\mathbb{\Gamma}) \otimes C[\mathbb{\Gamma}])$ — recall that we have $(\Delta \otimes \mathrm{id})(W) = W_{13}W_{23}$, $(\mathrm{id} \otimes \Delta)(W) = W_{13}W_{12}$, $(S \otimes \mathrm{id})(W) = W^* = (\mathrm{id} \otimes S^{-1})(W)$, and also, applying $S^{-1} \otimes S$: $(\mathrm{id} \otimes S)(W) = (S^{-1} \otimes \mathrm{id})(W)$. Recall also that we have a canonical conditional expectation $E : \mathbb{C}[\mathbb{\Gamma}] \to \mathbb{C}[\mathbb{\Lambda}]$, see [29]. Following [30] we consider the linear bijection $\mathcal{F}_{\mathbb{\Lambda}}^{-1} : \mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}] \to c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ given by $\mathcal{F}_{\mathbb{\Lambda}}^{-1}(x) = (\mathrm{id} \otimes \epsilon E S(x))(W)$, which corresponds to (the inverse of) the Fourier transform when $\mathbb{\Lambda} = \{e\}$. Using Theorem 3.10 one can check the following explicit formula for the inverse map $\mathcal{F}_{\mathbb{\Lambda}} : c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \to \mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}]$:

$$\mathcal{F}_{\mathbb{\Lambda}}(a) = (a\mu \otimes \mathrm{id})(W) \otimes_{\mathbb{C}[\mathbb{\Lambda}]} 1.$$

On the other hand the space $\mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}]$ has a natural prehilbertian structure given by $(x \mid y) = \epsilon E(x^* y)$. One can then easily check that $\mathcal{F}_{\mathbb{\Lambda}}$ is an isometry with respect to this scalar product on $\mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}]$ and the one of Definition 3.13 on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.

3.1.3 | The constants κ

In contrast to the classical case, where they trivialize, the constants κ of Definition 3.8 play an important role in the theory of quantum group Hecke algebras (or in the quantum version of the Clifford theory). In this subsection we give more details about them, which will, however, not be used in the rest of the article (except in Proposition 3.22).

Let us first note that related constants already appeared in Lemma 3.5 of [11], which states the existence, for every $\alpha \in I(\mathbb{\Gamma})$, of a constant $\eta_\alpha > 0$ such that

$$(\mathrm{id} \otimes \mathrm{qTr}_{\bar{\alpha}})\Delta(p_{\mathbb{\Lambda}}) = \eta_\alpha p_{[\alpha]},$$

where $[\alpha] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$. As we have $h_L p_{\bar{\alpha}} = \dim_q(\alpha)\,\mathrm{qTr}_{\bar{\alpha}}$, comparing the formulas shows that we have in fact $\eta_\alpha = \kappa_{\bar{\alpha}} / \dim_q(\alpha)$.

The next example shows that we do not have $\kappa_{\bar{\alpha}} = \kappa_\alpha$ in general.

Example 3.15. We consider the quantum subgroup $\mathbb{\Lambda} \subset \mathbb{\Gamma} = \mathbb{Z} * \mathbb{\Lambda}$, where $\mathbb{\Lambda}$ is any discrete quantum group. Denote by a the generating corepresentation of $\mathbb{Z}$ and take $v \in I(\mathbb{\Lambda})$. Consider $\alpha = v \otimes a$, which is an irreducible corepresentation of $\mathbb{\Gamma}$. We have $\bar{\alpha} \otimes \alpha \simeq \bigoplus_{w \subset \bar{v} \otimes v} a^{-1} \otimes w \otimes a$, hence $\kappa_\alpha = 1$, and $\alpha \otimes \bar{\alpha} = v \otimes \bar{v}$ hence $\kappa_{\bar{\alpha}} = \dim_q(v)^2$.

If $\mathbb{\Lambda}$ is non-classical, one can choose v such that $\kappa_{\bar{\alpha}} = \dim_q(v)^2 > 1 = \kappa_\alpha$, and if $\mathbb{\Lambda}$ is finite, we are in the setting of Hecke pairs. For $\mathbb{\Lambda}$ finite and non-classical we can further consider $\mathbb{\Lambda}^\infty = \varinjlim \mathbb{\Lambda}^n$ which is still commensurated inside $\mathbb{\Gamma}^\infty$. For $w \in \mathrm{Corep}(\mathbb{\Gamma})$ we denote $w^{(k)} \in \mathrm{Corep}(\mathbb{\Gamma}^\infty)$ the corepresentation corresponding to w and the kth copy of $\mathbb{\Gamma}$ in $\mathbb{\Gamma}^\infty$. Then for any $v \in I(\mathbb{\Lambda})$ such that $\dim_q(v) > 1$ and $\alpha = \bigotimes_{k=1}^n (v \otimes a)^{(k)} \in I(\mathbb{\Gamma}^\infty)$ we have $\kappa_{\bar{\alpha}}/\kappa_\alpha = \dim_q(v)^{2n}$ which is not bounded.

If one allows non-commensurated quantum subgroups, one can take for $\mathbb{\Lambda}$ the dual of $SU(2)$ and we see again that the ratios $\kappa_{\bar{\alpha}}/\kappa_\alpha$ are not bounded when α varies. Notice also that $v = \alpha \otimes \bar{a}$, $\kappa_v = \dim_q(v)^2$, and $\kappa_\alpha = 1$. This shows that there is no control on the ratios $\kappa_\gamma/\kappa_\alpha$ either when $\gamma \subset \alpha \otimes \beta$ with $\beta \in I(\mathbb{\Gamma})$ fixed.

We shall now give another interpretation of the constants $\kappa_{\alpha,\beta}$ in terms of the Corep($\mathbb{\Gamma}$)-module category Corep($\mathbb{\Gamma}/\mathbb{\Lambda}$). Let us first introduce some more relevant notation. For $\pi \in$ Corep($\mathbb{\Gamma}$) and $\alpha \in I(\mathbb{\Gamma})$ we denote $M_{\alpha,\pi} = \mathrm{Hom}(\alpha, \pi)$, so that $H_\pi \simeq \bigoplus_{\alpha\in I(\mathbb{\Gamma})} H_\alpha \otimes M_{\alpha,\pi}$ canonically. When $\pi = \beta \otimes \gamma$ we denote $C_\alpha^{\beta,\gamma} = M_{\alpha,\pi}$, so that the dimensions $c_\alpha^{\beta,\gamma} = \dim(C_\alpha^{\beta,\gamma})$ are the structure constants of the based Grothendieck ring of Corep($\mathbb{\Gamma}$).

We can extend this notation to objects of Corep($\mathbb{\Gamma}/\mathbb{\Lambda}$). For $i \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$, $\pi \in$ Corep($\mathbb{\Gamma}$) we thus denote $M_{i,\pi} = \mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(i, \pi)$ where π is identified to an object in Corep($\mathbb{\Gamma}/\mathbb{\Lambda}$) by restriction. Similarly we denote $C_i^{\beta,j} = \mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{\Lambda}}(i, \beta \otimes j)$ for $i, j \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\beta \in$ Corep($\mathbb{\Gamma}$).

Recall that the modular group of the left Haar weight is implemented by the (possibly unbounded) modular element $F \in c(\mathbb{\Gamma})$ via the formula $\sigma_t^L(f) = F^{it} f F^{-it}$. For $\pi \in$ Corep($\mathbb{\Gamma}$) we denote $F_\pi = \pi(F) \in B(H_\pi)$ and we recall the definition of the quantum dimension $\dim_q(\pi) = \mathrm{Tr}(F_\pi) = \mathrm{Tr}(F_\pi^{-1})$. Decomposing $H_\pi \simeq \bigoplus_\alpha H_\alpha \otimes M_{\alpha,\pi}$ we have $F_\pi = \sum F_\alpha \otimes \mathrm{id}$.

In general F does not belong to $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ but we still have an induced modular structure on this subalgebra. More precisely, take $\alpha \in I(\mathbb{\Gamma})$ and decompose $H_\alpha \simeq \bigoplus_{i\in I(\mathbb{\Gamma}/\mathbb{\Lambda})} H_i \otimes M_{i,\alpha}$ in Corep($\mathbb{\Gamma}/\mathbb{\Lambda}$). Since σ_t^L stabilizes $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ by Lemma 3.1, $\mathrm{Ad}(F_\alpha^{it})$ stabilizes $\alpha(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})) \simeq \bigoplus_i B(H_i) \otimes \mathrm{id} \subset B(H_\alpha)$, hence also $\alpha(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}))' \simeq \bigoplus_i \mathrm{id} \otimes B(M_{i,\alpha})$. It follows that for i such that $M_{i,\alpha} \neq 0$ there exist unique positive elements $F_i \in B(H_i)$, $L_{i,\alpha} \in B(M_{i,\alpha})$ such that $\mathrm{Tr}(F_i) = \mathrm{Tr}(F_i^{-1})$, $\mathrm{Tr}(L_{i,\alpha}) = \mathrm{Tr}(L_{i,\alpha}^{-1})$ and $F_\alpha = \mathrm{diag}_i(F_i \otimes L_{i,\alpha})$.

Furthermore, decomposing $H_\beta \otimes H_j \simeq \bigoplus_i H_i \otimes C_i^{\beta,j}$, we claim the existence of a corresponding decomposition $F_\beta \otimes F_j = \mathrm{diag}_i(F_i \otimes D_i^{\beta,j})$. Indeed, consider the canonical isomorphisms

$$H_\beta \otimes H_\alpha \simeq \bigoplus\nolimits_j H_\beta \otimes H_j \otimes M_{j,\alpha} \simeq \bigoplus\nolimits_{i,j} H_i \otimes C_i^{\beta,j} \otimes M_{j,\alpha}.$$

Since $\Delta(F) = F \otimes F$, we have $F_{\beta\otimes\alpha} = F_\beta \otimes F_\alpha = \mathrm{diag}(F_\beta \otimes F_j \otimes L_{j,\alpha})$. In particular $\mathrm{Ad}(F_{\beta\otimes\alpha}^{it})$ stabilizes $B(H_\beta) \otimes \alpha(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})) \simeq \bigoplus_j B(\bigoplus_i H_i \otimes C_i^{\beta,j}) \otimes \mathrm{id}$, and also $(\beta \otimes \alpha)(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})) \simeq \bigoplus_i B(H_i) \otimes \mathrm{id} \otimes \mathrm{id}$; hence, it stabilizes the relative commutant $\bigoplus_{i,j} \mathrm{id} \otimes B(C_i^{\beta,j}) \otimes \mathrm{id}$ as well. This shows the existence of a unique positive element $D_i^{\beta,j} \in B(C_i^{\beta,j})$ such that $\mathrm{Tr}(D_i^{\beta,j}) = \mathrm{Tr}((D_i^{\beta,j})^{-1})$ and $F_\beta \otimes F_\alpha = \mathrm{diag}_{i,j}(F_i \otimes D_i^{\beta,j} \otimes L_{j,\alpha})$. We have then also $F_\beta \otimes F_j = \mathrm{diag}_i(F_i \otimes D_i^{\beta,j})$.

We summarize the conclusions of this discussion in the next proposition.

Proposition 3.16. *For i, $j \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$, $\alpha, \beta \in I(\mathbb{\Gamma})$ we denote $F_i \in B(H_i)$, $L_{i,\alpha} \in B(M_{i,\alpha})$, $D_i^{\beta,j} \in B(C_i^{\beta,j})$ the unique positive elements, normalized by the identity* $\mathrm{Tr}(A) = \mathrm{Tr}(A^{-1})$, *such that $F_\alpha = \mathrm{diag}(F_i \otimes L_{i,\alpha})$ and $F_\beta \otimes F_i = \mathrm{diag}(F_j \otimes D_j^{\beta,i})$. The element F_i does not depend on the choice of $\alpha \in I(\mathbb{\Gamma})$ such that $M_{i,\alpha} \neq 0$. We introduce the quantum dimensions, quantum multiplicities, and quantum structure coefficients as follows:*

$$\dim_q(i) = \mathrm{Tr}(F_i), \quad \mathrm{qmult}(i, \alpha) = \mathrm{Tr}(L_{i,\alpha}), \quad \mathrm{qc}_i^{\beta,j} = \mathrm{Tr}(D_i^{\beta,j}).$$

Proof. The element F_i does not depend on α because $\mathrm{Ad}(F_i^{it})$ corresponds for every $t \in \mathbb{R}$ to the restriction of σ_t^L to $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})q_i$, where q_i denotes the relevant minimal central projection of $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. □

By considering the case of a quantum subgroup $\mathbb{A} \subset \mathbb{\Gamma} = \mathbb{A} \times \mathbb{A}'$ with $\mathbb{A}$, $\mathbb{A}'$ non-unimodular one sees that the elements F_i, $L_{i,\alpha}$, $D_i^{\beta,j}$ can be non-trivial, and that their traces are in general different from the classical dimensions $\dim(i) = \dim(H_i)$, the classical multiplicity $\mathrm{mult}(i,\alpha) = \dim(\mathrm{Hom}(i,\alpha))$, and the classical coefficients $c_i^{\beta,j} = \dim(\mathrm{Hom}(i, \beta \otimes j))$.

We denote by $\mathbb{C}[\mathbb{\Gamma}]$, $\mathbb{C}[\mathbb{\Gamma}/\mathbb{A}]$ the free vector spaces generated by $I(\mathbb{\Gamma})$, $I(\mathbb{\Gamma}/\mathbb{A})$, endowed with the scalar products such that $I(\mathbb{\Gamma})$, $I(\mathbb{\Gamma}/\mathbb{A})$ are orthonormal bases. To any representation $\pi \in \mathrm{Corep}(\mathbb{\Gamma}/\mathbb{A})$ corresponds naturally an element $\pi = \sum_{i\in I(\mathbb{\Gamma}/\mathbb{A})} \mathrm{mult}(i,\pi) i \in \mathbb{Z}[\mathbb{\Gamma}/\mathbb{A}]$. When π comes from $\mathrm{Corep}(\mathbb{\Gamma})$, one can use quantum multiplicities and we denote

$$\tilde{\pi} = \sum_{i\in I(\mathbb{\Gamma}/\mathbb{A})} \mathrm{qmult}(i,\pi) i \in \mathbb{R}[\mathbb{\Gamma}/\mathbb{A}].$$

Note that we have $\tilde{\pi} = \pi$ in $\mathbb{C}[\mathbb{\Gamma}]$ if $\mathbb{\Gamma}$ is unimodular.

Proposition 3.17. *For α, $\beta \in I(\mathbb{\Gamma})$ we have $\kappa_{\bar{\alpha},\beta} = (\tilde{\alpha} \mid \tilde{\beta})$, where $(\cdot \mid \cdot)$ is the scalar product of $\mathbb{C}[\mathbb{\Gamma}/\mathbb{A}]$. In particular $\kappa_\alpha = \|\tilde{\alpha}\|^2$.*

Proof. Recall the identification $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{A}}(\alpha,\beta) = (\bar{\alpha} \otimes \beta)_{\mathbb{A}}$ via the map $T \mapsto (\mathrm{id} \otimes T)t_\alpha$, $T \in \mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{A}}(\alpha,\beta)$. We have $(F_{\bar{\alpha}} \otimes F_\beta)(\mathrm{id} \otimes T)t_\alpha = (\mathrm{id} \otimes F_\beta T F_\alpha^{-1})t_\alpha$. Denote by $\mathcal{F} \in B(\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{A}}(\alpha,\beta))$ the map given by $\mathcal{F}(T) = F_\beta T F_\alpha^{-1}$ for $T \in \mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{A}}(\alpha,\beta) \subset B(H_\alpha, H_\beta)$, so that we have $\kappa_{\bar{\alpha},\beta} = \mathrm{Tr}(\mathcal{F})$. Decomposing $H_\alpha \simeq \bigoplus H_i \otimes M_{i,\alpha}$, $H_\beta \simeq \bigoplus H_i \otimes M_{i,\beta}$ we have $\mathrm{Hom}_{\mathbb{\Gamma}/\mathbb{A}}(\alpha,\beta) = \bigoplus_i \mathrm{id}_i \otimes B(M_{i,\alpha}, M_{i,\beta})$. Using the corresponding decomposition of the F-matrices we have $\mathcal{F}(T) = \sum_i L_{i,\beta} T_i L_{i,\alpha}^{-1}$ for $T = \mathrm{diag}(\mathrm{id} \otimes T_i)$. Computing $\mathrm{Tr}(\mathcal{F})$ in a basis of matrix units this yields $\kappa_{\bar{\alpha},\beta} = \sum_i \mathrm{Tr}(L_{i,\beta})\,\mathrm{Tr}(L_{i,\alpha}^{-1}) = \sum_i \mathrm{qmult}(i,\beta)\,\mathrm{qmult}(i,\alpha) = (\tilde{\alpha} \mid \tilde{\beta})$. □

We have then the following result generalizing (and resulting from) Corollary 3.12.

Proposition 3.18. *The vector $\tilde{\alpha} / \dim_q(\alpha) \in \mathbb{C}[\mathbb{\Gamma}/\mathbb{A}]$ only depends on $[\alpha]$.*

Proof. We have to show that for $i \in I(\mathbb{\Gamma}/\mathbb{A})$, $\alpha \in I(\mathbb{\Gamma})$, the number $\mathrm{qmult}(i,\alpha)/\dim_q(\alpha)$ depends only on i and the class $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{A}$. Denote by $q_i \in c_c(\mathbb{\Gamma}/\mathbb{A}) \subset \ell^\infty(\mathbb{\Gamma})$ the minimal central projection of $c_c(\mathbb{\Gamma}/\mathbb{A})$ corresponding to $i \in I(\mathbb{\Gamma}/\mathbb{A})$. We have $(q_i)_\alpha = q_i p_\alpha$ and so by Corollary 3.11 the number $\kappa^{-1} h_L(q_i p_\alpha)$ only depends on $[\alpha]$. On the other hand we can compute it using the canonical identification $H_\alpha \simeq \bigoplus_i H_i \otimes M_{i,\alpha}$ as follows:

$$\begin{aligned}\kappa_\alpha^{-1} h_L(q_i p_\alpha) &= \kappa_\alpha^{-1} \dim_q(\alpha)\,\mathrm{Tr}(F_\alpha^{-1} p_\alpha q_i) = \kappa_\alpha^{-1} \dim_q(\alpha)(\mathrm{Tr} \otimes \mathrm{Tr})(F_i^{-1} \otimes L_{i,\alpha}^{-1}) \\ &= \dim_q(i) \frac{\dim_q(\alpha)^2}{\kappa_\alpha} \frac{\mathrm{qmult}(i,\alpha)}{\dim_q(\alpha)}.\end{aligned}$$

The assertion follows since $\dim_q(\alpha)^2/\kappa_\alpha$ only depends on $[\alpha]$ by Corollary 3.12. □

3.2 | The Hecke algebra

3.2.1 | The convolution product

Now we introduce the Hecke convolution product. Note that, thanks to Theorem 3.10, the different expressions for $a * b$ given in the next definition are indeed equal. By comparing (3.5) and (3.7) one sees that they do not depend on the choices of $\alpha \in [\alpha]$, $\beta \in [\beta]$.

Definition 3.19. Let $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. According to Theorem 3.10 we can define $a * b \in c(\mathbb{\Gamma})$ as follows:

$$a * b = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes \mathrm{id})\Delta(b) = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(h_R S(a_\alpha) \otimes \mathrm{id})\Delta(b) \tag{3.5}$$

$$= \sum_{[\alpha],[\beta]} \kappa_\alpha^{-1}\kappa_{\bar\beta}^{-1}(S^{-1}(a_\alpha)h_R \otimes \mathrm{id} \otimes h_L S^{-1}(b_\beta))\Delta^2(p_{\mathbb{\Lambda}}) \tag{3.6}$$

$$= \sum_{[\beta]\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1}(\mathrm{id} \otimes h_L S^{-1}(b_\beta))\Delta(a) = \sum_{[\beta]\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1}(\mathrm{id} \otimes S(b_\beta)h_L)\Delta(a). \tag{3.7}$$

In the classical case $\mathbb{\Gamma} = \Gamma$ we have $\kappa_\alpha = 1$ for all $\alpha \in \Gamma$, and we recover the classical formulae for the Hecke convolution product [24]. Note that one can in fact define $a * b \in c(\mathbb{\Gamma})$ for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ (respectively, $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$) using (3.5) (respectively, (3.7)). The following results are immediate from the definition:

Proposition 3.20. *For any $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ we have $\Delta(a * b) = (a * \otimes\mathrm{id})\Delta(b) = (\mathrm{id}\otimes * b)\Delta(a)$. In particular we have $a * b \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$ if $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \cap c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $a * b \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ if $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.*

One can also express the convolution product using the functionals μ, ν from Definition 3.13, for instance, we have $a * b = ((\mu a)S \otimes \mathrm{id})\Delta(b)$ for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, and $a * b = (\mathrm{id} \otimes S(b)\mu)\Delta(a)$ for $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. From the first expression and Proposition 3.20 we then immediately obtain the following statement.

Proposition 3.21. *For any $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have $a * 1 = \mu(a)1$, where μ is the functional of Definition 3.13. Moreover μ is $\mathbb{\Gamma}$-invariant: we have $(\mathrm{id} \otimes \mu)\Delta(a) = \mu(a)1$ for any $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.*

In the next proposition we give a description of the convolution product between $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = S(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ using only the structure of the $\mathbb{\Gamma}$-invariant subalgebra $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \subset \ell^\infty(\mathbb{\Gamma})$, without explicit reference to the quantum subgroup $\mathbb{\Lambda}$. Note that by Proposition 3.18 the fractions appearing in the expression (3.8) only depend on i (and not on the choice of β_i).

Proposition 3.22. *Choose for each $i \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$ an element $\beta_i \in I(\mathbb{\Gamma})$ such that $\mathrm{qmult}(i,\beta_i) \neq 0$ and consider the linear forms $\varphi_i : \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \to \mathbb{C}$, $\varphi_i(a) = \mathrm{Tr}(F_i^{-1})\,\mathrm{Tr}(F_i^{-1}a_i)$. Then for all $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ we have*

$$a * b = \sum_{i\in I(\mathbb{\Gamma}/\mathbb{\Lambda})} \frac{\dim_q(\beta_i)\,\mathrm{qmult}(i,\beta_i)}{\kappa_{\beta_i}\dim_q(i)}(\mathrm{id} \otimes S(b)\varphi_i)\Delta(a). \tag{3.8}$$

Proof. We start from the last expression in Equation (3.7). We shall compute the linear form $\sum_{[\beta]} \kappa_{\bar\beta}^{-1} S(b_\beta) h_L$ on the matrix blocks of $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ using the canonical identification $p_{\bar\beta} c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq \bigoplus_i B(H_i) \otimes \mathrm{id}_{M(i,\bar\beta)} \subset B(H_{\bar\beta})$. For $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have

$$\begin{aligned}\sum_{[\beta]} \kappa_{\bar\beta}^{-1} S(b_\beta) h_L(a) &= \sum_{[\beta]} \sum_i \kappa_{\bar\beta}^{-1} \operatorname{Tr}(F_{\bar\beta}^{-1})(\operatorname{Tr}\otimes\operatorname{Tr})(F_i^{-1} a_i S(b)_i \otimes L_{i,\bar\beta}^{-1}) \\ &= \sum_{[\beta]} \sum_i \kappa_{\bar\beta}^{-1} \dim_q(\bar\beta) \operatorname{Tr}(F_i^{-1} a_i S(b)_i)\, \mathrm{qmult}(i,\bar\beta).\end{aligned}$$

Putting $\bar\beta = \beta_i$ for each $i \in I(\mathbb{\Gamma}/\mathbb{\Lambda})$ we obtain (3.8). □

Proposition 3.23. *If $a, b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}} := c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, then $a * b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$ as well. The convolution production defined in this way on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$ is bilinear, associative, with unit element $p_{\mathbb{\Lambda}}$. We have $\sigma_t^R(a * b) = \sigma_t^R(a) * \sigma_t^R(b)$ for any $t \in \mathbb{R}$. One obtains similarly a convolution product on $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})^{\mathbb{\Lambda}} = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \cap c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Moreover, the map $a \mapsto a^\sharp := S(a^*)$ is an antimultiplicative, antilinear involution exchanging both convolution algebras.*

Proof. Since $(c_c(\mathbb{\Gamma}) \otimes 1)\Delta(c_c(\mathbb{\Gamma}/\mathbb{\Lambda})) = c_c(\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, it follows from (3.5) that $a * b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ if $a, b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$. It follows from (3.5) and Theorem 3.10 that $p_{\mathbb{\Lambda}}$ is a unit for the convolution product.

To prove associativity we will use (3.7) to compute $b * c$ with $b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. This makes sense in the multiplier algebra $c(\mathbb{\Gamma})$. Indeed, fix $\delta \in I(\mathbb{\Gamma})$ and note that if $(p_\delta \otimes h_L S^{-1}(c_\gamma))\Delta(b)$ is non-zero for $\gamma \in I(\Gamma)$, there exists β such that $b_\beta \neq 0$ and $\beta \subset \delta \otimes \bar\gamma$. Then $\bar\gamma \subset \bar\delta \otimes \beta$, that is, $[\gamma] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ is conjugate to the class in $I(\mathbb{\Gamma})/\mathbb{\Lambda}$ of a subobject of $\bar\delta \otimes \beta$. Since there is only a finite number of classes $[\beta] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ such that $b_\beta \neq 0$, there is only a finite number of classes $[\gamma] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ which yield a non-zero term on the right-hand side of (3.7).

Now for $a, b, c \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$ we can write, using classes $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, $[\gamma] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$:

$$\begin{aligned} a * (b * c) &= \sum_{[\alpha],[\gamma]} \kappa_\alpha^{-1}\kappa_{\bar\gamma}^{-1} (S^{-1}(a_\alpha) h_R \otimes \mathrm{id})\Delta((\mathrm{id} \otimes h_L S^{-1}(c_\gamma))\Delta(b)) \\ &= \sum_{[\alpha],[\gamma]} \kappa_\alpha^{-1}\kappa_{\bar\gamma}^{-1} (S^{-1}(a_\alpha) h_R \otimes \mathrm{id} \otimes h_L S^{-1}(c_\gamma))\Delta^2(b) \\ &= \sum_{[\alpha],[\gamma]} \kappa_\alpha^{-1}\kappa_{\bar\gamma}^{-1} (\mathrm{id} \otimes h_L S^{-1}(c_\gamma))\Delta((S^{-1}(a_\alpha) h_R \otimes \mathrm{id})\Delta(b)) = (a * b) * c. \end{aligned}$$

Since $\Delta\sigma_t^R = (\sigma_t^R \otimes \sigma_t^R)\Delta$, the modular automorphisms leave $B(H_\alpha)$ invariant and $S\sigma_t^R = \sigma_t^R S$ we have, for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $b \in c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$:

$$\begin{aligned} \sigma_t^R(a) * \sigma_t^R(b) &= \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1} (h_R \otimes \mathrm{id})[(\sigma_t^R \otimes \sigma_t^R)\Delta(b)(\sigma_t^R S^{-1}(a_\alpha) \otimes 1)] \\ &= \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1} (h_R \otimes \sigma_t^R)[\Delta(b)(S^{-1}(a_\alpha) \otimes 1)] = \sigma_t^R(a * b). \end{aligned}$$

We compute similarly for the involution, using first (3.5) and then (3.7):

$$a^\sharp * b^\sharp = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(a_{\bar\alpha}^* h_R \otimes \mathrm{id})\Delta(S(b^*)) = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1}(S\otimes a_{\bar\alpha}^* h_R S)\Delta(b^*)$$

$$= \sum_{[\alpha]\in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\alpha}^{-1}[\mathrm{id}\otimes (\overline{a_\alpha^* h_R S})\Delta(b)]^\sharp = (b*a)^\sharp,$$

since $\overline{a_\alpha^* h_R S} = S(a_\alpha)h_L$. □

We can finally make the following definition. Recall that we have $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})\cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda})$ where $\mathbb{\Gamma}'$ is the commensurator of $\mathbb{\Lambda}$ in $\mathbb{\Gamma}$, and $\mathbb{\Gamma}' = \mathbb{\Gamma}$ if $(\mathbb{\Gamma},\mathbb{\Lambda})$ is a Hecke pair.

Definition 3.24. Let $\mathbb{\Lambda}$ be a quantum subgroup of a discrete quantum group $\mathbb{\Gamma}$. The *Hecke algebra* associated with $(\mathbb{\Gamma},\mathbb{\Lambda})$ is $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda}) := c_c(\mathbb{\Gamma}/\mathbb{\Lambda})\cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ equipped with the convolution product $*$ and the involution $\cdot^\sharp$.

Remark 3.25. Note that $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ depends only on the pair $(\mathbb{\Gamma}',\mathbb{\Lambda})$, where $\mathbb{\Gamma}'$ is the commensurator of $\mathbb{\Lambda}$ in $\mathbb{\Gamma}$. The algebra $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ is also equipped with more structure coming from the ambient space $c(\mathbb{\Gamma})$: the 'pointwise' product and involution, the scaling and modular groups $(\tau_t)_{t\in\mathbb{R}}$, $(\sigma_t^R)_{t\in\mathbb{R}}$, $(\sigma_t^L)_{t\in\mathbb{R}}$, see Lemma 3.1. Moreover all these one-parameter groups act by automorphisms of $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$, which can be shown similarly as it was done for $(\sigma_t^R)_{t\in\mathbb{R}}$ in the last proposition.

Example 3.26. Let us mention some 'trivial' examples where $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ is non-trivial. We will investigate more interesting examples in Subsection 3.3. If $\mathbb{\Lambda}$ is finite or of finite index in $\mathbb{\Gamma}$, but different from $\mathbb{\Gamma}$, then obviously $\mathbb{\Gamma}=\mathbb{\Gamma}'$ and $\mathbb{C}p_{\mathbb{\Lambda}} \subsetneq \mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$. If $\mathbb{\Lambda}$ is finite and $\mathbb{\Gamma}$ infinite, $\dim(\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})) = +\infty$.

Consider now the case when $\mathbb{\Lambda}$ is normal in $\mathbb{\Gamma}$, that is, $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. Then $\Delta(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})) \subset \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})\bar\otimes\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$, so that we have a discrete quantum group $\mathbb{\Gamma}/\mathbb{\Lambda}$ underlying the quantum quotient space. The dual $\mathbb{H}$ of $\mathbb{\Gamma}/\mathbb{\Lambda}$ identifies with a normal subgroup of the compact quantum group $\mathbb{G}$ dual to $\mathbb{\Gamma}$, with restriction morphism $\rho : C_u(\mathbb{G}) \to C_u(\mathbb{H})$, such that given $\alpha\in\mathrm{Corep}(\mathbb{\Gamma})$ we have $\alpha\in\mathrm{Corep}(\mathbb{\Lambda})$ if and only if $(\mathrm{id}\otimes\rho)(\alpha)$ is trivial — or, equivalently, contains the 1-dimensional corepresentation. It is then easy to check that $\alpha\sim\beta$ if and only if $\mathbb{1}\subset(\mathrm{id}\otimes\rho)(\bar\alpha\otimes\beta)$ if and only if $\mathbb{1}\subset(\mathrm{id}\otimes\rho)(\beta\otimes\bar\alpha)$ if and only if $\alpha \backsim \beta$. In particular $I(\mathbb{\Gamma})/\mathbb{\Lambda} = \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda} = \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ and $L([\![\alpha]\!]) = R([\![\alpha]\!]) = 1$ for all $\alpha\in I(\mathbb{\Gamma})$, so that $(\mathbb{\Gamma},\mathbb{\Lambda})$ is a Hecke pair.

By uniqueness and the invariance result of Proposition 3.21, the functional μ,ν of Definition 3.13 are the left, respectively, right Haar weights of $\mathbb{\Gamma}/\mathbb{\Lambda}$, normalized by the condition $\mu(p_{\mathbb{\Lambda}}) = \nu(p_{\mathbb{\Lambda}}) = 1$. Moreover, from the formula $a*b = (\mathrm{id}\otimes S(b)\mu)\Delta(a)$ we recognize the usual convolution product of the discrete quantum group algebra $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ (or its opposite, depending on conventions), transported from the product of $\mathbb{C}[\mathbb{\Gamma}/\mathbb{\Lambda}]$ via the Fourier transform.

3.2.2 | Hecke operators

Now we want to identify the Hecke algebra with an algebra of equivariant endomorphisms of $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. We turn $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ into a right $\mathbb{C}[\mathbb{\Gamma}]$-module via the formula $a\cdot x = (x\otimes\mathrm{id})\Delta(a)$, where $a\in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $x\in\mathbb{C}[\mathbb{\Gamma}]$ and we view $\mathbb{C}[\mathbb{\Gamma}]$ as linear subspace of $c_c(\mathbb{\Gamma})^*$ via evaluation. By definition, a linear map $F : c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \to c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ is $\mathbb{\Gamma}$-equivariant if and only if it is $\mathbb{C}[\mathbb{\Gamma}]$-linear, and we write $\mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ for the space of $\mathbb{\Gamma}$-equivariant linear endomorphisms of $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Restricting the action of $\mathbb{C}[\mathbb{\Gamma}]$ to $\mathbb{C}[\mathbb{\Lambda}]$ we obtain the associated space of fixed points $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}} = \{a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \mid (p_{\mathbb{\Lambda}} \otimes 1)\Delta(a) = p_{\mathbb{\Lambda}} \otimes a\} = \{a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \mid a \cdot x = \hat{\epsilon}(x)a\} = c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.

Proposition 3.27. *Let $\mathbb{\Lambda}$ be a quantum subgroup of $\mathbb{\Gamma}$. The map $\mathrm{ev}_{\mathbb{\Lambda}} : F \mapsto f := F(p_{\mathbb{\Lambda}})$ defines an antimultiplicative isomorphism from $\mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ to $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$, with inverse bijection T given by $T(f)(a) = a * f$, $f \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$, $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.*

Proof. Let $F \in \mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$. For all $g \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\alpha \in I(\mathbb{\Gamma})$ we have $(p_\alpha \otimes \mathrm{id})\Delta(g) \in c_c(\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and equivariance of F means that we have $(\mathrm{id} \otimes F)((p_\alpha \otimes \mathrm{id})\Delta(g)) = (p_\alpha \otimes \mathrm{id})\Delta(F(g))$. Applying this to $g = p_{\mathbb{\Lambda}}$ we obtain $(\mathrm{id} \otimes F)((p_\alpha \otimes \mathrm{id})\Delta(p_{\mathbb{\Lambda}})) = (p_\alpha \otimes \mathrm{id})\ \Delta(f)$. Since $p_{\mathbb{\Lambda}}$ is also left invariant under $\mathbb{\Lambda}$, for $\alpha = \lambda \in I(\mathbb{\Lambda})$ we obtain $(p_\lambda \otimes \mathrm{id})\Delta(f) = (\mathrm{id} \otimes F)(p_\lambda \otimes p_{\mathbb{\Lambda}}) = p_\lambda \otimes f$ and we have indeed $f \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$. The same equivariance formula can also be written $F((a_\alpha h_R \otimes \mathrm{id})\Delta(p_{\mathbb{\Lambda}})) = (a_\alpha h_R \otimes \mathrm{id})\Delta(f)$ for any $a \in c(\mathbb{\Gamma})$ and $\alpha \in I(\mathbb{\Gamma})$.

Moreover, take $a \in p_{[\alpha]}c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Using Theorem 3.10 and the previous equivariance formula we can write

$$\begin{aligned} F(a) &= \kappa_\alpha^{-1} F((S^{-1}(a_\alpha)h_R \otimes \mathrm{id})\Delta(p_{\mathbb{\Lambda}})) \\ &= \kappa_\alpha^{-1}(S^{-1}(a_\alpha)h_R \otimes \mathrm{id})\Delta(f) = a * f = T(f)(a). \end{aligned}$$

This shows that T is a left inverse of $\mathrm{ev}_{\mathbb{\Lambda}}$.

Conversely starting from $f \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$ we can consider $T(f) : a \mapsto a * f$. We already noticed at Proposition 3.23 that $T(f)(a) \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, and after Definition 3.19 that $T(f)$ is equivariant with respect to the left $\mathbb{\Gamma}$-action induced by Δ. Since $p_{\mathbb{\Lambda}}$ is the unit of the convolution product, T is a right inverse of $\mathrm{ev}_{\mathbb{\Lambda}}$. Finally T is antimultiplicative by associativity of the convolution product. □

We now use the prehilbertian structure on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ obtained from the functional μ and consider the corresponding subspace of adjointable operators in $\mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$. Note that the formula (3.3) for $(a \mid b)$ given in Definition 3.13 also makes sense for $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, or for $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. We use this in the following proposition. Note also that $\|p_{\mathbb{\Lambda}}\|_{\mathbb{\Gamma}/\mathbb{\Lambda}} = 1$.

Proposition 3.28. *For $a, b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $c \in c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ we have $(a \mid b * c) = (a * c^\sharp \mid b)$.*

Proof. We compute, using (3.3) and (3.5):

$$\begin{aligned} (a * c^\sharp \mid b) &= \sum_{[\beta]} \kappa_\beta^{-1}(b_\beta h_L)((a * c^\sharp)^*) = \sum_{[\alpha],[\beta]} \kappa_\alpha^{-1}\kappa_\beta^{-1}(\overline{S^{-1}(a_\alpha)h_R} \otimes b_\beta h_L)\Delta(c^{\sharp *}) \\ &= \sum_{[\alpha],[\beta]} \kappa_\alpha^{-1}\kappa_\beta^{-1}(b_\beta h_L \otimes \overline{S^{-1}(a_\alpha)h_R})(S^{-1} \otimes S^{-1})\Delta(c) \\ &= \sum_{[\alpha],[\beta]} \kappa_\alpha^{-1}\kappa_\beta^{-1}(h_R S(b_\beta) \otimes S^2(a_\alpha^*)h_L)\Delta(c) = \sum_{[\alpha]} \kappa_\alpha^{-1}(h_L a_\alpha^*)(b * c) = (a \mid b * c). \end{aligned}$$

We also used the identities $\overline{S^{-1}(a_\alpha)h_R} \circ S^{-1} = S^2(a_\alpha^*)h_L = h_L a_\alpha^*$ which are easy to check. □

In the next theorem, which offers an alternative description of the Hecke algebra, we write $\mathrm{End}'(c_c(\mathbb{\Gamma}/\mathbb{\Lambda})) \subset \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ for the subspace of adjointable maps, that is, of maps T for which there exists $S \in \mathrm{End}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ satisfying $(a \mid Tb) = (Sa \mid b)$ for all $a, b \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$.

Theorem 3.29. *Let $f \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}}$. Then we have $T(f) \in \mathrm{End}'(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ if and only if $f \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. As a result, T implements an antimultiplicative $*$-isomorphism between $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ and $\mathrm{End}'_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$. If $(\mathbb{\Gamma},\mathbb{\Lambda})$ is a Hecke pair, all maps in $\mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda}))$ are adjointable.*

Proof. In view of Proposition 3.27 it essentially suffices to understand the relevant $*$-structures. By Proposition 3.28, if $T(f)$ is adjointable, then its adjoint is $S : a \mapsto a * f^\sharp$. Taking $a = p_{\mathbb{\Lambda}}$ we obtain $f^\sharp \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, hence $f \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. The converse implication and the statement about T are then clear. If $(\mathbb{\Gamma},\mathbb{\Lambda})$ is a Hecke pair, we have $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^{\mathbb{\Lambda}} = c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) \cap c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. □

Now we investigate the question whether Hecke operators extend to bounded operators on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$, the completion of $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ with respect to the norm $\|\cdot\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$ arising from Definition 3.13. In the setting of discrete quantum groups we prove in Theorem 3.32 that this is equivalent to a combinatorial property of the constants κ_α introduced in Definition 3.30 below. Surprisingly we could not prove directly that this property always hold for Hecke pairs. However, we will show later in Section 4.3, using the Schlichting completion, that this is indeed the case, by showing that Hecke operators are always bounded.

Definition 3.30. Given the inclusion $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ and $\beta \in I(\mathbb{\Gamma})$ we say that β satisfies property (RT) if there exists a constant C_β such that $\kappa_\gamma \leqslant C_\beta \kappa_\alpha$ for all $\alpha, \gamma \in I(\mathbb{\Gamma})$ such that $\gamma \subset \alpha \otimes \beta$. We say that the inclusion $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ satisfies property (RT) if the property (RT) holds for each $\beta \in I(\mathbb{\Gamma}')$, where $\mathbb{\Gamma}'$ denotes the commensurator of $\mathbb{\Lambda}$ in $\mathbb{\Gamma}$.

Property (RT) is of course always verified in the classical case since then $\kappa_\alpha = 1$ for all $\alpha \in I(\mathbb{\Gamma})$. In general it does not hold for all corepresentations, as shown by the (non-commensurated) Example 3.15.

Lemma 3.31. *For $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, $[\beta] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ we have*

$$p_{[\alpha]} * p_{[\beta]} = \frac{\dim_q(\beta)}{\kappa_{\bar\beta}} \sum_{\delta \in I(\mathbb{\Gamma})} \frac{\dim_q(\delta \otimes \bar\beta)_{[\alpha]}}{\dim_q(\delta)} p_\delta, \tag{3.9}$$

where $(\delta \otimes \bar\beta)_{[\alpha]}$ denotes the span of the subobjects of $\delta \otimes \bar\beta$ isomorphic to elements of $[\alpha]$.

Proof. Note first that in the proof of Lemma 3.9 we showed that for each $\gamma, \delta, \mu \in I(\mathbb{\Gamma})$ we have

$$(h_R p_\gamma \otimes p_\delta)\Delta(p_\mu) = \dim_q(\gamma)\dim_q(\mu)\dim_q(\delta)^{-1} c_\mu^{\gamma,\delta} p_\delta,$$

where $c_\mu^{\gamma,\delta} = \dim(\mathrm{Hom}(\mu, \gamma\otimes\delta))$. Applying the antipode to this formula yields

$$(p_\delta \otimes h_L p_\gamma)\Delta(p_\mu) = \dim_q(\gamma)\dim_q(\mu)\dim_q(\delta)^{-1} c_{\bar\mu}^{\bar\gamma,\bar\delta} p_\delta.$$

Thus for each $[\alpha] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, $[\beta] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ we obtain

$$\begin{aligned} p_{[\alpha]} * p_{[\beta]} &= \kappa_\alpha^{-1}\kappa_{\bar\beta}^{-1}(h_R p_{\bar\alpha} \otimes \mathrm{id} \otimes h_L p_{\bar\beta})\Delta^2(p_{\mathbb{\Lambda}}) \\ &= \kappa_\alpha^{-1}\kappa_{\bar\beta}^{-1} \textstyle\sum_{\delta\in I(\mathbb{\Gamma}),\lambda\in I(\mathbb{\Lambda})}(h_R p_{\bar\alpha} \otimes p_\delta \otimes h_L p_{\bar\beta})\Delta^2(p_\lambda) \\ &= \kappa_\alpha^{-1}\kappa_{\bar\beta}^{-1} \textstyle\sum_{\delta\in I(\mathbb{\Gamma}),\lambda\in I(\mathbb{\Lambda}),\gamma\in I(\mathbb{\Gamma})}(h_R p_{\bar\alpha} \otimes p_\delta)\circ\Delta\circ(p_\gamma \otimes h_L p_{\bar\beta})(\Delta(p_\lambda)) \\ &= \kappa_\alpha^{-1}\kappa_{\bar\beta}^{-1} \textstyle\sum_{\delta,\lambda,\gamma} \dim_q(\bar\alpha)\dim_q(\delta)^{-1} c_\gamma^{\bar\alpha,\delta}\dim_q(\bar\beta)\dim_q(\lambda) c_{\bar\lambda}^{\beta,\bar\gamma} p_\delta \\ &= \frac{\dim_q(\alpha)}{\kappa_\alpha}\frac{\dim_q(\beta)}{\kappa_{\bar\beta}} \sum_{\delta\in I(\mathbb{\Gamma})} \frac{\dim_q(\bar\alpha\otimes\delta\otimes\bar\beta)_{\mathbb{\Lambda}}}{\dim_q(\delta)}\, p_\delta. \end{aligned}$$

We used the identity $\sum_{\gamma\in I(\mathbb{\Gamma})} c_\gamma^{\bar\alpha,\delta} c_{\bar\lambda}^{\beta,\bar\gamma} = \sum_{\gamma\in I(\mathbb{\Gamma})} c_\gamma^{\bar\alpha,\delta} c_\lambda^{\gamma,\bar\beta} = \dim \mathrm{Hom}(\lambda, \bar\alpha\otimes\delta\otimes\bar\beta)$, so that adding the quantum dimensions $\dim_q(\lambda)$ over $\lambda \in I(\mathbb{\Lambda})$ with these multiplicities yields $\dim_q(\bar\alpha\otimes\delta\otimes\bar\beta)_{\mathbb{\Lambda}}$.

Notice that the term corresponding to δ vanishes unless $\delta \subset \alpha\otimes\lambda\otimes\beta$ for some $\lambda \in I(\mathbb{\Lambda})$. Moreover, decomposing into irreducibles $\delta\otimes\bar\beta = \bigoplus c_{\alpha'}^{\delta,\bar\beta}\alpha'$ we can write, using Corollary 3.12:

$$\begin{aligned} \dim_q(\bar\alpha\otimes\delta\otimes\bar\beta)_{\mathbb{\Lambda}} &= \textstyle\sum_{\alpha'} c_{\alpha'}^{\delta,\bar\beta}\dim_q(\bar\alpha\otimes\alpha')_{\mathbb{\Lambda}} \\ &= \textstyle\sum_{\alpha'\in[\alpha]} c_{\alpha'}^{\delta,\bar\beta}\kappa_\alpha \dfrac{\dim_q(\alpha')}{\dim_q(\alpha)} = \dfrac{\kappa_\alpha}{\dim_q(\alpha)}\dim_q(\delta\otimes\bar\beta)_{[\alpha]} \end{aligned}$$

and the formula follows. □

Theorem 3.32. *Let $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ be an inclusion of discrete quantum groups. The operator $T(b)$ is bounded with respect to $\|\cdot\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$ for every $b \in \mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ if and only if the inclusion $\mathbb{\Lambda}\subset\mathbb{\Gamma}$ satisfies property (RT).*

Proof. Assume first that $T(b)$ is bounded with respect to $\|\cdot\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$ for all $b \in \mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$. In particular $T(p_\tau)$ is bounded for every $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma}')/\mathbb{\Lambda}$. Consider $\xi_{[\alpha]} = \kappa_\alpha^{1/2} p_{[\alpha]}/\dim_q(\alpha)$, which has norm 1 with respect to $\|\cdot\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$. By definition of the scalar product, and writing τ as a disjoint union of finitely many left classes $[\beta]$ we have

$$\begin{aligned} (\xi_{[\gamma]} \mid \xi_{[\alpha]} * p_\tau) &= \frac{\kappa_\alpha^{1/2}\kappa_\gamma^{1/2}}{\dim_q(\alpha)\dim_q(\gamma)}(p_{[\gamma]}|p_{[\alpha]} * p_\tau) = \frac{\kappa_\alpha^{1/2}\kappa_\gamma^{1/2}}{\dim_q(\alpha)\dim_q(\gamma)}\kappa_\gamma^{-1} h_L(p_\gamma(p_{[\alpha]} * p_\tau)) \\ &= \frac{\kappa_\alpha^{1/2}\kappa_\gamma^{1/2}}{\dim_q(\alpha)\dim_q(\gamma)}\kappa_\gamma^{-1}\sum_{[\beta]\subset\tau} h_L(p_\gamma(p_{[\alpha]} * p_{[\beta]})). \end{aligned}$$

Using (3.9) in the preceding lemma we can further compute:

$$(\xi_{[\gamma]} \mid \xi_{[\alpha]} * p_\tau) = \sum_{[\beta]\subset\tau} \frac{\kappa_\alpha^{1/2}\kappa_\gamma^{1/2}}{\dim_q(\alpha)\dim_q(\gamma)}\kappa_\gamma^{-1}\frac{\dim_q(\beta)}{\kappa_{\bar\beta}}\frac{\dim_q(\gamma\otimes\bar\beta)_{[\alpha]}}{\dim_q(\gamma)} h_L(p_\gamma)$$

$$= \sum_{[\beta]\subset\tau} \frac{\dim_q(\beta)}{\kappa_{\bar\beta}} \frac{\kappa_\alpha^{1/2}}{\dim_q(\alpha)} \frac{\dim_q(\gamma\otimes\bar\beta)_{[\alpha]}}{\kappa_\gamma^{1/2}}.$$

Finally we decompose $(\gamma\otimes\bar\beta)$ into irreducible subobjects α' and select the ones in $[\alpha]$. Using Corollary 3.12 this yields

$$(\xi_{[\gamma]} \mid \xi_{[\alpha]} * p_\tau) = \sum_{[\beta]\subset\tau} \frac{\dim_q(\beta)}{\kappa_{\bar\beta}} \sum_{\alpha'\in[\alpha]} c_{\alpha'}^{\gamma,\bar\beta} \frac{\kappa_\alpha^{1/2}}{\dim_q(\alpha)} \frac{\dim_q(\alpha')}{\kappa_\gamma^{1/2}} = \sum_{[\beta]\subset\tau} \frac{\dim_q(\beta)}{\kappa_{\bar\beta}} \sum_{\alpha'\in[\alpha]} c_{\alpha'}^{\gamma,\bar\beta} \frac{\kappa_{\alpha'}^{1/2}}{\kappa_\gamma^{1/2}}.$$

As a result we have, for any $\alpha', \gamma \in I(\mathbb{\Gamma})$ and $\beta\in I(\mathbb{\Gamma}')$ such that $\alpha'\subset\gamma\otimes\bar\beta$:

$$\frac{\kappa_{\alpha'}}{\kappa_\gamma} \leqslant \frac{\kappa_{\bar\beta}^2 \, \|T(p_{[\![\beta]\!]})\|^2}{\dim_q(\beta)^2}.$$

This shows the existence of the constants C_β and the direct implication.

Conversely, assume that (RT) holds. Taking $b\in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda})$, we can assume that $b\in p_\tau c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda})$ with $\tau\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma}')/\mathbb{\Lambda}$, and we then have a decomposition $b=\sum_{[\beta]\subset\tau} p_{[\beta]}b$ with $L(\tau)$ terms, where $[\beta]\in\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$. We first fix classes $[\alpha], [\gamma]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}$. For $a\in p_{[\alpha]}c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $c\in p_{[\gamma]}c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have

$$\begin{aligned}(c \mid a*b) &= \sum_{[\beta]\subset\tau} \kappa_{\bar\beta}^{-1}(c \mid (\mathrm{id}\otimes S(b_\beta)h_L)\Delta(a)) \\ &= \sum_{[\beta]\subset\tau,\alpha'\in[\alpha]} \kappa_{\bar\beta}^{-1}\kappa_\gamma^{-1}(h_L c_\gamma^* \otimes S(b_\beta)h_L)\Delta(a_{\alpha'}).\end{aligned}$$

The sum is in fact finite since its terms vanish unless $\alpha'\subset\gamma\otimes\bar\beta$.

Now we use the case when $\mathbb{\Lambda}$ is the trivial subgroup, denoting as $\|\cdot\|_{\mathbb{\Gamma}}$ the hermitian norm on $c_c(\mathbb{\Gamma})$ and $T_{\mathbb{\Gamma}}$ the homomorphism from $c_c(\mathbb{\Gamma})$ to $\mathrm{End}(c_c(\mathbb{\Gamma}))$. We know that $T_{\mathbb{\Gamma}}(d)$ extends to a bounded operator on $\ell^2(\mathbb{\Gamma})$ for any $d\in c_c(\mathbb{\Gamma})$, because it is an operator from the right regular representation of $\mathbb{\Gamma}$. Fix a choice of representatives β for the classes $[\beta]$ and put $M=\max_{[\beta]\subset\tau}\|T_{\mathbb{\Gamma}}(b_\beta)\|$ (so that M might depend on the choice made). We then have

$$|(h_L c_\gamma^*\otimes S(b_\beta)h_L)\Delta(a_{\alpha'})| = |(c_\gamma \mid T_{\mathbb{\Gamma}}(b_\beta)(a_{\alpha'}))_{\mathbb{\Gamma}}| \leqslant M\|c_\gamma\|_{\mathbb{\Gamma}}\|a_{\alpha'}\|_{\mathbb{\Gamma}}.$$

We have moreover $\|c_\gamma\|_{\mathbb{\Gamma}} = \sqrt{\kappa_\gamma}\|c\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$, $\|a_{\alpha'}\|_{\mathbb{\Gamma}} = \sqrt{\kappa_{\alpha'}}\|a\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}$ by definition.

As a result we can write:

$$|(c \mid a*b)| \leqslant M\sum_{[\beta]\subset\tau}\sum_{\alpha'\in[\alpha],\alpha'\subset\gamma\otimes\bar\beta} \kappa_{\bar\beta}^{-1}\kappa_\gamma^{-1/2}\kappa_{\alpha'}^{1/2}\|c\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}\|a\|_{\mathbb{\Gamma}/\mathbb{\Lambda}} \leqslant K\|c\|_{\mathbb{\Gamma}/\mathbb{\Lambda}}\|a\|_{\mathbb{\Gamma}/\mathbb{\Lambda}},$$

where $K=M\sum_{[\beta]\subset\tau}\kappa_{\bar\beta}^{-1}C_{\bar\beta}^{1/2}\dim(\beta)^2$ only depends on b (and not on $[\alpha], [\gamma], a, c$). Here we used property (RT) and the fact from Lemma 2.1 that $\gamma\otimes\bar\beta$ has at most $\dim(\beta)^2$ irreducible subobjects.

Let use write $[\gamma]\#[\alpha]$ if there exist $c\in p_{[\gamma]}c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $a\in p_{[\alpha]}c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ such that $(c\mid a*b)\neq 0$. The arguments above also show that $[\gamma]\#[\alpha]$ implies $\gamma\subset\alpha\otimes\lambda\otimes\beta$ for some $\lambda\in I(\mathbb{\Lambda})$ and $\beta\in\tau$. Writing τ as a disjoint union of $R(\tau)$ classes $[\beta_i]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}$, this inclusion implies $\gamma\subset\alpha\otimes\beta_i\otimes\mu$ for some i and $\mu\in I(\mathbb{\Lambda})$. Since $\alpha\otimes\beta_i$ contains at most $\dim(\beta_i)^2$ irreducibles, this shows that, once

$[\alpha]$ is fixed, we can have $[\gamma]\#[\alpha]$ for at most $\sum \dim(\beta_i)^2$ classes $[\gamma]$. Similarly, since $(c \mid a * b) = (c * b^\sharp \mid a)$, if $[\gamma]$ is fixed, we can have $[\gamma]\#[\alpha]$ for at most $\sum \dim(\beta'_j)^2$ classes $[\alpha]$, where we have now written τ as a disjoint union of $L(\tau)$ classes $[\beta'_j] \in \mathbb{A}\backslash I(\mathbb{\Gamma})$. Let us denote by N the maximum of these two sums, which depends only on τ.

Finally we can use Cauchy–Schwarz to write, for any $a, b \in c_c(\mathbb{\Gamma}/\mathbb{A})$:

$$|(c \mid a * b)| \leqslant \sum_{[\gamma]\#[\alpha]} |(p_{[\gamma]}c \mid (p_{[\alpha]}a) * b)| \leqslant K \sum_{[\gamma]\#[\alpha]} \|p_{[\gamma]}c\|_{\mathbb{\Gamma}/\mathbb{A}} \|p_{[\alpha]}a\|_{\mathbb{\Gamma}/\mathbb{A}}$$

$$\leqslant K\left(\sum_{[\gamma]\#[\alpha]} \|p_{[\gamma]}c\|^2_{\mathbb{\Gamma}/\mathbb{A}}\right)^{1/2}\left(\sum_{[\gamma]\#[\alpha]} \|p_{[\alpha]}a\|^2_{\mathbb{\Gamma}/\mathbb{A}}\right)^{1/2} \leqslant KN\|c\|_{\mathbb{\Gamma}/\mathbb{A}}\|a\|_{\mathbb{\Gamma}/\mathbb{A}}.$$

This shows that $T(b)$ is bounded with $\|T(b)\| \leqslant KN$. □

3.2.3 | Modular structure

Definition 3.33. The *canonical state* on $\mathcal{H}(\mathbb{\Gamma}, \mathbb{A})$ is given by the formula

$$\omega(f) := \epsilon(f) = (p_{\mathbb{A}} \mid T(f)p_{\mathbb{A}}), \quad f \in \mathcal{H}(\mathbb{\Gamma}, \mathbb{A}).$$

Since the sesquilinear form $(\cdot \mid \cdot)$ is positive-definite, ω is faithful, that is, $\omega(f^\sharp * f) = 0$ implies $f = 0$. To investigate the modular properties of ω we first construct a quantum analog of the classical modular function. We consider the restrictions of the functionals μ, ν introduced in Definition 3.13 to $c_c(\mathbb{A}\backslash\mathbb{\Gamma}'/\mathbb{A})$. These forms are faithful by Proposition 3.5, and so we can make the following definition.

Definition 3.34. The *modular element* associated with $(\mathbb{\Gamma}, \mathbb{A})$ is the unique element $\nabla \in c(\mathbb{A}\backslash\mathbb{\Gamma}'/\mathbb{A})$ such that $\nu(a) = \mu(\nabla a)$ for all $a \in c_c(\mathbb{A}\backslash\mathbb{\Gamma}'/\mathbb{A})$.

More concretely, fix $\alpha \in I(\mathbb{\Gamma}')$. Since $(a \mapsto p_\alpha a)$ is faithful on $p_{[\![\alpha]\!]}c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$ we can consider the positive linear forms μ_α, ν_α defined on the finite-dimensional C^*-algebra $p_\alpha c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$ by the following identities, for every $a \in p_{[\![\alpha]\!]}c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$:

$$\mu_\alpha(a_\alpha) = \sum\{\kappa_\delta^{-1} h_L(a_\delta) \mid [\delta] \subset [\![\alpha]\!], [\delta] \in I(\mathbb{\Gamma})/\mathbb{A}\},$$

$$\nu_\alpha(a_\alpha) = \sum\{\kappa_{\bar\delta}^{-1} h_R(a_\delta) \mid [\delta] \subset [\![\alpha]\!], [\delta] \in \mathbb{A}\backslash I(\mathbb{\Gamma})\}.$$

Then ∇ is characterized by the identities $\nu_\alpha(a_\alpha) = \mu_\alpha(\nabla_\alpha a_\alpha)$, $a \in p_{[\![\alpha]\!]}c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$, $[\![\alpha]\!] \in \mathbb{A}\backslash I(\mathbb{\Gamma}')/\mathbb{A}$. Note that, according to the next lemma, we also have $\nu(a) = \mu(a\nabla)$. Naturally $\nabla = \nabla^*$.

Lemma 3.35. *We have* $p_{\mathbb{A}}\nabla = p_{\mathbb{A}}$, $\sigma_t^R(\nabla) = \nabla$ *for any* $t \in \mathbb{R}$, $S(\nabla) = \nabla^{-1} = S^{-1}(\nabla)$.

Proof. Since h_R and h_L are both $\sigma_t^R = \sigma_{-t}^L$ invariant, this is also the case of μ_α and ν_α, hence we have $\nu_\alpha(a) = \nu_\alpha(\sigma_{-t}^R(a)) = \mu_\alpha(\nabla_\alpha \sigma_{-t}^R(a)) = \mu_\alpha(\sigma_t^R(\nabla_\alpha)a)$ and we conclude by uniqueness that $\sigma_t^R(\nabla) = \nabla$. This, together with the definition of μ and the fact that $(\sigma_t^R)_{t\in\mathbb{R}}$ is the modular automorphism group for h_R, also implies that $\mu(a\nabla) = \mu(\nabla a)$ for all $a \in c_c(\mathbb{A}\backslash\mathbb{\Gamma}'/\mathbb{A})$. Similarly, noting that $\nu S = \mu = \nu S^{-1}$ we can write $\nu(S(\nabla)a) = \mu(S^{-1}(a)\nabla) = \mu(\nabla S^{-1}(a)) = \nu(S^{-1}(a)) = \mu(a)$ and we conclude that $S(\nabla) = \nabla^{-1}$. We proceed in the same way with S^{-1}. □

The next result shows that the operator ∇ indeed plays the role of the modular operator for the canonical state on the Hecke algebra (or rather its relevant von Neumann algebraic completion).

Theorem 3.36. *The maps $\theta_t : a \mapsto \sigma^R_t(\nabla^{it}a) = \nabla^{it}\sigma^R_t(a)$, $t \in \mathbb{R}$, $a \in \mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ define a 1-parameter group of $*$-automorphisms of $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ and ω is a θ-KMS$_1$ state.*

Proof. Using the fact that ϵ is the counit and equalities (3.5), (3.7) we obtain the following expressions:

$$\omega(a * b) = \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1} h_R(S(a_\alpha)b) = \sum_{[\beta]\in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\beta}^{-1} h_L(aS(b_\beta)).$$

We can then compute, for $a, b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda})$:

$$\begin{aligned}\omega(a * b) &= \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1} h_L(a_\alpha S(b_{\bar\alpha})) = \sum_{[\![\alpha]\!]\in \mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}} \mu_\alpha(a_\alpha S(b_{\bar\alpha}))\\ &= \sum_{[\![\alpha]\!]\in \mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}} \nu_\alpha(\nabla_\alpha^{-1} a_\alpha S(b_{\bar\alpha})) = \sum_{[\alpha]\in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})} \kappa_{\bar\alpha}^{-1} h_R(\nabla_\alpha^{-1} a_\alpha S(b_{\bar\alpha}))\\ &= \sum_{[\alpha]\in I(\mathbb{\Gamma})/\mathbb{\Lambda}} \kappa_\alpha^{-1} h_R(S(b_\alpha)\sigma^R_i(\nabla^{-1}a)) = \omega(b * \sigma^R_i(\nabla^{-1}a)).\end{aligned}$$

We also have the following variant of the last step of the computation: since h_R is σ^R-invariant and σ^R commutes with S, it is easy to check that $h_R(S(b_\alpha)\sigma^R_i(\nabla^{-1}a)) = h_R(S(\nabla\sigma^R_{-i}(b_\alpha))a)$, and this yields $\omega(a * b) = \omega((\nabla\sigma^R_{-i}(b)) * a)$.

From these properties we first deduce:

$$\begin{aligned}\omega(a * b * c) &= \omega(b * c * \sigma^R_i(\nabla^{-1}a)) = \omega(c * \sigma^R_i(\nabla^{-1}a) * \sigma^R_i(\nabla^{-1}b))\\ &= \omega(\nabla\sigma^R_{-i}(\sigma^R_i(\nabla^{-1}a) * \sigma^R_i(\nabla^{-1}b)) * c).\end{aligned}$$

By faithfulness of ω this yields $\sigma^R_i(\nabla^{-1}a) * \sigma^R_i(\nabla^{-1}b) = \sigma^R_i(\nabla^{-1}(a * b))$, hence by Proposition 3.23 we have $(\nabla^{-1}a) * (\nabla^{-1}b) = \nabla^{-1}(a * b)$. This implies $(\nabla^k a) * (\nabla^k b) = \nabla^k(a * b)$ for all $k \in \mathbb{Z}$ and, by the usual argument, $(\nabla^z a) * (\nabla^z b) = \nabla^z(a * b)$ for all $z \in \mathbb{C}$. It follows that the maps θ_t are multiplicative for the convolution product. They are also compatible with the involution since $\sigma^R_t S = S\sigma^R_t$ and $(\nabla^{it}a)^\sharp = \nabla^{it\sharp}a^\sharp = \nabla^{it}a^\sharp$ for real t, using the property $S(\nabla^{-1}) = \nabla$. □

Corollary 3.37. *We have $\Delta(\nabla) = \nabla \otimes \nabla$.*

Proof. We use the property $(\nabla a) * (\nabla b) = \nabla(a * b)$ for $a, b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}'/\mathbb{\Lambda})$, established in the proof of the previous proposition. Since $S(\nabla) = \nabla^{-1}$, we can write

$$\begin{aligned}\nabla^{-1}(\nabla a * \nabla b) &= \sum \kappa_\alpha^{-1}(h_R S(\nabla a_\alpha) \otimes \nabla^{-1})\Delta(\nabla b)\\ &= \sum \kappa_\alpha^{-1}(h_R S(a_\alpha) \otimes \mathrm{id})[(\nabla^{-1} \otimes \nabla^{-1})\Delta(\nabla)\Delta(b)]\end{aligned}$$

$$= (a * b) = \sum \kappa_\alpha^{-1}(h_R S(a_\alpha) \otimes \mathrm{id})[\Delta(b)].$$

Since $\Delta(c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})) \subset \mathcal{M}(c_c(\mathbb{A}\backslash\mathbb{\Gamma}) \otimes c_c(\mathbb{\Gamma}/\mathbb{A}))$ and $h_R p_{\bar\alpha}$ is faithful on $p_{[\bar\alpha]} c_c(\mathbb{A}\backslash\mathbb{\Gamma})$ by Proposition 3.5, we can conclude that $(\nabla^{-1} \otimes \nabla^{-1})\Delta(\nabla) = 1 \otimes 1$. □

We shall now give an explicit formula for the modular function ∇ in terms of the structure of the inclusion $\mathbb{A} \subset \mathbb{\Gamma}$. This will involve quantum analogs $\tilde L_\alpha$, $\tilde R_\alpha$ of the counting functions L, R which arise from the interplay between the modular structure of the Haar weight h_R and the structure of the quantum quotient space $\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A}$.

For every $\alpha \in I(\mathbb{\Gamma})$, we have a unique h_L-preserving (respectively, h_R-preserving) conditional expectation from $p_\alpha c(\mathbb{\Gamma}) = B(H_\alpha)$ onto the subalgebra $p_\alpha c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$. We consider the following related maps.

Definition 3.38. We denote by E^L_α (respectively, E^R_α) the unique map $c(\mathbb{\Gamma}) \to p_{[\![\alpha]\!]} c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$ such that $h_L(p_\alpha E^L_\alpha(a)b) = h_L(p_\alpha ab)$ (respectively, $h_R(p_\alpha E^R_\alpha(a)b) = h_R(p_\alpha ab)$) for all $b \in c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$, $a \in c(\mathbb{\Gamma})$.

In the classical case, $E^L_\alpha(f)$ is the constant function on $[\![\alpha]\!]$, equal to the value $f(\alpha)$. Let us record the following property of these maps in connection with Woronowicz' modular element.

Lemma 3.39. *We have* $E^R_\alpha(F^{-2}) = E^L_\alpha(F^2)^{-1}$.

Proof. Recall that $h_R(a) = h_L(F^2 a)$ for all $a \in c_c(\mathbb{\Gamma})$. In particular we have, for $a \in c_c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$ and $\alpha \in I(\mathbb{\Gamma})$:

$$h_R(a_\alpha) = h_L(F^2 a_\alpha) = h_L(E^L_\alpha(F^2)a_\alpha) = h_R(F^{-2}E^L_\alpha(F^2)a_\alpha) = h_R(E^R_\alpha(F^{-2})E^L_\alpha(F^2)a_\alpha),$$

since $E^L_\alpha(F^2)a_\alpha \in p_\alpha c_c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$. As h_R is faithful on $p_\alpha c_c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$, we can infer that $p_\alpha E^R_\alpha(F^{-2}) = (p_\alpha E^L_\alpha(F^2))^{-1}$ and we conclude by Proposition 3.5. □

Definition 3.40. Fix $\alpha \in I(\mathbb{\Gamma}')$ and choose elements $\delta_i \sim \alpha$ (respectively, $\epsilon_j \backsim \alpha$) such that $[\![\alpha]\!]$ is the disjoint union of the classes $[\delta_i] \in \mathbb{A}\backslash I(\mathbb{\Gamma})$ (respectively, $[\epsilon_j] \in I(\mathbb{\Gamma})/\mathbb{A}$). We define the following elements of $p_{[\![\alpha]\!]} c_c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$:

$$\tilde L_\alpha = \sum_i \frac{\dim_q(\delta_i)^2}{\kappa_{\bar\delta_i}} E^L_{\delta_i}(F^2), \quad \tilde R_\alpha = \sum_j \frac{\dim_q(\epsilon_j)^2}{\kappa_{\epsilon_j}} E^R_{\epsilon_j}(F^{-2}).$$

Remark 3.41. Note that we have $F \in c(\mathbb{A}\backslash\mathbb{\Gamma}/\mathbb{A})$ if and only if $\mathbb{A}$ is unimodular (that is, $F|_{\mathbb{A}} = I$), since $\Delta(F) = F \otimes F$. In this case we have $E^L_\alpha(F^t) = E^R_\alpha(F^t) = p_{[\![\alpha]\!]}F^t$ for all $t \in \mathbb{R}$, and hence, writing $F_{[\![\alpha]\!]} = p_{[\![\alpha]\!]}F$:

$$\tilde L_\alpha = \left(\sum_i \frac{\dim_q(\delta_i)^2}{\kappa_{\bar\delta_i}}\right) F^2_{[\![\alpha]\!]}, \quad \tilde R_\alpha = \left(\sum_j \frac{\dim_q(\epsilon_j)^2}{\kappa_{\epsilon_j}}\right) F^{-2}_{[\![\alpha]\!]}.$$

Since $\dim_q(\delta_i)^2/\kappa_{\bar\delta_i}$ only depends on the class $[\delta_i] \in \mathbb{A}\backslash I(\mathbb{\Gamma})$, we can drop the constraint $\delta_i \sim \alpha$ in the definition of $\tilde L_\alpha$ and we see in particular that $\tilde L_\alpha$, $\tilde R_\alpha$ only depend on $[\![\alpha]\!]$ in this case. If we have moreover $\kappa_\delta = \kappa_{\bar\delta}$ for all $\delta \in I(\mathbb{\Gamma})$, the terms $\dim_q(\delta_i)^2/\kappa_{\bar\delta_i}$ only depend on $[\![\delta_i]\!] = [\![\alpha]\!]$ and hence we have

$$\tilde{L}_\alpha = L([\![\alpha]\!])\,\kappa_\alpha^{-1}\dim_q(\alpha)^2 F^2_{[\![\alpha]\!]}, \quad \tilde{R}_\alpha = R([\![\alpha]\!])\,\kappa_\alpha^{-1}\dim_q(\alpha)^2 F^{-2}_{[\![\alpha]\!]}.$$

On the other hand, let us consider the case when $p_{[\![\alpha]\!]}c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda}) = \mathbb{C}p_{[\![\alpha]\!]}$. Then we have $E^L_\alpha(F^2) = p_{[\![\alpha]\!]} = E^R_\alpha(F^{-2})$ (as, for example, $h_L(F^2_\alpha) = h_L(p_\alpha)$), so that

$$\tilde{L}_\alpha = \left(\sum_i \frac{\dim_q(\delta_i)^2}{\kappa_{\bar{\delta}_i}}\right)p_{[\![\alpha]\!]}, \quad \tilde{R}_\alpha = \left(\sum_j \frac{\dim_q(\epsilon_j)^2}{\kappa_{\epsilon_j}}\right)p_{[\![\alpha]\!]},$$

with the same simplification as above if $\kappa_\delta = \kappa_{\bar{\delta}}$ for all δ.

Proposition 3.42. *We have $\nabla_\alpha = p_\alpha \tilde{R}_\alpha^{-1}E^L_\alpha(F^2)^{-1}\tilde{L}_\alpha$. In particular, if $\mathbb{\Lambda}$ is unimodular and $\kappa_\delta = \kappa_{\bar{\delta}}$ for all $\delta \in I(\mathbb{\Gamma})$, we obtain the 'semi-classical' formula $\nabla_\alpha = \frac{L([\![\alpha]\!])}{R([\![\alpha]\!])}F^2_\alpha$.*

Proof. Take $\alpha \in I(\mathbb{\Gamma})$ and $a \in p_{[\![\alpha]\!]}c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$. We choose elements $\epsilon_j \in [\![\alpha]\!]$ such that $[\![\alpha]\!]$ is the disjoint union of the classes $[\epsilon_j] \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ and $\epsilon_j \backsim \alpha$ for all i.

By Corollary 3.11 we have

$$\kappa_{\bar{\epsilon}_j}^{-1}h_L(a_{\epsilon_j}) = \kappa_{\bar{\epsilon}_j}^{-1}h_R(p_{\epsilon_j}F^{-2}a) = \kappa_{\bar{\epsilon}_j}^{-1}h_R(p_{\epsilon_j}E^R_{\epsilon_j}(F^{-2})a) = \kappa_{\bar{\alpha}}^{-1}h_R(p_\alpha E^R_{\epsilon_j}(F^{-2})a).$$

Recall moreover that we have $\kappa_{\bar{\epsilon}}/\kappa_{\bar{\alpha}} = (\dim_q(\epsilon)/\dim_q(\alpha))^2$ when $\epsilon \backsim \alpha$. Hence we can write

$$\begin{aligned}\mu_\alpha(a_\alpha) &= \textstyle\sum_j \kappa_{\epsilon_j}^{-1}h_L(a_{\epsilon_j}) = \sum_j \frac{\kappa_{\bar{\epsilon}_j}}{\kappa_{\epsilon_j}}\kappa_{\bar{\epsilon}_j}^{-1}h_L(a_{\epsilon_j}) = \sum_j \frac{\kappa_{\bar{\epsilon}_j}}{\kappa_{\epsilon_j}}\kappa_{\bar{\alpha}}^{-1}h_R(E^R_{\epsilon_j}(F^{-2})a_\alpha)\\ &= \dim_q(\alpha)^{-2}h_R(\tilde{R}_\alpha a_\alpha) = \dim_q(\alpha)^{-2}h_L(F^2\tilde{R}_\alpha a_\alpha) = \dim_q(\alpha)^{-2}h_L(E^L_\alpha(F^2)\tilde{R}_\alpha a_\alpha).\end{aligned}$$

We proceed similarly on the other side with classes $[\delta_i] \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ and $\delta_i \sim \alpha$:

$$\begin{aligned}\nu_\alpha(a_\alpha) &= \textstyle\sum_i \frac{\kappa_{\delta_i}}{\kappa_{\bar{\delta}_i}}\kappa_{\delta_i}^{-1}h_R(a_{\delta_i}) = \sum_i \frac{\kappa_{\delta_i}}{\kappa_{\bar{\delta}_i}}\kappa_{\delta_i}^{-1}h_L(F^2a_{\delta_i}) = \sum_i \frac{\kappa_{\delta_i}}{\kappa_{\bar{\delta}_i}}\kappa_{\delta_i}^{-1}h_L(E^L_{\delta_i}(F^2)a_{\delta_i})\\ &= \textstyle\sum_i \frac{\kappa_{\delta_i}}{\kappa_{\bar{\delta}_i}}\kappa_\alpha^{-1}h_L(E^L_{\delta_i}(F^2)a_\alpha) = \dim_q(\alpha)^{-2}h_L(\tilde{L}_\alpha a_\alpha).\end{aligned}$$

This yields the result by definition of ∇. □

3.3 | Examples: HNN extensions

Let $\mathbb{\Gamma}_0$ be a discrete quantum group with two quantum subgroups $\mathbb{\Lambda}_\epsilon \subset \mathbb{\Gamma}_0$ ($\epsilon = \pm 1$). Following [8], we start with an isomorphism between the two quantum subgroups, described via a Hopf $*$-algebra isomorphism $\theta : \mathbb{C}[\mathbb{\Lambda}_1] \to \mathbb{C}[\mathbb{\Lambda}_{-1}]$ and we form $\mathbb{\Gamma} = HNN(\mathbb{\Gamma}_0, \theta)$. Recall that $\mathbb{C}[\mathbb{\Gamma}]$ is generated by $\mathbb{C}[\mathbb{\Gamma}_0]$ and a group-like unitary w such that $w^\epsilon b w^{-\epsilon} = \theta^\epsilon(b)$ for $b \in \mathbb{C}[\mathbb{\Lambda}_\epsilon]$. We denote by $E_\epsilon : \mathbb{C}[\mathbb{\Gamma}_0] \to \mathbb{C}[\mathbb{\Lambda}_\epsilon]$ the canonical conditional expectations. The algebra $\mathbb{C}[\mathbb{\Gamma}]$ is the direct sum of the subspaces

$$\mathbb{C}[\mathbb{F}]_n = \{x_0 w^{\epsilon_1} x_1 \cdots w^{\epsilon_n} x_n \mid x_i \in \mathbb{C}[\mathbb{F}_0], \epsilon_i = \pm 1, E_{\epsilon_i}(x_i) = 0 \text{ whenever } \epsilon_{i+1} \neq \epsilon_i\}.$$

The subspaces $\mathbb{C}[\mathbb{F}]_n$, $n \geqslant 1$ span the kernel of the canonical conditional expectation $E_0 : \mathbb{C}[\mathbb{F}] \to \mathbb{C}[\mathbb{F}_0] = \mathbb{C}[\mathbb{F}]_0$ and the Haar state of $\mathbb{F}$ is $h = h_0 \circ E_0$ — see [8]. It follows in particular that $I(\mathbb{F})$ is partitioned into the subsets

$$I(\mathbb{F})_n = \{\alpha \subset \alpha_0 \otimes w^{\epsilon_1} \otimes \alpha_1 \otimes \cdots \otimes w^{\epsilon_n} \otimes \alpha_n \mid \\ \alpha_i \in I(\mathbb{F}_0), \epsilon_i = \pm 1, \alpha_i \notin I(\mathbb{A}_{\epsilon_i}) \text{ whenever } \epsilon_{i+1} \neq \epsilon_i\}.$$

Proposition 3.43. *Assume that the quantum subgroups $\mathbb{A}_\epsilon$ have finite index in $\mathbb{F}_0$ and at least one of them is distinct from $\mathbb{F}_0$. Then $\mathbb{F}_0$ is commensurated in $\mathbb{F}$, not normal, and of infinite index.*

Proof. Write $I(\mathbb{F}_0)/\mathbb{A}_\epsilon = \{[\gamma_{\epsilon,0}], \dots, [\gamma_{\epsilon,p}]\}$ with $\gamma_{\epsilon,0} = 1$. Then any $\alpha \in I(\mathbb{F})_n$ is contained in a representation $\gamma_{-\epsilon_1,k_1} \otimes w^{\epsilon_1} \otimes \gamma_{-\epsilon_2,k_2} \otimes w^{\epsilon_2} \otimes \cdots \otimes w^{\epsilon_n} \otimes \alpha_n$ with $k_i \neq 0$ if $\epsilon_i \neq \epsilon_{i+1}$. Indeed, starting from $\alpha \subset \alpha_0 \otimes w^{\epsilon_1} \otimes \alpha_1 \otimes \cdots \otimes w^{\epsilon_n} \otimes \alpha_n$ as previously, write $\alpha_0 \subset \gamma_{-\epsilon_1,k_1} \otimes \lambda$ with $\lambda \in I(\mathbb{A}_{-\epsilon_1})$. Observe moreover that $\lambda \otimes w^{\epsilon_1} \simeq w^{\epsilon_1} \otimes \theta^{-\epsilon_1}(\lambda)$ and decompose $\theta^{-\epsilon_1}(\lambda) \otimes \alpha_1$ into irreducible subobjects α_1'. Since α is irreducible, it appears as a subobject of one of the corresponding corepresentations $\gamma_{-\epsilon_1,k_1} \otimes w^{\epsilon_1} \otimes \alpha_1' \otimes w^{\epsilon_2} \cdots \otimes w^{\epsilon_n} \otimes \alpha_n$. Moreover since $\theta^{-\epsilon_1}(\lambda) \in I(\mathbb{A}_{\epsilon_1})$, we have $\alpha_1 \notin I(\mathbb{A}_{\epsilon_1}) \Rightarrow \alpha_1' \notin I(\mathbb{A}_{\epsilon_1})$. Iterating the procedure we see that α_1' can also be chosen among the representatives $\gamma_{-\epsilon_2,k}$, etc.

In particular it follows that $I(\mathbb{F})_n/\mathbb{F}_0$ — and similarly $\mathbb{F}_0 \backslash I(\mathbb{F})_n$ — is finite. Since $I(\mathbb{F})_n$ is clearly saturated with respect to the equivalence relations $\sim$, $\backsim$ relative to $\mathbb{F}_0$, this shows that $\mathbb{F}_0$ is commensurated in $\mathbb{F}$. It is never of finite index in $\mathbb{F}$ since the subsets $I(\mathbb{F})_n$, $n \geqslant 1$, are non-empty. Finally, assuming, for example, $\mathbb{A}_1 \neq \mathbb{F}_0$ and taking $\alpha \in I(\mathbb{F}_0) \setminus I(\mathbb{A}_1)$, then we get that $w \otimes \alpha \otimes w^*$ belongs to $I(\mathbb{F})_2$. If we had $c(\mathbb{F}/\mathbb{F}_0) = c(\mathbb{F}_0 \backslash \mathbb{F})$, then $w \otimes \alpha$ would be equivalent to a corepresentation of the form $\beta \otimes w$ with $\beta \in I(\mathbb{F}_0)$, but then $w \otimes \alpha \otimes w^* \simeq \beta$ would belong to $I(\mathbb{F})_0$. Hence $\mathbb{F}_0$ is not normal in $\mathbb{F}$. ◻

Denote by $N(\mathbb{F} \curvearrowright \mathbb{F}/\mathbb{A})$ the weak closure of $\{(\mathrm{id} \otimes \varphi)\Delta(a) \mid a \in c_0(\mathbb{F}/\mathbb{A}), \varphi \in c_0(\mathbb{F}/\mathbb{A})^*\}$ in $\ell^\infty(\mathbb{F})$. Recall from [11] that the action of $\mathbb{F}$ on $\mathbb{F}/\mathbb{A}$ is called faithful if we have $N(\mathbb{F} \curvearrowright \mathbb{F}/\mathbb{A}) = \ell^\infty(\mathbb{F})$.

Lemma 3.44. *Let $\mathbb{A} \subset \mathbb{F}$ be a quantum subgroup. Assume that for every non-trivial $\alpha \in I(\mathbb{F})$ we can find $\gamma_\alpha \in I(\mathbb{F})$ such that no subobjects of $\alpha \otimes \gamma_\alpha$ belong to $[\gamma_\alpha] \in I(\mathbb{F})/\mathbb{A}$. Then the action of $\mathbb{F}$ on $\mathbb{F}/\mathbb{A}$ is faithful.*

Proof. We denote p_0 the minimal central projection p_α corresponding to the trivial corepresentation $\alpha = 1$. It is also the central support of the counit ϵ of $c_c(\mathbb{F})$. The condition on α, γ_α can also be written $(p_\alpha \otimes p_{\gamma_\alpha})\Delta(p_{[\gamma_\alpha]}) = 0$. On the other hand we have $(p_0 \otimes p_{\gamma_\alpha})\Delta(p_{[\gamma_\alpha]}) = (p_0 \epsilon \otimes p_{\gamma_\alpha})\Delta(p_{[\gamma_\alpha]}) = p_0 \otimes p_{\gamma_\alpha} p_{[\gamma_\alpha]} = p_0 \otimes p_{\gamma_\alpha}$. For $\alpha \neq 1$ we denote $x_\alpha = \dim_q(\gamma_\alpha)^{-2}(\mathrm{id} \otimes h_L p_{\gamma_\alpha})\Delta(p_{[\gamma_\alpha]})$. We have $p_\alpha x_\alpha = 0$, $p_0 x_\alpha = p_0$, and $\|x_\alpha\| \leqslant 1$ since $\dim_q(\gamma)^{-2} h_L p_\gamma$ is a state. Introduce a total order on $I(\mathbb{F})$ and put $y_F = \prod_{\alpha \in F} x_\alpha$ for $F \subset I(\mathbb{F})$ finite, $1 \notin F$. Then the net $(y_F)_F$ converges to p_0 in the weak topology. Indeed the elements of the net are uniformly bounded in norm, and for $F \subset I(\mathbb{F})$ finite, $1 \notin F$, we have $p_0 y_F = p_0$ and $p_\beta y_F = 0$ as soon as $\beta \in F$.

Since $x_\alpha \in N(\mathbb{\Gamma} \curvearrowright \mathbb{\Gamma}/\mathbb{\Lambda})$ for all $\alpha \in I(\mathbb{\Gamma})$, $\alpha \neq 1$, it follows that $p_0 \in N(\mathbb{\Gamma} \curvearrowright \mathbb{\Gamma}/\mathbb{\Lambda})$, which implies $N(\mathbb{\Gamma} \curvearrowright \mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\mathbb{\Gamma})$ by [11, Proposition 2.10]. ☐

We still denote by $\theta : I(\mathbb{\Lambda}_1) \to I(\mathbb{\Lambda}_{-1})$ the map induced by θ on irreducible representations. We define by induction $\operatorname{Dom}\theta^k \subset I(\mathbb{\Gamma}_0)$, for $k \in \mathbb{Z}$, by putting $\operatorname{Dom}\theta^0 = I(\mathbb{\Gamma}_0)$ and $\operatorname{Dom}\theta^{(n+1)\epsilon} = \{\alpha \in \operatorname{Dom}\theta^{n\epsilon} \mid \theta^{n\epsilon}(\alpha) \in I(\mathbb{\Lambda}_\epsilon)\}$ for $n \in \mathbb{N}$, $\epsilon = \pm 1$.

Proposition 3.45. *Assume that $\bigcap_{k\in\mathbb{Z}} \operatorname{Dom}\theta^k = \{1\}$. Then the action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Gamma}_0$ is faithful.*

Proof. We apply Lemma 3.44. Take $\alpha \in I(\mathbb{\Gamma})$ non-trivial. If $\alpha \notin I(\mathbb{\Gamma}_0)$, we can just take $\gamma = 1$. Hence we can assume $\alpha \in \mathbb{\Gamma}_0$, $\alpha \neq 1$. By assumption there exists $\epsilon \in \{\pm 1\}$, $n \in \mathbb{N}^*$ such that $\alpha \in \operatorname{Dom}\theta^{(n-1)\epsilon}$ but $\alpha \notin \operatorname{Dom}\theta^{n\epsilon}$. We take $\gamma_\alpha = w^{-n\epsilon}$. We have then $\alpha \otimes w^{-n\epsilon} = w^{-(n-1)\epsilon} \otimes \theta^{(n-1)\epsilon}(\alpha) \otimes w^{-\epsilon}$, which is irreducible and not equivalent to $w^{-n\epsilon} \otimes \beta$ with $\beta \in I(\mathbb{\Gamma}_0)$ — indeed β is in $I(\mathbb{\Gamma})_0$ but $w^\epsilon \otimes \theta^{(n-1)\epsilon}(\alpha) \otimes w^{-\epsilon}$ is in $I(\mathbb{\Gamma})_2$ since $\theta^{(n-1)\epsilon}(\alpha) \notin I(\mathbb{\Lambda}_\epsilon)$. ☐

Note that, in the classical case, if K is a central subgroup of Γ_0 contained in $\bigcap_{k\in\mathbb{Z}} \operatorname{Dom}\theta^k$, then it acts trivially on Γ/Γ_0.

Now we compute the modular function ∇ on generators, using Proposition 3.42. Clearly $\nabla_\alpha = p_\alpha$ for $\alpha \in I(\mathbb{\Gamma}_0) \subset I(\mathbb{\Gamma})$. The value at w is given as follows in terms of $\mathbb{\Gamma}_0$ and $\mathbb{\Lambda}_\epsilon$.

Proposition 3.46. *Assume that the quantum subgroups $\mathbb{\Lambda}_\epsilon$ have finite index in $\mathbb{\Gamma}_0$ and denote $I(\mathbb{\Gamma}_0)/\mathbb{\Lambda}_{-1} = \{[\epsilon_0], \dots, [\epsilon_p]\}$, $\mathbb{\Lambda}_1\backslash I(\mathbb{\Gamma}) = \{[\delta_0], \dots, [\delta_q]\}$. Then we have $\nabla_w = p_w \tilde{R}_w^{-1}\tilde{L}_w$ with*

$$\tilde{L}_w = \sum_{i=1}^{q} \frac{\dim_q(\delta_i \otimes \bar{\delta}_i)}{\dim_q(\delta_i \otimes \bar{\delta}_i)_{\mathbb{\Lambda}_1}} p_{[\![w]\!]} \quad \textit{and} \quad \tilde{R}_w = \sum_{j=1}^{p} \frac{\dim_q(\bar{\epsilon}_j \otimes \epsilon_j)}{\dim_q(\bar{\epsilon}_j \otimes \epsilon_j)_{\mathbb{\Lambda}_{-1}}} p_{[\![w]\!]}.$$

Proof. We have $[\![w]\!] = \bigcup_j [\epsilon_j \otimes w]$ and $\epsilon_j \otimes w \backsim w$, see the proof of Proposition 3.43. Note that $\epsilon_j \otimes w$ is irreducible (whereas $\epsilon_j \otimes w \otimes \alpha$ needs not to be): indeed $\epsilon_j \otimes w \otimes w^* \otimes \bar{\epsilon}_j = \epsilon_j \otimes \bar{\epsilon}_j$ contains the trivial representation only once. Moreover we claim that if $k \neq l$, then the classes $[\epsilon_k \otimes w]$, $[\epsilon_l \otimes w]$ are distinct, that is, $w^* \otimes \bar{\epsilon}_l \otimes \epsilon_k \otimes w$ has no subobject in $I(\mathbb{\Gamma}_0)$. Indeed by definition $\bar{\epsilon}_l \otimes \epsilon_k$ has no subobject in $\mathbb{\Lambda}_{-1}$; hence, the irreducible subobjects of $w^* \otimes \bar{\epsilon}_l \otimes \epsilon_k \otimes w$ belong to $I(\mathbb{\Gamma})_2$.

Then we can apply the definition of $\tilde{L}_w$, $\tilde{R}_w$ and Proposition 3.42. Note that we are in the case when $p_{[\![w]\!]}c(\mathbb{\Gamma}_0\backslash\mathbb{\Gamma}/\mathbb{\Gamma}_0) = \mathbb{C}p_{[\![w]\!]}$ since $\dim(w) = 1$, see Remark 3.41. For $\alpha = \epsilon \otimes w$, $\epsilon \in I(\mathbb{\Gamma}_0)$ as above, we have $\kappa_\alpha = \dim_q(w^* \otimes \bar{\epsilon} \otimes \epsilon \otimes w)_{\mathbb{\Gamma}_0} = \dim_q(\bar{\epsilon} \otimes \epsilon)_{\mathbb{\Lambda}_{-1}}$ — whereas $\kappa_{\bar{\alpha}} = \dim_q(\epsilon \otimes \bar{\epsilon}) = \dim_q(\bar{\epsilon} \otimes \epsilon)$. This gives the formula for $\tilde{R}_w$, and the one for $\tilde{L}_w$ follows by symmetry. ☐

Example 3.47. One can construct quantum examples as follows. Take two finite quantum groups $\Sigma_{\pm 1}$, for instance, duals of classical finite groups. Form the restricted product $\mathbb{\Gamma}_0 = \prod'_{k\in\mathbb{Z}^*} \Sigma_{\operatorname{sgn}(k)}$, which is the dual of a profinite group if $\Sigma_{\pm 1}$ is the dual of a finite classical group. If one group Σ_ϵ is not classical, $\mathbb{\Gamma}_0$ is a unimodular non-classical discrete quantum group. Consider the finite index subgroups $\mathbb{\Lambda}_\epsilon = \prod'_{k\in\mathbb{Z}^*, k\neq\epsilon} \Sigma_{\operatorname{sgn}(k)}$. We have evident isomorphisms $\mathbb{\Lambda}_\epsilon \simeq \mathbb{\Gamma}$ obtained by shifting the copies of Σ_ϵ toward $k = 0$ in the restricted product. We denote by $\theta : \mathbb{C}[\mathbb{\Lambda}_1] \to \mathbb{C}[\mathbb{\Lambda}_{-1}]$ the corresponding isomorphism.

Denoting $I_\epsilon = \operatorname{Corep}(\Sigma_\epsilon)$, we have natural identifications $I(\mathbb{\Gamma}_0) \simeq \prod'_k I_{\operatorname{sgn}(k)}$, $\mathbb{\Lambda}_\epsilon\backslash I(\mathbb{\Gamma}_0) = I(\mathbb{\Gamma}_0)/\mathbb{\Lambda}_\epsilon \simeq I_\epsilon$. For $\gamma \in I_{-1}$ we have $(\bar{\gamma} \otimes \gamma)_{\mathbb{\Lambda}_{-1}} = \{1\}$ hence $\tilde{R}_w = \sum_{\gamma\in I_{-1}} \dim(\gamma)^2 = \#\Sigma_{-1}$ and similarly $\tilde{L}_w = \#\Sigma_1$, where we denote $\#\Sigma = \dim(c(\Sigma))$. As a result the modular function ∇ of the Hecke pair $(\mathbb{\Gamma}, \mathbb{\Gamma}_0)$ is non-trivial as soon as Σ_1, Σ_{-1} have different dimensions/cardinals. If one of $\Sigma_{\pm 1}$ is non-classical (for example, the dual of a non-abelian finite group), the HNN extension $\mathbb{\Gamma}$ is neither classical, nor co-classical (but it is unimodular).

An element $\alpha = (\alpha_k)_{k\in\mathbb{Z}^*}$ of $I(\mathbb{\Gamma}_0)$ is in $\mathrm{Dom}\,\theta^{n\epsilon}$, $n \in \mathbb{N}^*$, if and only if we have $\alpha_\epsilon = \cdots = \alpha_{n\epsilon} = 1$. Hence $\bigcap_{k\in\mathbb{Z}} \mathrm{Dom}\,\theta^k = \{1\}$ and the action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Gamma}_0$ is faithful. Observe also that $\mathbb{\Gamma}$ is finitely generated although $\mathbb{\Gamma}_0$ is not: indeed, denoting $\Sigma^{(k)}$ the copy of $\Sigma_{\mathrm{sgn}(k)}$ in $\mathbb{\Gamma}_0$ we have $\theta^{-\epsilon}(\Sigma^{(n\epsilon)}) = \Sigma^{((n+1)\epsilon)}$, so that $\mathbb{\Gamma}$ is generated by $\Sigma^{(1)}$, $\Sigma^{(-1)}$ and w.

Example 3.48. One can also construct quantum examples by taking for $\mathbb{\Gamma}_0$ the dual of a compact group G, and using quantum subgroups $\mathbb{\Lambda}_\epsilon$ associated with quotients $H_\epsilon = G/K_\epsilon$. The index of $\mathbb{\Lambda}_\epsilon$ in $\mathbb{\Gamma}$ is finite if and only if K_ϵ is finite. If G is connected, the subgroups K_ϵ must then be central, and we have $\#I(\mathbb{\Gamma}_0)/\mathbb{\Lambda}_\epsilon = \#K_\epsilon$.

Assume that G is a connected compact Lie group. Then the fundamental group of H_ϵ remembers the cardinality of the kernel K_ϵ, and since we assume H_1 and H_{-1} to be isomorphic, we will always have $L([\![w]\!]) = \#K_1 = \#K_{-1} = R([\![w]\!])$ in this case. Similarly, subobjects of $\bar{\gamma} \otimes \gamma$ factor through the center $Z(G)$ for any $\gamma \in I(\mathbb{\Gamma}_0) \subset \mathrm{Rep}(G)$, hence always belong to $I(\mathbb{\Lambda}_\epsilon)$ so that we have $\dim_q(\bar{\gamma} \otimes \gamma)_{\mathbb{\Lambda}_\epsilon} = \dim_q(\bar{\gamma} \otimes \gamma)$ and $\tilde{L}_w = \tilde{R}_w = (\#K_1)p_w$. Moreover, in most simple Lie groups the center is cyclic so that $\#K_\epsilon$ determines K_ϵ and $\mathbb{\Lambda}_1 \simeq \mathbb{\Lambda}_{-1}$ implies in fact $\mathbb{\Lambda}_1 = \mathbb{\Lambda}_{-1}$. This does not mean that the Hecke algebra will be completely trivial. One can also take for θ a non-inner automorphism of H to make the construction more interesting, so that the resulting $\mathbb{\Gamma}$ looks like a variant of the partial crossed-product construction.

A typical case is given by $G = SU(n) \twoheadrightarrow H_1 = H_{-1} = PSU(n)$, to be compared with the 'classical case' of the Baumslag–Solitar group $BS(n,n) = HNN(\mathbb{Z}, \mathrm{id} : n\mathbb{Z} \to n\mathbb{Z})$. On the other hand $Z(\mathrm{Spin}(4k)) = (\mathbb{Z}/2\mathbb{Z})^2$, so that the dual of $SO(4k)$ can be realized in two different ways as a subgroup of the dual of $\mathrm{Spin}(4k)$. Of course one can also look at $SO(3) \times SU(2)$ which is a quotient of $SU(2) \times SU(2)$ in two different ways. In all these cases we have $\nabla = 1$ because $\tilde{L}_w = \tilde{R}_w$.

Note that since K_ϵ is contained in any maximal torus of G, the above construction is compatible with q-deformations — K_ϵ remains a quantum subgroup of the compact quantum group $\mathbb{G}_q$ corresponding to G. However, we still get $\nabla = 1$, by essentially the same reasoning since the fusion ring of $\mathbb{G}_q$ is the same as the one of G.

4 | COMPACT OPEN QUANTUM HECKE PAIRS

In this section we introduce Hecke algebras in the setting of locally compact (algebraic) quantum groups with compact open quantum subgroups. We then describe a generalized Schlichting completion, which allows us to subsume the Hecke algebras from Section 3 in this setting, and deduce some analytic consequences. Finally, we describe how to pass from arbitrary Hecke pairs to their reduced versions, and exhibit some new examples of algebraic quantum groups.

4.1 | Compact open Hecke algebras

Let us fix an algebraic quantum group $\mathbb{G}$ together with an algebraic quantum subgroup $\mathbb{H} \subset \mathbb{G}$. Recall that this is determined by a non-zero central projection $p_{\mathbb{H}} \in \mathcal{O}_c(\mathbb{G})$ such that $\Delta(p_{\mathbb{H}})(1 \otimes$

$p_{\mathbb{H}}) = p_{\mathbb{H}} \otimes p_{\mathbb{H}}$. Throughout this section we normalize the Haar functionals of $\mathcal{O}_c(\mathbb{G})$ in such a way that $\varphi(p_{\mathbb{H}}) = \psi(p_{\mathbb{H}}) = 1$.

The definition of compactly supported functions on the quantum homogeneous space $\mathbb{G}/\mathbb{H}$ is easier than in the discrete case, since the relevant invariant functions on $\mathbb{G}$ are also compactly supported. Namely, we define $c_c(\mathbb{G}/\mathbb{H}) = \mathcal{O}_c(\mathbb{G})^{\mathbb{H}} = \{f \in \mathcal{O}_c(\mathbb{G}) \mid \Delta(f)(1 \otimes p_{\mathbb{H}}) = f \otimes p_{\mathbb{H}}\}$. It is shown in [18] that this algebra is a direct sum of matrix algebras, that is, it corresponds to a 'discrete' quantum space. We denote by $c_0(\mathbb{G}/\mathbb{H})$ the closure of $c_c(\mathbb{G}/\mathbb{H})$ in $C_0(\mathbb{G})$ and $\ell^2(\mathbb{G}/\mathbb{H})$ its closure in $L^2(\mathbb{G})$. It can be shown that $c_0(\mathbb{G}/\mathbb{H})$ is a quantum homogeneous space in the sense of [26], cf. [10, Proposition 6.2, Theorem 6.4]. One can define $c_c(\mathbb{H}\backslash\mathbb{G})$, $c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ exactly in the same way, as well as the corresponding c_0 and ℓ^2 spaces.

To define the Hecke algebra $\mathcal{H}(\mathbb{G}, \mathbb{H})$ it suffices to restrict the natural convolution product of $\mathcal{O}_c(\mathbb{G})$ to the space of $\mathbb{H}$-bi-invariant functions. This product is transported from the dual-multiplier Hopf algebra $\mathcal{D}(\mathbb{G})$ via the Fourier transform $\mathcal{F} : \mathcal{O}_c(\mathbb{G}) \to \mathcal{D}(\mathbb{G})$ determined by $(\mathcal{F}(f), h) = \varphi(hf)$. Explicitly we have, for $f, g \in \mathcal{O}_c(\mathbb{G})$:

$$f * g = (f\varphi \otimes \mathrm{id})(S^{-1} \otimes \mathrm{id})\Delta(g) = (\mathrm{id} \otimes \varphi S^{-1}(g))\Delta(f).$$

Together with the $*$-structure $f^\sharp := \mathcal{F}^{-1}(\mathcal{F}(f)^*)$, not to be confused with the given $*$-structure on $\mathcal{O}_c(\mathbb{G})$, this turns $\mathcal{O}_c(\mathbb{G})$ into a $*$-algebra. Explicitly we have $f^\sharp = S(f)^*\delta$, where $\delta \in \mathcal{M}(\mathcal{O}_c(\mathbb{G}))$ is the modular element of $\mathbb{G}$.

The left regular representation of the dual algebra $\lambda : \mathcal{D}(\mathbb{G}) \to B(L^2(\mathbb{G}))$ is then given by $\lambda(\mathcal{F}(f))(\Lambda(g)) = \Lambda(f * g)$, see [31, Section 4.2.2]. We also have the right regular representation $\rho : \mathcal{D}(\mathbb{G}) \to B(L^2(\mathbb{G}))$ given by $\rho(\mathcal{F}(f)) = \hat{J}\lambda(\mathcal{F}(f))^*\hat{J}$. Here $\hat{J}$ is the modular conjugation operator for $\hat{\varphi}$, the dual left Haar weight also given by the formula $\hat{\varphi}(\mathcal{F}(f)) = \epsilon(f)$.

Explicitly we have $\rho(\mathcal{F}(f))(\Lambda(g)) = \Lambda(g * \hat{\sigma}_{-i/2}(f))$ for all $f, g \in \mathcal{O}_c(\mathbb{G})$. Here, by slight abuse of notation, we write $\hat{\sigma}_{-i/2}(f)$ instead of $\mathcal{F}^{-1}(\hat{\sigma}_{-i/2}(\mathcal{F}(f))$, where $(\hat{\sigma}_t)_{t\in\mathbb{R}}$ is the modular group of $\hat{\varphi}$. Note that the map $f \mapsto \hat{\sigma}_{-i/2}(f)$ is an algebra isomorphism from $(\mathcal{O}_c(\mathbb{G}), *)$ to $\mathcal{D}(\mathbb{G})$. We have $\delta p_{\mathbb{H}} = p_{\mathbb{H}}$ because $\mathbb{H}$ is compact and $\hat{\sigma}_t(p_{\mathbb{H}}) = p_{\mathbb{H}}$ for all $t \in \mathbb{R}$ because the restriction of $\hat{\varphi} : \mathcal{D}(\mathbb{G}) \to \mathbb{C}$ to $\mathcal{D}(\mathbb{H})$ is the left Haar weight of $\hat{\mathbb{H}}$.

Lemma 4.1. *We have* $c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H}) = p_{\mathbb{H}} * \mathcal{O}_c(\mathbb{G}) * p_{\mathbb{H}}$. *In particular* $c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ *is closed under the convolution product* $*$ *and the involution* $\sharp$.

Proof. Let $f \in \mathcal{O}_c(\mathbb{G})$. Using the definition of the convolution product we calculate $p_{\mathbb{H}} * f = (p_{\mathbb{H}}\varphi \otimes \mathrm{id})(S^{-1} \otimes \mathrm{id})\Delta(f) = (h \otimes \mathrm{id})(\pi_{\mathbb{H}}S^{-1} \otimes \mathrm{id})\Delta(f) = (h\pi_{\mathbb{H}} \otimes \mathrm{id})\Delta(f)$, where h is the Haar functional of $\mathcal{O}(\mathbb{H})$ and $\pi_{\mathbb{H}} : \mathcal{O}_c(\mathbb{G}) \to \mathcal{O}(\mathbb{H})$ the restriction map. It follows that $p_{\mathbb{H}} * f = f$ if and only if $f \in c_c(\mathbb{H}\backslash\mathbb{G})$. Similarly one checks $f \in c_c(\mathbb{G}/\mathbb{H})$ if and only if $f * p_{\mathbb{H}} = f$. Since $p_{\mathbb{H}}$ is a projection in the convolution algebra, this yields the claim. □

This allows us to give the following definition.

Definition 4.2. The Hecke algebra of $(\mathbb{G}, \mathbb{H})$ is the $*$-algebra $\mathcal{H}(\mathbb{G}, \mathbb{H}) = c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ with the convolution product and $*$-structure $\sharp$ as above.

We obtain a non-degenerate $*$-representation of $\mathcal{H}(\mathbb{G}, \mathbb{H})$ on $\ell^2(\mathbb{G}/\mathbb{H})$ by considering the restriction of the right regular representation. Here we use that for $f \in c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ and $g \in c_c(\mathbb{G}/\mathbb{H})$ the element $g * f$ is again contained in $c_c(\mathbb{G}/\mathbb{H})$, so that $\rho(\mathcal{F}(f))$ indeed maps $\ell^2(\mathbb{G}/\mathbb{H})$ to itself. We also observe that $\rho \circ \mathcal{F} : \mathcal{H}(\mathbb{G}, \mathbb{H}) \to B(\ell^2(\mathbb{G}/\mathbb{H}))$ is antimultiplicative for the convolution product, that is, $\rho(\mathcal{F}(f * g)) = \rho(\mathcal{F}(g))\rho(\mathcal{F}(f))$ for all $f, g \in \mathcal{H}(\mathbb{G}, \mathbb{H})$.

In the same way as in the discrete case we view $c_c(\mathbb{G}/\mathbb{H})$ as a $\mathcal{D}(\mathbb{G})$-module, see the discussion before Proposition 3.27, and obtain the space $\mathrm{End}_{\mathbb{G}}(c_c(\mathbb{G}/\mathbb{H}))$ of $\mathcal{D}(\mathbb{G})$-module maps.

Proposition 4.3. *We have (mutually inverse) antimultiplicative algebra isomorphisms*

$$T : \mathcal{H}(\mathbb{G}, \mathbb{H}) \to \mathrm{End}_{\mathbb{G}}(c_c(\mathbb{G}/\mathbb{H})), \quad T(f)(h) = h * f \qquad \textit{and}$$
$$T^{-1} : \mathrm{End}_{\mathbb{G}}(c_c(\mathbb{G}/\mathbb{H})) \to \mathcal{H}(\mathbb{G}, \mathbb{H}) \cong c_c(\mathbb{G}/\mathbb{H})^{\mathbb{H}}, \quad T^{-1}(F) = F(p_{\mathbb{H}}).$$

Proof. Since $T(f)$ for $f \in \mathcal{H}(\mathbb{G}, \mathbb{H})$ commutes with the left convolution action of $\mathcal{D}(\mathbb{G})$, it is clear that T is well defined, and it is obvious from the definition that T is antimultiplicative.

Let us verify that T is an isomorphism by verifying that the above formula for T^{-1} yields indeed its inverse. Well-definedness is again easy to check, and using left $\mathbb{H}$-invariance of $f \in \mathcal{H}(\mathbb{G}, \mathbb{H}) = c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ we get

$$(T^{-1}T)(f) = T(f)(p_{\mathbb{H}}) = p_{\mathbb{H}} * f = f.$$

Conversely, given $F \in \mathrm{Hom}_{\mathbb{G}}(c_c(\mathbb{G}/\mathbb{H}), c_c(\mathbb{G}/\mathbb{H}))$ we compute

$$\begin{aligned}(TT^{-1})(F)(h) &= (TT^{-1})(F)(h * p_{\mathbb{H}}) = h * (TT^{-1})(F)(p_{\mathbb{H}}) \\ &= h * p_{\mathbb{H}} * T^{-1}(F) = h * F(p_{\mathbb{H}}) = F(h * p_{\mathbb{H}}) = F(h)\end{aligned}$$

for all $h \in c_c(\mathbb{G}/\mathbb{H})$, using that $c_c(\mathbb{G}/\mathbb{H}) \subset \mathcal{O}_c(\mathbb{G})$ because $\mathbb{H} \subset \mathbb{G}$ is compact open. □

4.2 | Example: Quantum doubles

Let us consider the situation where $\mathbb{H}$ is a compact quantum group and $\mathbb{G} = \mathbb{H} \bowtie \hat{\mathbb{H}}$ the quantum double of $\mathbb{H}$. Recall that $\mathbb{G}$ is the locally compact quantum group given by the von Neumann algebra $L^\infty(\mathbb{G}) = L^\infty(\mathbb{H})\bar{\otimes}\mathcal{L}(\mathbb{H})$, equipped with the coproduct

$$\Delta_{\mathbb{G}} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{ad}(W) \otimes \mathrm{id})(\Delta \otimes \hat{\Delta}),$$

where $\mathrm{ad}(W)$ is conjugation with the multiplicative unitary $W \in L^\infty(\mathbb{H})\bar{\otimes}\mathcal{L}(\mathbb{H})$. This is a special case of the generalized quantum doubles studied in [2].

In fact, the quantum double of a compact quantum group is naturally an algebraic quantum group in the sense of Van Daele, which allows us to give algebraic descriptions of almost all the data involved [7, 31]. More precisely, if we write $\mathcal{O}(\mathbb{H})$ for the polynomial function algebra of $\mathbb{H}$ as before and $\mathcal{D}(\mathbb{H})$ for the algebraic convolution algebra, then $W \in \mathcal{M}(\mathcal{O}(\mathbb{H}) \odot \mathcal{D}(\mathbb{H}))$, and $\mathcal{O}_c(\mathbb{G}) = \mathcal{O}(\mathbb{H}) \odot \mathcal{D}(\mathbb{H})$, equipped with the comultiplication given by the formula above, defines an algebraic quantum group. It contains $\mathbb{H}$ naturally as an algebraic compact open quantum subgroup, and the corresponding group-like projection is $p_{\mathbb{H}} = 1 \otimes p_0$, where $p_0 \in \mathcal{D}(\mathbb{H})$ is the central support of the counit. One checks that

$$c_c(\mathbb{H}\backslash\mathbb{G}) = 1 \otimes \mathcal{D}(\mathbb{H}) \qquad c_c(\mathbb{G}/\mathbb{H}) = W^*(1 \otimes \mathcal{D}(\mathbb{H}))W$$

inside $\mathcal{O}_c(\mathbb{G})$, and the space of $\mathbb{H}$-bi-invariant functions on $\mathbb{G}$ is

$$c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H}) \cong \{x \in \mathcal{D}(\mathbb{H}) \mid W^*(1 \otimes x)W = 1 \otimes x\} = Z(\mathcal{D}(\mathbb{H})),$$

that is, $c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ identifies with the center of the algebraic convolution algebra of $\mathbb{H}$ with respect to its ordinary product.

By definition, the Hecke algebra $\mathcal{H}(\mathbb{G}, \mathbb{H})$ is $c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ equipped with the restriction of the convolution product on $\mathcal{O}_c(\mathbb{G})$. In the present situation, it is more convenient to describe the $*$-subalgebra $\mathcal{F}(c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})) \subset \mathcal{D}(\mathbb{G})$ obtained via the Fourier transform $\mathcal{F} : \mathcal{O}_c(\mathbb{G}) \to \mathcal{D}(\mathbb{G}), \mathcal{F}(f)(h) = \varphi(hf)$. As discussed in [31, Chapter 4], we can identify $\mathcal{D}(\mathbb{G}) = \mathcal{D}(\mathbb{H}) \bowtie \mathcal{O}(\mathbb{H})$, which is the algebraic tensor product $\mathcal{D}(\mathbb{H}) \otimes \mathcal{O}(\mathbb{H})$ equipped with the twisted multiplication

$$(x \bowtie f)(y \bowtie g) := xy_{(2)}(y_{(1)}, f_{(1)}) \bowtie (\hat{S}(y_{(3)}), f_{(3)})f_{(2)}g,$$

for $x, y \in \mathcal{D}(\mathbb{H}), f, g \in \mathcal{O}(\mathbb{H})$. The $*$-structure on $\mathcal{D}(\mathbb{G})$ is defined in such a way that both $\mathcal{D}(\mathbb{H}) \bowtie 1$ and $1 \bowtie \mathcal{O}(\mathbb{H})$ are $*$-subalgebras of $\mathcal{M}(\mathcal{D}(\mathbb{G}))$, and the natural skew-pairing between $\mathcal{D}(\mathbb{G})$ and $\mathcal{O}_c(\mathbb{G})$ is

$$(y \bowtie g, f \otimes x) = (y, f)(g, x), \quad x, y \in \mathcal{D}(\mathbb{H}), f, g \in \mathcal{O}(\mathbb{H}),$$

again following the conventions in [31]. The left and right invariant Haar functional on $\mathcal{O}_c(\mathbb{G})$ is given by $\varphi = \hat{h} \otimes h_R$, where $\hat{h}$ is the Haar state of $\mathcal{O}(\mathbb{H})$ and h_R the right Haar functional on $\mathcal{D}(\mathbb{H})$, compare [31, Proposition 4.19]. As is well known, using the Fourier transform $\mathcal{F}$ we can identify $Z(\mathcal{D}(\mathbb{H})) \subset (\mathcal{O}_c(\mathbb{G}), *)$ with the $*$-subalgebra $(p_0 \bowtie 1)(1 \bowtie \mathcal{O}(\mathbb{H}))\,(p_0 \bowtie 1) = p_0 \bowtie \mathcal{O}(\mathbb{H})^{\mathrm{ad}} \subset \mathcal{D}(\mathbb{G})$, where

$$\begin{aligned}\mathcal{O}(\mathbb{H})^{\mathrm{ad}} &= \{f \in \mathcal{O}(\mathbb{H}) \mid f_{(2)} \otimes \hat{h}(f_{(1)}S^{-1}(f_{(3)})) = f \otimes 1\} \\ &= \{f \in \mathcal{O}(\mathbb{H}) \mid \Delta^{\mathrm{cop}}(f) = \Delta(f)\},\end{aligned}$$

with the product and $*$-structure induced from $\mathcal{O}(\mathbb{H})$. This is precisely the algebra of characters inside $\mathcal{O}(\mathbb{H})$.

We note that, with suitable adjustments, similar computations go through for generalized quantum doubles built out of compact and discrete quantum groups.

4.3 | The Schlichting completion

Let $\mathbb{\Gamma}$ be a discrete quantum group and $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ be a quantum subgroup. If $\mathbb{\Lambda} \subset \mathbb{\Gamma}$ is almost normal (see Definition 3.3), we shall construct a pair $(\mathbb{G}, \mathbb{H})$ consisting of an algebraic quantum group $\mathbb{G}$ and a compact open quantum subgroup $\mathbb{H} \subset \mathbb{G}$, playing the role of the Schlichting completion of the Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$ [23].

More precisely, our strategy is as follows. We first define the algebra $\mathcal{O}_c(\mathbb{G})$ as a subalgebra of $\ell^\infty(\mathbb{\Gamma})$ using the 'discrete' Hecke convolution product, see Definition 4.5. The key point of the construction consists then in proving that this algebra is a multiplier Hopf $*$-algebra, and more specifically, that the coproduct takes its values in the appropriate subspace of $\mathcal{M}(\mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G}))$, see Proposition 4.6.

It is then easy to see that the projection $p_{\mathbb{\Lambda}}$ corresponds to a CQG algebra $\mathcal{O}_c(\mathbb{H}) \subset \mathcal{O}_c(\mathbb{G})$, and that $c_c(\mathbb{G}/\mathbb{H}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ as subspaces of $\ell^\infty(\mathbb{\Gamma})$, see Propositions 4.10 and 4.11. Using the Haar functional of $\mathcal{O}_c(\mathbb{H})$ and the $\mathbb{\Gamma}$-invariant functional μ on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ one can then construct the integrals of $\mathcal{O}_c(\mathbb{G})$, so that $\mathbb{G}$ is in fact a locally compact quantum group by [17]. We end the section by making the connection between the 'discrete' and 'compact open' Hecke algebras $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ and $\mathcal{H}(\mathbb{G}, \mathbb{H})$.

Lemma 4.4. *We say that $\gamma \in I(\mathbb{\Gamma})$ is in the support of $x \in \ell^\infty(\mathbb{\Gamma})$ if $p_\gamma x \neq 0$. Then given $\gamma \in I(\mathbb{\Gamma})$, $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ such that $\gamma \in \mathrm{Supp}(a * b)$, there exist $\alpha \in \mathrm{Supp}(a)$, $\beta \in \mathrm{Supp}(b)$ and $\lambda \in I(\mathbb{\Lambda})$ such that $\gamma \subset \alpha \otimes \lambda \otimes \beta$.*

Proof. If $\gamma \in \mathrm{Supp}\, a * b$, at least one term of the sum (3.6) has γ in its support. Hence there exist $\alpha \in \mathrm{Supp}(a)$, $\beta \in \mathrm{Supp}(b)$ such that the corepresentation $\bar{\alpha} \otimes \gamma \otimes \bar{\beta}$ contains an element $\lambda \in I(\mathbb{\Lambda})$. By Frobenius reciprocity a non-zero morphism $\lambda \to \bar{\alpha} \otimes \gamma \otimes \bar{\beta}$ induces a non-zero morphism $\alpha \otimes \lambda \otimes \beta \to \gamma$ and we are done. □

Similarly one can define the support $\mathrm{Supp}(\varphi)$ of a linear functional $\varphi \in c_c(\mathbb{\Gamma})^*$ as the set of elements $\gamma \in I(\mathbb{\Gamma})$ such that $p_\gamma \varphi \neq 0$. If φ has finite support, it extends uniquely to a normal functional $\varphi \in \ell^\infty(\mathbb{\Gamma})_*$.

Definition 4.5. Given a Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$ we denote by $\mathcal{O}_c(\mathbb{G})$ (respectively, $C_0(\mathbb{G})$) the subalgebra (respectively, the C^*-subalgebra) of $\ell^\infty(\mathbb{\Gamma})$ generated by the elements $a * b$ with $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$.

In the following lemmas we will always assume that $\mathcal{O}_c(\mathbb{G})$ and $C_0(\mathbb{G})$ arise in the above way from a Hecke pair. Note that by definition of the convolution product, for example, (3.5), $a * b$ is a finite sum of elements of $\ell^\infty(\mathbb{\Gamma})$ so that it is indeed in $\ell^\infty(\mathbb{\Gamma})$. It is easy to check, using both expressions in (3.5), that $a^* * b^* = (a * b)^*$, so that $\mathcal{O}_c(\mathbb{G})$ is in fact a $*$-subalgebra of $\ell^\infty(\mathbb{\Gamma})$ and $C_0(\mathbb{G})$ is its norm closure.

In view of the definition of $a * b$ one can also say that the algebra $\mathcal{O}_c(\mathbb{G})$ is generated by elements of the form $(\mathrm{id} \otimes \varphi)\Delta(a)$ where $a \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $\varphi \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^*$ have both finite support over $I(\mathbb{\Gamma})/\mathbb{\Lambda}$, that is, $p_\tau a = 0$, $p_\tau \varphi = 0$ for all but a finite number of right classes τ. We denote $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^\vee$ the space of these finitely supported functionals. It follows from this description that the weak closure of $\mathcal{O}_c(\mathbb{G})$ in $\ell^\infty(\mathbb{\Gamma})$ is the so-called cokernel of the $\mathbb{\Gamma}$-action on $\mathbb{\Gamma}/\mathbb{\Lambda}$, which is known to be a Baaj–Vaes subalgebra [11, Definition 2.8, Proposition 2.9]. Below we prove a more precise, C^*-algebraic version of this property, specific to the case of Hecke pairs.

For $x \in C_0(\mathbb{G})$ we can consider $\Delta(x)$ which is a priori an element of $\ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$.

Proposition 4.6. *We have $\Delta(\mathcal{O}_c(\mathbb{G}))(1 \otimes \mathcal{O}_c(\mathbb{G}))$, $\Delta(\mathcal{O}_c(\mathbb{G}))(\mathcal{O}_c(\mathbb{G}) \otimes 1) \subset \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G})$ and $\Delta(C_0(\mathbb{G}))(1 \otimes C_0(\mathbb{G}))$, $\Delta(C_0(\mathbb{G}))(C_0(\mathbb{G}) \otimes 1) \subset C_0(\mathbb{G}) \otimes C_0(\mathbb{G})$.*

Proof. The assertions about $C_0(\mathbb{G})$ follow by density from the ones for $\mathcal{O}_c(\mathbb{G})$, since Δ is continuous. Let us prove that $\Delta(\mathcal{O}_c(\mathbb{G}))(\mathcal{O}_c(\mathbb{G}) \otimes 1) \subset \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G})$. By multiplicativity it suffices to consider elements of the form $\Delta(x)(y \otimes 1)$ with $x = a * b$, $y = c * d$, $a, c \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b, d \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, and by linearity we can assume $a = p_{[\alpha]}a$, $b = p_{[\beta]}b$, $c = p_{[\gamma]}c$, $d = p_{[\delta]}d$ using left or right classes as appropriate.

As a first step, consider a linear functional $\varphi \in c_c(\mathbb{\Gamma})^*$ with finite support and compute, using (3.5) and coassociativity:

$$(\mathrm{id} \otimes \varphi)\Delta(x) = \kappa_\alpha^{-1}(h_R S(a_\alpha) \otimes \mathrm{id} \otimes \varphi)\Delta^2(b) = a * ((\mathrm{id} \otimes \varphi)\Delta(b)).$$

Since $\Delta(\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})) \subset \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$, we have $(\mathrm{id} \otimes \varphi)\Delta(b) \subset \ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$. Take moreover $\tau \in \mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ such that $(p_\tau \otimes \varphi)\Delta(b) \neq 0$. Then there exist $\nu \in \tau$, $\mu \in \mathrm{Supp}(\varphi)$ and $\lambda \in I(\mathbb{\Lambda})$ such that $\lambda \otimes \beta$ and $\nu \otimes \mu$ have a common irreducible subobject. By Frobenius reciprocity this implies $\nu \subset \lambda \otimes \beta \otimes \bar{\mu}$. Since $\mathrm{Supp}(\varphi)$ is finite, this shows that τ belongs to a finite subset of $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})$ (depending on $\mathrm{Supp}(\varphi)$ and β). Hence we have $(\mathrm{id} \otimes \varphi)\Delta(b) \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ and $(\mathrm{id} \otimes \varphi)\Delta(x) \in \mathcal{O}_c(\mathbb{G})$.

Second step. Observe that we have by (3.7) and coassociativity:

$$\Delta(a * b)(y \otimes 1) = \kappa_{\bar{\beta}}^{-1}(\mathrm{id} \otimes \mathrm{id} \otimes h_L S^{-1}(b_\beta))(\mathrm{id} \otimes \Delta)[\Delta(a)(y \otimes 1)].$$

Recall that we have $\Delta(a) \in \ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. Moreover, take $\tau \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$ such that $\Delta(a)(y \otimes p_\tau) \neq 0$. Then there exist $\mu \in \mathrm{Supp}(y)$, $\nu \in \tau$, $\lambda \in I(\mathbb{\Lambda})$ such that $\alpha \otimes \lambda$ and $\mu \otimes \nu$ have a common irreducible subobject. By Frobenius reciprocity this implies $\nu \subset \bar{\mu} \otimes \alpha \otimes \lambda$. According to Lemma 4.4 there exists $\lambda' \in I(\mathbb{\Lambda})$ such that $\mu \subset \gamma \otimes \lambda' \otimes \delta$. Since double classes in $\mathbb{\Lambda}\backslash I(\mathbb{\Gamma})/\mathbb{\Lambda}$ are finite unions of right classes, it follows that τ belongs to a finite subset $P \subset I(\mathbb{\Gamma})/\mathbb{\Lambda}$.

Note that $\ell^\infty(\mathbb{P}) := \sum_{\tau\in P} p_\tau \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ is a finite-dimensional subspace of $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. It follows that there is a finite family of vectors $t_i \in \ell^\infty(\mathbb{P})$ and elements $s_i \in \ell^\infty(\mathbb{\Gamma})$, such that $\Delta(a)(y \otimes 1) = \sum s_i \otimes t_i$. We have then, according to the above equation: $\Delta(a * b)(y \otimes 1) = \sum s_i \otimes (t_i * b) \in \ell^\infty(\mathbb{\Gamma}) \odot (\ell^\infty(\mathbb{P}) * b)$.

Choose now a finite, linearly independent family of vectors $x_k = a_k * b$ in the finite-dimensional subspace $\ell^\infty(\mathbb{P}) * b \subset \mathcal{O}_c(\mathbb{G})$, and elements $z_k \in \ell^\infty(\mathbb{\Gamma})$ such that $\Delta(x)(y \otimes 1) = \sum z_k \otimes x_k$. Choose a corresponding family of linear forms with finite support $\varphi_k \in c_c(\mathbb{\Gamma})^*$ such that $\varphi_l(x_k) = \delta_{k,l}$ for all k, l; we have then $z_k = ((\mathrm{id} \otimes \varphi_k)\Delta(x))y$. Applying the first step to φ_k we get $(\mathrm{id} \otimes \varphi_k)\Delta(x) \in \mathcal{O}_c(\mathbb{G})$, hence $z_k \in \mathcal{O}_c(\mathbb{G})$. □

Recall that the antipode of $\mathbb{\Gamma}$ is well defined as a map $S : c(\mathbb{\Gamma}) \to c(\mathbb{\Gamma})$ or $c_c(\mathbb{\Gamma}) \to c_c(\mathbb{\Gamma})$, but not in general from $\ell^\infty(\mathbb{\Gamma})$ to itself. It exchanges the subspaces $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ of $c(\mathbb{\Gamma})$, which are also subspaces of $\ell^\infty(\mathbb{\Gamma})$.

Proposition 4.7. *We have $S(\mathcal{O}_c(\mathbb{G})) \subset \mathcal{O}_c(\mathbb{G})$. Equipped with the restriction of Δ, $\mathcal{O}_c(\mathbb{G})$ (respectively, $C_0(\mathbb{G})$) is a multiplier Hopf-$*$-algebra (respectively, a bicancellative Hopf-C^*-algebra).*

Proof. For any $a \in c_c(\mathbb{\Gamma})/\mathbb{\Lambda}$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ we have $S(b^*) * S(a^*) = S((a * b)^*) = S(a^* * b^*)$, see the last part of the proof of Proposition 3.23 where bi-invariance of a, b is not used. Replacing a, b by their adjoints and using the fact that S exchanges $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$ we see that S stabilizes the canonical generating subspace of the algebra $\mathcal{O}_c(\mathbb{G})$. Since S is antimultiplicative, it stabilizes $\mathcal{O}_c(\mathbb{G})$.

We also obtain a character $\epsilon : \mathcal{O}_c(\mathbb{G}) \to \mathbb{C}$ by restricting the counit of $\ell^\infty(\mathbb{\Gamma})$. From the fact that $\mathbb{\Gamma}$ is a discrete quantum group, we know that $(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta$ on the level of $\ell^\infty(\mathbb{\Gamma})$, which implies that ϵ is a counit for $\mathcal{O}_c(\mathbb{G})$. Using that S is an algebra antiautomorphism of $\mathcal{O}_c(\mathbb{G})$ one then checks easily that $\mathcal{O}_c(\mathbb{G})$ is a multiplier Hopf $*$-algebra. Upon taking completions it follows that $C_0(\mathbb{G})$ is a bicancellative Hopf C^*-algebra. □

Note that the central projection $p_{\mathbb{\Lambda}} \in \ell^\infty(\mathbb{\Gamma})$ belongs to $\mathcal{O}_c(\mathbb{G})$ since $p_{\mathbb{\Lambda}} * p_{\mathbb{\Lambda}} = p_{\mathbb{\Lambda}}$. It moreover satisfies the property $\Delta(p_{\mathbb{\Lambda}})(1 \otimes p_{\mathbb{\Lambda}}) = p_{\mathbb{\Lambda}} \otimes p_{\mathbb{\Lambda}} = \Delta(p_{\mathbb{\Lambda}})(p_{\mathbb{\Lambda}} \otimes 1)$, in $\ell^\infty(\mathbb{\Gamma})$ hence also in $\mathcal{O}_c(\mathbb{G})$.

Definition 4.8. We denote $p_{\mathbb{H}} = p_{\mathbb{\Lambda}} \in \mathcal{O}_c(\mathbb{G})$ and $\mathcal{O}(\mathbb{H}) = p_{\mathbb{H}}\mathcal{O}_c(\mathbb{G})$, and let $C(\mathbb{H})$ be the norm closure of $\mathcal{O}(\mathbb{H})$ in $\ell^\infty(\mathbb{\Gamma})$.

We now describe the connection of the construction developed above to the classical Schlichting completion, as described, for example, in [12].

Proposition 4.9. *Let $(\mathbb{\Gamma}, \mathbb{\Lambda}) = (\Gamma, \Lambda)$ be a classical discrete Hecke pair, with Schlichting completion (G, H). The canonical map $\Gamma \to G$ with dense image induces an embedding $C_0(G) \subset \ell^\infty(\Gamma)$. Under this identification we have $C_0(G) = C_0(\mathbb{G})$ and $\mathcal{O}_c(G) = \mathcal{O}_c(\mathbb{G})$, and similarly $C(H) = C(\mathbb{H})$ and $\mathcal{O}(H) = \mathcal{O}(\mathbb{H})$.*

Proof. Observe that $a * b$ for $a \in c_c(\Gamma/\Lambda)$, $b \in c_c(\Lambda\backslash\Gamma)$ is right invariant under the action of the intersection of the point stabilizers in Γ of all points in the finite set $\mathrm{Supp}(b) \subset \Lambda\backslash\Gamma$. Since this group contains the intersection of finitely many-point stabilizers of the action of Γ on Γ/Λ, we see that $a * b \in \mathcal{O}_c(G)$, compare the description of the latter in [12]. Hence we get $\mathcal{O}_c(\mathbb{G}) \subset \mathcal{O}_c(G)$. Conversely, an element g of $\mathcal{O}(G)$ can be written as finite sum of characteristic functions on Γ/Γ_F for finite sets $F \subset \Gamma/\Lambda$. Modulo left translation by Γ we can assume that g is the characteristic function of the point Γ_F in Γ/Γ_F. We can write this as product of all $a_\gamma * f_\gamma$, where $a_\gamma = \delta_{\gamma\Lambda} \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $f_\gamma = \mathrm{ev}_{\gamma\Lambda} \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^*$ for $\gamma \in F$. This yields the equality $\mathcal{O}_c(G) = \mathcal{O}_c(\mathbb{G})$.

From the fact that both $C_0(G)$ and $C_0(\mathbb{G})$ are completions of $\mathcal{O}_c(G) = \mathcal{O}_c(\mathbb{G})$ inside $B(\ell^2(\Gamma))$ we get $C_0(G) = C_0(\mathbb{G})$. Finally, the claim about the canonical subgroups follows from $\mathcal{O}(H) = p_\Lambda\mathcal{O}_c(G) = p_\Lambda\mathcal{O}_c(\mathbb{G}) = \mathcal{O}(\mathbb{H})$. □

We now return to the general setup of quantum Hecke pairs.

Proposition 4.10. *Equipped with the restriction $\Delta_{\mathbb{H}}$ of $(p_{\mathbb{H}} \otimes p_{\mathbb{H}})\Delta$, $\mathcal{O}(\mathbb{H})$ (respectively, $C(\mathbb{H})$) is a CQG algebra (respectively, a Woronowicz C^*-algebra).*

Proof. Recall that the comultiplication on $\mathcal{O}_c(\mathbb{G})$ is implemented on the Hilbert space level by conjugation with the multiplicative unitary for $\mathbb{\Gamma}$, that is, for $f \in \ell^\infty(\mathbb{\Gamma})$ we have $\Delta(f) = W^*(1 \otimes f)W$ in $B(\ell^2(\mathbb{\Gamma}) \otimes \ell^2(\mathbb{\Gamma}))$. In particular, the comultiplication of $\mathcal{O}(\mathbb{H})$ extends continuously to a unital $*$-homomorphism $\Delta_{\mathbb{H}} : C(\mathbb{H}) \to C(\mathbb{H}) \otimes C(\mathbb{H})$. Since we already know that $\mathcal{O}(\mathbb{H})$ is a Hopf $*$-algebra, the cancellation conditions for $C(\mathbb{H})$ are satisfied. Hence $C(\mathbb{H})$ is a Woronowicz C^*-algebra.

This implies that there is a Haar functional on $\mathcal{O}(\mathbb{H})$, obtained by restricting from the Haar state of $C(\mathbb{H})$. We conclude that $\mathcal{O}(\mathbb{H})$ is a CQG algebra, compare [14]. □

So the corresponding compact quantum group $\mathbb{H}$ is an 'algebraic' compact open quantum subgroup of $\mathbb{G}$, with restriction map induced by the projection $p_{\mathbb{H}} = p_{\mathbb{\Lambda}}$. We denote by h its Haar functional. We can identify the corresponding homogeneous space $c_c(\mathbb{G}/\mathbb{H}) = \{a \in \mathcal{O}_c(\mathbb{G}) \mid (1 \otimes p_{\mathbb{H}})(\Delta(a)) = a \otimes p_{\mathbb{H}}\}$ as follows.

Proposition 4.11. *We have $c_c(\mathbb{G}/\mathbb{H}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ as subspaces of $\ell^\infty(\mathbb{\Gamma})$. The coproduct restricts to an algebraic action $c_c(\mathbb{G}/\mathbb{H}) \to \mathcal{M}(\mathcal{O}_c(\mathbb{G}) \odot c_c(\mathbb{G}/\mathbb{H}))$, in particular we have $\Delta(c_c(\mathbb{G}/\mathbb{H}))(\mathcal{O}_c(\mathbb{G}) \otimes 1) \subset \mathcal{O}_c(\mathbb{G}) \odot c_c(\mathbb{G}/\mathbb{H})$.*

If μ is a $\mathbb{\Gamma}$-invariant functional on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, that is, $(\mathrm{id} \otimes \mu)\,((p_\alpha \otimes 1)\Delta(x)) = \mu(x)p_\alpha$ for all $x \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\alpha \in I(\mathbb{\Gamma})$, then μ is at the same time a $\mathbb{G}$-invariant functional on $c_c(\mathbb{G}/\mathbb{H})$, that is, $(\mathrm{id} \otimes \mu)((y \otimes 1)\Delta(x)) = \mu(x)y$ for all $x \in c_c(\mathbb{G}/\mathbb{H})$, $y \in \mathcal{O}_c(\mathbb{G})$.

Proof. Take $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. Then we have $a = a * p_{\mathbb{\Lambda}}$ hence $a \in \mathcal{O}_c(\mathbb{G})$. By definition of $c(\mathbb{\Gamma}/\mathbb{\Lambda})$ we have $(1 \otimes p_{\mathbb{\Lambda}})\Delta(a) = a \otimes p_{\mathbb{\Lambda}}$ hence $a \in c_c(\mathbb{G}/\mathbb{H})$. For the converse inclusion, take $x \in c_c(\mathbb{G}/\mathbb{H})$. In particular we have $(1 \otimes p_{\mathbb{\Lambda}})\Delta(x) = x \otimes p_{\mathbb{\Lambda}}$, so $x \in c(\mathbb{\Gamma}/\mathbb{\Lambda})$. It remains to prove that x has finite support in this algebra, that is, $p_\tau x = 0$ for all but a finite number of classes $\tau \in I(\mathbb{\Gamma})/\mathbb{\Lambda}$. It is clearly sufficient to prove this for $x = a * b$ with $a \in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$, $b \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma})$, since taking products of such elements reduces the support. But this results from Lemma 4.4 and the Hecke condition: if $\gamma \in \mathrm{Supp}(a * b)$, then $\gamma \subset \alpha \otimes \mu$ with $\alpha \in \mathrm{Supp}(a)$ and $\mu \in [\![\beta]\!]$, $\beta \in \mathrm{Supp}(b)$; writing $[\![\beta]\!]$ as a finite union of right classes $[\gamma]$ and decomposing $\alpha \otimes \gamma$ into a finite number of irreducibles δ we see that $\mathrm{Supp}(a * b)$ included in the union of the finite number of right classes $[\delta]$.

For $x \in c_c(\mathbb{G}/\mathbb{H})$, $y \in \mathcal{O}_c(\mathbb{G})$ we have $\Delta(x)(y \otimes 1) \subset \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G})$ by Proposition 4.6, since $x \in \mathcal{O}_c(\mathbb{G})$. It remains to show that $z = (y\varphi \otimes \mathrm{id})\Delta(x) \in c_c(\mathbb{G}/\mathbb{H})$ for any $\varphi \in c_c(\mathbb{\Gamma})^*$ with finite support. But we can write

$$(1 \otimes p_{\mathbb{\Lambda}})\Delta(z) = (\varphi \otimes \mathrm{id} \otimes \mathrm{id})((y \otimes 1)\Delta \otimes \mathrm{id})[(1 \otimes p_{\mathbb{\Lambda}})\Delta(x)],$$

where all terms belong to the corresponding algebraic tensor products, and since $(1 \otimes p_{\mathbb{\Lambda}})\,\Delta(x) = x \otimes p_{\mathbb{\Lambda}}$ we recognize $(1 \otimes p_{\mathbb{\Lambda}})\Delta(z) = z \otimes p_{\mathbb{\Lambda}}$. The last assertion is trivial because we can check the equality $(\mathrm{id} \otimes \mu)((y \otimes 1)\Delta(x)) = \mu(x)y$ by multiplying by an arbitrary central projection p_α. □

Note that $(\mathrm{id} \otimes hp_{\mathbb{H}})\Delta(x)$ is well defined in $\mathcal{O}_c(\mathbb{G})$ for any $x \in \mathcal{O}_c(\mathbb{G})$ since $(\mathrm{id} \otimes p_{\mathbb{H}})\Delta(x) \in \mathcal{O}_c(\mathbb{G}) \odot p_{\mathbb{H}}\mathcal{O}_c(\mathbb{G}) = \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}(\mathbb{H})$.

Proposition 4.12. *For any $x \in \mathcal{O}_c(\mathbb{G})$ we have $(\mathrm{id} \otimes hp_{\mathbb{H}})\Delta(x) \in c_c(\mathbb{G}/\mathbb{H})$. If μ is a $\mathbb{G}$-invariant functional on $c_c(\mathbb{G}/\mathbb{H})$, then $\varphi : x \mapsto \mu[(\mathrm{id} \otimes hp_{\mathbb{H}})\Delta(x)]$ defines a left invariant functional on $\mathcal{O}_c(\mathbb{G})$.*

Proof. Define a map $T : \mathcal{O}_c(\mathbb{G}) \to \mathcal{O}_c(\mathbb{G})$ by setting $T(x) := (\mathrm{id} \otimes hp_{\mathbb{H}})\Delta(x)$, $x \in \mathcal{O}_c(\mathbb{G})$. Let us check that $T(x) \in c_c(\mathbb{G}/\mathbb{H})$. We have $(\Delta \otimes \mathrm{id})\Delta(x) = (\mathrm{id} \otimes \Delta)\Delta(x)$ in $\ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})\bar{\otimes}\ell^\infty(\mathbb{\Gamma})$. Multiplying on the left by $1 \otimes p_{\mathbb{H}} \otimes p_{\mathbb{H}}$ we obtain, since $(p_{\mathbb{H}} \otimes p_{\mathbb{H}})\,\Delta(1 - p_{\mathbb{H}}) = 0$:

$$(1 \otimes p_{\mathbb{H}} \otimes 1)(\Delta \otimes \mathrm{id})((1 \otimes p_{\mathbb{H}})\Delta(x)) = (\mathrm{id} \otimes \Delta_{\mathbb{H}})((1 \otimes p_{\mathbb{H}})\Delta(x)).$$

Note that both sides of the identity now lie in $\mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}(\mathbb{H}) \odot \mathcal{O}(\mathbb{H})$. Applying $\mathrm{id} \otimes \mathrm{id} \otimes h$ we obtain $(1 \otimes p_{\mathbb{H}})\Delta(T(x)) = (\mathrm{id} \otimes p_{\mathbb{H}}h)((1 \otimes p_{\mathbb{H}})\Delta(x)) = T(x) \otimes p_{\mathbb{H}}$, by invariance of h.

On the other hand, starting again from the coassociativity relation and multiplying by $y \otimes 1 \otimes p_{\mathbb{H}}$ we get the following identity in $\mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{G}) \odot \mathcal{O}_c(\mathbb{H})$:

$$((y \otimes 1)\Delta \otimes \mathrm{id})((1 \otimes p_{\mathbb{H}})\Delta(x)) = (\mathrm{id} \otimes (1 \otimes p_{\mathbb{H}})\Delta)((y \otimes 1)\Delta(x)).$$

Applying $\mathrm{id} \otimes \mathrm{id} \otimes h$ we obtain $(y \otimes 1)\Delta(T(x)) = (\mathrm{id} \otimes T)((y \otimes 1)\Delta(x))$. In other words, $T : \mathcal{O}_c(\mathbb{G}) \to c_c(\mathbb{G}/\mathbb{H})$ is equivariant with respect to the left $\mathbb{G}$-actions.

So we can indeed define $\varphi = \mu \circ T$, and it is left invariant if μ is invariant on $c_c(\mathbb{G}/\mathbb{H})$:

$$y\mu T(x) = (\mathrm{id} \otimes \mu)((y \otimes 1)\Delta(T(x))) = (\mathrm{id} \otimes \mu T)((y \otimes 1)\Delta(x)),$$

using the previous equivariance identity for T. □

Theorem 4.13. *Suppose that $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a quantum Hecke pair. Then $\mathbb{G}$ introduced in Definition 4.5 is an algebraic quantum group. We call the pair $(\mathbb{G}, \mathbb{H})$ the Schlichting completion of $(\mathbb{\Gamma}, \mathbb{\Lambda})$.*

Proof. We apply Proposition 4.12 to the functional μ on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ from Definition 3.13, which is invariant by Proposition 3.21, and defines at the same time an invariant functional on $c_c(\mathbb{G}/\mathbb{H})$ by Proposition 4.11. We have $\varphi(p_{\mathbb{H}}) = \mu(p_{\mathbb{\Lambda}}) = 1$, hence φ does not vanish. Finally it is positive because $\mu \otimes h$ is positive; indeed μ and h have both C^*-algebraic realizations — recall that μ is a sum of positive forms on each matrix factor of $c_c(\mathbb{G}/\mathbb{H}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$. □

Now we compare the Hecke algebras of a Hecke pair and of its Schlichting completion. This will provide an analytic proof that the 'discrete' Hecke operators on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$ are bounded.

Proposition 4.14. *Let $(\mathbb{G}, \mathbb{H})$ be the Schlichting completion of a quantum Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$. Then we have canonical identifications*

$$\mathcal{H}(\mathbb{G}, \mathbb{H}) \simeq \mathrm{End}_{\mathbb{G}}(c_c(\mathbb{G}/\mathbb{H})) = \mathrm{End}_{\mathbb{\Gamma}}(c_c(\mathbb{\Gamma}/\mathbb{\Lambda})) \simeq \mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda}),$$

compatible with the multiplications. This identification is compatible with the $*$-structures if we identify a bi-invariant function $f \in c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$, viewed as element of $\mathcal{H}(\mathbb{G}, \mathbb{H})$, with $\hat{\sigma}_{-i/2}(f) \in c_c(\mathbb{\Lambda}\backslash\mathbb{\Gamma}/\mathbb{\Lambda})$, viewed as element of $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$.

Proof. The first and last identifications are given by Propositions 4.3 and 3.28, respectively. The identity in the middle is given by Proposition 4.11. □

Proposition 4.15. *We have a canonical unitary isomorphism $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq \ell^2(\mathbb{G}/\mathbb{H})$ induced by the equality $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{G}/\mathbb{H})$ in $\ell^\infty(\mathbb{\Gamma})$. The Hecke algebra $\mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ acts by bounded operators on $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda})$.*

Proof. The identification $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{G}/\mathbb{H})$ in Proposition 4.11 is isometric since by construction the restriction of φ to $c_c(\mathbb{G}/\mathbb{H}) = c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ coincides with μ. Hence it induces a unitary isomorphism $\ell^2(\mathbb{\Gamma}/\mathbb{\Lambda}) \simeq \ell^2(\mathbb{G}/\mathbb{H})$. Now the action of $f \in \mathcal{H}(\mathbb{\Gamma}, \mathbb{\Lambda})$ on $c_c(\mathbb{\Gamma}/\mathbb{\Lambda}) = c_c(\mathbb{G}/\mathbb{H})$ agrees with the convolution on the right by $\hat{\sigma}_{i/2}(f) \in c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H})$ by Proposition 4.14. Hence it suffices to observe that the latter is obtained by restriction of the right regular representation of $\mathcal{D}(\mathbb{G})$ on $L^2(\mathbb{G})$, which acts by bounded operators. □

This, together with the Theorem 3.32, yields an analytical proof of Property (RT) from Definition 3.30 for Hecke pairs. It should be possible to give a categorical proof as well.

Corollary 4.16. *Property (RT) is satisfied by any Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$.*

Finally we record the connection between the modular group of a discrete Hecke pair $(\mathbb{\Gamma}, \mathbb{\Lambda})$, see Theorem 3.36, and the modular group of its Schlichting completion $\mathbb{G}$.

Proposition 4.17. *The modular group of the canonical state ω on $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ agrees with the restriction of the modular group of $\hat{\varphi}$ to $\mathcal{F}(c_c(\mathbb{H}\backslash\mathbb{G}/\mathbb{H}))\subset\mathcal{D}(\mathbb{G})$ via the Fourier transform.*

Proof. Recall that the element in $\mathcal{H}(\mathbb{\Gamma},\mathbb{\Lambda})$ corresponding to $f\in\mathcal{H}(\mathbb{G},\mathbb{H})$ is $\tilde{f}:=\hat{\sigma}_{-i/2}(f)$. We have then

$$\begin{aligned}\omega(\tilde{f}) &= (p_{\mathbb{\Lambda}}\mid\tilde{f}) = (p_{\mathbb{H}}\mid\hat{\sigma}_{-i/2}(f)) = \varphi(\pi_{\mathbb{H}}(\hat{\sigma}_{-i/2}(f)) \\ &= \epsilon(\pi_{\mathbb{H}}(\hat{\sigma}_{-i/2}(f)) = \epsilon(\hat{\sigma}_{-i/2}(f)) = \hat{\varphi}(\hat{\sigma}_{-i/2}(\mathcal{F}(f))) = \hat{\varphi}(\mathcal{F}(f)).\end{aligned}$$

Note that we use in the fourth equality the fact that $\pi_{\mathbb{H}}(\hat{\sigma}_{-i/2}(f))$ is $\mathbb{H}$-invariant, hence constant. □

4.4 | Reduction procedure

Starting from a discrete Hecke pair $(\mathbb{\Gamma},\mathbb{\Lambda})$, it can well happen that the Schlichting completion $\mathbb{G}$ is in fact discrete or even trivial. This is connected to the faithfulness of the action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Lambda}$ and to the reduction procedure that we describe now.

Suppose that $\mathbb{\Gamma}$ is a discrete quantum group with a quantum subgroup $\mathbb{\Lambda}$ and the corresponding projection $p_{\mathbb{\Lambda}}\in\ell^\infty(\mathbb{\Gamma})$. By $c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^\vee$ we denote finitely supported functionals on $\mathbb{\Gamma}/\mathbb{\Lambda}$ as in the previous subsection.

Recall from [11, Definition 2.8] that the cokernel of an action of a discrete quantum group $\mathbb{\Gamma}$ on a C^*-algebra $A=C_0(\mathbb{X})$, given by a $*$-homomorphism $\alpha : C_0(\mathbb{X})\to M(c_0(\mathbb{\Gamma})\otimes C_0(\mathbb{X}))$, is defined as the following weak closure in $\ell^\infty(\mathbb{\Gamma})$:

$$N(\mathbb{\Gamma}\curvearrowright\mathbb{X}) = \{(\mathrm{id}\otimes\mu)\alpha(a); a\in A, \mu\in A^*\}''. \tag{4.1}$$

Here we are concerned with the case $C_0(\mathbb{X})=c_0(\mathbb{\Gamma}/\mathbb{\Lambda})$, with α being the appropriate restriction of Δ.

Definition 4.18. We say that the pair $(\mathbb{\Gamma},\mathbb{\Lambda})$ is reduced if the canonical action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Lambda}$ is faithful, that is, the cokernel $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ coincides with $\ell^\infty(\mathbb{\Gamma})$.

Note that in the definition (4.1) of the cokernel, when $\mathbb{X}=\mathbb{\Gamma}/\mathbb{\Lambda}$, we can also work with $a\in\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\mu\in\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})_*$, or with $a\in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})$ and $\mu\in c_c(\mathbb{\Gamma}/\mathbb{\Lambda})^\vee$. In particular we see, following the discussion after Definition 4.5, that a Hecke pair $(\mathbb{\Gamma},\mathbb{\Lambda})$ is reduced if and only if $\mathbb{\Gamma}$ embeds into its Schlichting completion $(\mathbb{G},\mathbb{H})$, that is, the canonical map $\iota : \mathcal{O}_c(\mathbb{G})\to\ell^\infty(\mathbb{\Gamma})$ has strictly dense image.

It is shown in [11, Proposition 2.9] that the cokernel $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ is a Baaj–Vaes subalgebra of $\ell^\infty(\mathbb{\Gamma})$, so that there exists a discrete quantum group $\tilde{\mathbb{\Gamma}}$ such that $\ell^\infty(\tilde{\mathbb{\Gamma}})=N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ (with the comultiplication given simply by the restriction). Putting $a=p_{\mathbb{\Lambda}}$ and taking for μ the restriction of the counit ϵ, we see that $p_{\mathbb{\Lambda}}$ belongs also to $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ and thus defines a quantum subgroup $\tilde{\mathbb{\Lambda}}$ of $\tilde{\mathbb{\Gamma}}$ such that $\ell^\infty(\tilde{\mathbb{\Lambda}})=p_{\mathbb{\Lambda}}\ell^\infty(\tilde{\mathbb{\Gamma}})$.

When $\mathbb{\Lambda}$ is normal in $\mathbb{\Gamma}$, it follows, for example, from the proof of [27, Theorem 2.11] (or from results of [10]) that $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})=\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$. In this case it is easy to see that $\tilde{\mathbb{\Lambda}}=\{e\}$: as $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$

is the space of left slices of $\Delta(p_{\mathbb{\Lambda}})$ by [10, Theorem 3.3], we have that

$$\begin{aligned} p_{\mathbb{\Lambda}}\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) &= p_{\mathbb{\Lambda}}\{(\omega\otimes\mathrm{id})\Delta(p_{\mathbb{\Lambda}}) \mid \omega\in\ell^1(\mathbb{\Gamma})\}'' \\ &= \{(\omega\otimes\mathrm{id})((1\otimes p_{\mathbb{\Lambda}})\Delta(p_{\mathbb{\Lambda}})) \mid \omega\in\ell^1(\mathbb{\Gamma})\}'' = \mathbb{C}p_{\mathbb{\Lambda}}. \end{aligned}$$

Proposition 4.19. *Let* $(\mathbb{\Gamma},\mathbb{\Lambda})$ *and* $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ *be as above. Then* $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ *(called further the* reduction *of* $(\mathbb{\Gamma},\mathbb{\Lambda})$*) is reduced,* $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) = \ell^\infty(\tilde{\mathbb{\Gamma}}/\tilde{\mathbb{\Lambda}})$*,* $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = \ell^\infty(\tilde{\mathbb{\Lambda}}\backslash\tilde{\mathbb{\Gamma}})$*. Moreover if* $\theta : \ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})_+ \to \mathbb{R}_+$ *is an nsf weight, then it is* $\mathbb{\Gamma}$*-invariant if and only if it is* $\tilde{\mathbb{\Gamma}}$*-invariant.*

Proof. Denote by $\tilde{\alpha}$ the action of $\tilde{\mathbb{\Gamma}}$ on $\ell^\infty(\tilde{\mathbb{\Gamma}}/\tilde{\mathbb{\Lambda}})$. Note that this is again given by the (suitable restriction of) the coproduct of $\ell^\infty(\mathbb{\Gamma})$. Thus to see that $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ is reduced, it suffices to show that $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda}) = N(\tilde{\mathbb{\Gamma}}\curvearrowright\tilde{\mathbb{\Gamma}}/\tilde{\mathbb{\Lambda}})$; this in turn will follow once we establish the latter part of the proposition.

Note that

$$\ell^\infty(\tilde{\mathbb{\Gamma}}/\tilde{\mathbb{\Lambda}}) = \{a\in N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda}) \mid (1\otimes p_{\mathbb{\Lambda}})\Delta(a) = p_{\mathbb{\Lambda}}\otimes a\}.$$

Thus it suffices to show that we have $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \subset N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$. That, however, follows immediately as we can simply put $\mu=\epsilon$ in the definition of $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$.

The equality of the left coset spaces would follow in a similar manner once we observe that $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) \subset N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$. This is true as $N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ is R-invariant — and $\ell^\infty(\mathbb{\Lambda}\backslash\mathbb{\Gamma}) = R(\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}))$. The last statement is then obvious (the invariance condition is literally the same). □

Note that the inclusion $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda}) \subset N(\mathbb{\Gamma}\curvearrowright\mathbb{\Gamma}/\mathbb{\Lambda})$ can be informally understood as the classically obvious fact that the kernel of the action of $\mathbb{\Gamma}$ on $\mathbb{\Gamma}/\mathbb{\Lambda}$ is contained in $\mathbb{\Lambda}$.

Proposition 4.20. *Let* $(\mathbb{\Gamma},\mathbb{\Lambda})$ *be as above and let* $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ *be its reduction. Then* $(\mathbb{\Gamma},\mathbb{\Lambda})$ *satisfies the Hecke condition if and only if* $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ *does, and if this is the case, the corresponding Hecke algebras are isomorphic.*

Proof. The first statement follows from Proposition 3.7, as the proposition above implies that the actions of $\mathbb{\Lambda}$ on $\ell^\infty(\mathbb{\Gamma}/\mathbb{\Lambda})$ and of $\tilde{\mathbb{\Lambda}}$ on $\ell^\infty(\tilde{\mathbb{\Gamma}}/\tilde{\mathbb{\Lambda}})$ are given by the same von Neumann algebraic morphism and the notion of finite orbits does not formally involve any quantum group structure. The second statement follows now from the identification of Hecke algebras as certain commutants with respect to these actions. □

We can now characterize the Hecke pairs that give rise to non-discrete Schlichting completions as follows.

Lemma 4.21. *Let* $(\mathbb{\Gamma},\mathbb{\Lambda})$ *be a Hecke pair,* $(\tilde{\mathbb{\Gamma}},\tilde{\mathbb{\Lambda}})$ *its reduction, and* $(\mathbb{G},\mathbb{H})$ *its Schlichting completion. Then* $\mathbb{G}$ *is discrete if and only if* $\tilde{\mathbb{\Lambda}}$ *is finite.*

Proof. Recall that we have by construction the strictly dense (equivalently, so-dense) inclusion $C_0(\mathbb{G}) \subset \ell^\infty(\tilde{\mathbb{\Gamma}})$. Multiplying by $p_{\mathbb{\Lambda}} = p_{\tilde{\mathbb{\Lambda}}} = p_{\mathbb{H}}$ we see that $C(\mathbb{H})$ is strictly dense (equivalently, so-dense) in $\ell^\infty(\tilde{\mathbb{\Lambda}})$, so that $\tilde{\mathbb{\Lambda}}$ is finite if and only if $\mathbb{H}$ is finite.

Now if $\mathbb{H}$ is finite, [10, Proposition 4.5] implies that $\mathbb{G}$ is discrete. On the other hand, if $\mathbb{G}$ is discrete, $\mathbb{H}$, being an open (hence also closed by [10, Theorem 3.6]) quantum subgroup of $\mathbb{G}$, must be discrete by [4, Theorem 6.2]. Finally a discrete and compact quantum group must be finite. □

In particular if $(\mathbb{\Gamma}, \mathbb{\Lambda})$ is a reduced Hecke pair, the associated Schlichting completion is discrete if and only if $\mathbb{\Lambda}$ is finite. This shows, with the help of Lemma 3.44, that the examples of quantum Hecke pairs discussed in the previous section lead to non-discrete Schlichting completions.

Corollary 4.22. *The Schlichting completions associated with the HNN Hecke pairs of Example 3.47 are non-discrete locally compact quantum groups, with non-trivial modular group as soon as* $\#\Sigma_1 \neq \#\Sigma_{-1}$.

Note that the scaling constant of these quantum groups equals 1, since they arise from algebraic quantum groups.

We address now the reduction procedure for compact open Hecke pairs. Suppose that $\mathbb{G}$ is an algebraic quantum group and that $\mathbb{H}$ is an algebraic compact open quantum subgroup of $\mathbb{G}$, given by a projection $p_{\mathbb{H}} \in \mathcal{O}_c(\mathbb{G})$. We shall check that the procedure described above again yields a reduction of the pair $(\mathbb{G}, \mathbb{H})$.

Consider the $*$-algebra generated as follows:

$$\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H}) :=* \text{-alg}\{(\mathrm{id} \otimes \mu)(\Delta(a)) \mid a \in c_c(\mathbb{G}/\mathbb{H}), \mu \in c_c(\mathbb{G}/\mathbb{H})^\vee\},$$

where $c_c(\mathbb{G}/\mathbb{H})^\vee$ denotes the finitely supported functionals on $c_c(\mathbb{G}/\mathbb{H}))$; note that as each of these can be written in the form νb with $\nu \in c_c(\mathbb{G}/\mathbb{H}))^\vee$ and $b \in c_c(\mathbb{G}/\mathbb{H})$, the formula above makes sense. Note also that we can identify $c_c(\mathbb{G}/\mathbb{H}))^\vee$ with elements of the form $\psi b|_{c_c(\mathbb{G}/\mathbb{H})}$ for $b \in c_c(\mathbb{G}/\mathbb{H}))$, where ψ is the right Haar weight of $\mathbb{G}$.

Proposition 4.23. *The algebra* $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ *defines an algebraic quantum group* (*with the structure inherited from* $\mathcal{O}_c(\mathbb{G})$). *Moreover* $p_{\mathbb{H}} \in \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ *and* $c_c(\mathbb{G}/\mathbb{H}) \subset \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$.

Proof. We will use the map $T : \mathcal{O}_c(\mathbb{G}) \to c_c(\mathbb{G}/\mathbb{H})$ given by the formula

$$T(a) = (\mathrm{id} \otimes h_{\mathbb{H}})(\Delta(a)(1 \otimes p_{\mathbb{H}})), \qquad a \in \mathcal{O}_c(\mathbb{G}).$$

Note that this map extends to a C^*-algebraic conditional expectation (preserving the right invariant Haar weight) from $C_0(\mathbb{G})$ onto $c_0(\mathbb{G}/\mathbb{H})$, with the property $T(\mathcal{O}_c(\mathbb{G})) = c_c(\mathbb{G}/\mathbb{H})$. This, together with [28, Remarks, p. 342] shows the following facts:

$$c_c(\mathbb{G}/\mathbb{H})^\vee = \{\omega|_{c_c(\mathbb{G}/\mathbb{H})} \mid \omega \in \widehat{\mathcal{O}_c(\mathbb{G})}\}, \quad \text{and}$$

$$c_c(\mathbb{G}/\mathbb{H})^\vee = \{b\psi|_{c_c(\mathbb{G}/\mathbb{H})} \mid b \in c_c(\mathbb{G}/\mathbb{H})\} = \{b\varphi|_{c_c(\mathbb{G}/\mathbb{H})} \mid b \in c_c(\mathbb{G}/\mathbb{H})\}$$

$$= \{\varphi b|_{c_c(\mathbb{G}/\mathbb{H})} \mid b \in c_c(\mathbb{G}/\mathbb{H})\},$$

where φ is the left-invariant weight.

Using the properties of T and the fact that $\mathcal{O}_c(\mathbb{G})$ is a multiplier Hopf algebra one can show the following fact: $\mathcal{O}_c(\mathbb{G}) \odot c_c(\mathbb{G}/\mathbb{H}) = \Delta(c_c(\mathbb{G}/\mathbb{H}))(\mathcal{O}_c(\mathbb{G}) \otimes 1)$. This implies (via the arguments of [28, Proposition 4.2]) that convolving a functional $\omega \in \widehat{\mathcal{O}_c(\mathbb{G})}$ and a functional $\mu \in c_c(\mathbb{G}/\mathbb{H})^\vee$ yields $\omega \star \mu \in c_c(\mathbb{G}/\mathbb{H})^\vee$.

We need to show that for every $a, b \in \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ we have (for example) $\Delta(a)(b \otimes 1) \in \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H}) \otimes \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$. To this end it suffices to prove that for all $\omega \in \widehat{\mathcal{O}_c(\mathbb{G})}$,$a \in c_c(\mathbb{G}/\mathbb{H})$ and $\mu \in c_c(\mathbb{G}/\mathbb{H})^\vee$ the elements $(\omega \otimes \mathrm{id})(\Delta((\mathrm{id} \otimes \mu)(\Delta(a)))(b \otimes 1))$ and $(\mathrm{id} \otimes \omega)(\Delta((\mathrm{id} \otimes \mu)(\Delta(a))(b \otimes 1))$ belong to $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$; and the latter amount to noting that $c_c(\mathbb{G}/\mathbb{H})$ is right-invariant (for the first expression) and exploiting the convolution statement of the previous paragraph (for the second expression).

Furthermore, $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ is S-invariant. This is an easy consequence of the strong invariance of the left Haar weight, which says that for all $a, b \in \mathcal{O}_c(\mathbb{G})$ we have

$$S((\mathrm{id} \otimes \phi)(\Delta(a^*)(1 \otimes b)) = (\mathrm{id} \otimes \phi)((1 \otimes a^*)(\Delta(b)),$$

combined with the statements in the beginning of the proof. This suffices to complete the proof that $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ is a multiplier Hopf $*$-algebra, and in fact an algebraic quantum group, as we can just use the invariant weights of $\mathcal{O}_c(\mathbb{G})$.

Then it suffices to show that as we have $\epsilon(p_{\mathbb{H}}) = 1$ we also have $\epsilon = \epsilon p_{\mathbb{H}}$, so that $\epsilon|_{c_c(\mathbb{G}/\mathbb{H})}$ is finitely supported. This implies that $c_c(\mathbb{G}/\mathbb{H}) \subset \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$. □

Definition 4.24. Let $(\mathbb{G}, \mathbb{H})$ be as above, denote the algebraic quantum group corresponding to $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ by $\tilde{\mathbb{G}}$, and its compact quantum subgroup given by $p_{\mathbb{H}} \in \mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H})$ by $\tilde{\mathbb{H}}$. We call the pair $(\tilde{\mathbb{G}}, \tilde{\mathbb{H}})$ the reduction of $(\mathbb{G}, \mathbb{H})$ and say that a pair $(\mathbb{G}, \mathbb{H})$ as above is reduced if $\mathcal{A}(\mathbb{G} \curvearrowright \mathbb{G}/\mathbb{H}) = \mathcal{O}_c(\mathbb{G})$.

Proposition 4.25. *Let $(\mathbb{G}, \mathbb{H})$ and $(\tilde{\mathbb{G}}, \tilde{\mathbb{H}})$ be as above. Then $(\tilde{\mathbb{G}}, \tilde{\mathbb{H}})$ is reduced, $c_c(\mathbb{G}/\mathbb{H}) = c_c(\tilde{\mathbb{G}}/\tilde{\mathbb{H}})$ and $c_c(\mathbb{H}\backslash\mathbb{G}) = c_c(\tilde{\mathbb{H}}\backslash\tilde{\mathbb{G}})$. Moreover a functional $\theta : c_c(\mathbb{G}/\mathbb{H}) \to \mathbb{C}$ is $\mathbb{G}$-invariant if and only if it is $\tilde{\mathbb{G}}$-invariant.*

Proof. Follows exactly the same lines as in Proposition 4.19. □

Observe that the Schlichting completion is constructed specifically so that the resulting pair $(\mathbb{G}, \mathbb{H})$ is reduced. Further we will call $(\mathbb{G}, \mathbb{H})$ a *Schlichting pair* whenever $\mathbb{G}$ is an algebraic quantum group, $\mathbb{H}$ is an algebraic compact open quantum subgroup of $\mathbb{G}$, and the pair $(\mathbb{G}, \mathbb{H})$ is reduced.

Suppose that we have two locally compact quantum groups $\mathbb{G}_1$, $\mathbb{G}_2$ with respective open quantum subgroups $\mathbb{H}_1$, $\mathbb{H}_2$ corresponding to projections $P_1 \in C_b(\mathbb{G}_1)$, $P_2 \in C_b(\mathbb{G}_2)$. We say that a morphism from $\mathbb{G}_1$ to $\mathbb{G}_2$, described via a Hopf-C^*-algebra morphism $\pi : C_0(\mathbb{G}_2) \to C_b(\mathbb{G}_1)$, maps $\mathbb{H}_1$ to $\mathbb{H}_2$ if $\pi(P_2) \geqslant P_1$. One may check, using [10, Corollary 3.8], that indeed one obtains then (by restriction and multiplying by P_1) a quantum group morphism from $\mathbb{H}_1$ to $\mathbb{H}_2$.

The following abstract characterization of the Schlichting completion for classical groups appears in [25, Proposition 4.1]. The injectivity of the map ι' corresponds in the classical case to the density of the image of Γ in G', and the identity $\iota'(p_{H'}) = p_\Lambda$, to the fact that Λ is the preimage of H'.

Proposition 4.26. *Let $(\mathbb{\Gamma}, \mathbb{\Lambda})$ be a Hecke pair and $(\mathbb{G}, \mathbb{H})$ its Schlichting completion, with the canonical embedding $\iota : \mathcal{O}_c(\mathbb{G}) \to \ell^\infty(\mathbb{\Gamma})$ defining the morphism from $\mathbb{\Gamma}$ to $\mathbb{G}$. Then for any other Schlichting pair $(\mathbb{G}', \mathbb{H}')$ and any morphism from $\mathbb{\Gamma}$ to $\mathbb{G}'$ mapping $\mathbb{\Lambda}$ to $\mathbb{H}'$ and given by an* injective *map $\iota' : \mathcal{O}_c(\mathbb{G}') \to \ell^\infty(\mathbb{\Gamma})$, there exists a unique morphism from $\mathbb{G}$ to $\mathbb{G}'$, described by a map $\sigma : \mathcal{O}_c(\mathbb{G}') \to \mathcal{O}_c(\mathbb{G})$, such that $\iota \circ \sigma = \iota'$. If in addition we assume that $\iota'(p_{\mathbb{H}'}) = p_{\mathbb{\Lambda}}$, then the morphism from $\mathbb{G}$ to $\mathbb{G}'$ is an isomorphism.*

Proof. A moment of thought shows that it suffices to show that $\iota'(\mathcal{O}_c(\mathbb{G}')) \subset \iota(\mathcal{O}_c(\mathbb{G}))$. Ignoring the injective embedding maps, and using the fact that both $(\mathbb{G}, \mathbb{H})$ and $(\mathbb{G}', \mathbb{H}')$ are Schlichting pairs, it suffices to note that $c_c(\mathbb{G}'/\mathbb{H}') \subset c_c(\mathbb{G}/\mathbb{H})$. But this follows as we have $P \leqslant P'$ (again viewing both as projections in $\ell^\infty(\mathbb{\Gamma})$). The second part follows similarly. □

ACKNOWLEDGEMENTS

A.S. was partially supported by the National Science Center (NCN) grant no. 2020/39/I/ST1/01566. R.V. was partially supported by the Agence Nationale de la Recherche (ANR) grant ANR-19-CE40-0002 and the CEFIPRA project 6101-1. C.V. was supported by EPSRC grant EP/T03064X/1. This project was started and concluded during visits of C.V. and R.V. at IMPAN Warsaw. The second and third author would like to thank the first author for the kind hospitality.

JOURNAL INFORMATION

The *Journal of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. C. Anantharaman-Delaroche, *Approximation properties for coset spaces and their operator algebras*, The varied landscape of operator theory, Theta Ser. Adv. Math., vol. 17, Theta, Bucharest, 2014, pp. 23–45.
2. S. Baaj and S. Vaes, *Double crossed products of locally compact quantum groups*, J. Inst. Math. Jussieu **4** (2005), no. 1, 135–173.
3. J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Math. (N.S.) **1** (1995), no. 3, 411–457.
4. M. Daws, P. Kasprzak, A. Skalski, and P. M. Sołtan, *Closed quantum subgroups of locally compact quantum groups*, Adv. Math. **231** (2012), no. 6, 3473–3501.
5. K. De Commer and M. Yamashita, *Tannaka-Kreĭn duality for compact quantum homogeneous spaces. I. General theory*, Theory Appl. Categ. **28** (2013), no. 31, 1099–1138.
6. K. De Commer, P. Kasprzak, A. Skalski, and P. M. Sołtan, *Quantum actions on discrete quantum spaces and a generalization of Clifford's theory of representations*, Israel J. Math. **226** (2018), no. 1, 475–503.
7. L. Delvaux and A. Van Daele, *The Drinfeld double of multiplier Hopf algebras*, J. Algebra **272** (2004), no. 1, 273–291.
8. P. Fima, *K-amenability of HNN extensions of amenable discrete quantum groups*, J. Funct. Anal. **265** (2013), no. 4, 507–519.
9. M. Izumi, *Non-commutative Poisson boundaries and compact quantum group actions*, Adv. Math. **169** (2002), no. 1, 1–57.
10. M. Kalantar, P. Kasprzak, and A. Skalski, *Open quantum subgroups of locally compact quantum groups*, Adv. Math. **303** (2016), 322–359.
11. M. Kalantar, P. Kasprzak, A. Skalski, and R. Vergnioux, *Noncommutative Furstenberg boundary*, Anal. PDE **15** (2022), no. 3, 795–842.
12. S. Kaliszewski, M. B. Landstad, and J. Quigg, *Hecke C^*-algebras, Schlichting completions and Morita equivalence*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 3, 657–695.
13. P. Kasprzak and P. M. Sołtan, *Quantum groups with projection and extensions of locally compact quantum groups*, J. Noncommut. Geom. **14** (2020), no. 1, 105–123.
14. A. Klimyk and K. Schmüdgen, *Quantum groups and their representations*, Texts and Monographs in Physics, Springer, Berlin, 1997.
15. J. Kustermans, *The analytic structure of algebraic quantum groups*, J. Algebra **259** (2003), no. 2, 415–450.

16. J. Kustermans and S. Vaes, *Locally compact quantum groups*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 6, 837–934.
17. J. Kustermans and A. Van Daele, *C^*-algebraic quantum groups arising from algebraic quantum groups*, Internat. J. Math. **8** (1997), no. 8, 1067–1139.
18. M. B. Landstad and A. Van Daele, *Compact and discrete subgroups of algebraic quantum groups I*, Preprint, arXiv:math/0702458.
19. M. B. Landstad and A. Van Daele, *Groups with compact open subgroups and multiplier Hopf $*$-algebras*, Expo. Math. **26** (2008), no. 3, 197–217.
20. R. Meyer, S. Roy, and S. a. L. Woronowicz, *Homomorphisms of quantum groups*, Münster J. Math. **5** (2012), 1–24.
21. S. Neshveyev and L. Tuset, *Compact quantum groups and their representation categories*, Cours Spécialisés [Specialized Courses], vol. 20, Société Mathématique de France, Paris, 2013.
22. S. Neshveyev and M. Yamashita, *Categorically Morita equivalent compact quantum groups*, Doc. Math. **23** (2018), 2165–2216.
23. G. Schlichting, *Operationen mit periodischen Stabilisatoren*, Arch. Math. (Basel) **34** (1980), no. 2, 97–99.
24. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, NJ, 1971, Kanô Memorial Lectures, No. 1.
25. K. Tzanev, *Hecke C^*-algebras and amenability*, J. Operator Theory **50** (2003), no. 1, 169–178.
26. S. Vaes, *A new approach to induction and imprimitivity results*, J. Funct. Anal. **229** (2005), no. 2, 317–374.
27. S. Vaes and L. Vainerman, *On low-dimensional locally compact quantum groups*, Locally compact quantum groups and groupoids (Strasbourg, 2002), IRMA Lect. Math. Theor. Phys., vol. 2, de Gruyter, Berlin, 2003, pp. 127–187.
28. A. Van Daele, *An algebraic framework for group duality*, Adv. Math. **140** (1998), no. 2, 323–366.
29. R. Vergnioux, *K-amenability for amalgamated free products of amenable discrete quantum groups*, J. Funct. Anal. **212** (2004), no. 1, 206–221.
30. R. Vergnioux and C. Voigt, *The K-theory of free quantum groups*, Math. Ann. **357** (2013), no. 1, 355–400.
31. C. Voigt and R. Yuncken, *Complex semisimple quantum groups and representation theory*, Lect. Notes Math., vol. 2264, Springer, Cham, 2020.
32. S. a. L. Woronowicz, *Compact quantum groups*, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884.