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Infrared scaling for a graviton condensate

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Abstract

The coupling between gravity and matter provides an intriguing length scale in the infrared for theories of gravity within Einstein-Hilbert action and beyond. In particular, we will show that such an infrared length scale is determined by the number of gravitons $N_g \gg 1$ associated to a given mass in the non-relativistic limit. After tracing out the matter degrees of freedom, the graviton vacuum is found to be in a displaced vacuum with an occupation number of gravitons $N_g \gg 1$. In the infrared, the length scale appears to be $L = \sqrt{N_g}\ell_p$, where L is the new infrared length scale, and ℓ_p is the Planck length. In a specific example, we have found that the infrared length scale is greater than the Schwarzschild radius for a slowly moving in-falling thin shell of matter. We will argue that the appearance of such an infrared length scale in higher curvature theories of gravity, such as in quadratic and cubic curvature theories of gravity, is also expected. Furthermore, we will show that gravity is fundamentally different from the electromagnetic interaction where the number of photons, N_p, is the *fine structure constant* after tracing out an electron wave function.

1. Introduction

It is believed that the gravitational interaction is mediated by the spin-2 graviton, which can be canonically quantised around a weak curvature background [1]. A massless graviton in four spacetime dimensions will have both 2-on-shell and 6-off-shell degrees of freedom [2]. The

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former is responsible for describing independent dynamical modes such as gravitational waves, while the latter describes how the force is being mediated between the two masses. Despite all our efforts the discovery of a graviton remains a challenging problem, see [3–5]. However, the quantum nature of a graviton leaves indelible mark in both classical and quantum systems [6–12].

Despite of the weakness in the gravitational interaction, gravity is unique among the other known fundamental interactions of nature that it generates a new length scale in the presence of a self-gravitating matter [13]. The gravitational radius for a given non-rotating mass, M, is given by $r_g = 2GM$, which is known as the Schwarzschild radius or the gravitational radius, while the Planck length, which is determined solely by fundamental constants,[1] is given by $\ell_p = \sqrt{G}$. In Ref. [14], the idea has been proposed from the corpuscular nature of a black hole – a black hole is a condensate of gravitons [14–17], whose occupation number can be denoted here by $N_g \sim (M/M_p)^2 \gg 1$ for any mass $M \geq M_p$. The large occupancy leads to not only weakening of the gravitational strength by $\sim 1/\sqrt{N_g}$, but also leads to classicalization of a black hole, therefore, recovers the classical black hole spacetime geometry outside the Schwarzschild radius. In this regard, we might imagine that N_g would dictate how classical the space time of a black hole would appear to be for a far away observer. Thus the new length scale appears in the infrared; $L \sim \sqrt{N_g}/M_p \gg \ell_p$ for $N_g \gg 1$.

Recently, a very similar result were obtained by us in a quantum system [18], where both matter and gravity were treated as a quantum entity in a perturbative regime. We found that by *tracing out* the non-relativistic self-gravitating matter of mass M, the graviton vacuum state is found to be that of a displaced vacuum, like a coherent state with the occupation number similar to that of $N_g \sim (M/M_p)^2$. For $N_g \gg 1$, the gravitons can be thought of as a condensate of mass M. For a light subatomic particle, such as that of an electron, the number of gravitons by tracing out the electron is much less than unity, $N_g \sim (m_e/M_p)^2 \sim 10^{-44} \ll 1$.[2,3]

The aim of this paper will be two-fold. First of all, we will argue that gravity is unique in this regard as it provides an infrared scale. This infrared length scale may even be larger than that gravitational radius, $L \geq r_g = 2GM$, we will provide an example of this. A similar analysis in the quantum electrodynamics (QED) does not yield any such infrared length scale. In particular, we will show that by *integrating out an electron* in QED, the photon vacuum is that of a displaced vacuum (similar to the case of a graviton), but the occupation number of photon in this case is always bounded below unity for foreseeable energies. In fact, the occupation number of photons is proportional to the fine structure constant. Indeed, the fine structure evolves with the energy in the ultraviolet, but it remains below unity for energies below the Planck energy. Moreover, we will further show that Bekenstein's entropy bound in our case is always satisfied [26], i.e. Bekenstein's entropy is always bounded by the energy and the distance scale.

The second goal will be to generalise our earlier results of Ref. [18] by including both relativistic/non-relativistic effects while integrating out the energy momentum tensor for the matter

[1] We are working in natural units $c = \hbar = 1$, and $\epsilon_0 = 1$. The metric signature is given by $(-, +, +, +)$ and Einstein summation convetion will be used in the text.

[2] Interestingly, the electron cannot be described by a gravitational metric, such as a Reissner-Nordström or a Kerr metric [19]. The metric is inherently a classical notion.

[3] There is another proposal, known as the fuzz ball paradigm [20], where it has been argued that the new scale in gravity will arise naturally from the quantum fluctuations in the gravitational degrees of freedom [21], in particular by taking all microscopic states of string theory, namely the fuzz ball states [20]. The fuzz-ball paradigm is one of the popular contenders to resolve the black hole information-loss paradox. The idea here is that an astrophysical black hole can have a radius few Planck length greater than then the gravitational radius, i.e. $r_{bh} = r_g(1 + \epsilon)$, where $\epsilon < 0.5 r_g = 3Gm$ to avoid having an event horizon. There are already astrophysical constraints on ϵ, see [23,24].

field. To illustrate, we will consider an in-falling thin shell of matter, in an adiabatic approximation, where the vacuum changes slowly. We will show that up to the leading order in the Newton's constant, G, by integrating out the energy momentum tensor of an in-falling thin shell, we will obtain a large occupation number of gravitons. In fact, we will further show that the infrared scale in this example is *slightly larger than the Schwarzschild radius* of the corresponding mass M, i.e., $L \geq r_g = 2GM$. Moreover, we will argue that surprisingly such an infrared scaling persists for higher curvature theories of gravity as well, but now the emergence of a new infrared length scale is *different* from that of general relativity. This occurs due to the fact that higher curvature theories of gravity brings in a new mass scale, $M_s \leq M_p$.

In section 2, we will show how by integrating out the electron, we will obtain that the photon vacuum to be displaced and the corresponding occupation number would scales as that of the fine structure constant. In section 3, we will generalise our earlier results of Ref. [18] for the relativistic energy momentum tensor for an arbitrary geometric configuration, and find the corresponding number of gravitons. In section 4, we consider an example of an in falling thin shell of matter and compute the graviton occupation number by integrating out the thin shell of matter. In section 5, we will discuss the infrared length scale in theories of gravity within general relativity and in higher curvature theories of gravity.

2. Number of photons and the fine structure

Let us now consider the example within QED. Working in a Coulomb gauge we will expand the electromagnetic field as:

$$A^i(\boldsymbol{x}) = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} a_{\boldsymbol{k},\lambda} \mathrm{e}^i_\lambda(\boldsymbol{n}) e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{H.c.}, \tag{1}$$

where $i = 1, 2, 3$, $\boldsymbol{x} = (x, y, z)$, $\omega_k = k$, $k = \|\boldsymbol{k}\|$, $\boldsymbol{n} = \boldsymbol{k}/\|\boldsymbol{k}\|$, $a_{\boldsymbol{k},\lambda}$ is the annihilation operator, and e^i_λ denote the basis vectors for the two polarisations, $\lambda = 1, 2$. The completeness relation is given by:

$$P^{ij}(\boldsymbol{n}) \equiv \sum_\lambda \mathrm{e}^i_\lambda(\boldsymbol{n}) \mathrm{e}^j_\lambda(\boldsymbol{n}) = \delta^{ij} - \boldsymbol{n}^i \boldsymbol{n}^j. \tag{2}$$

The interaction Hamiltonian is given by

$$H_{\text{int}} = \int d\boldsymbol{x} A^i(\boldsymbol{x}) J_i(\boldsymbol{x}), \tag{3}$$

where J_i are the components of the current density. We will now proceed by taking the mean-field approximation

$$J_i(\boldsymbol{x}) \to \langle J_i(\boldsymbol{x}) \rangle, \tag{4}$$

where $\langle J_i(\boldsymbol{x}) \rangle = \text{tr}[\rho J_i(\boldsymbol{x})]$ is the expectation value of the current density, and ρ is a generic matter state.[4] From Eq. (1) and (3), we find

[4] We summarize here the general procedure we will follow for computing the number of photons/gravitons. We first take the mean-field approximation of the electromagnetic/gravitational interaction Hamiltonian H_{int} by *tracing out* the matter state $H_{\text{int}} \to \langle H_{\text{int}} \rangle$, where $\langle \cdot \rangle = \text{tr}[\rho \cdot]$ indicates the trace with respect to the matter state ρ. We find that the ground state of the electromagnetic/gravitational field becomes displaced depending on the values of $\langle J_i(\boldsymbol{x}) \rangle$ and $\langle T_{ij}(\boldsymbol{x}) \rangle$, where $J_i(\boldsymbol{x})$ and $T_{ij}(\boldsymbol{x})$ denote the current density and the stress-energy tensor, respectively. In the computa-

$$H_{\rm int} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} a_{\boldsymbol{k},\lambda} \mathrm{e}^i_\lambda(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k}) + \mathrm{H.c.}, \tag{5}$$

where we have introduced the Fourier transform

$$\tilde{J}_i(\boldsymbol{k}) = \int d\boldsymbol{x} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \langle J_i(\boldsymbol{x})\rangle. \tag{6}$$

We will work in the basis where $\mathrm{e}^i_\lambda(\boldsymbol{n})$ is real-valued, but $\tilde{J}_i(\boldsymbol{k})$ can in general be a c-number. However, we can always absorb any global phase from the $\sum_i \mathrm{e}^i_\lambda(\boldsymbol{n})\tilde{J}_i(\boldsymbol{k}) = |C(\boldsymbol{k})|e^{i\theta(\boldsymbol{k})}$ by a redefinition of the modes, i.e. $a_{\boldsymbol{k},\lambda}e^{i\theta(\boldsymbol{k})} \to a_{\boldsymbol{k},\lambda}$ (and which leaves invariant kinetic term for a photon in a Coulomb gauge; $\sim a^\dagger_{\boldsymbol{k},\lambda}a_{\boldsymbol{k},\lambda}$)

$$\begin{aligned} H_p &= \int d\boldsymbol{k}\omega_k a^\dagger_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda} \\ &= \sum_\lambda \int d\boldsymbol{k} \frac{\omega_k}{4}\left[P^2_{\boldsymbol{k},\lambda} + Y^2_{\boldsymbol{k},\lambda}\right], \end{aligned} \tag{7}$$

where

$$Y_{\boldsymbol{k},\lambda} = a^\dagger_{\boldsymbol{k},\lambda} + a_{\boldsymbol{k},\lambda}\,, \tag{8}$$

$$P_{\boldsymbol{k},\lambda} = i(a^\dagger_{\boldsymbol{k},\lambda} - a_{\boldsymbol{k},\lambda})\,, \tag{9}$$

follow the commutation relation $[a_{\boldsymbol{k},\lambda}, a^\dagger_{\boldsymbol{k}'\lambda'}] = \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}$. From Eq. (5), we find

$$H_{\rm int} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} |\mathrm{e}^i_\lambda(\boldsymbol{n})\tilde{J}_i(\boldsymbol{k})|(a^\dagger_{\boldsymbol{k},\lambda} + a_{\boldsymbol{k},\lambda}). \tag{10}$$

Specifically, combining the interaction term with the kinetic term of the electromagnetic field, Eqs. (7) and (10), we find:

$$H_{tot} = \int \frac{d\boldsymbol{k}}{(2\pi)^3} \frac{\omega_k}{4}\left[P^2_{\boldsymbol{k},\lambda} + (Y_{\boldsymbol{k},\lambda} - \alpha_{\boldsymbol{k},\lambda})^2\right], \tag{11}$$

where now

$$\alpha_{\boldsymbol{k},\lambda} \equiv \sqrt{\frac{2}{\omega_k^3}} |\mathrm{e}^i_\lambda(\boldsymbol{n})\tilde{J}_i(\boldsymbol{k})|. \tag{12}$$

Note that the electromagnetic field $a_{\boldsymbol{k},\lambda}$ is in a ground state, centred around $\alpha_{\boldsymbol{k},\lambda}$, which is described by a *displaced coherent state* of a photon [25]:

$$|\alpha_{\boldsymbol{k},\lambda}\rangle = D(\alpha_{\boldsymbol{k},\lambda})|0\rangle = e^{\alpha_{\boldsymbol{k},\lambda}\left[a^\dagger_{\boldsymbol{k},\lambda} + a_{\boldsymbol{k},\lambda}\right]}|0\rangle \tag{13}$$

We are assuming that the electromagnetic field is in the ground state of the displaced harmonic trap, and the vacuum is *stable* and obeys *adiabatic* conditions. Indeed, a different choice of the vacuum for $A_i(\boldsymbol{x})$ will not change significantly the final result as long as the state remains centred and confined around the same minimum and obeys adiabaticity, given by $|\alpha_{\boldsymbol{k},\lambda}\rangle$.

tion we do not specify directly the matter state ρ, but only make generic symmetry and dimensional consideration about the expectation values $\langle J_i(\boldsymbol{x})\rangle$ and $\langle T_{ij}(\boldsymbol{x})\rangle$. Assuming such ground states for the electromagnetic/gravitational field (i.e. displaced coherent states) we then estimate the corresponding number of photons/gravitons, N_p and N_g, respectively.

For such a displaced quantum state we can compute the expectation values $\langle\cdot\rangle$. We will now compute the number operator $N_{\boldsymbol{k},\lambda} = a^\dagger_{\boldsymbol{k},\lambda} a_{\boldsymbol{k},\lambda}$, where $\langle N_{\boldsymbol{k},\lambda}\rangle = |\alpha_{\boldsymbol{k},\lambda}|^2$, and the total number of photons is summation of all $\boldsymbol{k}$, λ modes:

$$N_p \equiv \sum_\lambda \int d\boldsymbol{k} |\alpha_{\boldsymbol{k},\lambda}|^2 = \sum_\lambda \int d\boldsymbol{k} \frac{|\mathrm{e}^i_\lambda(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k})|^2}{4\pi^3 \omega_k^3}. \tag{14}$$

Let us then perform the sum over the polarizations. Exploiting the completeness relation in Eq. (2), we find

$$\begin{aligned}\sum_\lambda |\mathrm{e}^i_\lambda(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k})|^2 &= \sum_{i,j,\lambda} \mathrm{e}^i_\lambda(\boldsymbol{n}) \mathrm{e}^j_\lambda(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k}) [\tilde{J}_j(\boldsymbol{k})]^* \\ &= \sum_{i,j} P^{ij}(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k}) [\tilde{J}_j(\boldsymbol{k})]^*. \end{aligned} \tag{15}$$

By inserting Eq. (15) back in Eq. (14), we finally find:

$$N_p = \int d\boldsymbol{k} \frac{1}{4\pi^3 \omega_k^3} P^{ij}(\boldsymbol{n}) \tilde{J}_i(\boldsymbol{k}) [\tilde{J}_j(\boldsymbol{k})]^*. \tag{16}$$

At this point of the calculation we have not made any assumptions on the current density and thus Eq. (16) is still completely general.

Let us now make some simplifying assumptions. We assume that the Fourier transform of the current can be split into the radial and angular components

$$\tilde{J}_i(\boldsymbol{k}) = R_i(\omega_k) \Omega_i(\boldsymbol{n}), \tag{17}$$

where $\omega_k = k$. We then insert back Eq. (17) into Eq. (16), and use $d\boldsymbol{k} = dk d\boldsymbol{n} = \frac{\omega_k^2}{c} d\omega_k d\boldsymbol{n}$, to find

$$N_p = \int d\omega_k \frac{R_i(\omega_k)[R_j(\omega_k)]^*}{4\pi^3 \omega_k} \int d\boldsymbol{n} P^{ij}(\boldsymbol{n}) \Omega_i(\boldsymbol{n}) [\Omega_j(\boldsymbol{n})]^*. \tag{18}$$

Without loss of generality, let us furthermore assume that we get a nonzero contribution only for $i = j = 3$, and $R(\omega_k) \equiv R_i(\omega_k)$. The number of photons after tracing out an electron simplifies to

$$N_p = \int d\omega_k \frac{|R(\omega_k)|^2}{4\pi^3 \omega_k} \underbrace{\int d\boldsymbol{n} P^{ii}(\boldsymbol{n}) |\Omega_i(\theta, \phi)|^2}_{\sim \mathcal{O}(1)}. \tag{19}$$

By assuming that the angular part will be non-vanishing, we are thus left with

$$N_p \sim \int d\omega_k \frac{|R(\omega_k)|^2}{4\pi^3 \omega_k}. \tag{20}$$

We now make the assumption that

$$R(\omega_k) \sim e L \omega_k, \tag{21}$$

where L is the side of the box containing the current with total charge e and introduce the frequency cutoff

$$\bar{\omega} = \frac{2\pi}{L}. \tag{22}$$

Here we have in mind the following: $L\omega_k$ is a velocity and thus $R(\omega_k) \sim eL\omega_k$ is a current. We can consider other examples too, for example, for an in falling charged sphere we will have

$$R(\omega_k) \sim i\frac{eL\omega_k}{3} \tag{23}$$

and $\Omega_3(\theta,\phi) = 1$, which gives $\int d\boldsymbol{n} P_{33}(\boldsymbol{n})|\Omega_3(\theta,\phi)|^2 = \frac{8\pi}{3}$.

In the above equations the length L was introduced from dimensional analysis. However, it can be shown that such a length-scale L also arises in specific examples. For example, one can consider a thin charged shell of radius R_0 and compute the Fourier transform of the currents $\tilde{J}_i(\boldsymbol{k})$ (Appendix A). Moreover, by expanding the currents to order $\mathcal{O}(\omega_k)$ we find that the only non-vanishing component is given by $\tilde{J}_3(k) \approx i\frac{e\omega_k R_0}{3}$. From Eqs. (17) and (23) we can then conclude that L can be identified with R_0 in such an example.

Now, if we combine all these results, we find from Eq. (20):

$$N_p \sim \frac{e^2L^2}{4\pi^3}\int_0^{\bar{\omega}} d\omega_k\omega_k. \tag{24}$$

We thus finally find:

$$N_p \sim \frac{e^2}{4\pi} = \alpha_{em} \ll 1. \tag{25}$$

We have obtained an interesting result; by tracing out the current driven by one electron, the photon occupation number is *just* that of the fine structure constant (see also [27] for a computation of the number of longitudinal photons). The QED interactions will allow the fine structure constant to evolve with energies. Up to all relevant energies, i.e., say up to the Planck scale, the fine structure constant remains $\alpha_{em}(\mu) \ll 1$, where μ is the momentum scale.

We can gain further insight into this problem by recalling that the Bekenstin's entropy [26] for any system is always bounded by the energy and the distance scale, i.e. $S_{BEK} \leq 2\pi ER$. We can check this in our case; say for instance, we can estimate the entropy for a thin shell. The energy of a charge carrying thin shell of radius R is given by, $E \sim \frac{e^2}{8\pi R}$. Therefore, the entropy is given by:

$$S_{BEK} \sim e^2 \sim \alpha_{em} \sim N_p. \tag{26}$$

Bekenstein's entropy is now related to the number of photons obtained by tracing out the charge. What it also suggests that the electron is inherently a quantum system, it can never be classicalised for a foreseeable energy, e.g. the Planck scale [28].

3. Tracing out energy momentum tensor – relativistic treatment

In this section, we will study the graviton occupation number N_g, in a long wavelength limit, where we can perturb the metric by:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{27}$$

where $\mu,\nu = 0,1,2,3$. Note that here we have perturbed the metric around Minkowski background, but we will mention below other choices of the background.

We will start the gravitational field in the transverse traceless (TT) gauge in the asymptotically flat region of space time:

$$h^{ij}(\boldsymbol{x}) = \sum_\lambda \int d\boldsymbol{k}\sqrt{\frac{G}{\pi^2\omega_k}}g_{\boldsymbol{k},\lambda}\mathrm{e}_\lambda^{ij}(\boldsymbol{n})e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \text{H.c.}, \tag{28}$$

where $i, j = 1, 2, 3$, G is the Newton's constant, $\omega_k = k$, $k = \|\boldsymbol{k}\|$, $\boldsymbol{n} = \boldsymbol{k}/\|\boldsymbol{k}\|$, $g_{\boldsymbol{k},\lambda}$ is the annihilation operator, and e_λ^{ij} denote the basis tensors for the two polarisations, $\lambda = 1, 2$. We can write the Hamiltonian governed by the kinetic term of the massless graviton field to be:

$$\begin{aligned} H_{\text{grav}} &= \int d\boldsymbol{k}\,\omega_k g^\dagger_{\boldsymbol{k},\lambda}g_{\boldsymbol{k},\lambda} \\ &= \sum_\lambda \int d\boldsymbol{k}\frac{\omega_k}{4}\left[P^2_{\boldsymbol{k},\lambda} + Y^2_{\boldsymbol{k},\lambda}\right], \end{aligned} \tag{29}$$

where

$$Y_{\boldsymbol{k},\lambda} = g_{\boldsymbol{k},\lambda} + g^\dagger_{\boldsymbol{k},\lambda} \qquad P_{\boldsymbol{k},\lambda} = i(g^\dagger_{\boldsymbol{k},\lambda} - g_{\boldsymbol{k},\lambda}). \tag{30}$$

They operators follow the commutation relation $[g_{\boldsymbol{k},\lambda}, g^\dagger_{\boldsymbol{k}'\lambda'}] = \delta(\boldsymbol{k}-\boldsymbol{k}')\delta_{\lambda\lambda'}$. The minimal coupling between graviton and matter dictates the interaction Hamiltonian $\int\sqrt{-g}d^4xG^{\mu\nu}T_{\mu\nu}$, and can be written in the TT-gauge as:

$$H_{\text{int}} = -\frac{1}{2}\int d\boldsymbol{x}h^{ij}(\boldsymbol{x})T_{ij}(\boldsymbol{x}), \tag{31}$$

where $\boldsymbol{x} = (x, y, z)$, and T_{ij} are the components of the stress-energy tensor. We will now take the mean-field approximation, where we take

$$T_{ij}(\boldsymbol{x}) \to \langle T_{ij}(\boldsymbol{x})\rangle, \tag{32}$$

where $\langle T_{ij}(\boldsymbol{x})\rangle = \text{tr}[\rho T_{ij}(\boldsymbol{x})]$ is the expectation value of the stress-energy tensor, and ρ is a generic matter state 4.

From Eq. (28) and (31), we find

$$H_{\text{int}} = -\frac{1}{2}\sum_\lambda\int d\boldsymbol{k}\sqrt{\frac{G}{\pi^2\omega_k}}g_{\boldsymbol{k},\lambda}\mathrm{e}_\lambda^{ij}(\boldsymbol{n})\tilde{T}_{ij}(\boldsymbol{k}) + \text{H.c.}, \tag{33}$$

where we have introduced the Fourier transform

$$\tilde{T}_{ij}(\boldsymbol{k}) = \int d\boldsymbol{x}\langle T_{ij}(\boldsymbol{x})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}. \tag{34}$$

We will work in the basis where $\mathrm{e}_\lambda^{ij}(\boldsymbol{n})$ is a real-valued, but $\tilde{T}_{ij}(\boldsymbol{k})$ in general can be a c-number. We can always absorb any global phase from the $\sum_{ij}\mathrm{e}_\lambda^{ij}(\boldsymbol{n})\tilde{T}_{ij}(\boldsymbol{k}) = |A(\boldsymbol{k})|e^{i\theta(\boldsymbol{k})}$ by a redefinition of the modes, i.e. $g_{\boldsymbol{k},\lambda}e^{i\theta(\boldsymbol{k})} \to g_{\boldsymbol{k},\lambda}$ (and which leaves invariant kinetic term $\sim g^\dagger_{\boldsymbol{k},\lambda}g_{\boldsymbol{k},\lambda}$). From Eq. (33), we thus find

$$H_{\text{int}} = -\frac{1}{2}\sum_\lambda\int d\boldsymbol{k}\sqrt{\frac{G}{\pi^2\omega_k}}|\mathrm{e}_\lambda^{ij}(\boldsymbol{n})\tilde{T}_{ij}(\boldsymbol{k})|(g_{\boldsymbol{k},\lambda} + g^\dagger_{\boldsymbol{k},\lambda}). \tag{35}$$

By combining the interaction term with the kinetic term of the gravitational field, we obtain:

$$H = \sum_\lambda\int d\boldsymbol{k}\frac{\omega_k}{4}\left[P^2_{\boldsymbol{k},\lambda} + (Y_{\boldsymbol{k},\lambda} - \alpha_{\boldsymbol{k},\lambda})^2\right], \tag{36}$$

where now

$$\alpha_{\boldsymbol{k},\lambda} \equiv \sqrt{\frac{G}{\pi^2 \omega_k^3}} |\mathrm{e}_{\lambda}^{ij}(\boldsymbol{n}) \tilde{T}_{ij}(\boldsymbol{k})|. \tag{37}$$

Assuming that the gravitational field is in the ground state of Eq. (36), i.e. in a displaced coherent state, similar to the electromagnetic case [25]:

$$|\alpha_{\boldsymbol{k},\lambda}\rangle = D(\alpha_{\boldsymbol{k},\lambda})|0\rangle = e^{\alpha_{\boldsymbol{k},\lambda}\left[g_{\boldsymbol{k},\lambda}^{\dagger} + g_{\boldsymbol{k},\lambda}\right]}|0\rangle \,, \tag{38}$$

we can compute the number operator $N_{\boldsymbol{k},\lambda} = g_{\boldsymbol{k},\lambda}^{\dagger} g_{\boldsymbol{k},\lambda}$, where $\langle N_{\boldsymbol{k},\lambda}\rangle = |\alpha_{\boldsymbol{k},\lambda}|^2$, and the total number of gravitons by summing over all $\boldsymbol{k}$, λ modes:

$$N_g \equiv \sum_{\lambda} \int d\boldsymbol{k} |\alpha_{\boldsymbol{k},\lambda}|^2 = \sum_{\lambda} \int d\boldsymbol{k} \frac{G}{\pi^2 \omega_k^3} |\mathrm{e}_{\lambda}^{ij}(\boldsymbol{n}) \tilde{T}_{ij}(\boldsymbol{k})|^2, \tag{39}$$

where we sum over the polarizations and momenta of each mode. Let us first perform the sum over the polarizations[5] by exploiting the completeness relation, we find

$$N_g = \int d\boldsymbol{k} \frac{G}{\pi^2 \omega_k^3} P^{iji'j'}(\boldsymbol{n}) \tilde{T}_{ij}(\boldsymbol{k}) [\tilde{T}_{i'j'}(\boldsymbol{k})]^*. \tag{40}$$

This is the generalisation of our earlier computation [18], where we have not made any assumptions on the stress-energy tensor and thus Eq. (40) is applicable for any matter, i.e. a relativistic or a non-relativistic equation of state.

4. In-falling shell

Since we have the most general expression, let us consider a special example of a radially in-falling thin shell of matter, whose stress-energy tensor is given by, see [38]:

$$T_{ij}(\boldsymbol{R}) \equiv \varepsilon \delta(R - R_0) n_i(\theta, \phi) n_j(\theta, \phi), \tag{41}$$

where ε is a surface energy density, $\boldsymbol{R} = (R, \theta, \phi)$ is the 3-vector expressed in spherical coordinates, and $\boldsymbol{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ is the unit vector. The radius of the thin shell is $R_0 \equiv R_0(t)$ with $\dot{R}_0 < 0$ ($\dot{R}_0 > 0$) corresponding to an in falling (outgoing) shell), where dot denotes time derivative with respect to time t. The surface energy density, here assumed homogeneous for simplicity, can be written in terms of an effective mass M:

$$\varepsilon = \frac{M}{4\pi R_0(t)^2}. \tag{42}$$

Inside the shell the gravitational potential is zero, while outside the shell the gravitational metric potential behaves as a Schwarzschild metric if R_0 is fixed, if not, it should be similar to the in falling shell of a Vaidya metric [38].

To simplify our computations, and just to capture the leading order effect in G, we will work in the regime where the gravitational potential is negligible, i.e. $R_0 \gg r_g = 2GM$. The 3D Fourier transform of the stress-energy tensor for the static shell is defined as:

[5] We recall the basis tensors satisfy the completeness relation: $P^{ijkl} \equiv \sum_{\lambda} \mathrm{e}_{\lambda}^{ij}(\boldsymbol{n}) \mathrm{e}_{\lambda}^{kl}(\boldsymbol{n}) = P^{ik}P^{jl} + P^{il}P^{jk} - P^{ij}P^{kl}$ where $P^{ij} \equiv P^{ij}(\boldsymbol{n}) = \delta^{ij} - \boldsymbol{n}^i \boldsymbol{n}^j$.

$$\tilde{T}_{ij}(\boldsymbol{k}) \equiv \int d\boldsymbol{R}\, T_{ij}(\boldsymbol{R}) e^{i\boldsymbol{k}\cdot\boldsymbol{R}}. \tag{43}$$

We can evaluate such integrations using spherical coordinates and by choosing to align the $\boldsymbol{k}$ vector with the z axis (Appendix B). Specifically, inserting Eq. (41) in Eq. (43) we then find after expanding up to order $\mathcal{O}(k^4)$,

$$\tilde{T}_{11}(k) = \tilde{T}_{22}(k) = M(\frac{1}{3} - \frac{R_0^2 k^2}{30} + ...), \tag{44}$$

$$\tilde{T}_{33}(k) = M(\frac{1}{3} - \frac{R_0^2 k^2}{10} + ...). \tag{45}$$

Here we will be interested in the regime $R_0(t)k \ll 1$ where higher order terms can be neglected. Using $k = \omega_k$ this corresponds to the UV cut-off frequency, $\omega_k \ll 1/R_0(t)$. Note that although $R_0(t)$ has an explicit time dependence, here we will only consider that the in-falling shell is moving very slowly and the corresponding vacuum for the shell remains adiabatic. Violation of adiabaticity in the vacuum will lead to particle creation and the break down of our key assumptions.

This interaction Hamiltonian considered here is consistent with the perturbations around Minkowski spacetime $\eta_{\mu\nu}$ in Eq. (27). As we will see below, this will lead to $N_g \sim \mathcal{O}(h_{\mu\nu}^2) \sim \mathcal{O}(G)$. In the case of an in-falling shell, the metric is not Minkowski even for $\omega_k \ll 1/R_0$, $R_0 k \ll 1$. In the static case, the metric outside the shell is that of the Schwarzschild (for static) or Vaidya (dynamic) metric. Therefore, the correct expansion of the perturbations should be: $g_{\mu\nu} = \tilde{g}_{\mu\nu} + \tilde{h}_{\mu\nu}$, where $\tilde{g}_{\mu\nu}$ is the background metric.

However, in the linearised limit, the leading order term in the metric remains that of Minkowski one for $r \sim R_0 \gg r_g = 2GM$, so any correction due to $2GM/R$ contribution in the metric will yield higher order corrections in G, i.e. due to $\sqrt{-g}$ contribution in the interaction Hamiltonian, $\int \sqrt{-g} d^4 x G^{\mu\nu} T_{\mu\nu}$. Hence, the above mentioned corrections due to either Schwarzschild or Vaidya remain sub-leading in $N_g \sim \mathcal{O}(G^2)$.

Therefore, working at the leading order in G, we can split Eq. (40) into the radial and angular parts:

$$N_g = \frac{G}{\pi^2} \int \frac{d\omega_k}{\omega_k} |\tilde{T}(\omega_k)|^2 \int d\boldsymbol{n}\, P^{iji'j'}(\boldsymbol{n}) F_{ij}(\boldsymbol{n}) F_{i'j'}(\boldsymbol{n}), \tag{46}$$

where we have used $d\boldsymbol{k} = dk d\boldsymbol{n} = \omega_k^2 d\omega_k d\boldsymbol{n}$, and $\tilde{T}(\omega_k)$ and $F_{ij}(\boldsymbol{n})$ will be specified below. The expansion of the stress energy tensor from Eq. (45) can be treated perturbatively – computing first the contribution to order $\mathcal{O}(1)$, and then to order $\mathcal{O}(k^2)$.

From Eq. (45), at the lowest order $\mathcal{O}(1)$, we have the terms

$$\tilde{T}(\omega_k) = \frac{M}{3}, \qquad F_{ij} = \delta_{ij}, \tag{47}$$

which however leads to a vanishing contribution in Eq. (40) as we have $\sum_{i,j} \int d\boldsymbol{n}\, P^{iijj}(\boldsymbol{n}) = 0$ due to symmetry considerations. From Eq. (45), the quadratic terms, $\mathcal{O}(k^2)$, give

$$\tilde{T}(\omega_k) = \frac{M R_0^2 \omega_k^2}{30}, \tag{48}$$

which can be seen as the relativistic counterpart of a harmonic oscillator potential energy. In addition, the only nonzero angular terms are $F_{11} = 1$, $F_{22} = 1$, but $F_{33} = 3$ (the asymmetry of

F_{33} with respect to F_{11}, F_{22} originates from the choice of the alignment of the $\boldsymbol{k}$ vector with the z axis). After some straightforward but tedious algebra we eventually find

$$N_g = \frac{G}{\pi^2 c^5}\int \frac{d\omega_k}{\omega_k}|\tilde{T}(\omega_k)|^2 \int d\boldsymbol{n} P^{iji'j'}(\boldsymbol{n})F_{ij}(\boldsymbol{n})F_{i'j'}(\boldsymbol{n}). \tag{49}$$

We now introduce an UV cutoff in Fourier space, i.e. $\bar{\omega} = \frac{2\pi}{R_0}$ (matching the approximations in Eq. (45)). From Eq. (49), we then readily find

$$N_g = \frac{32 G R_0^4 M^2}{3375\pi}\int_0^{\bar{\omega}} d\omega_k \omega_k^3. \tag{50}$$

Finally, by evaluating the frequency integral, and using $\bar{\omega} = \frac{2\pi}{R}$, we obtain the occupation number at the leading order in G to be:

$$N_g = \frac{128\pi^3}{3375} G M^2, \tag{51}$$

which is very similar to what we had obtained earlier in the non-relativistic setup [18]. The numerical factors are indeed different, due to the geometry, but GM^2 factor remains the same.

Recalling our discussions in section 5, we can establish a new length scale in the infrared. Since, $l_P = \sqrt{G}$, then with the help of Eq. (51) and Eq. (58), we obtain:

$$L \leq \sqrt{N_g} l_P = \sqrt{\frac{128 \times 16 \times \pi^4}{3375}} GM \sim 3.8 r_g, \tag{52}$$

which signifies that the quantum effects, such as the quantum fluctuations of the virtual gravitons play an interesting role as the shell crosses $\sim 3.8 r_g$. This is a significant result, which matches the expectations found in the analysis of the fuzz-ball scenario where the fuzz-ball micro states played an important role even before the black hole horizon started forming [21]. Note that we have not computed the amplitude of the total probability here in Eq. (58). However, the exponent being more sensitive, we expect that the emergence of a new scale in gravity is inevitable whenever there exists a large number of states, i.e. $N_g \gg 1$. The actual numerical factor $3.8 r_g$ may change, but within an order of magnitude of order G our conclusions will remain intact. Appearance of a new scale in the infrared is a welcoming sign, in particular, it enforces us to rethink our understanding of a traditional classical black hole with an event horizon [20]. Given the future advancements in observational gravitational waves, our analysis prompts us to study various consequences of a compact object devoid of any classical horizon, see [23,24,39].

For the rest of the paper, we will consider the physical effects due to higher curvature contributions, such as $\mathcal{R}^2$ in the gravitational action Eq. (55).

5. Infrared scale for gravitons

Let us begin by considering the gravitational action which also allows higher order derivative and curvature terms.

$$S = S_{EH} + S_q\,, \tag{53}$$

$$S_{EH} = \frac{1}{16\pi G}\int d^4x \sqrt{-g} R\,, \tag{54}$$

$$S_q = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\left[\alpha \mathcal{R}^2 + \beta \mathcal{R}^3 + \cdots\right], \tag{55}$$

where $M_p^2 = 1/(16\pi G)$, α, β are dimension full constants. Since gravity is a massless theory; therefore space time diffeomorphism invariance allows higher curvature and higher derivative (more than two covariant derivatives) contributions. In general, we may expect [29]:

$$\mathcal{R}^2 = Rf(\Box_s)R + \cdots , \tag{56}$$

where $\Box_s = \Box/M_s^2$ operator will bring a new scale M_s, where $M_s \leq M_p$. For all practical purposes we can take $M_s = M_p$. We will show that despite of all these scales, there exists a unique scale in the infrared[6] as pointed out in Refs. [13,14,18,20,21].

Let us introduce a characteristic length scale L, so that one has $\partial x \sim L$ and $1/\partial x \sim 1/L$; in the same way all curvature tensors will scale as $\mathcal{R} \sim 1/L^2$, $\mathcal{R}^2 \sim 1/L^4, \cdots$. Therefore, Einstein-Hilbert action will contribute:

$$S_{EH} \sim M_p^2 L^2 . \tag{57}$$

To understand the appearance of this new scale in the infrared, there is another intuitive way to proceed [21,34] (see also [35] for a discussion motived by the AdS/CFT calculations).

Let us recall the arguments proposed in Ref. [21,22]. The fluctuations in energy for a given mass is given by; $E \sim M$, if the fluctuations exist for a time scale $T \sim L$, then the action will be given by: $S \sim ET$. In [21,22], it was argued that for a black hole with a mass M will also accompany the virtual excitations of the vacuum. Typically, these fluctuations are *exponentially suppressed*, nevertheless, the number of states available in the case of a black hole is exponentially large, as in the case of a fuzz ball. Take only Einstein-Hilbert contribution in the gravitational action, we will get $S_{EH} \sim ET \sim M_p^2 L^2$. For the black hole case, $L = T = r_g = 2GM$. Therefore, the total probability for the existence of a black hole must also take into account of the gravitational states available, given by $\mathcal{N} \approx e^{N_g}$, and the gravitational action. Collectively, we can express the total probability to be:

$$P_T \sim e^{N_g} e^{-S} \sim e^{N_g - M_p^2 L^2} \sim \mathcal{O}(1) . \tag{58}$$

In [21,22], the number of states available for a black hole was due to the fuzz-ball states. In our case, the number of quantum states available is mainly due to the graviton states, very similar to [13,14,16,18]. The total probability becomes $P_T \sim \mathcal{O}(1)$, provided

$$L \leq \frac{\sqrt{N_g}}{M_p} \Longrightarrow M_{eff} \sim \frac{M_p}{\sqrt{N_g}}, \tag{59}$$

where we have taken $L \sim M_{eff}^{-1}$. In this regard fuzz-ball [20] and the corpuscular hypotheses [13,14,16,18] gave very similar results. Both the hypotheses saturate the bound for N_g for the gravitational radius; $L = r_g = 2GM$. Although, the fuzz-ball hypothesis will go beyond these steps, and will bring a new infrared scale which can be even larger than the gravitational radius. We will discuss them in the next section.

To evaluate N_g, we will take inspiration from our recent computation in Ref. [18], where we have shown that it is indeed possible to estimate N_g by tracing out the non-relativistic matter,

[6] Note that the graviton propagator depends on the action and the background, see [29–31]. Higher derivative contributions to gravity also bring ghosts in the graviton propagator [2,31]. In order to avoid ghosts, either we restrict our action to only two derivatives or all infinite covariant derivatives, as shown in papers [29,32]. Such class of gravitational theories are known as infinite derivative theory of gravity (IDG). Infinite derivatives do not have a point support, see [33], and therefore introduces non-local interactions.

and we had shown that the minimal coupling between the matter and the gravity would suffice to show that the gravitons are in a displaced coherent state, whose occupation is given by the same as that of Bekenstein's entropy bound.

$$N_g = S_{BEK} = \frac{Area}{4G} \sim GM^2 \sim \left(\frac{M}{M_p}\right)^2 . \tag{60}$$

By substituting in Eq. (58), we obtain the infrared scale in gravity to be similar to the Schwarzschild radius, i.e. $L \sim r_g$, as we have discussed above. It is worth highlighting that the gravitational entropy is indeed holographic in nature also, see [36,37].

It is worth comparing out that the electromagnetic case has no infrared length scale. A similar analysis will suggest that the electromagnetic action has no explicit length scale dependence, unlike gravity, where the interaction strength $\sqrt{G} \sim 1/M_p$, and possess a length scale. This naturally forbids appearance of an infrared scaling in the case of a photon.

A natural question arises; could we evaluate N_g for other equations of state, or for other geometries. Our previous computation in Ref. [18] was performed in a non-relativistic setting. We wish to now consider these issues and compute N_g, where we assume that the graviton vacuum is always *stable* and obeys *adiabatic* condition, by tracing out the energy momentum tensor of the matter sector.

Let us consider terms which are proportional to $\mathcal{R}^2$ in S_q in Eq. (55).[7] Our previous arguments from section 5 have suggested that on dimensional grounds, the total gravitational action including Einstein-Hilbert term and the quadratic piece will give us:

$$S \sim M_p^2 L^2 \left[1 + \frac{1}{M_s^2 L^2}\right] . \tag{61}$$

If we demand that $S_q > S_{EH}$ in Eq. (55) for a certain length scale L, then this would mean $L < 1/M_s$. Now, if we demand that the virtual excitations of gravitons for a given mass M ought to play a significant role, then

$$P_T \sim e^{N_g} e^{-M_p^2 L^2 [1 + \frac{1}{M_s^2 L^2}]} \sim \mathcal{O}(1) . \tag{62}$$

Therefore, if the quadratic in curvature term were to dominate over Einstein-Hilbert term then the probability would become order one, provided

$$M_s = \frac{M_p}{\sqrt{N_g}} \tag{63}$$

[8]To be consistent, we would need $L < M_s^{-1}$. Therefore, we will have the following relationship for an in falling thin shell;

$$\frac{1}{L} \geq M_s = \frac{M_p}{\sqrt{N_g}} \Longrightarrow L \leq 3.8 r_g . \tag{64}$$

[7] The ghost free quadratic curvature action with analytic operators is given by: $S = 1/(16\pi G) \int d^4x \sqrt{-g}[R + \alpha(Rf_1(\Box_s)R + R_{\mu\nu} f_2(\Box_s) R^{\mu\nu} + R_{\mu\nu\lambda\sigma} f_3(\Box_s) R^{\mu\nu\lambda\sigma})]$. The whole action can be made ghost-free in Minkowski [29] and in maximally symmetric backgrounds. In Minkowski spacetime, the ghost-free condition demands that $2f_1 + f_2 + 2f_3 = 0$.

[8] This is indeed a very interesting relationship, as we had shown in a completely different context; how a new scale appears in gravity but in the context o fa non-local gravitational interaction [39].

Let us now check what would happen if we were to demand the domination of the cubic order terms in the curvature over all the other contributions for some length scale L, then

$$S \sim M_p^2 L^2 \left[1 + \frac{1}{M_s^2 L^2} + \frac{1}{M_s^4 L^4} + \cdots \right]. \tag{65}$$

Indeed, all the higher curvature terms do become important for $L \leq M_s^{-1}$, barring any fine-tuned cancellations. Let's take the cubic term first. If the cubic contribution dominates overall, then Eq. (58), (62) would suggest

$$L \sim \frac{M_p}{M_s^2} \frac{1}{\sqrt{N_g}}. \tag{66}$$

Indeed, now we have two new parameters to constrain M_s and L for a given N_g. Let us suppose, conservatively, we take $M_s \sim M_p/\sqrt{N_g}$, then we would obtain the same conclusion that the infrared scale of gravity becomes $L \leq \sqrt{N_g}/M_p$, same as that of Eq. (63), since $L \sim M_s^{-1}$. If M_s is considered to be the string scale, then the hierarchy between M_s and M_p is related to the gravitational states.

All these results point towards one very crucial fact that irrespective of any higher-order curvature and/or higher derivative corrections, there must appear a new scale of gravity in the infrared, which has a universal feature given by Eq. (59), i.e. $M_{eff} \sim M_p/\sqrt{N_g}$, where N_g denotes the number of graviton states associated with the mass M [21].

6. Conclusion

In this paper we have found two results. First, by tracing out the charged source, i.e. the electron, we have found that the photon vacuum is displaced. This is analogous to the displaced coherent state of a photon vacuum with an occupation number of photons, N_p, which scales as the fine structure constant. Since the fine structure constant remains less than one, it implies that the electron remains a quantum system for energies below the Planck scale. Furthermore, the photon number is always bounded by the Bekenstein's entropy bound. All the computations are based on the adiabatic evolution of the charged source and the photon vacuum.

The second result, we have shown that the gravitational interaction with the matter is entirely different as expected. By tracing out the matter, we found that the graviton vacuum is also displaced. Still, now the occupation number of gravitons is proportional to the Area. The current result generalises our previous result [18], where we have generalised the computation for an arbitrary energy momentum tensor and beyond Einstein-Hilbert action. Motivated by [21,22], we have found that by including the large degeneracy provided by the occupation number of the gravitons in the displaced vacuum, the infrared scale emerges. This infrared scale can be larger than the gravitational radius. In fact, in the simple toy model we have studied, i.e. an in-falling thin shell of matter, the emergence of the infrared length scales appears to be $L \leq 3.8 r_g$. We have further noticed that the appearance of this infrared scale in gravity persists even if we go beyond Einstein-Hilbert action. Apparently, such a new scale is always determined by the large occupation number of gravitons in the displaced vacuum, see Eq. (66).

Our results prompt us to investigate further open questions such as the new scale of gravity in a generic collapsing geometry, particularly in the context of cosmology [40], and in the formation of an ultra compact object. Would the appearance of a new scale alleviate cosmological Big Bang singularity or resolve black hole singularity? Would the appearance of a new scale in gravity alter the way we view traditional black holes? Would there be associated observational signatures which can be falsifiable by future gravitational wave detectors? All these questions remain outstanding, indeed they go beyond the scope of the current paper, and deserves a detailed investigation.

CRediT authorship contribution statement

Sougato Bose: Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Anupam Mazumdar:** Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing. **Marko Toroš:** Conceptualization, Formal analysis, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Thin charged shell of matter

In the case of a charged shell, we start from the current density:

$$J_i(\boldsymbol{R}) = \frac{e}{4\pi R_0^2}\delta(R-R_0)n_i(\theta,\phi). \tag{A.1}$$

We are interested in the Fourier transform

$$\tilde{J}_i(\boldsymbol{k}) = \frac{e}{4\pi R_0^2}\int_0^\infty dR \int_{-1}^{1} d\cos(\theta)\int_0^{2\pi} d\phi R^2\delta(R-R_0)n_i(\theta,\phi)e^{ikR\cos(\theta)} \tag{A.2}$$

Performing the integrations we find the following nonzero terms

$$\tilde{J}_3(k) = -\frac{ie(kR_0\cos(kR_0)-\sin(kR_0))}{k^2R_0^2} \tag{A.3}$$

where $k \equiv |\boldsymbol{k}| = \omega_k$ is the radial component of the wave vector.

Appendix B. Thin neutral shell of matter

We start from the stress-energy tensor:

$$\tilde{T}_{ij}(\boldsymbol{k}) = \varepsilon\int_0^\infty dR \int_{-1}^{1} d\cos(\theta)\int_0^{2\pi} d\phi R^2\delta(R-R_0)n_i(\theta,\phi)n_j(\theta,\phi)e^{ikR\cos(\theta)}. \tag{B.1}$$

Performing the integrations, and inserting Eq. (42), we find:

$$\tilde{T}_{11}(k) = M\frac{\sin(kR_0) - kR_0\cos(kR_0)}{k^3R_0^3}, \tag{B.2}$$

$$\tilde{T}_{22}(k) = M\frac{\sin(kR_0) - kR_0\cos(kR_0)}{k^3R_0^3} \tag{B.3}$$

$$\tilde{T}_{33}(k) = M\frac{\left(k^2R_0^2 - 2\right)\sin(kR_0) + 2kR_0\cos(kR_0)}{k^3{R_0}^3} \tag{B.4}$$

where $k \equiv |\boldsymbol{k}| = \omega_k$ is the radial component of the wave vector.

References

[1] S.N. Gupta, Phys. Rev. 96 (1954) 1683–1685.
[2] P. Van Nieuwenhuizen, On ghost-free tensor Lagrangians and linearized gravitation, Nucl. Phys. B 60 (1973) 478;
T. Biswas, T. Koivisto, A. Mazumdar, Nonlocal theories of gravity: the flat space propagator, arXiv:1302.0532 [gr-qc].
[3] G. Baym, T. Ozawa, Proc. Natl. Acad. Sci. USA 106 (2009) 3035–3040.
[4] F.J. Dyson, The World on a String? Review of the Fabric of the Cosmos: Space, Time, and the Texture of Realitiy by Brian Greene, New York Review of Books, vol. 51, Number 8, May 13 2004;
T. Rothman, S. Boughn, Found. Phys. 36 (2006) 1801–1825.
[5] A. Ashoorioon, P.S. Bhupal Dev, A. Mazumdar, Mod. Phys. Lett. A 29 (30) (2014) 1450163, https://doi.org/10.1142/S0217732314501636, arXiv:1211.4678 [hep-th].
[6] Y.J. Ng, H. van Dam, Found. Phys. 30 (2000) 795–805;
W.A. Christiansen, Y.J. Ng, H. van Dam, Phys. Rev. Lett. 96 (2006) 051301.
[7] G. Amelino-Camelia, J.R. Ellis, N.E. Mavromatos, D.V. Nanopoulos, S. Sarkar, Nature 393 (1998) 763–765;
G. Amelino-Camelia, Living Rev. Relativ. 16 (2013) 5.
[8] M. Parikh, F. Wilczek, G. Zahariade, Int. J. Mod. Phys. D 29 (14) (2020) 2042001.
[9] M. Toroš, T.W. Van De Kamp, R.J. Marshman, M.S. Kim, A. Mazumdar, S. Bose, Phys. Rev. Res. Int. 3 (2) (2021) 023178, https://doi.org/10.1103/PhysRevResearch.3.023178, arXiv:2007.15029 [gr-qc].
[10] S. Bose, A. Mazumdar, G.W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. Geraci, P. Barker, M.S. Kim, G. Milburn, Phys. Rev. Lett. 119 (24) (2017) 240401, https://doi.org/10.1103/PhysRevLett.119.240401, arXiv:1707.06050 [quant-ph].
[11] C. Marletto, V. Vedral, Phys. Rev. Lett. 119 (2017) 240402.
[12] R.J. Marshman, A. Mazumdar, S. Bose, Phys. Rev. A 101 (5) (2020) 052110, https://doi.org/10.1103/PhysRevA.101.052110, arXiv:1907.01568 [quant-ph].
[13] R. Ruffini, S. Bonazzola, Phys. Rev. 187 (1969) 1767.
[14] G. Dvali, C. Gomez, Fortschr. Phys. 61 (2013) 742–767;
G. Dvali, C. Gomez, arXiv:1212.0765 [hep-th];
G. Dvali, D. Flassig, C. Gomez, A. Pritzel, N. Wintergerst, Phys. Rev. D 88 (12) (2013) 124041, arXiv:1307.3458;
G. Dvali, C. Gomez, Eur. Phys. J. C 74 (2014) 2752, arXiv:1207.4059;
G. Dvali, C. Gomez, J. Cosmol. Astropart. Phys. 1401 (2014) 023, arXiv:1312.4795.
[15] G. Dvali, C. Gomez, R.S. Isermann, D. Lüst, S. Stieberger, Nucl. Phys. B 893 (2015).
[16] R. Casadio, A. Giugno, A. Orlandi, Phys. Rev. D 91 (12) (2015) 124069, arXiv:1504.05356 [gr-qc];
R. Casadio, A. Giugno, A. Giusti, Phys. Lett. B 763 (2016) 337, arXiv:1606.04744 [hep-th];
R. Casadio, arXiv:2103.14582 [gr-qc].
[17] G. Dvali, J. High Energy Phys. 03 (2021).
[18] S. Bose, A. Mazumdar, M. Toroš, Gravitons in a box, Phys. Rev. D 104 (6) (2021) 066019, https://doi.org/10.1103/PhysRevD.104.066019, arXiv:2104.12793 [gr-qc].
[19] A. Burinskii, The Dirac-Kerr electron, Gravit. Cosmol. 14 (2008) 109–122, https://doi.org/10.1134/S0202289308020011, arXiv:hep-th/0507109 [hep-th].
[20] S.D. Mathur, The fuzzball proposal for black holes: an elementary review, Fortschr. Phys. 53 (2005) 793–827, https://doi.org/10.1002/prop.200410203, arXiv:hep-th/0502050 [hep-th];

S.D. Mathur, The information paradox: a pedagogical introduction, Class. Quantum Gravity 26 (2009) 224001, https://doi.org/10.1088/0264-9381/26/22/224001, arXiv:0909.1038 [hep-th];
O. Lunin, S.D. Mathur, AdS / CFT duality and the black hole information paradox, Nucl. Phys. B 623 (2002) 342–394, https://doi.org/10.1016/S0550-3213(01)00620-4, arXiv:hep-th/0109154 [hep-th].
[21] S.D. Mathur, The VECRO hypothesis, https://doi.org/10.1142/S0218271820300098, arXiv:2001.11057 [hep-th].
[22] S.D. Mathur, Tunneling into fuzzball states, Gen. Relativ. Gravit. 42 (2010) 113–118, https://doi.org/10.1007/s10714-009-0837-3, arXiv:0805.3716 [hep-th].
[23] B. Guo, S. Hampton, S.D. Mathur, Can we observe fuzzballs or firewalls?, J. High Energy Phys. 07 (2018) 162, https://doi.org/10.1007/JHEP07(2018)162, arXiv:1711.01617 [hep-th].
[24] E. Maggio, L. Buoninfante, A. Mazumdar, P. Pani, How does a dark compact object ringdown?, Phys. Rev. D 102 (6) (2020) 064053, https://doi.org/10.1103/PhysRevD.102.064053, arXiv:2006.14628 [gr-qc].
[25] C.C. Gerry, P.L. Knight, Introductory Quantum Optics, Cambridge University Press, 2005.
[26] J.D. Bekenstein, Phys. Rev. D 23 (1981) 287.
[27] W. Mück, Eur. Phys. J. C 73 (2013) 2679;
W. Mück, Eur. Phys. J. C 75 (2015) 585.
[28] M. Gockeler, R. Horsley, V. Linke, P.E.L. Rakow, G. Schierholz, H. Stuben, Is there a Landau pole problem in QED?, Phys. Rev. Lett. 80 (1998) 4119–4122, https://doi.org/10.1103/PhysRevLett.80.4119, arXiv:hep-th/9712244 [hep-th].
[29] T. Biswas, E. Gerwick, T. Koivisto, A. Mazumdar, Towards singularity and ghost free theories of gravity, Phys. Rev. Lett. 108 (2012) 031101, https://doi.org/10.1103/PhysRevLett.108.031101, arXiv:1110.5249 [gr-qc];
T. Biswas, A. Mazumdar, W. Siegel, Bouncing universes in string-inspired gravity, J. Cosmol. Astropart. Phys. 03 (2006) 009, https://doi.org/10.1088/1475-7516/2006/03/009, arXiv:hep-th/0508194 [hep-th].
[30] E. D'Hoker, D.Z. Freedman, S.D. Mathur, A. Matusis, L. Rastelli, Nucl. Phys. B 562 (1999) 330–352, https://doi.org/10.1016/S0550-3213(99)00524-6, arXiv:hep-th/9902042 [hep-th].
[31] T. Biswas, T. Koivisto, A. Mazumdar, Nonlocal theories of gravity: the flat space propagator, arXiv:1302.0532 [gr-qc].
[32] E.T. Tomboulis, Superrenormalizable gauge and gravitational theories, arXiv:hep-th/9702146;
L. Modesto, Superrenormalizable quantum gravity?, Phys. Rev. D 86 (2012) 044005.
[33] L. Buoninfante, A.S. Koshelev, G. Lambiase, A. Mazumdar, Classical properties of non-local, ghost- and singularity-free gravity, J. Cosmol. Astropart. Phys. 09 (2018) 034, https://doi.org/10.1088/1475-7516/2018/09/034, arXiv:1802.00399 [gr-qc].
[34] S.D. Mathur, arXiv:0805.3716 [hep-th];
S.D. Mathur, Int. J. Mod. Phys. D 18 (2009) 2215, arXiv:0905.4483 [hep-th].
[35] Kathryn M. Zurek, On vacuum fluctuations in quantum gravity and interferometer arm fluctuations, arXiv:2012.05870 [hep-th].
[36] G. t Hooft, Conf. Proc. C 930308 (1993) 284–296.
[37] L. Susskind, J. Math. Phys. 36 (1995) 6377–6396.
[38] E. Poisson, A relativist's toolkit: the mathematics of black-hole mechanics, https://doi.org/10.1017/CBO9780511606601.
[39] A.S. Koshelev, A. Mazumdar, Do massive compact objects without event horizon exist in infinite derivative gravity?, Phys. Rev. D 96 (8) (2017) 084069, https://doi.org/10.1103/PhysRevD.96.084069, arXiv:1707.00273 [gr-qc];
L. Buoninfante, A. Mazumdar, Nonlocal star as a blackhole mimicker, Phys. Rev. D 100 (2) (2019) 024031, https://doi.org/10.1103/PhysRevD.100.024031, arXiv:1903.01542 [gr-qc].
[40] M. Cadoni, R. Casadio, A. Giusti, W. Mück, M. Tuveri, Phys. Lett. B 776 (2018) 242, https://doi.org/10.1016/j.physletb.2017.11.058;
M. Cadoni, R. Casadio, A. Giusti, M. Tuveri, Phys. Rev. D 97 (2018) 044047, https://doi.org/10.1103/PhysRevD.97.044047;
M. Cadoni, M. Tuveri, Phys. Rev. D 100 (2019) 024029, https://doi.org/10.1103/PhysRevD.100.024029.