

[Fairen, M.](#) and [Valeri, D.](#) (2023) Double multiplicative Poisson vertex algebras. *International Mathematics Research Notices*, 2023(17), pp. 14991-15072. (doi: [10.1093/imrn/rnac245](https://doi.org/10.1093/imrn/rnac245))

This is the author version of the work. There may be differences between this version and the published version. You are advised to consult the published version if you wish to cite from it:

<https://doi.org/10.1093/imrn/rnac245>

<https://eprints.gla.ac.uk/279196/>

Deposited on 15 September 2022

Enlighten – Research publications by members of the University of Glasgow
<http://eprints.gla.ac.uk>

DOUBLE MULTIPLICATIVE POISSON VERTEX ALGEBRAS

MAXIME FAIRON AND DANIELE VALERI

ABSTRACT. We develop the theory of double multiplicative Poisson vertex algebras. These structures, defined at the level of associative algebras, are shown to be such that they induce a classical structure of multiplicative Poisson vertex algebra on the corresponding representation spaces. Moreover, we prove that they are in one-to-one correspondence with local lattice double Poisson algebras, a new important class among Van den Bergh's double Poisson algebras. We derive several classification results, and we exhibit their relation to non-abelian integrable differential-difference equations. A rigorous definition of double multiplicative Poisson vertex algebras in the non-local and rational cases is also provided.

CONTENTS

1. Introduction	1
2. Preliminaries	3
3. Double multiplicative Poisson vertex algebras	7
4. Double multiplicative Poisson vertex algebra structure on an algebra of (non-commutative) difference functions	12
5. Relation to representation spaces	23
6. Connection to Integrable Systems	25
7. Non-local and rational double multiplicative Poisson vertex algebras	35
References	46

1. INTRODUCTION

Given a unital associative algebra \mathcal{V} , Van den Bergh [VdB1] introduced the structure of a double bracket on \mathcal{V} as a map

$$\{\{-, -\} : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}, \quad (a, b) \mapsto \{\{a, b\}\},$$

which is linear in both arguments and which enjoys properties of derivation and skewsymmetry, see Subsection 3.1 for the definition. The importance of double brackets can then be realised through representation theory as follows. Denoting by \mathbb{k} the base field of \mathcal{V} , we can form the representation space $\text{Rep}(\mathcal{V}, N)$ parametrised by representations of \mathcal{V} over \mathbb{k}^N , $N \geq 1$. The coordinate ring of this affine scheme is generated by the functions a_{ij} , where $(a_{ij})_{1 \leq i, j \leq N}$ is the matrix-valued function on $\text{Rep}(\mathcal{V}, N)$ corresponding to $a \in \mathcal{V}$. Then, it was observed by Van den Bergh that the operation $\{-, -\}$ on $\text{Rep}(\mathcal{V}, N)$ satisfying

$$\{a_{ij}, b_{kl}\} = \{\{a, b\}\}'_{kj} \{\{a, b\}\}''_{il}, \quad a, b \in \mathcal{V}, \quad 1 \leq i, j, k, l \leq N, \quad (1.1)$$

(here, we use a strong version of Sweedler's notation : $d' \otimes d'' := d \in \mathcal{V} \otimes \mathcal{V}$) defines a unique skewsymmetric biderivation over $\text{Rep}(\mathcal{V}, N)$. This gives an example of application of the Kontsevich-Rosenberg principle [Ko, KR], which states that the non-commutative version P_{nc} of a property P defined over commutative algebras should give back P when we go from an associative algebra \mathcal{V} to (the coordinate ring of) its representation spaces $\text{Rep}(\mathcal{V}, N)$. Furthermore, it was possible to generalise this construction using (1.1) to introduce non-commutative versions of Lie algebras [S, ORS, DSKV], (quasi-)Poisson algebras [VdB1], Lie-Rinehart algebras [VdB2], or Poisson vertex algebras [DSKV] and Courant-Dorfman algebras [FH]. This paper is devoted to pursue the Kontsevich-Rosenberg principle even further using (1.1) by bringing to

light the non-commutative version of multiplicative Poisson vertex algebras, which have recently been introduced by De Sole, Kac, Valeri and Wakimoto [DSKVW1, DSKVW2].

To understand the objects at stake, recall that a Poisson algebra is a commutative algebra V endowed with a Poisson bracket $\{-, -\}$. In other words, V is equipped with a Lie bracket that is compatible with the commutative product on V , see Definition 2.1. Let us assume that V admits an infinite order automorphism $S \in \text{Aut}(V)$ such that it commutes with the Poisson bracket, i.e.

$$S \circ \{-, -\} = \{-, -\} \circ (S \times S), \quad (1.2)$$

and such that, for $a, b \in V$, we have $\{S^n(a), b\} = 0$, for all but finitely many $n \in \mathbb{Z}$. In this case, we call V a local lattice Poisson algebra. Then, it was observed in [DSKVW1] that the Poisson bracket on V can be equivalently understood in terms of a bilinear operation

$$\{-\lambda-\} : V \times V \rightarrow V[\lambda^{\pm 1}], \quad (1.3)$$

(here λ should be seen as a formal parameter) defined for any $a, b \in V$ by

$$\{a_\lambda b\} = \sum_{n \in \mathbb{Z}} \lambda^n \{S^n(a), b\}. \quad (1.4)$$

Note that only finitely many terms are non-zero in the right-hand side of (1.4). We refer to Proposition 2.3 for a precise statement. Due to the defining properties of the Poisson bracket and the compatibility with S given by (1.2), the map (1.3) inherits several useful properties: it is skewsymmetric (2.2a), sesquilinear (2.1a), and it satisfies Leibniz rules (2.1b)–(2.1c) as well as an analogue of Jacobi identity (2.2b). In full generalities, an operation on V of the form (1.3) satisfying these assumptions is called a multiplicative Poisson vertex algebra, see Definition 2.2. Hence, this proves a correspondence, that can be depicted as follows:

$$\{\text{local lattice Poisson algebras}\} \xleftrightarrow{1-1} \{\text{multiplicative Poisson vertex algebras}\} \quad (1.5)$$

As mentioned earlier, there is a non-commutative version of Poisson algebras due to Van den Bergh [VdB1]. Our definition of a non-commutative version of multiplicative Poisson vertex algebras is motivated by the correspondence (1.5). Namely, if we replace Poisson algebras by Van den Bergh’s double Poisson algebras in (1.5), we end up with the notion of a double multiplicative Poisson vertex algebra. The main ingredient used in the definition is then a double multiplicative λ -bracket, which is a suitable “double” generalization of (1.4). This is a bilinear map

$$\{\{-\lambda-\}\} : \mathcal{V} \times \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})[\lambda^{\pm 1}],$$

defined over an associative algebra \mathcal{V} endowed with an automorphism S , see Definition 3.8. Hence, in Proposition 3.14, we get the correspondence

$$\{\text{local lattice double Poisson algebras}\} \xleftrightarrow{1-1} \{\text{double multiplicative Poisson vertex algebras}\}$$

which is the “double” analogue of (1.5). This analogy is precisely stated in Corollary 5.5, where we show that the non-commutative correspondence (Proposition 3.14) implies the commutative correspondence (Proposition 2.3) when going to representation spaces, i.e. when going from \mathcal{V} to $V = \mathbb{k}[\text{Rep}(\mathcal{V}, N)]$. That result crucially depends on Van den Bergh’s work [VdB1], and the fact that a double multiplicative Poisson vertex algebra induces on representation spaces a structure of multiplicative Poisson vertex algebra through a mapping of the form (1.1).

The relation that we have just outlined to the commutative theory developed in [DSKVW1, DSKVW2] is crucial for applications. Indeed, it is shown in [DSKVW1, DSKVW2] how multiplicative Poisson vertex algebras are useful to understand the structure of differential-difference equations. In our case, the non-commutative version of this theory allows us to understand the structure of non-abelian (i.e. matrix-valued) differential-difference equations. We are able to construct several families of integrable hierarchies of differential-difference equations in that way, see e.g. §6.4.2. This important application to integrable systems motivates us in the same way to investigate the double multiplicative Poisson vertex algebra structures on algebras

of non-commutative difference functions extending the algebra of non-commutative difference polynomials in $\ell \geq 1$ variables $u_i := u_{i,0}$

$$\mathcal{R}_\ell = \mathbb{k}\langle u_{i,n} \mid 1 \leq i \leq \ell, n \in \mathbb{Z} \rangle, \quad S(u_{i,n}) = u_{i,n+1}.$$

We perform a classification of double multiplicative Poisson vertex algebra structures on \mathcal{R}_1 and \mathcal{R}_2 , see Proposition 4.8 and Theorem 4.14 respectively.

Layout of the paper. In Section 2, we review the correspondence between Poisson algebras and multiplicative Poisson vertex algebras. We also introduce some operations induced on tensor products of an algebra. In Section 3, we state the key definition of a double multiplicative Poisson vertex algebra, before deriving several properties and examples. Next, we provide classification results for double multiplicative Poisson vertex algebras as part of Section 4. Then, we explain in Section 5 how a double multiplicative Poisson vertex algebra structure on an algebra \mathcal{V} induces a multiplicative Poisson vertex algebra structure on the associated (commutative) algebra $\mathcal{V}_N = \mathbb{k}[\text{Rep}(\mathcal{V}, N)]$. In Section 6, we apply our theory to the study of differential-difference equations. Finally, in Section 7 we outline how to modify double multiplicative Poisson vertex algebras in the non-local or rational cases, and we provide several examples.

Relation to the work of Casati-Wang. While we were working on this project, we became aware that a parallel investigation on double multiplicative Poisson vertex algebras was carried out independently by Casati and Wang [CW2]. For the reader's convenience, let us outline the main differences between these two works. Firstly, Casati and Wang [CW2] introduced double multiplicative Poisson vertex algebras on the space of non-commutative Laurent polynomials with an infinite order automorphism, while we work in the more general setup of algebras of non-commutative difference functions and we provide classification results in Section 4. Their motivation stems from the integrability of non-abelian difference equations, which we also consider in Section 6, though we do not study the several non-local examples gathered in [CW2, Sect. 6]. Secondly, they compare the formalism of double multiplicative λ -brackets to the (difference version of the) θ -formalism of Olver and Sokolov [OS]. In particular, their comparison uses a graded version of double multiplicative λ -brackets, which we do not consider. In the present work, we exclusively use the formalism of double multiplicative Poisson vertex algebras for computations, and we present a deeper study of their algebraic structure. For example, we give a correspondence with lattice double Poisson algebras in §3.3, and we explain in Section 5 that double multiplicative Poisson vertex algebras induce *usual* multiplicative Poisson vertex algebra structures (cf. [DSKVW1, DSKVW2]) on their representation spaces, in agreement with the Kontsevich-Rosenberg principle.

Acknowledgments. We wish to thank Sylvain Carpentier for useful and stimulating discussions. We are also grateful to Matteo Casati for sharing with us a preliminary version of the work [CW2] and for enlightening discussions on this topic. M.F. is supported by a Rankin-Sneddon Research Fellowship of the University of Glasgow. D.V. acknowledges the financial support of the project MMNLP (Mathematical Methods in Non Linear Physics) of the INFN.

2. PRELIMINARIES

Throughout the paper \mathbb{k} denotes a field of characteristic zero (assumed to be algebraically closed for computations), and unadorned tensor products are taken over \mathbb{k} .

2.1. Commutative Poisson structures. In this subsection, we follow [DSKVW1]. All algebras are unital commutative algebras over \mathbb{k} .

Definition 2.1. A *Poisson algebra* is an algebra V endowed with a Poisson bracket, i.e. a linear map

$$\{-, -\} : V \otimes V \rightarrow V, \quad a \otimes b \mapsto \{a, b\},$$

which is skewsymmetric, i.e. $\{a, b\} = -\{b, a\}$, satisfies the Left and right Leibniz rules

$$\{a, bc\} = \{a, b\}c + b\{a, c\}, \quad \{ab, c\} = \{a, c\}b + a\{b, c\},$$

and the Jacobi identity

$$\{a, \{b, c\}\} - \{b, \{a, c\}\} - \{\{a, b\}, c\} = 0,$$

for all $a, b, c \in V$.

A *lattice Poisson algebra* is a Poisson algebra V with an infinite order automorphism $S \in \text{Aut}(V)$, namely for all $a, b \in V$,

$$S(ab) = S(a)S(b) \quad \text{and} \quad S(\{a, b\}) = \{S(a), S(b)\}.$$

It is called local if, for every $a, b \in V$, $\{S^n(a), b\} = 0$ for all but finitely many $n \in \mathbb{Z}$.

Definition 2.2. Let V be an algebra endowed with an automorphism $S \in \text{Aut}(V)$.

A *multiplicative λ -bracket* on V is a linear map

$$\{-_\lambda -\} : V \otimes V \rightarrow V[\lambda^{\pm 1}], \quad a \otimes b \mapsto \{a_\lambda b\},$$

such that for any $a, b, c \in V$,

$$\{S(a)_\lambda b\} = \lambda^{-1} \{a_\lambda b\}, \quad \{a_\lambda S(b)\} = \lambda S(\{a_\lambda b\}), \quad (\text{sesquilinearity}) \quad (2.1a)$$

$$\{a_\lambda bc\} = \{a_\lambda b\}c + b\{a_\lambda c\}, \quad (\text{left Leibniz rule}) \quad (2.1b)$$

$$\{ab_\lambda c\} = \{a_{\lambda x} c\} \left(\left|_{x=S} b \right. \right) + \left(\left|_{x=S} a \right. \right) \{b_{\lambda x} c\}. \quad (\text{right Leibniz rule}) \quad (2.1c)$$

We say that V is a *multiplicative Poisson vertex algebra* if it admits a multiplicative λ -bracket $\{-_\lambda -\}$ satisfying

$$\{a_\lambda b\} = - \left|_{x=S} \{b_{\lambda^{-1}x^{-1}} a\}, \quad (\text{skewsymmetry}) \quad (2.2a)$$

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} - \{\{a_\lambda b\}_{\lambda\mu} c\} = 0. \quad (\text{Jacobi identity}) \quad (2.2b)$$

In the above formulas, given an element $a(\lambda) = \sum_{k \in \mathbb{Z}} a_k \lambda^k \in V[\lambda^{\pm 1}]$, we use the notation

$$a(\lambda x) \left(\left|_{x=S} b \right. \right) = \sum_{k \in \mathbb{Z}} a_k S^k(b) \lambda^k.$$

Furthermore, let us set $\text{mRes}_\lambda a(\lambda) = a_0$. The next result can be found in [DSKVW1, §3.1].

Proposition 2.3. *If V is a multiplicative Poisson vertex algebra with multiplicative λ -bracket $\{-_\lambda -\}$ and automorphism $S \in \text{Aut}(V)$, then V is a local lattice Poisson algebra with the Poisson bracket*

$$\{a, b\} = \text{mRes}_\lambda \{a_\lambda b\}, \quad a, b \in V. \quad (2.3)$$

Conversely, if V is a local lattice Poisson algebra with Poisson bracket $\{-, -\}$ and automorphism $S \in \text{Aut}(V)$, then we can endow it with a structure of multiplicative Poisson vertex algebra with the multiplicative λ -bracket

$$\{a_\lambda b\} := \sum_{n \in \mathbb{Z}} \lambda^n \{S^n(a), b\}, \quad a, b \in V. \quad (2.4)$$

Remark 2.4. If V is simply a vector space with an invertible endomorphism S , we can define the notion of a *local lattice Lie algebra* from Definition 2.1 by forgetting the derivation rules. Similarly, a *multiplicative Lie conformal algebra* is obtained from Definition 2.2 by omitting the Leibniz rules (2.1b)–(2.1c). Then, Proposition 2.3 can be weakened to an equivalence between these two structures, and this result originally appeared in [GKK].

2.2. Operations on an algebra. Let \mathcal{V} be a unital associative algebra over a field \mathbb{k} of characteristic 0.

2.2.1. *Basic operations.* We introduce several notations following [VdB1, DSKV]. Given $n \geq 2$, we can form the tensor product $\mathcal{V}^{\otimes n}$, which we see as an associative algebra for

$$(a_1 \otimes \dots \otimes a_n)(b_1 \otimes \dots \otimes b_n) = a_1 b_1 \otimes \dots \otimes a_n b_n.$$

When $n = 2$, we use a strong version of Sweedler's notation

$$A = \sum_l A'_l \otimes A''_l =: A' \otimes A'' \in \mathcal{V} \otimes \mathcal{V}$$

to denote elements. The permutation endomorphism $(-)^{\sigma}$ on $\mathcal{V} \otimes \mathcal{V}$ is given by

$$(a \otimes b)^{\sigma} = b \otimes a. \quad (2.5)$$

In full generalities for $n \geq 2$, we introduce the permutation

$$\mathcal{V}^{\otimes n} \rightarrow \mathcal{V}^{\otimes n} : a_1 \otimes \dots \otimes a_n \mapsto (a_1 \otimes \dots \otimes a_n)^{\sigma} := a_n \otimes a_1 \otimes \dots \otimes a_{n-1}. \quad (2.6)$$

We introduce the outer and inner bimodule structures on $\mathcal{V} \otimes \mathcal{V}$ by

$$a A b = a A' \otimes A'' b, \quad a * A * b = A' b \otimes a A'', \quad a, b \in \mathcal{V}, A \in \mathcal{V} \otimes \mathcal{V}. \quad (2.7)$$

We define left and right \mathcal{V} -module structures on $\mathcal{V}^{\otimes n}$ as follows. For $0 \leq i \leq n-1$,

$$\begin{aligned} b *_i (a_1 \otimes \dots \otimes a_n) &= a_1 \otimes \dots \otimes a_i \otimes b a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_n, \\ (a_1 \otimes \dots \otimes a_n) *_i b &= a_1 \otimes \dots \otimes a_{n-i-1} \otimes a_{n-i} b \otimes a_{n-i+1} \otimes \dots \otimes a_n. \end{aligned}$$

For $i = 0$, these are just the multiplication on the left of the left-most component, and on the right of the right-most component. In that case, we omit to write $*_0$, so that $bA = b *_0 A$ and $Ab = A *_0 b$ for any $A \in \mathcal{V}^{\otimes n}$. We set $*_{i+n} = *_i$ to define the operation for any $i \in \mathbb{Z}$.

Next, we introduce tensor product rules as maps $\mathcal{V} \otimes \mathcal{V}^{\otimes n} \rightarrow \mathcal{V}^{\otimes(n+1)}$. For $0 \leq i \leq n-1$,

$$\begin{aligned} b \otimes_i (a_1 \otimes \dots \otimes a_n) &= a_1 \otimes \dots \otimes a_i \otimes b \otimes a_{i+1} \otimes \dots \otimes a_n, \\ (a_1 \otimes \dots \otimes a_n) \otimes_i b &= a_1 \otimes \dots \otimes a_{n-i} \otimes b \otimes a_{n-i+1} \otimes \dots \otimes a_n, \end{aligned}$$

We omit the subscript for $i = 0$ from now on.

Finally, there is an associative product \bullet on $\mathcal{V} \otimes \mathcal{V}$ defined by

$$A \bullet B = A' B' \otimes B'' A''. \quad (2.8)$$

The permutation σ defined in (2.5) is an antihomomorphism for this product:

$$(A \bullet B)^{\sigma} = B^{\sigma} \bullet A^{\sigma}. \quad (2.9)$$

From (2.7) we get that the inner and outer bimodule structures of $\mathcal{V} \otimes \mathcal{V}$ are related to the associative product in (2.8) as follows

$$A' B A'' = A \bullet B = B'' * A * B'.$$

We can define three possible left and right module structures for $(\mathcal{V}^{\otimes 2}, \bullet)$ on $\mathcal{V}^{\otimes 3}$, denoted by \bullet_i , $i = 1, 2, 3$, as follows

$$\begin{aligned} A \bullet_1 (x \otimes y \otimes z) &= x \otimes A' y \otimes z A'', & (x \otimes y \otimes z) \bullet_1 A &= x \otimes y A' \otimes A'' z, \\ A \bullet_2 (x \otimes y \otimes z) &= A' x \otimes y \otimes z A'', & (x \otimes y \otimes z) \bullet_2 A &= x A' \otimes y \otimes A'' z, \\ A \bullet_3 (x \otimes y \otimes z) &= A' x \otimes y A'' \otimes z, & (x \otimes y \otimes z) \bullet_3 A &= x A' \otimes A'' y \otimes z, \end{aligned} \quad (2.10)$$

The following result appeared in [DSKV].

Lemma 2.5. (a) *The \bullet_i left (and right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ are indeed actions, i.e. they are associative with respect to the \bullet -product of $V^{\otimes 2}$:*

$$A \bullet_i (B \bullet_i X) = (A \bullet B) \bullet_i X \quad \text{and} \quad (X \bullet_i A) \bullet_i B = X \bullet_i (A \bullet B),$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.

(b) The left \bullet_i and the right \bullet_j actions commute for every $i, j = 1, 2, 3$ such that $|i - j| \neq 2$ ¹:

$$A \bullet_i (X \bullet_j B) = (A \bullet_i X) \bullet_j B$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$.

(c) The \bullet_1 and \bullet_3 left (resp. right) actions of $V^{\otimes 2}$ on $V^{\otimes 3}$ commute:

$$A \bullet_1 (B \bullet_3 X) = B \bullet_3 (A \bullet_1 X) \text{ and } (X \bullet_1 A) \bullet_3 B = (X \bullet_3 B) \bullet_1 A,$$

for every $A, B \in V^{\otimes 2}$ and $X \in V^{\otimes 3}$. (In general, the \bullet_i and \bullet_j left (resp. right) actions do NOT commute if $|i - j| = 1$.)

2.2.2. Extending derivations and homomorphisms. Consider a derivation $\partial \in \text{Der}(\mathcal{V}, M)$ where M is a \mathcal{V} -bimodule. We can extend ∂ to $\mathcal{V}^{\otimes m}$ by acting only on one copy of \mathcal{V} as

$$\partial_{(i)} : \mathcal{V}^{\otimes m} \rightarrow \mathcal{V}^{\otimes(i-1)} \otimes M \otimes \mathcal{V}^{\otimes(m-i)}, \quad \partial_{(i)}(a_1 \otimes \dots \otimes a_m) = a_1 \otimes \dots \otimes \partial(a_i) \otimes \dots \otimes a_m, \quad (2.11)$$

for any $1 \leq i \leq m$. In particular, we denote the induced derivations $\partial_{(1)}, \partial_{(m)}$ on the leftmost and rightmost factors by ∂_L, ∂_R respectively, i.e.

$$\begin{aligned} \partial_L : \mathcal{V}^{\otimes m} &\rightarrow M \otimes \mathcal{V}^{\otimes(m-1)}, & \partial_R : \mathcal{V}^{\otimes n} &\rightarrow \mathcal{V}^{\otimes(m-1)} \otimes M, \\ \partial_L(a_1 \otimes \dots \otimes a_m) &= \partial(a_1) \otimes \dots \otimes a_m, & \partial_R(a_1 \otimes \dots \otimes a_m) &= a_1 \otimes \dots \otimes \partial(a_m). \end{aligned} \quad (2.12)$$

We also extend ∂ to $\mathcal{V}^{\otimes m}$ by

$$\begin{aligned} \partial &:= \sum_{i=1}^m \partial_{(i)} : \mathcal{V}^{\otimes m} \longrightarrow \oplus_{i=1}^m (\mathcal{V}^{\otimes(i-1)} \otimes M \otimes \mathcal{V}^{\otimes(m-i)}), \\ \partial(a_1 \otimes \dots \otimes a_m) &= \sum_{i=1}^n a_1 \otimes \dots \otimes a_{i-1} \otimes \partial(a_i) \otimes a_{i+1} \otimes \dots \otimes a_m. \end{aligned} \quad (2.13)$$

When $M = \mathcal{V}^{\otimes n}$, we call ∂ and n -fold derivation. For a 2-fold derivation $\partial \in \text{Der}(\mathcal{V}, \mathcal{V}^{\otimes 2})$ which satisfies by definition

$$\partial(ab) = a\partial(b)' \otimes \partial(b)'' + \partial(a)' \otimes \partial(a)''b,$$

we get for example that

$$\begin{aligned} \partial(a \otimes b) &= \partial_L(a \otimes b) + \partial_R(a \otimes b), \quad \text{where} \\ \partial_L(a \otimes b) &= \partial(a)' \otimes \partial(a)'' \otimes b, \quad \partial_R(a \otimes b) = a \otimes \partial(b)' \otimes \partial(b)'' . \end{aligned} \quad (2.14)$$

We will use the natural bimodule structure on $\text{Der}(\mathcal{V}, \mathcal{V}^{\otimes 2})$ induced by the inner bimodule structure of $\mathcal{V} \otimes \mathcal{V}$:

$$(a\partial b)(f) = a * \partial(f) * b = \partial(f)'b \otimes a\partial(f)'', \quad a, b, f \in \mathcal{V}, \partial \in \text{Der}(\mathcal{V}, \mathcal{V}^{\otimes 2}).$$

Given a collection of 2-fold derivations $\partial_1, \dots, \partial_\ell$ in $\text{Der}(\mathcal{V}, \mathcal{V}^{\otimes 2})$, we say that they are *linearly independent* if the identity $(a_i, b_i \in \mathcal{V})$

$$\sum_{i=1}^{\ell} a_i \partial_i b_i = 0, \quad (2.15)$$

implies $a_i = 0$ or $b_i = 0$, for every $i = 1, \dots, \ell$. If ℓ is infinite, we require that this condition is satisfied for any finite subset of $\{\partial_i\}_{i=1}^{\ell}$.

Assume that $S \in \text{Hom}(\mathcal{V})$ is an algebra homomorphism. Then we can extend S to $\mathcal{V}^{\otimes m}$ by

$$S(a_1 \otimes \dots \otimes a_m) = S(a_1) \otimes \dots \otimes S(a_m). \quad (2.16)$$

In particular, if $S \in \text{Aut}(\mathcal{V})$, this defines an automorphism of $\text{Aut}(\mathcal{V}^{\otimes m})$. As in the case of derivations, we can adapt the construction to act on one copy of \mathcal{V} only, or to consider an arbitrary algebra homomorphism $\mathcal{V}_1 \rightarrow \mathcal{V}_2$.

¹There is a misprint in Lemma 2.5(b) in [DSKV] and the further assumption $|i - j| \neq 2$ is omitted.

2.2.3. Representation algebra. Let $N \in \mathbb{N}^\times$. We define \mathcal{V}_N as the commutative algebra generated by symbols a_{ij} for $a \in \mathcal{V}$ and $1 \leq i, j \leq N$, which are subject to the relations

$$1_{ij} = \delta_{ij}, \quad (ab)_{ij} = \sum_{1 \leq k \leq N} a_{ik} b_{kj}, \quad (\alpha a + \beta b)_{ij} = \alpha a_{ij} + \beta b_{ij},$$

for any $a, b \in \mathcal{V}$, $\alpha, \beta \in \mathbb{k}$, $1 \leq i, j \leq N$. We call \mathcal{V}_N the N -th representation algebra of \mathcal{V} . Clearly, \mathcal{V}_N is finitely generated if \mathcal{V} has this property. Recall that \mathcal{V}_N is the coordinate ring of the representation scheme $\text{Rep}(\mathcal{V}, N)$ parametrised by representations of \mathcal{V} on \mathbb{k}^N . If $\partial \in \text{Der}(\mathcal{V})$, it induces a derivation of \mathcal{V}_N from its definition on generators as $\partial(a_{ij}) = (\partial(a))_{ij}$. If $S \in \text{Aut}(\mathcal{V})$, it induces an automorphism of \mathcal{V}_N as $S(a_{ij}) = (S(a))_{ij}$.

3. DOUBLE MULTIPLICATIVE POISSON VERTEX ALGEBRAS

3.1. Review on double Poisson algebras. In this subsection we let \mathcal{V} be a unital associative algebra over \mathbb{k} . We review the notion of double bracket and double Poisson algebra introduced in [VdB1]. Example 3.4 is taken from [P], while Example 3.5 is a special case of [VdB1, §6.3] for a one-loop quiver.

Definition 3.1. A *double bracket* (or *2-fold bracket*) on \mathcal{V} is a linear map

$$\{\!\!\{-, -\}\!\!\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}, \quad a \otimes b \mapsto \{\!\!\{a, b\}\!\!\},$$

such that for all $a, b, c \in \mathcal{V}$

$$\{\!\!\{a, b\}\!\!\} = -\{\!\!\{b, a\}\!\!\}^\sigma, \quad (\text{cyclic skewsymmetry}) \quad (3.1a)$$

$$\{\!\!\{a, bc\}\!\!\} = \{\!\!\{a, b\}\!\!\} c + b \{\!\!\{a, c\}\!\!\}, \quad (\text{left Leibniz rule}) \quad (3.1b)$$

$$\{\!\!\{ab, c\}\!\!\} = \{\!\!\{a, c\}\!\!\} *_1 b + a *_1 \{\!\!\{b, c\}\!\!\}. \quad (\text{right Leibniz rule}) \quad (3.1c)$$

Given a double bracket, we introduce the maps

$$\begin{aligned} \{\!\!\{a, b' \otimes b''\}\!\!\}_L &= \{\!\!\{a, b'\}\!\!\} \otimes b'', & \{\!\!\{a, b' \otimes b''\}\!\!\}_R &= b' \otimes \{\!\!\{a, b''\}\!\!\}, \\ \{\!\!\{a' \otimes a'', b\}\!\!\}_L &= \{\!\!\{a', b\}\!\!\} \otimes_1 a'', & \{\!\!\{a' \otimes a'', b\}\!\!\}_R &= a' \otimes_1 \{\!\!\{a'', b\}\!\!\}. \end{aligned}$$

Definition 3.2. A *double Poisson algebra* is an algebra \mathcal{V} endowed with a double bracket such that for all $a, b, c \in \mathcal{V}$

$$\{\!\!\{a, \{\!\!\{b, c\}\!\!\}\!\!\}_L - \{\!\!\{b, \{\!\!\{a, c\}\!\!\}\!\!\}_R - \{\!\!\{\{\!\!\{a, b\}\!\!\}, c\}\!\!\}_L = 0. \quad (\text{Jacobi identity}) \quad (3.2)$$

In that case, we say that $\{\!\!\{-, -\}\!\!\}$ is a *double Poisson bracket*.

Remark 3.3. In Definition 3.2, we chose the condition (3.2) which is given in [DSKV] and is equivalent to the original condition of Van den Bergh [VdB1]:

$$\{\!\!\{a, \{\!\!\{b, c\}\!\!\}\!\!\}_L + (\{\!\!\{b, \{\!\!\{c, a\}\!\!\}\!\!\}_L)^\sigma + (\{\!\!\{c, \{\!\!\{a, b\}\!\!\}\!\!\}_L)^\sigma = 0.$$

Example 3.4. Let $\mathcal{V} = \mathbb{k}[u]$. It is shown in [P, VdB1] that a double bracket $\{\!\!\{-, -\}\!\!\}$ on \mathcal{V} is a double Poisson bracket if and only if it satisfies

$$\{\!\!\{u, u\}\!\!\} = \alpha(u \otimes 1 - 1 \otimes u) + \beta(u^2 \otimes 1 - 1 \otimes u^2) + \gamma(u^2 \otimes u - u \otimes u^2),$$

where $\alpha, \beta, \gamma \in \mathbb{k}$ satisfy $\beta^2 = \alpha\gamma$.

Example 3.5. Let $\mathcal{V} = \mathbb{k}\langle u, v \rangle$. Then $\{\!\!\{u, u\}\!\!\} = 0 = \{\!\!\{v, v\}\!\!\}$, $\{\!\!\{v, u\}\!\!\} = 1 \otimes 1$ defines a double Poisson bracket.

Double Poisson brackets can be seen as a non-commutative version of Poisson brackets due to the next result, where we use the notations from §2.2.3.

Theorem 3.6. Assume that $\{\!\!\{-, -\}\!\!\}$ is a double bracket on \mathcal{V} . Then there is a unique skewsymmetric biderivation on \mathcal{V}_N which satisfies for any $a, b \in \mathcal{V}$, $1 \leq i, j \leq N$,

$$\{a_{ij}, b_{kl}\} = \{\!\!\{a, b\}\!\!\}'_{kj} \{\!\!\{a, b\}\!\!\}''_{il}. \quad (3.3)$$

Furthermore, if $\{\!\!\{-, -\}\!\!\}$ is a double Poisson bracket, then $(\mathcal{V}_N, \{-, -\})$ is a Poisson algebra.

3.2. Definition and first properties. In the sequel we assume that \mathcal{V} is a unital associative algebra endowed with an infinite order automorphism $S \in \text{Aut}(\mathcal{V})$.

Definition 3.7. A *double multiplicative λ -bracket* on \mathcal{V} is a linear map

$$\{\{-\lambda-\}\} : \mathcal{V} \otimes \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})[\lambda^{\pm 1}], a \otimes b \mapsto \{\{a_\lambda b\}\}$$

such that

$$\{\{S(a)_\lambda b\}\} = \lambda^{-1} \{\{a_\lambda b\}\}, \quad \{\{a_\lambda S(b)\}\} = \lambda S(\{\{a_\lambda b\}\}), \quad (\text{sesquilinearity}) \quad (3.4a)$$

$$\{\{a_\lambda bc\}\} = \{\{a_\lambda b\}\} c + b \{\{a_\lambda c\}\}, \quad (\text{left Leibniz rule}) \quad (3.4b)$$

$$\{\{ab_\lambda c\}\} = \{\{a_{\lambda x} c\}\} *_1 \left(\left|_{x=S} b \right. \right) + \left(\left|_{x=S} a \right. \right) *_1 \{\{b_{\lambda x} c\}\}. \quad (\text{right Leibniz rule}) \quad (3.4c)$$

In (3.4c) and further we use the following notation (cf. Subsection 2.1): for $P(\lambda) = \sum_n p_n \lambda^n \in (\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}]$ and $a, b \in \mathcal{V}$, we let

$$a|_{x=S} P(\lambda x) b = \sum_n a(\lambda S)^n (p_n b), \quad (3.5)$$

namely, we substitute the variable x by the automorphism S acting on the terms enclosed in parenthesis. Note that the two equations in (3.4a) imply

$$S \{\{a_\lambda b\}\} = \{\{S(a)_\lambda S(b)\}\}, \quad a, b \in \mathcal{V}. \quad (3.6)$$

Given a double multiplicative λ -bracket, we introduce the maps

$$\{\{a_\lambda b' \otimes b''\}\}_L = \{\{a_\lambda b'\}\} \otimes b'', \quad \{\{a_\lambda b' \otimes b''\}\}_R = b' \otimes \{\{a_\lambda b''\}\}, \quad (3.7a)$$

$$\{\{a' \otimes a''_\lambda b\}\}_L = \{\{a'_\lambda x b\}\} \otimes_1 \left(\left|_{x=S} a'' \right. \right), \quad (3.7b)$$

$$\{\{a' \otimes a''_\lambda b\}\}_R = \left(\left|_{x=S} a' \right. \right) \otimes_1 \{\{a''_\lambda x b\}\}. \quad (3.7c)$$

Definition 3.8. A *double multiplicative Poisson vertex algebra* is an algebra \mathcal{V} endowed with a multiplicative λ -bracket such that

$$\{\{a_\lambda b\}\} = - \left|_{x=S} \{\{b_{\lambda^{-1}x^{-1}a}\}\}^\sigma, \quad (\text{skewsymmetry}) \quad (3.8a)$$

$$\{\{a_\lambda \{\{b_\mu c\}\}\}\}_L - \{\{b_\mu \{\{a_\lambda c\}\}\}\}_R - \left\{ \left\{ \{\{a_\lambda b\}\}_{\lambda\mu} c \right\} \right\}_L = 0. \quad (\text{Jacobi identity}) \quad (3.8b)$$

Remark 3.9. Under (3.8a) the rules (3.4b) and (3.4c) are equivalent.

The following result will be useful for computations.

Lemma 3.10. (a) *The sesquilinearity relations*

$$\{\{S(A)_\lambda B\}\}_{L(\text{resp. } R)} = \lambda^{-1} \{\{A_\lambda B\}\}_{L(\text{resp. } R)},$$

$$\{\{A_\lambda S(B)\}\}_{L(\text{resp. } R)} = \lambda S \{\{A_\lambda B\}\}_{L(\text{resp. } R)},$$

hold if either A or B lies in $\mathcal{V} \otimes \mathcal{V}$, and the other one lies in \mathcal{V} .

(b) *For every $a \in \mathcal{V}$ and $B, C \in \mathcal{V}^{\otimes 2}$, we have*

$$\begin{aligned} \{\{a_\lambda B \bullet C\}\}_L &= B \bullet_2 \{\{a_\lambda C\}\}_L + \{\{a_\lambda B\}\}_L \bullet_1 C, \\ \{\{a_\lambda B \bullet C\}\}_R &= B \bullet_2 \{\{a_\lambda C\}\}_R + \{\{a_\lambda B\}\}_R \bullet_3 C, \\ \{\{B \bullet C_{\lambda a}\}\}_L &= \{\{B_{\lambda x} a\}\}_L \bullet_3 (\left|_{x=S} C \right.) + \{\{C_{\lambda x} a\}\}_L \bullet_1 (\left|_{x=S} B^\sigma \right.). \end{aligned} \quad (3.9)$$

In the last equation we are using the notation (3.5).

Proof. Straightforward. □

3.2.1. *Property of Jacobi identity.* Given a double multiplicative λ -bracket on \mathcal{V} , introduce the map

$$\begin{aligned} \{-\lambda - \mu -\} : \mathcal{V}^{\otimes 3} &\rightarrow \mathcal{V}^{\otimes 3}[\lambda^{\pm 1}, \mu^{\pm 1}], \\ \{a_\lambda b_\mu c\} &:= \{a_\lambda \{b_\mu c\}\}_L - \{b_\mu \{a_\lambda c\}\}_R - \left\{ \{a_\lambda b\}_{\lambda\mu} c \right\}_L. \end{aligned} \quad (3.10)$$

A direct comparison with (3.8b) yields that a double multiplicative λ -bracket which is skewsymmetric and such that the map (3.10) vanishes yields, by definition, a double multiplicative Poisson vertex algebra structure on \mathcal{V} .

Lemma 3.11. *Given a skewsymmetric double multiplicative λ -bracket on \mathcal{V} , we have*

$$\left\{ \{b_\mu a\}_{\lambda\mu} c \right\}_L = - \left\{ \{a_\lambda b\}_{\lambda\mu}^\sigma c \right\}_L.$$

Proof. Using skewsymmetry (3.8a) and the first identity in Lemma 3.10(a), we have

$$\left\{ \{b_\mu a\}_{\lambda\mu} c \right\}_L = - \left\{ \left(\left|_{x=S} \{a_{\mu^{-1}x^{-1}b}\}^\sigma \right)_{\lambda\mu} c \right\}_L = - \left\{ \{a_\lambda b\}_{\lambda\mu}^\sigma c \right\}_L,$$

as desired. \square

As an application of this lemma, remark that we can equivalently define (3.10) as

$$\{a_\lambda b_\mu c\} := \{a_\lambda \{b_\mu c\}\}_L - \{b_\mu \{a_\lambda c\}\}_R + \left\{ \{b_\mu a\}_{\lambda\mu}^\sigma c \right\}_L. \quad (3.11)$$

The following properties of the operation (3.10) can also be proven.

Lemma 3.12. *Given a double multiplicative λ -bracket $\{-\lambda -\}$ on \mathcal{V} , we have*

$$\{a_\lambda b_\mu S(c)\} = \lambda_\mu S(\{a_\lambda b_\mu c\}), \quad (3.12)$$

$$\{a_\lambda b_\mu cd\} = c \{a_\lambda b_\mu d\} + \{a_\lambda b_\mu c\} d. \quad (3.13)$$

Furthermore, if $\{-\lambda -\}$ is skewsymmetric, we have

$$\{a_\lambda b_\mu c\} = \left|_{x=S} \{b_\mu c_{\lambda^{-1}\mu^{-1}x^{-1}a}\}^\sigma, \quad (3.14)$$

In particular, given a subset $\mathcal{K}_0 \subset \mathcal{V}$ such that the elements of $\mathcal{K} = \{S^i(u) \mid u \in \mathcal{K}_0, i \in \mathbb{Z}\}$ generates \mathcal{V} as an associative algebra, then the map (3.10) vanishes identically on \mathcal{V} if and only if we have $\{a_\lambda b_\mu c\} = 0$ for any $a, b, c \in \mathcal{K}_0$.

Proof. The proof goes along the lines of Lemma 3.4 in [DSKV]. It is easy to get (3.12) by combining sesquilinearity (3.4a) (only for the second argument) with the definition of the map (3.10). In the same way, we can obtain (3.13) from the left Leibniz rule (3.4b).

To check (3.14), we note the following identities which require skewsymmetry (3.8a):

$$\begin{aligned} \{a_\lambda \{b_\mu c\}\}_L &= - \left(\left|_{x=S} \{ \{b_\mu c\}'_{\lambda^{-1}x^{-1}a} \}^\sigma \right) \otimes \{b_\mu c\}'' \right) \\ &= - \left|_{x=S} \left(\{ \{b_\mu c\}'_{\lambda^{-1}x^{-1}y} \}^\sigma \otimes \left|_{y=S} \{b_\mu c\}'' \right) \right) \\ &= - \left|_{x=S} \left(\{ \{b_\mu c\}'_{\lambda^{-1}x^{-1}y} \} \otimes_1 \left|_{y=S} \{b_\mu c\}'' \right) \right)^\sigma \\ &= - \left|_{x=S} \{ \{b_\mu c\}_{\lambda^{-1}x^{-1}} \}^\sigma_L, \\ \{b_\mu \{a_\lambda c\}\}_R &= - \left(\left|_{x=S} \{c_{\lambda^{-1}x^{-1}a}\}'' \right) \otimes \{b_\mu \left|_{x=S} \{c_{\lambda^{-1}x^{-1}a}\}' \} \right) \right) \\ &= - \left|_{x=S} \left(\{ \{b_\mu \{c_{\lambda^{-1}(x\mu)^{-1}a}\}' \} \} \otimes \{c_{\lambda^{-1}(x\mu)^{-1}a}\}'' \right)^\sigma \\ &= - \left|_{x=S} \{ \{b_\mu \{c_{\lambda^{-1}\mu^{-1}x^{-1}a}\} \}^\sigma_L, \\ \left\{ \{b_\mu a\}_{\lambda\mu}^\sigma c \right\}_L &= \left(\left|_{x=S} \{b_\mu a\}' \right) \otimes \{ \{b_\mu a\}''_{\lambda\mu x} \}^\sigma \right)^\sigma \\ &= - \left|_{x=S} \left(\{b_\mu a\}' \otimes \{c_{\lambda^{-1}\mu^{-1}x^{-1}} \{b_\mu a\}'' \} \right)^\sigma \\ &= - \left|_{x=S} \{c_{\lambda^{-1}\mu^{-1}x^{-1}} \{b_\mu a\}\}^\sigma_R. \end{aligned}$$

Thus, writing $\{\{a_\lambda b_\mu c\}\}$ through (3.11), we get that (3.14) holds after writing the right-hand side with (3.10).

For the second part of the lemma, note that as a consequence of (3.14) we can write

$$\begin{aligned} \{\{a_\lambda S(b)_\mu c\}\} &= \mu^{-1} \{\{a_\lambda b_\mu c\}\}, \quad \{\{S(a)_\lambda b_\mu c\}\} = \lambda^{-1} \{\{a_\lambda b_\mu c\}\}, \\ \{\{a_\lambda b d_\mu c\}\} &= (\mid_{x=S} b) *_2 \{\{a_\lambda d_\mu c\}\} + \{\{a_\lambda b_\mu c\}\} *_1 (\mid_{x=S} d), \end{aligned} \quad (3.15)$$

$$\{\{a d_\lambda b_\mu c\}\} = (\mid_{x=S} a) *_1 \{\{d_\lambda b_\mu c\}\} + \{\{a_\lambda b_\mu c\}\} *_2 (\mid_{x=S} d). \quad (3.16)$$

Since we have derivation and sesquilinearity rules in the three arguments of the map (3.10), this operation is completely determined by its value on the elements of \mathcal{K}_0 . In particular, the map (3.10) vanishes if and only if it does when evaluated on the elements of \mathcal{K}_0 . \square

3.3. Relation to local lattice double Poisson algebras. We introduce here the notion of a local lattice double Poisson algebra, which is equivalent to that of a double multiplicative Poisson vertex algebra.

Definition 3.13. A *lattice double Poisson algebra* is a double Poisson algebra \mathcal{V} with an infinite order automorphism $S \in \text{Aut}(\mathcal{V})$, namely $(a, b \in \mathcal{V})$

$$S(ab) = S(a)S(b) \quad \text{and} \quad S(\{\{a, b\}\}) = \{\{S(a), S(b)\}\}.$$

It is called *local* if, for every $a, b \in \mathcal{V}$, we have

$$\{\{S^n(a), b\}\} = 0, \quad \text{for all but finitely many values of } n \in \mathbb{Z}. \quad (3.17)$$

For an element $a(\lambda) = \sum a_k \lambda^k \in \mathcal{V}^{\otimes n}[\lambda^{\pm 1}]$, $n \geq 1$, we define its *multiplicative residue* by

$$\text{mRes}_\lambda a(\lambda) = a_0.$$

Proposition 3.14. If \mathcal{V} is a double multiplicative Poisson vertex algebra with double multiplicative λ -bracket $\{\{-_\lambda -\}\}$ and automorphism $S \in \text{Aut}(\mathcal{V})$, then \mathcal{V} is a local lattice double Poisson algebra with the double Poisson bracket

$$\{\{a, b\}\} = \text{mRes}_\lambda \{\{a_\lambda b\}\}, \quad a, b \in \mathcal{V}. \quad (3.18)$$

Conversely, if \mathcal{V} is a local lattice double Poisson algebra with double Poisson bracket $\{\{-, -\}\}$ and automorphism $S \in \text{Aut}(\mathcal{V})$, then we can endow it with a structure of a double multiplicative Poisson vertex algebra with the double multiplicative λ -bracket

$$\{\{a_\lambda b\}\} := \sum_{n \in \mathbb{Z}} \lambda^n \{\{S^n(a), b\}\}, \quad a, b \in \mathcal{V}. \quad (3.19)$$

Proof. Applying mRes_λ to both sides of equation (3.6) it follows that $S \{\{a, b\}\} = \{\{S(a), S(b)\}\}$. Cyclic skewsymmetry of the double bracket (3.18) follows by applying mRes_λ to both sides of (3.8a) and using the fact that $\text{mRes}_\lambda a(\lambda) = \text{mRes}_\lambda a(\lambda^{-1})$. Left and right Leibniz rules (3.1b)-(3.1c), respectively Jacobi identity, for the double bracket (3.18) follow by applying $\text{mRes}_\lambda \text{mRes}_\mu$ to (3.4b)-(3.4c), respectively (3.8b).

Conversely, let $\{\{a_\lambda b\}\}$, $a, b \in \mathcal{V}$ be defined by (3.19). Note that $\{\{a_\lambda b\}\} \in (\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}]$ by (3.17). Moreover, we have

$$\{\{S(a)_\lambda b\}\} = \sum_{n \in \mathbb{Z}} \lambda^n \{\{S^{n+1}(a), b\}\} = \lambda^{-1} \sum_{n \in \mathbb{Z}} \lambda^{n+1} \{\{S^{n+1}(a), b\}\} = \lambda^{-1} \{\{a_\lambda b\}\},$$

and, using the fact that S is an automorphism,

$$\{\{a_\lambda S(b)\}\} = \sum_{n \in \mathbb{Z}} \lambda^n \{\{S^n(a), S(b)\}\} = \lambda S \sum_{n \in \mathbb{Z}} \lambda^{n-1} \{\{S^{n-1}(a), b\}\} = \lambda S \{\{a_\lambda b\}\},$$

proving sesquilinearity (3.4a). Next, we have

$$\begin{aligned} \{\{ab_\lambda c\}\} &= \sum_n \lambda^n (S^n(a) *_1 \{\{S^n(b), c\}\} + \{\{S^n(a), c\}\} *_1 S^n(b)) \\ &= \sum_n (\lambda y)^n \left(\left(\Big|_{y=S} a \right) *_1 \{\{S^n(b), c\}\} + \{\{S^n(a), c\}\} *_1 \left(\Big|_{y=S} b \right) \right) \\ &= \left(\Big|_{y=S} a \right) *_1 \{\{b_{\lambda y} c\}\} + \{\{a_{\lambda y} c\}\} *_1 \left(\Big|_{y=S} b \right), \end{aligned}$$

which proves the right Leibniz rule (3.4c) for (3.19). The left Leibniz rule (3.4b) can be proven similarly. Finally, we prove the Jacobi identity for (3.19). Note that

$$\begin{aligned} &\{\{a_\lambda \{\{b_\mu c\}\}\}_L - \{\{b_\mu \{\{a_\lambda c\}\}\}_R \\ &= \sum_{n,m} \lambda^m \mu^n (\{\{S^m(a), \{\{S^n(b), c\}\}\}_L - \{\{S^n(b), \{\{S^m(a), c\}\}\}_R) \\ &= \sum_{n,m} \lambda^m \mu^n \{\{\{S^m(a), S^n(b)\}\}, c\}_L \quad \text{by (3.2)} \\ &= \sum_n \mu^n \{\{\{a_\lambda S^n(b)\}\}, c\}_L. \end{aligned}$$

Hence, by (3.4a), we have

$$\begin{aligned} &\{\{a_\lambda \{\{b_\mu c\}\}\}_L - \{\{b_\mu \{\{a_\lambda c\}\}\}_R \\ &= \sum_n \mu^n \lambda^n \{\{S^n(\{\{a_\lambda b\}'\}) \otimes S^n(\{\{a_\lambda b\}''\}), c\}_L \\ &= \sum_n \mu^n \lambda^n \{\{S^n(\{\{a_\lambda b\}'\}), c\}\} \otimes_1 S^n(\{\{a_\lambda b\}''\}) \\ &= \sum_n (\lambda \mu y)^n \{\{S^n(\{\{a_\lambda b\}'\}), c\}\} \otimes_1 \left(\Big|_{y=S} \{\{a_\lambda b\}''\} \right) = \{\{\{\{a_\lambda b\}_{\lambda \mu} c\}\}\}_L. \end{aligned}$$

This concludes the proof. \square

We can get several examples of double multiplicative Poisson vertex algebras using Proposition 3.14.

Example 3.15. Let V be a double Poisson algebra with double Poisson bracket $\{\{-, -\}\}$. Consider the unital associative algebra $\mathcal{V} = \mathbb{k}[S, S^{-1}] \otimes V$. In other words, \mathcal{V} is isomorphic to the direct sum of infinitely many copies of V and the automorphism S is the “shift” operator. We can endow \mathcal{V} with a lattice double Poisson algebra structure with

$$\{\{S^m \otimes a, S^n \otimes b\}\} = \delta_{m,n} (S^n \otimes S^n) \{\{a, b\}\}, \quad a, b \in V. \quad (3.20)$$

Example 3.16. As a special case of Example 3.15 consider the double Poisson algebra V from Example 3.4 with $\alpha = 1$ and $\beta = \gamma = 0$. Then $\mathcal{V} = \mathbb{k}[S, S^{-1}] \otimes V = \mathbb{k}\langle u_i \mid i \in \mathbb{Z} \rangle$, where $S(u_i) = u_{i+1}$, $i \in \mathbb{Z}$. The double Poisson bracket (3.20) on generators then reads

$$\{\{u_i, u_j\}\} = \delta_{ij} (u_j \otimes 1 - 1 \otimes u_j).$$

Using Proposition 3.14, we get a double multiplicative Poisson vertex algebra structure on \mathcal{V} defined on generators by

$$\{\{u_i \lambda u_j\}\} = \lambda^{j-i} (u_j \otimes 1 - 1 \otimes u_j) = (\lambda S)^{j-i} (u_i \otimes 1 - 1 \otimes u_i),$$

and extended using sesquilinearity and the Leibniz rules.

Example 3.17. Fix $\ell \geq 1$, and consider the algebra $\mathcal{V} = \mathbb{k}\langle u_{r,i}, v_{r,i} \mid 1 \leq r \leq \ell, i \in \mathbb{Z} \rangle$ with the automorphism S defined by $S(u_{r,i}) = u_{r,i+1}$ and $S(v_{r,i}) = v_{r,i+1}$. We can endow \mathcal{V} with a lattice double Poisson algebra structure (cf. Example 3.5) with

$$\{\{u_{r,i}, u_{s,j}\}\} = 0 = \{\{v_{r,i}, v_{s,j}\}\}, \quad \{\{v_{r,i}, u_{s,j}\}\} = \delta_{rs} \delta_{ij} 1 \otimes 1.$$

By Proposition 3.14, we get a double multiplicative Poisson vertex algebra structure on \mathcal{V} defined on generators by

$$\{\!\{u_{r,i}\lambda u_{s,j}\}\!\} = 0, \quad \{\!\{v_{r,i}\lambda v_{s,j}\}\!\} = 0, \quad \{\!\{v_{r,i}\lambda u_{s,j}\}\!\} = \delta_{rs}\lambda^{j-i}1 \otimes 1,$$

and extended using sesquilinearity and Leibniz rules.

3.3.1. Finite order automorphism. The construction of this section still holds in the case when S is an automorphism of finite order $e \geq 1$. In that case, from the sesquilinearity axiom (2.1a), we get that the following relation should be satisfied for every $a, b \in \mathcal{V}$:

$$\{a_\lambda b\} = \{S^{-e}(a)_\lambda b\} = \lambda^e \{a_\lambda b\}.$$

Hence, if S is an automorphism of finite order $e \geq 1$, a double multiplicative λ -bracket is a map $\{\!\{-\lambda-\}\!\} : \mathcal{V} \otimes \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})[\lambda]/\langle \lambda^e - 1 \rangle$, satisfying (3.4a), (3.4b) and (3.4c). Then we still have Example 3.15 where $\mathbb{k}[S, S^{-1}]$ should be replaced by $\mathbb{k}[S]/\langle S^e - 1 \rangle$. Furthermore, all results of this and the next sections extend to this framework with little changes.

Example 3.18. Fix $e \geq 1$ and consider the algebra $\mathcal{V} = \mathbb{k}\langle u_j \mid j \in \mathbb{Z}/e\mathbb{Z} \rangle$ with the automorphism S of \mathcal{V} given by $u_i \rightarrow u_{i+1}$. On \mathcal{V} we can define the double Poisson bracket

$$\{\!\{u_i, u_j\}\!\} = \delta_{ij}(u_j \otimes 1 - 1 \otimes u_j),$$

which can be obtained from e copies of Example 3.4 with $\alpha = 1, \beta = \gamma = 0$. The automorphism S of \mathcal{V} has order e and commutes with $\{\!\{-, -\}\!\}$. Using Proposition 3.14, we get a double multiplicative Poisson vertex algebra structure on \mathcal{V} completely determined by

$$\{\!\{u_i \lambda u_j\}\!\} = \lambda^{[j-i]}(u_j \otimes 1 - 1 \otimes u_j),$$

where $0 \leq [j-i] < e$ is the remainder of $j-i$ modulo e . This example can be seen as a closed chain version of Example 3.16.

Example 3.19. For $\ell \geq 1$, we form the algebra $\mathcal{V} = \mathbb{k}\langle u_1, \dots, u_\ell, v_1, \dots, v_\ell \rangle$, which admits a double Poisson bracket by taking ℓ copies of Example 3.5 :

$$\{\!\{u_i, u_j\}\!\} = 0 = \{\!\{v_i, v_j\}\!\}, \quad \{\!\{v_i, u_j\}\!\} = \delta_{ij}1 \otimes 1.$$

If we consider the automorphism S of \mathcal{V} given by $u_i \mapsto v_i, v_i \mapsto -u_i$, we can show that ($k \geq 0$)

$$S^{2k+1}(u_i) = (-1)^k v_i, \quad S^{2k}(u_i) = (-1)^k u_i, \quad S^{2k+1}(v_i) = (-1)^{k+1} u_i, \quad S^{2k}(v_i) = (-1)^k v_i.$$

Hence, S has order 4. Moreover, we have

$$S(\{\!\{v_i, u_j\}\!\}) = \delta_{ij}1 \otimes 1 = -\{\!\{u_i, v_j\}\!\} = \{\!\{S(v_i), S(u_j)\}\!\},$$

from which it follows that S commutes with $\{\!\{-, -\}\!\}$. Using Proposition 3.14, we get a double multiplicative Poisson vertex algebra structure on \mathcal{V} completely determined by

$$\{\!\{u_i \lambda u_j\}\!\} = \delta_{ij}(\lambda - \lambda^3)(1 \otimes 1), \quad \{\!\{v_i \lambda v_j\}\!\} = \delta_{ij}(\lambda - \lambda^3)(1 \otimes 1), \quad \{\!\{v_i \lambda u_j\}\!\} = \delta_{ij}(1 - \lambda^2)(1 \otimes 1).$$

4. DOUBLE MULTIPLICATIVE POISSON VERTEX ALGEBRA STRUCTURE ON AN ALGEBRA OF (NON-COMMUTATIVE) DIFFERENCE FUNCTIONS

4.1. The algebra of non-commutative difference operators. Let \mathcal{V} be a unital associative algebra with an automorphism S and consider the space $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$. We extend the associative product \bullet on $\mathcal{V} \otimes \mathcal{V}$ defined by (2.8) to an associative product on $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$ by letting, for $a, b \in \mathcal{V} \otimes \mathcal{V}$ and $m, n \in \mathbb{Z}$:

$$a S^m \bullet b S^n = (a \bullet S^m(b)) S^{m+n} = (a' S^m(b') \otimes S^m(b'') a'') S^{m+n}, \quad (4.1)$$

and extending it by linearity to $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$. We then call $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$ the algebra of scalar (non-commutative) difference operators. The action of a scalar difference operator $A(S) = \sum_{n \in \mathbb{Z}} a_n S^n \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$ on $f \in \mathcal{V}$ is given by

$$A(S)f = \sum_{n \in \mathbb{Z}} a'_n S^n(f) a''_n. \quad (4.2)$$

The adjoint of $A(S)$ is the difference operator

$$A^*(S) = \sum_{n \in \mathbb{Z}} S^{-n} \bullet a_n^\sigma = \sum_{n \in \mathbb{Z}} (S^{-n}(a_n'') \otimes S^{-n}(a_n')) S^{-n}, \quad (4.3)$$

where in the second identity we used (4.1) (the element $(1 \otimes 1)S^k \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$, $k \in \mathbb{Z}$, will be usually simply denoted by S^k). Using (2.9), it is immediate to check that

$$(A(S) \bullet B(S))^* = B^*(S) \bullet A^*(S), \quad A(S), B(S) \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]. \quad (4.4)$$

The symbol of a scalar difference operator $A(S) = \sum_{n \in \mathbb{Z}} a_n S^n \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$ is the Laurent polynomial

$$A(z) = \sum_{n \in \mathbb{Z}} a_n z^n \in (\mathcal{V} \otimes \mathcal{V})[z, z^{-1}]. \quad (4.5)$$

The formula for the symbol of products of scalar difference operators $A(S) \bullet B(S)$, and its adjoint $(A(S) \bullet B(S))^*$ is

$$(A \bullet B)(z) = A(zS) \bullet B(z), \quad (A \bullet B)^*(z) = B^*(zS) \bullet A^*(z). \quad (4.6)$$

More generally, for every $\ell \geq 1$, the space $\text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$ is an algebra with the product (4.1) extended componentwise using matrix multiplication: if $H(S) = (H_{ij}(S))_{i,j=1}^\ell$, $K(S) = (K_{ij}(S))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$, then $(H \bullet K)(S) = ((H \bullet K)_{ij}(S))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$, where

$$(H \bullet K)_{ij}(S) = \sum_{k=1}^\ell H_{ik}(S) \bullet K_{kj}(S) \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}].$$

We call it the algebra of (non-commutative) matrix difference operators over \mathcal{V} . The action of $H(S) = (H_{ij}(S))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$, where

$$H_{ij}(S) = \sum_{n \in \mathbb{Z}} H_{i,j;n} S^n \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}],$$

(note that the sum is finite) on a vector $F \in \mathcal{V}^\ell$ is given by extending (4.2) componentwise

$$(H(S)F)_i = \sum_{j=1}^\ell H_{ij}(S)F_j = \sum_{j=1}^\ell H'_{i,j;n} S^n (F_j) H''_{i,j;n}, \quad i = 1, \dots, \ell. \quad (4.7)$$

The adjoint of $H(S)$ is the matrix difference operator $H^*(S) = (H_{j,i}^*(S))_{i,j=1}^\ell$, where the scalar difference operator $H_{ji}^*(S)$ is obtained using (4.3). Equation (4.4) holds for $A(S), B(S) \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$ as well. The symbol of the matrix difference operator $H(S)$ is obtained using equation (4.5) componentwise.

4.2. Algebras of non-commutative difference functions and double multiplicative Poisson vertex algebras. Consider the algebra of non-commutative difference polynomials \mathcal{R}_ℓ in ℓ variables u_i , $i \in I = \{1, \dots, \ell\}$. It is the algebra of non-commutative polynomials in the indeterminates $u_{i,n}$,

$$\mathcal{R}_\ell = \mathbb{k}\langle u_{i,n} \mid i \in I, n \in \mathbb{Z} \rangle,$$

endowed with an automorphism S , defined on generators by $S(u_{i,n}) = u_{i,n+1}$, and partial derivatives $\frac{\partial}{\partial u_{i,n}} : \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$, for every $i \in I$ and $n \in \mathbb{Z}$, defined on monomials by

$$\frac{\partial}{\partial u_{i,n}}(u_{i_1,n_1} \dots u_{i_s,n_s}) = \sum_{k=1}^s \delta_{i_k,i} \delta_{n_k,n} u_{i_1,n_1} \dots u_{i_{k-1},n_{k-1}} \otimes u_{i_{k+1},n_{k+1}} \dots u_{i_s,n_s}, \quad (4.8)$$

which are commuting 2-fold derivations of \mathcal{R}_ℓ (using the terminology of [DSKV]) such that

$$S \circ \frac{\partial}{\partial u_{i,n}} = \frac{\partial}{\partial u_{i,n+1}} \circ S. \quad (4.9)$$

In (4.9), S is extended to $\mathcal{V}^{\otimes 2}$ using (2.16). Given the partial derivative $\frac{\partial}{\partial u_{i,n}}$, $i \in I$, $n \in \mathbb{Z}$, recall the derivations $\left(\frac{\partial}{\partial u_{i,n}}\right)_L, \left(\frac{\partial}{\partial u_{i,m}}\right)_R : \mathcal{V}^{\otimes 2} \rightarrow \mathcal{V}^{\otimes 3}$ defined by (2.12).

Lemma 4.1. *For any non-commutative difference polynomial $f \in \mathcal{R}_\ell$, and $i, j \in I$ and $n, m \in \mathbb{Z}$, the partial derivatives strongly commute, i.e. we have*

$$\left(\frac{\partial}{\partial u_{i,m}}\right)_L \frac{\partial f}{\partial u_{j,n}} = \left(\frac{\partial}{\partial u_{j,n}}\right)_R \frac{\partial f}{\partial u_{i,m}}.$$

Proof. Same as the proof of Lemma 2.6 in [DSKV]. \square

Definition 4.2. An algebra of difference functions in ℓ variables is a unital associative algebra \mathcal{V} , with an automorphism S , endowed with strongly commuting linearly independent (cf. (2.15)) 2-fold derivations $\frac{\partial}{\partial u_{i,n}} : \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}$, $i \in I = \{1, \dots, \ell\}$, $n \in \mathbb{Z}$, such that (4.9) holds and, for every $f \in \mathcal{V}$, we have $\frac{\partial f}{\partial u_{i,n}} = 0$ for all but finitely many choices of indices $(i, n) \in I \times \mathbb{Z}$.

An example of such an algebra is the algebra \mathcal{R}_ℓ , endowed with the 2-fold derivations defined in (4.8), or its localization by non-zero elements. Note that \mathcal{R}_ℓ is in fact an algebra of difference functions in m variables, where $1 \leq m \leq \ell$. In this case we should think of the variables u_i , $i > m$ as quasiconstants, i.e. they lie in the kernel of the 2-fold derivations defining the structure of algebra of difference functions of \mathcal{V} (see [DSK3]).

Theorem 4.3. (a) Any double multiplicative λ -bracket on \mathcal{R}_ℓ has the form ($f, g \in \mathcal{R}_\ell$):

$$\{f_\lambda g\} = \sum_{\substack{i,j \in I \\ m,n \in \mathbb{Z}}} \frac{\partial g}{\partial u_{j,n}} \bullet \lambda^n S^n \{u_{i\lambda} u_j\} \lambda^{-m} S^{-m} \bullet \left(\left| \frac{\partial f}{\partial u_{i,m}} \right|_{x=S} \right)^\sigma. \quad (4.10)$$

where \bullet denotes the associative product on $\mathcal{R}_\ell \otimes \mathcal{R}_\ell$ defined in (2.8), and we are using the notation (3.5).

- (b) Let \mathcal{V} be an algebra of difference functions in ℓ variables. Let $H(\lambda)$ be an $\ell \times \ell$ matrix with entries in $(\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}]$. We denote its entries by $H_{ij}(\lambda) = \{u_{j\lambda} u_i\}$, $i, j \in I$. Then formula (4.10) defines a double multiplicative λ -bracket on \mathcal{V} .
- (c) Equation (4.10) defines a structure of a double multiplicative Poisson vertex algebra on \mathcal{V} if and only if the skewsymmetry axiom (3.8a) and the Jacobi identity (3.8b) hold on the u_i 's.

Proof. Similar to the proof for the analogue result in [DSKV, Th.3.10]. \square

Remark 4.4. Let $H(S) = (H_{ij}(S)) \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}])$, where $H_{ij}(S)$ is the scalar difference operator with symbol $H_{ij}(\lambda) = \{u_{j\lambda} u_i\}$ (cf. (4.5)). Recalling the definition of the adjoint matrix difference operator $H^*(S)$ (cf. equation (4.3)), then skewsymmetry axiom (3.8a) on generators means that the matrix difference operator $H(S)$ is skew-adjoint, that is $H^*(S) = -H(S)$.

Definition 4.5. A matrix difference operator $H(S) = (H_{ij}(S))_{i,j=1}^\ell$, with entries $H_{ij}(S) \in (\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$, such that the double multiplicative λ -bracket (4.10) with $\{u_{j\lambda} u_i\} = H_{ij}(\lambda)$ satisfies skewsymmetry (3.8a) and Jacobi identity (3.8b) on the generators u_i 's of the algebra of difference functions \mathcal{V} is called a (local) *Poisson structure* on \mathcal{V} .

4.3. Double multiplicative Poisson vertex algebra structures on \mathcal{R}_1 . In this section we provide some classification results of double multiplicative Poisson vertex algebra structures on $\mathcal{R} := \mathcal{R}_1 = \mathbb{k}\langle u_i | i \in \mathbb{Z} \rangle$. Let

$$H(S) = \sum_{k \in \mathbb{Z}} f_k S^k \in (\mathcal{R} \otimes \mathcal{R})[S, S^{-1}]$$

be a difference operator with coefficients in $\mathcal{R} \otimes \mathcal{R}$ and define a double multiplicative λ -bracket on \mathcal{R} using the Master Formula (4.10) where

$$\{u_\lambda u\} = H(\lambda) = \sum_{k \in \mathbb{Z}} f_k \lambda^k. \quad (4.11)$$

We have that

$$-\llbracket u_{\lambda^{-1}S^{-1}u} \rrbracket^\sigma = - \sum_{k \in \mathbb{Z}} S^k f_{-k}^\sigma \lambda^k.$$

Hence, skewsymmetry holds on generators if and only if

$$f_k = -S^k f_{-k}^\sigma, \quad k \in \mathbb{Z}. \quad (4.12)$$

Next, using the Master Formula (4.10) and equation (4.11) we have that Jacobi identity on generators becomes the following equation

$$\begin{aligned} \sum_{i,j,k \in \mathbb{Z}} \left(\lambda^{i+k} \mu^j \left(\frac{\partial f_j'}{\partial u_i} \bullet S^i f_k \right) \otimes f_j'' - \lambda^j \mu^{i+k} f_j' \otimes \left(\frac{\partial f_j''}{\partial u_i} \bullet S^i f_k \right) \right. \\ \left. - \lambda^{i+j+k} \mu^{i+k} \left(f_k \bullet S^{i+k} \left(\frac{\partial f_j'}{\partial u_{-i}} \right)^\sigma \right) \otimes_1 S^{i+k} f_j'' \right) = 0. \end{aligned} \quad (4.13)$$

Equation (4.13) can also be rewritten as

$$\begin{aligned} \sum_{i,j,k \in \mathbb{Z}} \left(\lambda^{i+k} \mu^j \left(\frac{\partial}{\partial u_i} \right)_L (f_j) \bullet_3 S^i f_k - \lambda^j \mu^{i+k} \left(\frac{\partial}{\partial u_i} \right)_R (f_j) \bullet_1 S^i f_k \right. \\ \left. - \lambda^{i+j+k} \mu^{i+k} \left(S^{i+k} \left(\frac{\partial}{\partial u_{-i}} \right)_L (f_j) \bullet_3 f_k^\sigma \right)^{\sigma^2} \right) = 0. \end{aligned} \quad (4.14)$$

Arguing similarly to Lemma 2.6 in [DSKVW2], it is possible to show that

$$f_k = f_k(u, u_1, \dots, u_k), \quad k \geq 0. \quad (4.15)$$

Indeed, for any $i > k \geq 0$ we obtain that f_k is independent of u_i by looking at the terms in $\lambda^{i+N} \mu^k$ and $\lambda^k \mu^{N+i}$ with $N = \max\{i | f_i \neq 0\}$. We can obtain in the same way that for $i < -k \leq 0$, f_{-k} is independent of u_i ; this is equivalent to having f_k independent of u_i for $i < 0$.

Let us assume that $f_k = 0$, for $k \neq 0$, in (4.11). By (4.15) and (4.12) we have that

$$\llbracket u_\lambda u \rrbracket = f, \quad \text{where } f = f(u) = -f^\sigma.$$

In this case, the Jacobi identity (4.14) reads

$$\left(\frac{\partial}{\partial u} \right)_L (f) \bullet_3 f - \left(\frac{\partial}{\partial u} \right)_R (f) \bullet_1 f - \left(\left(\frac{\partial}{\partial u} \right)_L (f) \bullet_3 f^\sigma \right)^{\sigma^2} = 0.$$

This is the same condition defining double Poisson structures on $\mathbb{k}[u]$. By the results in [P, VdB1] (see Example 3.4) we have that

$$f = \alpha(u \otimes 1 - 1 \otimes u) + \beta(u^2 \otimes 1 - 1 \otimes u^2) + \gamma(u^2 \otimes u - u \otimes u^2),$$

where $\alpha, \beta, \gamma \in \mathbb{k}$ are such that $\beta^2 = \alpha\gamma$.

Next, we study the case when $f_k \neq 0$ in (4.11), for some $k \in \mathbb{Z}$. In the sequel, we will use the following result.

Lemma 4.6. *Let $f, g \in \mathcal{R} \otimes \mathcal{R}$ and $k \in \mathbb{Z}$.*

(a) *If*

$$\left(\frac{\partial}{\partial u_k} \right)_L (f) \bullet_3 g = \left(\left(\frac{\partial}{\partial u_k} \right)_L (g) \bullet_3 f^\sigma \right)^{\sigma^2},$$

then

$$f' = su_k + q, \quad g' = u_k r + p, \quad sp = qr,$$

where s, q, r, p lie in the kernel of $\frac{\partial}{\partial u_k}$.

(b) *If*

$$\left(\frac{\partial}{\partial u_k} \right)_R (f) \bullet_1 g = \left(\left(\frac{\partial}{\partial u_k} \right)_L (g^\sigma) \bullet_3 f^\sigma \right)^{\sigma^2},$$

then

$$f'' = u_k s + q, \quad g'' = ru_k + p, \quad ps = rq,$$

where s, q, r, p lie in the kernel of $\frac{\partial}{\partial u_k}$.

Proof. Let us prove part (a). Using (2.14), (2.6) and (2.10) we have

$$\left(\frac{\partial}{\partial u_k}\right)_L (f) \bullet_3 g = \left(\frac{\partial f'}{\partial u_k}\right)' g' \otimes g'' \left(\frac{\partial f'}{\partial u_k}\right)'' \otimes f''$$

and

$$\left(\left(\frac{\partial}{\partial u_k}\right)_L (g) \bullet_3 f^\sigma\right)^{\sigma^2} = f' \left(\frac{\partial g'}{\partial u_k}\right)'' \otimes g'' \otimes \left(\frac{\partial g'}{\partial u_k}\right)' f''.$$

Hence we need to solve the equation

$$\left(\frac{\partial f'}{\partial u_k}\right)' g' \otimes g'' \left(\frac{\partial f'}{\partial u_k}\right)'' \otimes f'' = f' \left(\frac{\partial g'}{\partial u_k}\right)'' \otimes g'' \otimes \left(\frac{\partial g'}{\partial u_k}\right)' f''. \quad (4.16)$$

This gives the conditions

$$\left(\frac{\partial f'}{\partial u_k}\right)'' \in \mathbb{k}, \quad \left(\frac{\partial g'}{\partial u_k}\right)' \in \mathbb{k},$$

from which we get $f' = su_k + q$ and $g' = u_k r + p$, with s, q, r, p in the kernel of $\frac{\partial}{\partial u_k}$. Hence, $\frac{\partial f'}{\partial u_k} = s \otimes 1$ and $\frac{\partial g'}{\partial u_k} = 1 \otimes r$. Substituting these expressions in (4.16) we get that s, q, r, p should satisfy

$$s(u_k r + p) = (su_k + q)r,$$

which implies $sp = qr$ and concludes the proof of the claim. Part (b) is proven similarly. \square

Let $g = g(u) \in \mathbb{k}[u] \otimes \mathbb{k}[u] \subset \mathcal{R} \otimes \mathcal{R}$ (that is $\frac{\partial g}{\partial u_n} = 0$, for every $n \neq 0$) and let $r(\lambda) \in \mathbb{k}[\lambda, \lambda^{-1}]$ be such that $r(\lambda^{-1}) = -r(\lambda)$. We consider the double multiplicative λ -bracket on \mathcal{R} defined by

$$\llbracket u \lambda u \rrbracket = g \bullet r(\lambda S) g^\sigma. \quad (4.17)$$

Proposition 4.7. *The λ -bracket (4.17) defines a double multiplicative Poisson vertex algebra structure on \mathcal{R} if and only if g is of the form (up to a constant multiple that can be absorbed in $r(\lambda)$)*

$$g = (\alpha u + \beta) \otimes (\alpha u + \beta), \quad \alpha, \beta \in \mathbb{k}. \quad (4.18)$$

Proof. The λ -bracket (4.17) is clearly skewsymmetric in view of the assumption on $r(\lambda)$ and (2.9). By a direct computation, using Lemma 3.10, equations (3.7a), (3.7b), the Master Formula (4.10), the assumption on $r(\lambda)$ and Lemma 2.5(b)-(c), the Jacobi identity on generators becomes

$$\begin{aligned} & \left(\left(\left(\frac{\partial}{\partial u} \right)_L (g) \bullet_3 g - \left(\frac{\partial}{\partial u} \right)_R (g) \bullet_1 g \right) \bullet_1 r(\lambda S) g^\sigma \right) \bullet_3 r(\mu S) g^\sigma \\ & + g \bullet_2 r(\lambda \mu S) \left(\left(\left(\frac{\partial}{\partial u} \right)_L (g^\sigma) \bullet_3 g - g^\sigma \bullet_2 \left(\left(\frac{\partial}{\partial u} \right)_L (g) \right)^{\sigma^2} \right) \bullet_3 r(\lambda S) g^\sigma \right) \\ & - g \bullet_2 r(\lambda \mu S) \left(\left(\left(\frac{\partial}{\partial u} \right)_R (g^\sigma) \bullet_1 g - g^\sigma \bullet_2 \left(\left(\frac{\partial}{\partial u} \right)_R (g) \right)^\sigma \right) \bullet_1 r(\mu S) g^\sigma \right) = 0. \end{aligned} \quad (4.19)$$

It is straightforward to check that if g is as in (4.18) then the LHS of (4.19) vanishes. On the other hand, let us assume that $r(\lambda)$ has order $N > 1$. Then, vanishing of the coefficient of $\lambda^{2N} \mu^N$ in the LHS of (4.19) gives

$$\left(\frac{\partial}{\partial u} \right)_L (g^\sigma) \bullet_3 g - g^\sigma \bullet_2 \left(\left(\frac{\partial}{\partial u} \right)_L (g) \right)^{\sigma^2} = 0. \quad (4.20)$$

Using the identity ($A \in \mathcal{R}^{\otimes 2}, B \in \mathcal{R}^{\otimes 3}$)

$$A^\sigma \bullet_2 B^{\sigma^2} = (B \bullet_3 A)^{\sigma^2}$$

we have that

$$g^\sigma \bullet_2 \left(\left(\frac{\partial}{\partial u} \right)_L (g) \right)^{\sigma^2} = \left(\left(\frac{\partial}{\partial u} \right)_L (g) \bullet_3 g \right)^{\sigma^2}. \quad (4.21)$$

From (4.20) and (4.21) it follows that $g \in \mathbb{k}[u] \otimes \mathbb{k}[u]$ must satisfy the equation

$$\left(\frac{\partial}{\partial u}\right)_L (g^\sigma) \bullet_3 g = \left(\left(\frac{\partial}{\partial u}\right)_L (g) \bullet_3 g\right)^{\sigma^2}.$$

By Lemma 4.6(a) with $k = 0$ and g^σ in place of f , and using the fact that $\ker \frac{\partial}{\partial u} \cap \mathbb{k}[u] = \mathbb{k}$, we have $g' = au + b$, $g'' = cu + d$, where $a, b, c, d \in \mathbb{k}$ satisfy $ad = bc$. Hence, up to a constant factor, it is necessary for g to be as in (4.18). This concludes the proof. \square

For $N \geq 1$, more generally, let us define a skewsymmetric double multiplicative λ -bracket on \mathcal{R} by

$$\{u_\lambda u\} = f\lambda^N - (\lambda S)^{-N} f^\sigma, \quad (4.22)$$

where $f \in \mathcal{R} \otimes \mathcal{R}$. The next result provides a classification of all the simplest non-trivial examples of double multiplicative Poisson vertex algebra structures on \mathcal{R} .

Proposition 4.8. *Fix $N \geq 1$. Then, (4.22) defines a double multiplicative Poisson vertex algebra structure on \mathcal{R} if and only if $f = g \bullet S^N g$, where g is as in (4.18).*

We will prove the classification result by checking the conditions given by the Jacobi identity (4.14) for $f_N := f$, $f_{-N} := -S^{-N}(f^\sigma)$ and $f_k = 0$ if $k \neq N, -N$. Recall that by (4.15), $f = f(u, u_1, \dots, u_N)$. First we need the following result.

Lemma 4.9. *If the double λ -bracket (4.22) satisfies Jacobi identity, then $f = f(u, u_N)$.*

Proof. For $1 \leq \alpha \leq N-1$, we get by looking at the terms in $\lambda^{N+\alpha}\mu^N$ and $\lambda^N\mu^{N+\alpha}$ in (4.14) that

$$\left(\frac{\partial}{\partial u_\alpha}\right)_L (f) \bullet_3 S^\alpha(f) = 0, \quad \left(\frac{\partial}{\partial u_\alpha}\right)_R (f) \bullet_1 S^\alpha(f) = 0.$$

More explicitly, we expand the above identities using (2.10) and get

$$\left(\frac{\partial f'}{\partial u_\alpha}\right)' S^\alpha(f') \otimes S^\alpha(f'') \left(\frac{\partial f'}{\partial u_\alpha}\right)'' \otimes f'' = 0, \quad f' \otimes \left(\frac{\partial f''}{\partial u_\alpha}\right)' S^\alpha(f') \otimes S^\alpha(f'') \left(\frac{\partial f''}{\partial u_\alpha}\right)'' = 0.$$

Since $f \neq 0$, $\frac{\partial f'}{\partial u_\alpha} = 0$ and $\frac{\partial f''}{\partial u_\alpha} = 0$ for $1 \leq \alpha \leq N-1$, so that $f = f'(u, u_N) \otimes f''(u, u_N)$. \square

Proof of Proposition 4.8. Vanishing of the coefficient of $\lambda^{2N}\mu^N$ in the LHS of (4.14) gives the equation

$$\left(\frac{\partial}{\partial u_N}\right)_L (f) \bullet_3 S^N f = \left(\left(\frac{\partial}{\partial u_N}\right)_L (S^N f) \bullet_3 f^\sigma\right)^{\sigma^2}.$$

By Lemma 4.6(a) with $k = N$ and $S^N f$ in place of g we get

$$f' = s(u)u_N + q(u), \quad S^N f' = u_N r(u_{2N}) + p(u_{2N}), \quad sp = qr,$$

where $s, q \in \mathbb{k}[u] = \ker \frac{\partial}{\partial u_N} \cap \mathbb{k}\langle u, u_N \rangle$ and $r, p \in \mathbb{k}[u_{2N}] = \ker \frac{\partial}{\partial u_N} \cap \mathbb{k}\langle u_N, u_{2N} \rangle$. The condition $sp = qr$ then implies that $f' = (\alpha u + \beta)(\alpha u_N + \beta)$ for some $\alpha, \beta \in \mathbb{k}$. Similarly, vanishing of the coefficient of $\lambda^N\mu^{2N}$ in the LHS of (4.14) and using (4.9) gives the equation

$$\left(\frac{\partial}{\partial u_N}\right)_R (f) \bullet_1 S^N f = \left(\left(\frac{\partial}{\partial u_N}\right)_L (S^N f^\sigma) \bullet_3 f^\sigma\right)^{\sigma^2},$$

which, again by Lemma 4.6(b) gives $f'' = (\gamma u_N + \delta)(\gamma u + \delta)$. Hence,

$$\begin{aligned} f &= f' \otimes f'' = (\alpha u + \beta)(\alpha u_N + \beta) \otimes (\gamma u_N + \delta)(\gamma u + \delta) \\ &= ((\alpha u + \beta) \otimes (\gamma u + \delta)) \bullet S^N ((\alpha u + \beta) \otimes (\gamma u + \delta)). \end{aligned} \quad (4.23)$$

Next, we show that $\alpha\delta = \beta\gamma$. By Proposition 4.7, this will conclude the proof of the claim. To do so, we look at the vanishing of the coefficient of $\lambda^N\mu^N$ in the LHS of (4.14). This gives the identity

$$\left(\frac{\partial}{\partial u}\right)_L (f) \bullet_3 f = \left(\frac{\partial}{\partial u}\right)_R (f) \bullet_1 f. \quad (4.24)$$

From (4.8) and (4.23) we have

$$\begin{aligned} \left(\frac{\partial}{\partial u}\right)_L(f) &= \alpha \otimes (\alpha u_N + \beta) \otimes (\gamma u_N + \delta)(\gamma u + \delta), \\ \left(\frac{\partial}{\partial u}\right)_R(f) &= (\alpha u + \beta)(\alpha u_N + \beta) \otimes (\gamma u_N + \delta) \otimes \gamma. \end{aligned} \quad (4.25)$$

Substituting equations (4.25) in (4.24) we get that f need to satisfy the identity

$$\begin{aligned} &\alpha(\alpha u + \beta)(\alpha u_N + \beta) \otimes (\gamma u_N + \delta)(\gamma u + \delta)(\alpha u_N + \beta) \otimes (\gamma u_N + \delta)(\gamma u + \delta) \\ &= (\alpha u + \beta)(\alpha u_N + \beta) \otimes (\gamma u_N + \delta)(\alpha u + \beta)(\alpha u_N + \beta) \otimes \gamma(\gamma u_N + \delta)(\gamma u + \delta), \end{aligned}$$

which is equivalent to $\alpha(\gamma u + \delta) = \gamma(\alpha u + \beta)$ and implies $\alpha\delta = \beta\gamma$. \square

Remark 4.10. In [CW2], it is shown that if \mathcal{R} is a double multiplicative Poisson vertex algebra for

$$\{u_\lambda u\} = f\lambda + g + (\lambda S)^{-1}f^\sigma,$$

then $g = 0$ and f is as in Proposition 4.8.

4.4. Double multiplicative Poisson vertex algebra structures on \mathcal{R}_2 . Let us consider $\mathcal{R}_2 = \mathbb{k}\langle u_i, v_i | i \in \mathbb{Z} \rangle$ with a double multiplicative Poisson vertex algebra structure such that $\{u_\lambda u\} = 0$ and $\{v_\lambda v\} = 0$. The following result gives a criterion for such structure, and it is proven in §4.4.1.

Proposition 4.11. *Assume that \mathcal{R}_2 is equipped with the skewsymmetric double multiplicative λ -bracket given by*

$$\{u_\lambda u\} = 0, \quad \{v_\lambda v\} = 0, \quad \{u_\lambda v\} = \sum_{k \in \mathbb{Z}} g_k \lambda^k \in \mathcal{R}_2 \otimes \mathcal{R}_2[\lambda^{\pm 1}]. \quad (4.26)$$

Then \mathcal{R}_2 is a double multiplicative Poisson vertex algebra if and only if

$$g_k = \sum_{a,b,c,d=0,1} K_{abcd}^k v^a u_k^b \otimes u_k^c v^d, \quad K_{abcd}^k \in \mathbb{k},$$

where the following conditions are satisfied:

- *for all $k, l \in \mathbb{Z}$ distinct and for any $a, b, c, d, a', b', c', d' \in \{0, 1\}$,*

$$K_{1bcd}^k K_{a'b'c'0}^l = K_{0bcd}^k K_{a'b'c'1}^l, \quad (4.27a)$$

$$K_{ab1d}^k K_{a'0c'd'}^l = K_{ab0d}^k K_{a'1c'd'}^l; \quad (4.27b)$$

- *for any $k \in \mathbb{Z}$ and for any $a, b, c, d \in \{0, 1\}$,*

$$K_{ab1\epsilon}^k K_{\epsilon 0cd}^k = K_{ab0\epsilon}^k K_{\epsilon 1cd}^k, \quad \forall \epsilon = 0, 1, \quad (4.28a)$$

$$K_{ab\epsilon 0}^k K_{1\epsilon cd}^k = K_{ab\epsilon 1}^k K_{0\epsilon cd}^k, \quad \forall \epsilon = 0, 1, \quad (4.28b)$$

$$K_{ab10}^k K_{10cd}^k = K_{ab01}^k K_{01cd}^k, \quad (4.28c)$$

$$K_{ab00}^k K_{11cd}^k = K_{ab11}^k K_{00cd}^k. \quad (4.28d)$$

Example 4.12. If only finitely many coefficients $\alpha_k := K_{1111}^k \in \mathbb{k}$ are non-zero, we get that

$$\{u_\lambda v\} = \sum_{k \in \mathbb{Z}} \alpha_k v u_k \otimes u_k v \lambda^k, \quad (4.29)$$

yields a double multiplicative Poisson vertex algebra. Indeed, all the conditions gathered in Proposition 4.11 are quadratic relations in which at least one factor on each side has an index 0. In the same way, for $\alpha_k := K_{0000}^k \in \mathbb{k}$

$$\{u_\lambda v\} = \sum_{k \in \mathbb{Z}} \alpha_k 1 \otimes 1 \lambda^k, \quad (4.30)$$

yields trivially a double multiplicative Poisson vertex algebra.

The double λ -brackets (4.29) and (4.30) are very similar to the two cases from the classification with one generator given in Proposition 4.8. The following example is quadratic and has no analogue in the case of an algebra generated by one element.

Example 4.13. For any $\alpha \in \mathbb{k}^\times$, the skewsymmetric double multiplicative λ -bracket given by

$$\{\!\{u_\lambda u\}\!\} = 0, \quad \{\!\{v_\lambda v\}\!\} = 0, \quad \{\!\{u_\lambda v\}\!\} = (v \otimes u_k + u_k \otimes v + \alpha v \otimes v + \alpha^{-1} u_k \otimes u_k) \lambda^k,$$

yields a double multiplicative Poisson vertex algebra. This can be obtained by checking the conditions from Proposition 4.11 where for fixed $k \in \mathbb{Z}$,

$$K_{1010}^k = 1 = K_{0101}^k, \quad K_{1001}^k = \alpha, \quad K_{0110}^k = \alpha^{-1}.$$

The four coefficients can not be chosen independently since, for example, (4.28c) yields

$$(K_{1010}^k)^2 = K_{1001}^k K_{0110}^k = (K_{0101}^k)^2.$$

Building on the previous example, the following result is proven in §4.4.2 and provides a classification when there is only one non-zero element g_k in (4.26).

Theorem 4.14. Assume that \mathcal{R}_2 is equipped with the skewsymmetric double multiplicative λ -bracket given by

$$\{\!\{u_\lambda u\}\!\} = 0, \quad \{\!\{v_\lambda v\}\!\} = 0, \quad \{\!\{u_\lambda v\}\!\} = g \lambda^k, \quad \text{for } g \in \mathcal{R}_2 \otimes \mathcal{R}_2, \quad k \in \mathbb{Z}. \quad (4.31)$$

Then \mathcal{R}_2 is a double multiplicative Poisson vertex algebra if and only if after a translation

$$(u, v) \mapsto (u + \mu, v + \nu), \quad \mu, \nu \in \mathbb{k}, \quad (4.32)$$

the element g satisfies exactly one of the following five conditions:

- (i) $g = a 1 \otimes 1$, $a \in \mathbb{k}$;
- (ii) $g = a v \otimes v$, $a \in \mathbb{k}^\times$;
- (iii) $g = a u_k \otimes u_k$, $a \in \mathbb{k}^\times$;
- (iv) $g = a v \otimes v + b [v \otimes u_k + u_k \otimes v] + \frac{b^2}{a} u_k \otimes u_k$, $a, b \in \mathbb{k}^\times$;
- (v) $g = a v u_k \otimes u_k v + b [v u_k \otimes 1 + 1 \otimes u_k v] + \frac{b^2}{a} 1 \otimes 1$, $a \in \mathbb{k}^\times$, $b \in \mathbb{k}$.

Note that the distinct cases can not be related through (4.32), but some are equivalent if one uses linear transformations. Indeed, cases (ii) and (iii) in Theorem 4.14 are related through

$$(u, v) \mapsto (v, u), \quad a \mapsto -a, \quad k \mapsto -k,$$

while cases (ii) and (iv) are equivalent under the map $(u, v) \mapsto (u, v - \frac{b}{a} u_k)$.

Remark 4.15. If we have a double λ -bracket satisfying Theorem 4.14 of the form

$$\{\!\{u_\lambda u\}\!\} = 0, \quad \{\!\{v_\lambda v\}\!\} = 0, \quad \{\!\{u_\lambda v\}\!\} = g_0 \in \mathcal{R}_2 \otimes \mathcal{R}_2,$$

then it defines a double Poisson bracket that is compatible with the automorphism S . Note that quadratic double Poisson brackets on free algebras are classified in [ORS]. Modulo a linear transformation, the three quadratic cases from Theorem 4.14 (with $k = 0$) are all equivalent to

$$\{\!\{u, u\}\!\} = 0, \quad \{\!\{v, v\}\!\} = 0, \quad \{\!\{u, v\}\!\} = v \otimes v,$$

which corresponds to Case 4 in [ORS, Theorem 1]. We also note that the quartic case from condition (v) in Theorem 4.14 with $k = 0$ is a new example of double Poisson bracket, to the best of our knowledge.

Remark 4.16. In view of the theory presented in Section 6, the interest of Proposition 4.11 lies in the fact that these λ -brackets give rise to commuting families of differential-difference equations as we have $m \{\!\{u_\lambda^k u^l\}\!\} = 0$ trivially. The same holds with v replacing u .

4.4.1. *Proof of Proposition 4.11.* We assume that $\{\{u_\lambda v\}\} = \sum_k g_k \lambda^k \neq 0$ from now on. We first remark that the Jacobi identity (3.8b) with $a = b = u$ and $c = v$ has only its first two terms which are non-zero since $\{\{u_\lambda u\}\} = 0$. Denoting it $\{\{u_\lambda u_\mu v\}\}$, we can then compute that

$$\begin{aligned} \{\{u_\lambda u_\mu v\}\} &= \sum_{k,l,n \in \mathbb{Z}} \lambda^{n+l} \mu^k \left(\frac{\partial}{\partial v_n} \right)_L (g_k) \bullet_3 S^n(g_l) \\ &\quad - \sum_{j,m,s \in \mathbb{Z}} \lambda^j \mu^{m+s} \left(\frac{\partial}{\partial v_m} \right)_R (g_j) \bullet_1 S^m(g_s), \end{aligned} \quad (4.33)$$

which must vanish. Similarly, we compute

$$\begin{aligned} \{\{v_\lambda v_\mu u\}\} &= \sum_{k,l,n \in \mathbb{Z}} \lambda^{n+l} \mu^k S^k \left(\frac{\partial}{\partial u_{n-k}} \right)_L (g_{-k}^\sigma) \bullet_3 S^{n+l}(g_{-l}^\sigma) \\ &\quad - \sum_{j,m,s \in \mathbb{Z}} \lambda^j \mu^{m+s} S^j \left(\frac{\partial}{\partial u_{m-j}} \right)_R (g_{-j}^\sigma) \bullet_1 S^{m+s}(g_{-s}^\sigma), \end{aligned} \quad (4.34)$$

which must also vanish.

Lemma 4.17. *For any $k \in \mathbb{Z}$, g_k depends only on v and u_k .*

Proof. Since $\{\{u_\lambda v\}\} \in \mathcal{R}_2 \otimes \mathcal{R}_2[\lambda^{\pm 1}]$, there exists

$$N_+ = \max\{k \mid g_k \neq 0\}, \quad N_- = \min\{k \mid g_k \neq 0\}.$$

We now adapt [DSKVW2, Lemma 2.6]. Denoting $g_k = g'_k \otimes g''_k$, we introduce for all k

$$i'_k = \max\{i \mid \frac{\partial g'_k}{\partial v_i} \neq 0\}, \quad j'_k = \min\{i \mid \frac{\partial g'_k}{\partial v_i} \neq 0\}.$$

Assuming that $i'_k > 0$, we see that the term in $\lambda^{i'_k + N_+} \mu^k$ in (4.33) is $\left(\frac{\partial}{\partial v_{i'_k}} \right)_L (g_k) \bullet_3 S^{i'_k}(g_{N_+})$, which is non-zero by assumption, a contradiction. Thus $i'_k \leq 0$. Similarly, if $j'_k < 0$, we look at the term in $\lambda^{j'_k + N_-} \mu^k$ in (4.33) and get a contradiction. Thus the dependence of g'_k on the (v_s) is only on $v = v_0$.

In the exact same way, introduce

$$i''_k = \max\{i \mid \frac{\partial g''_k}{\partial v_i} \neq 0\}, \quad j''_k = \min\{i \mid \frac{\partial g''_k}{\partial v_i} \neq 0\}.$$

If $i''_k > 0$ or $j''_k < 0$, we look at the terms in $\lambda^k \mu^{i''_k + N_+}$ or $\lambda^k \mu^{j''_k + N_-}$ in (4.33) and get contradictions. Thus g_k depends only on v and the (u_s) .

Next, we do the same with (4.34). For

$$r''_k = \max\{i \mid \frac{\partial g''_k}{\partial u_i} \neq 0\}, \quad s''_k = \min\{i \mid \frac{\partial g''_k}{\partial u_i} \neq 0\},$$

we note that if $r''_k > k$ or $s''_k < k$, then $k + r''_{-k} > 0$ or $k + s''_{-k} < 0$, so that by looking at the term in $\lambda^{k+r''_{-k}-N_-} \mu^k$ or $\lambda^{k+s''_{-k}-N_+} \mu^k$ in (4.34) we get contradictions. Similarly, for

$$r'_k = \max\{i \mid \frac{\partial g'_k}{\partial u_i} \neq 0\}, \quad s'_k = \min\{i \mid \frac{\partial g'_k}{\partial u_i} \neq 0\},$$

we get contradictions if $r'_k > k$ or $s'_k < k$ by looking at the terms in $\lambda^k \mu^{k+r'_{-k}-N_-}$ or $\lambda^k \mu^{k+s'_{-k}-N_+}$ in (4.34). Hence g_k can only depend on u_k . \square

Lemma 4.18. *We have that*

$$g_k = \sum_{a,b,c,d=0,1} K_{abcd}^k v^a u_k^b \otimes u_k^c v^d, \quad K_{abcd}^k \in \mathbb{k}.$$

Proof. Due to the previous lemma, we can write (4.33) as

$$\{u_\lambda u_\mu v\} = \sum_{k,l \in \mathbb{Z}} \lambda^l \mu^k \left[\left(\frac{\partial}{\partial v} \right)_L (g_k) \bullet_3 g_l - \left(\frac{\partial}{\partial v} \right)_R (g_l) \bullet_1 g_k \right]. \quad (4.35)$$

This vanishes if the following coefficient of $\lambda^l \mu^k$ is zero :

$$T_{l,k}^1 := \left(\frac{\partial g'_k}{\partial v} \right)' g'_l \otimes g''_l \left(\frac{\partial g'_k}{\partial v} \right)'' \otimes g''_k - g'_l \otimes \left(\frac{\partial g''_l}{\partial v} \right)' g'_k \otimes g''_k \left(\frac{\partial g''_l}{\partial v} \right)'' . \quad (4.36)$$

We can look at the first and third copies in the tensor product, and we see that to have cancellations we need

$$\left(\frac{\partial g'_k}{\partial v} \right)' \in \mathbb{K}, \quad \left(\frac{\partial g''_l}{\partial v} \right)'' \in \mathbb{K}.$$

These conditions mean that we must have for any k that

$$g_k = vp_{k,1}(u_k)v + vp_{k,2}(u_k) + p_{k,3}(u_k)v + p_{k,4}(u_k),$$

for some $p_{k,i}(z) \in \mathbb{K}[z] \otimes \mathbb{K}[z]$.

Next, we remark that we can write (4.34) as

$$\{v_\lambda v_\mu u\} = \sum_{k,l \in \mathbb{Z}} \lambda^l \mu^k \left[S^k \left(\frac{\partial}{\partial u_{-k}} \right)_L (g_{-k}^\sigma) \bullet_3 S^l (g_{-l}^\sigma) - S^l \left(\frac{\partial}{\partial u_{-l}} \right)_R (g_{-l}^\sigma) \bullet_1 S^k (g_{-k}^\sigma) \right]. \quad (4.37)$$

So the term in $\lambda^l \mu^k$ is

$$\begin{aligned} T_{l,k}^2 := & \left(\frac{\partial S^k(g_{-k}^\sigma)}{\partial u} \right)' S^l(g_{-l}^\sigma) \otimes S^l(g_{-l}^\sigma) \left(\frac{\partial S^k(g_{-k}^\sigma)}{\partial u} \right)'' \otimes S^k(g_{-k}^\sigma) \\ & - S^l(g_{-l}^\sigma) \otimes \left(\frac{\partial S^l(g_{-l}^\sigma)}{\partial u} \right)' S^k(g_{-k}^\sigma) \otimes S^k(g_{-k}^\sigma) \left(\frac{\partial S^l(g_{-l}^\sigma)}{\partial u} \right)'' . \end{aligned} \quad (4.38)$$

Looking at the first and third copies in the tensor product, we see that we need

$$\left(\frac{\partial S^k(g_{-k}^\sigma)}{\partial u} \right)' \in \mathbb{K}, \quad \left(\frac{\partial S^l(g_{-l}^\sigma)}{\partial u} \right)'' \in \mathbb{K}.$$

These conditions mean that we must have for any k that

$$g_{-k}^\sigma = u_{-k}q_{k,1}(v)u_{-k} + u_{-k}q_{k,2}(v) + q_{k,3}(v)u_{-k} + q_{k,4}(v),$$

for some $q_{k,i}(z) \in \mathbb{K}[z] \otimes \mathbb{K}[z]$. Gathering both conditions yield the result. \square

To finish the proof of the proposition, we remark that the two conditions (4.28c)–(4.28d) with fixed $k \in \mathbb{Z}$ and $a, b, c, d \in \{0, 1\}$ are equivalent to the two identities

$$K_{ab10}^k K_{10cd}^k + K_{ab11}^k K_{00cd}^k = K_{ab00}^k K_{11cd}^k + K_{ab01}^k K_{01cd}^k, \quad (4.39a)$$

$$K_{ab10}^k K_{10cd}^k + K_{ab00}^k K_{11cd}^k = K_{ab11}^k K_{00cd}^k + K_{ab01}^k K_{01cd}^k. \quad (4.39b)$$

Lemma 4.19. *We have $\{u_\lambda u_\mu v\} = 0$ if and only if the identities (4.27a), (4.28b) and (4.39b) hold.*

Proof. The vanishing of $\{u_\lambda u_\mu v\}$ is equivalent to the vanishing of each $T_{l,k}^1$ (4.36). Plugging the form of g_k in it, we get that

$$\sum_{a_i b_i c_i d_i = 0,1} K_{a_1 b_1 c_1 d_1}^k K_{a_2 b_2 c_2 d_2}^l \left[a_1 v^{a_2} u_l^{b_2} \otimes u_l^{c_2} v^{d_2} u_k^{b_1} \otimes u_k^{c_1} v^{d_1} - d_2 v^{a_2} u_l^{b_2} \otimes u_l^{c_2} v^{a_1} u_k^{b_1} \otimes u_k^{c_1} v^{d_1} \right].$$

So the factor appearing in front of the term $v^{a_2} u_l^{b_2} \otimes u_l^{c_2} v^e u_k^{b_1} \otimes u_k^{c_1} v^{d_1}$ is

$$K_{1b_1 c_1 d_1}^k K_{a_2 b_2 c_2 e}^l - K_{eb_1 c_1 d_1}^k K_{a_2 b_2 c_2 1}^l. \quad (4.40)$$

If $e = +1$, this factor must vanish, but this is always true. If $e = 0$ instead and $k \neq l$, (4.40) must vanish, and this is equivalent to (4.27a). If $e = 0$ and $k = l$, note that the terms with the

same $b_1 + c_2$ add up since the second factor of the tensor product becomes $u_k^{b_1+c_2}$. If $c_2 = b_1 = 0$ or $c_2 = b_1 = +1$, we get that (4.40) vanishes which is equivalent to (4.28b).

If $c_2 + b_1 = +1$, we sum up the coefficients (4.40) for $c_2 = 0, b_1 = +1$ and $c_2 = +1, b_1 = 0$, and this sum must vanish. This is (4.39b). \square

In the same way, we prove the next result :

Lemma 4.20. *We have $\{\{v_\lambda v_\mu u\}\} = 0$ if and only if the identities (4.27b), (4.28a) and (4.39a) hold.*

Proof. The vanishing of $\{\{v_\lambda v_\mu u\}\}$ is equivalent to the vanishing of each $T_{l,k}^2$ (4.38), which can be simplified as

$$\sum_{a_i b_i c_i d_i = 0,1} [K_{a_1 b_1 1 d_1}^k K_{a_2 0 c_2 d_2}^l - K_{a_1 b_1 0 d_1}^k K_{a_2 0 c_2 d_2}^l] u^{c_2} v_l^{d_2} \otimes v_l^{a_2} v_k^{d_1} \otimes v_k^{a_1} u^{b_1}. \quad (4.41)$$

The terms with $k \neq l$ must vanish and are equivalent to (4.27b). When $k = l$, analysing the cases for $a_2 + d_1 \in \{0, 1, 2\}$ gives (4.28a) and (4.39a). \square

This finishes the proof of Proposition 4.11.

4.4.2. *Proof of Theorem 4.14.* By Proposition 4.11, we can write

$$g = \sum_{a,b,c,d=0,1} K_{abcd} v^a u_k^b \otimes u_k^c v^d, \quad K_{abcd} \in \mathbb{k},$$

and the constants K_{abcd} satisfy (4.28a)–(4.28d) (with the index k omitted). We will repeatedly need the following identities, which are special cases of (4.28a)–(4.28d) containing squares:

$$K_{1011}^2 = K_{1111} K_{1001} = K_{1101}^2, \quad K_{0010}^2 = K_{0000} K_{0110} = K_{0100}^2, \quad (4.42a)$$

$$K_{1110}^2 = K_{1111} K_{0110} = K_{0111}^2, \quad K_{1000}^2 = K_{0000} K_{1001} = K_{0001}^2, \quad (4.42b)$$

$$K_{1100}^2 = K_{1111} K_{0000} = K_{0011}^2, \quad K_{1010}^2 = K_{1001} K_{0110} = K_{0101}^2. \quad (4.42c)$$

A. $K_{1111} = 0$. We must have

$$g = K_{0110} u_k \otimes u_k + K_{1001} v \otimes v + K_{1010} v \otimes u_k + K_{0101} u_k \otimes v \\ + K_{1000} v \otimes 1 + K_{0100} u_k \otimes 1 + K_{0010} 1 \otimes u_k + K_{0001} 1 \otimes v + K_{0000} 1 \otimes 1.$$

A.1. Assume furthermore that $K_{1001} = K_{0110} = 0$. All the coefficients except K_{0000} must be zero, and the latter can take an arbitrary value while (4.28a)–(4.28d) are satisfied. This is case (i).

A.2. Assume furthermore that $K_{0110} = 0, K_{1001} \neq 0$. We must have

$$g = K_{1001} v \otimes v + K_{1000} v \otimes 1 + K_{0001} 1 \otimes v + K_{0000} 1 \otimes 1.$$

By (4.28b), $K_{1000} K_{1001} = K_{1001} K_{0001}$ so that $K_{1000} = K_{0001}$. Up to making the translation $v \mapsto v - \frac{K_{1000}}{K_{1001}}$, we can assume that $g = K_{1001} v \otimes v + K_{0000} 1 \otimes 1$. Using (4.42b), $K_{0000} = 0$ and all the conditions (4.28a)–(4.28d) are satisfied for an arbitrary K_{1001} ; this is case (ii).

A.3. Assume furthermore that $K_{1001} = 0, K_{0110} \neq 0$. By an argument similar to **A.2**, we are in case (iii) of the statement after a translation.

A.4. If $K_{1001} \neq 0$ and $K_{0110} \neq 0$, we can adapt the previous arguments to reduce to the case

$$g = K_{0110} u_k \otimes u_k + K_{1001} v \otimes v + K_{1010} v \otimes u_k + K_{0101} u_k \otimes v,$$

after a translation. Note that the second identity in (4.42c) must hold, hence the four coefficients are non-zero. Moreover, we get from (4.28c) that $K_{1001} K_{1010} = K_{1001} K_{0101}$, from which $K_{1010} = K_{0101}$. We must then be in case (iv), and it is easy to see that such a form will always satisfy the conditions (4.28a)–(4.28d).

B. $K_{1111} \neq 0$. We have as special cases of (4.28a) and (4.28b) that

$$K_{1111} K_{1011} = K_{1101} K_{1111}, \quad K_{1110} K_{1111} = K_{1111} K_{0111}.$$

Hence after the translation $u \mapsto u - \frac{K_{1011}}{K_{1111}}$, $v \mapsto v - \frac{K_{0111}}{K_{1111}}$, we can assume that there is no term in g which is cubic in (u_k, v) . This in turn implies that

$$g = K_{1111} vu_k \otimes u_k v + K_{1100} vu_k \otimes 1 + K_{0011} 1 \otimes u_k v + K_{0000} 1 \otimes 1.$$

The remaining terms are subject to the first identity in (4.42c), as well as $K_{0000}K_{1100} = K_{0011}K_{0000}$ which is a special case of (4.28d). Therefore g must be of the form given in case (v), and it can be checked that g satisfies all the identities in (4.28a)–(4.28d).

5. RELATION TO REPRESENTATION SPACES

Given an associative algebra \mathcal{V} , recall that for $N \geq 1$ we can form the N -th representation algebra \mathcal{V}_N defined in §2.2.3. We also have that each $S \in \text{Aut}(\mathcal{V})$ induces an automorphism of \mathcal{V}_N from its definition on generators by $S(a_{ij}) = (S(a))_{ij}$. We will prove the analogue of Theorem 3.6 and [DSKV, §3.7] for multiplicative Poisson vertex algebras.

Theorem 5.1. *Assume that $\{\{-\lambda-\}\}$ is a double multiplicative λ -bracket on \mathcal{V} . Then there is a unique multiplicative λ -bracket on \mathcal{V}_N which satisfies for any $a, b \in \mathcal{V}$, $1 \leq i, j \leq N$,*

$$\{a_{ij} \lambda b_{kl}\} = \sum_{n \in \mathbb{Z}} (a_n b)'_{kj} (a_n b)''_{il} \lambda^n, \quad \text{where } \{a_\lambda b\} = \sum_{n \in \mathbb{Z}} ((a_n b)' \otimes (a_n b)'') \lambda^n. \quad (5.1)$$

Furthermore, if $(\mathcal{V}, \{\{-\lambda-\}\})$ is a double multiplicative Poisson vertex algebra, then $(\mathcal{V}_N, \{-\lambda-\})$ is a multiplicative Poisson vertex algebra.

Proof. We begin by proving the first part, which is similar to [DSKV, Prop. 3.20].

The operation $\{-\lambda-\}$ given by equation (5.1) is defined on generators and it is extended uniquely to all elements of \mathcal{V}_N by the Leibniz rules (2.1b)–(2.1c). To ensure that $\{-\lambda-\}$ is well-defined, we need to show

$$\{a_{ij} \lambda (bc)_{kl}\} = \sum_{u=1}^N \{a_{ij} \lambda b_{cu} c_{ul}\}, \quad (5.2)$$

and do the same with respect to the first entry. To see that (5.2) holds, we compute the left-hand side using (3.4b) as follows :

$$\begin{aligned} \{a_{ij} \lambda (bc)_{kl}\} &= \sum_{n \in \mathbb{Z}} (a_n bc)'_{kj} (a_n bc)''_{il} \lambda^n \\ &= \sum_{n \in \mathbb{Z}} \left[(a_n b)'_{kj} ((a_n b)'' c)_{il} + (b (a_n c)')_{kj} (a_n c)''_{il} \right] \lambda^n \\ &= \sum_{n \in \mathbb{Z}} \sum_{u=1}^N \left[(a_n b)'_{kj} (a_n b)''_{iu} c_{ul} + b_{ku} (a_n c)'_{uj} (a_n c)''_{il} \right] \lambda^n. \end{aligned}$$

The same result can easily be obtained for the right-hand side of (5.2) using the Leibniz rule.

To get that $\{-\lambda-\}$ defined by (5.1) is a multiplicative λ -bracket, it remains to check sesquilinearity (2.1a). This will follow if we can show that

$$\{S(a_{ij})_\lambda b_{kl}\} = \lambda^{-1} \{a_{ij} \lambda b_{kl}\}, \quad \{a_{ij} \lambda S(b_{kl})\} = \lambda S(\{a_{ij} \lambda b_{kl}\}). \quad (5.3)$$

For the first identity in (5.3), we have

$$\begin{aligned} \{S(a_{ij})_\lambda b_{kl}\} &= \{S(a)_{ij} \lambda b_{kl}\} = \sum_{n \in \mathbb{Z}} (S(a)_n b)'_{kj} (S(a)_n b)''_{il} \lambda^n \\ &= \sum_{n \in \mathbb{Z}} \lambda^{-1} (a_n b)'_{kj} (a_n b)''_{il} \lambda^n = \lambda^{-1} \{a_{ij} \lambda b_{kl}\}, \end{aligned}$$

where we used (3.4a) for the third equality. The second identity in (5.3) is checked in the same way.

If we have a double multiplicative Poisson vertex algebra, we use (3.8a) in the form

$$\sum_{n \in \mathbb{Z}} (a_n b)' \otimes (a_n b)'' \lambda^n = - \sum_{n \in \mathbb{Z}} \lambda^{-n} S^{-n} ((b_n a)'' \otimes (b_n a)')$$

to get that

$$\begin{aligned} \{a_{ij} \lambda b_{kl}\} &= \sum_{n \in \mathbb{Z}} (a_n b)'_{kj} (a_n b)''_{il} \lambda^n = - \sum_{n \in \mathbb{Z}} \lambda^{-n} S^{-n} ((b_n a)''_{kj} (b_n a)'_{il}) \\ &= - \sum_{n \in \mathbb{Z}} \lambda^{-n} S^{-n} ((b_n a)'_{il} (b_n a)''_{kj}) = - \left(\big|_{x=S} \{b_{kl} \lambda^{-1} x^{-1} a_{ij}\} \right), \end{aligned}$$

which gives that $\{-\lambda-\}$ defined by (5.1) satisfies the skewsymmetry property (2.2a). Then, we can conclude because Jacobi identity (2.2b) holds by Lemma 5.2 whenever (3.8b) does. \square

For the next lemma, we introduce some notations to go from $\mathcal{V}^{\otimes 3}$ to \mathcal{V}_N . For any $A = a' \otimes a'' \otimes a''' \in \mathcal{V}^{\otimes 3}$, and $1 \leq i, j, k, l, m, n \leq N$, we define

$$A_{ij,kl,mn} := a'_{ij} a''_{kl} a'''_{mn}.$$

We extend this operation in the obvious way to associate $A_{ij,kl,mn} \in \mathcal{V}_N[\lambda^{\pm 1}, \mu^{\pm 1}]$ to any $A \in \mathcal{V}^{\otimes 3}[\lambda^{\pm 1}, \mu^{\pm 1}]$.

Lemma 5.2. *For any $a, b, c \in \mathcal{V}$ and $1 \leq i, j, k, l, u, v \leq N$, if $\{\{-\lambda-\}\}$ is a double λ -bracket such that (3.8a) holds, then we have that*

$$\{a_{ij} \lambda \{b_{kl} \mu c_{uv}\}\} - \{b_{kl} \lambda \{a_{ij} \mu c_{uv}\}\} - \{\{a_{ij} \lambda b_{kl}\} \lambda \mu c_{uv}\} = \{\{a_{ij} \lambda b_{kl}\} \lambda \mu c_{uv}\} - \{\{b_{kl} \lambda a_{ij}\} \lambda \mu c_{uv}\},$$

where $\{-\lambda-\}$ is defined by (5.1), while $\{\{-\lambda-\mu-\}\}$ is given by (3.10).

Proof. Using the Leibniz rules for a multiplicative λ -bracket and the definition (5.1), we have

$$\begin{aligned} \{a_{ij} \lambda \{b_{kl} \mu c_{uv}\}\} &= \sum_{q \in \mathbb{Z}} \{a_{ij} \lambda (b_q c)'_{ul} (b_q c)''_{kv}\} \mu^q \\ &= \sum_{p, q \in \mathbb{Z}} \left[(a_p (b_q c)'_{ul})'_{ij} (a_p (b_q c)'_{ul})''_{kl} (b_q c)''_{kv} + (a_p (b_q c)''_{ul})'_{ij} (a_p (b_q c)''_{ul})''_{kl} (b_q c)'_{kv} \right] \lambda^p \mu^q. \end{aligned}$$

Similarly, $-\{b_{kl} \mu \{a_{ij} \lambda c_{uv}\}\}$ can be written as

$$\sum_{p, q \in \mathbb{Z}} \left[- (b_q (a_p c)'_{ul})'_{kl} (b_q (a_p c)'_{ul})''_{ij} (a_p c)''_{uv} - (b_q (a_p c)''_{ul})'_{kl} (b_q (a_p c)''_{ul})''_{ij} (a_p c)'_{uv} \right] \lambda^p \mu^q,$$

while $-\{\{a_{ij} \lambda b_{kl}\} \lambda \mu c_{uv}\}$ can be written as

$$\sum_{p, q \in \mathbb{Z}} \left[- ((a_p b)'_{qul})'_{ij} ((a_p b)'_{qul})''_{kl} S^q ((a_p b)''_{il}) - ((a_p b)''_{qul})'_{ij} ((a_p b)''_{qul})''_{kl} S^q ((a_p b)'_{il}) \right] \lambda^p \mu^q,$$

so that we get six terms for the left-hand side. Meanwhile, we can use (3.10) to get $\{\{a_{ij} \lambda b_{kl}\} \lambda \mu c_{uv}\}$, and we can write

$$\begin{aligned} \{\{a_{ij} \lambda b_{kl}\} \lambda \mu c_{uv}\} &= \sum_{p, q \in \mathbb{Z}} (a_p (b_q c)'_{ul})'_{ij} (a_p (b_q c)'_{ul})''_{kl} (b_q c)''_{kv} \lambda^p \mu^q \\ &\quad - \sum_{p, q \in \mathbb{Z}} (a_p c)'_{ul} (b_q (a_p c)''_{ul})'_{ij} (b_q (a_p c)''_{ul})''_{kl} \lambda^p \mu^q \\ &\quad - \sum_{p, q \in \mathbb{Z}} ((a_p b)'_{qul})'_{ij} S^q ((a_p b)''_{il}) ((a_p b)'_{qul})''_{kl} \lambda^{p+q} \mu^q. \end{aligned}$$

If we use (3.11) to expand $\{\{b_{kl} \lambda a_{ij}\} \lambda \mu c_{uv}\}$, we can write

$$\begin{aligned} -\{\{b_{kl} \lambda a_{ij}\} \lambda \mu c_{uv}\} &= - \sum_{p, q \in \mathbb{Z}} (b_q (a_p c)'_{ul})'_{kl} (b_q (a_p c)'_{ul})''_{ij} (a_p c)''_{uv} \lambda^p \mu^q \\ &\quad + \sum_{p, q \in \mathbb{Z}} (b_q c)'_{ul} (a_p (b_q c)''_{ul})'_{kl} (a_p (b_q c)''_{ul})''_{ij} \lambda^p \mu^q \\ &\quad - \sum_{p, q \in \mathbb{Z}} ((a_p b)''_{qul})'_{kl} S^q ((a_p b)'_{il}) ((a_p b)''_{qul})''_{ij} \lambda^{p+q} \mu^q. \end{aligned}$$

It now suffices to see that the left- and right-hand sides coincide since $S(d_{mn}) = S(d)_{mn}$ for any $d \in A$ and indices $1 \leq m, n \leq N$. \square

Example 5.3. Using \mathcal{V} from Example 3.16, we get for $N \geq 1$ the representation algebra $\mathcal{V}_N = \mathbb{k}[u_{m,ij} \mid m \in \mathbb{Z}, 1 \leq i, j \leq N]$ which is a multiplicative Poisson vertex algebra by Theorem 5.1. The automorphism on \mathcal{V}_N is given by $S(u_{m,ij}) = u_{m+1,ij}$, and the multiplicative λ -bracket satisfies

$$\{u_{m,ij}\lambda u_{n,kl}\} = \lambda^{n-m}(u_{n,kj}\delta_{il} - u_{n,il}\delta_{kj}).$$

Example 5.4. Combining Proposition 4.8 with $\mathcal{V} = \mathcal{R}_1$ and Theorem 5.1, we get for $N \geq 1$ that the representation algebra \mathcal{V}_N (as in Example 5.3) is a multiplicative Poisson vertex algebra for the multiplicative λ -bracket

$$\{u_{ij}\lambda u_{kl}\} = (uu_M)_{kj}(u_M u)_{il}\lambda^M - (uu_{-M})_{kj}(u_{-M} u)_{il}\lambda^{-M},$$

where $u_{ij} := u_{0,ij}$ and $M \geq 1$. For $M = N = 1$, if we set $u := u_{11}$ we get that the commutative polynomial algebra in one variable $\mathbb{k}[u_m \mid m \in \mathbb{Z}]$ is equipped with the following multiplicative λ -bracket

$$\{u\lambda u\} = u^2 u_1^2 \lambda - u^2 u_{-1}^2 \lambda^{-1}. \quad (5.4)$$

This can be seen as the “square” of the λ -bracket for the Volterra lattice [DSKVW2] given on $\mathbb{k}[v_s \mid s \in \mathbb{Z}]$ by

$$\{v\lambda v\} = vv_1\lambda - vv_{-1}\lambda^{-1}.$$

Indeed, we recover (5.4) for $u = v^2$ up to a factor.

Corollary 5.5. *The (non-commutative) correspondence between local lattice double Poisson algebras and double multiplicative Poisson vertex algebras from Proposition 3.14 induces the (commutative) correspondence between local lattice Poisson algebras and multiplicative Poisson vertex algebras from Proposition 2.3 on representation spaces.*

Proof. Fix $a, b \in \mathcal{V}$ for \mathcal{V} a local lattice double Poisson algebra with double Poisson bracket $\{\{-, -\}\}$. By Proposition 3.14, \mathcal{V} is a double multiplicative Poisson vertex algebra with multiplicative λ -bracket given in (2.4).

Using Van den Bergh’s work [VdB1], (1.1) defines a Poisson bracket on \mathcal{V}_N . It is easy to check that \mathcal{V}_N is a local lattice Poisson algebra by inducing S from \mathcal{V} to \mathcal{V}_N as in §2.2.3. Alternatively, we can use Theorem 5.1 to get a multiplicative λ -bracket on \mathcal{V}_N as follows. Recalling the first equation in (5.1), we have

$$\begin{aligned} \{a_{ij}\lambda b_{kl}\} &= \sum_{n \in \mathbb{Z}} \lambda^n (a_n b)'_{kj} (a_n b)''_{il} \stackrel{(3.19)}{=} \sum_{n \in \mathbb{Z}} \lambda^n \{\{S^n(a), b\}'_{kj}\} \{\{S^n(a), b\}''_{il}\} \\ &= \sum_{n \in \mathbb{Z}} \lambda^n \{S^n(a)_{ij}, b_{kl}\} = \sum_{n \in \mathbb{Z}} \lambda^n \{S^n(a_{ij}), b_{kl}\}, \end{aligned}$$

in agreement with (2.4).

Starting from the double multiplicative Poisson vertex algebra instead, we get in a similar fashion that the induced Poisson bracket on \mathcal{V}_N satisfies

$$\{a_{ij}, b_{kl}\} = \{\{a, b\}'_{kj}\} \{\{a, b\}''_{il}\} \stackrel{(3.18)}{=} (a_0 b)'_{kj} (a_0 b)''_{il} = \text{mRes}_\lambda \sum_{n \in \mathbb{Z}} (a_n b)'_{kj} (a_n b)''_{il} \lambda^n = \text{mRes}_\lambda \{a_{ij}\lambda b_{kl}\}.$$

This is consistent with (2.3). \square

Corollary 5.5 can be summarised in terms of the commutative diagram depicted in Figure 1.

6. CONNECTION TO INTEGRABLE SYSTEMS

It is shown in [DSKV] that double Poisson vertex algebras provide a convenient framework to study non-commutative partial differential equations. In this section we provide “multiplicative versions” of some of the results in [DSKV] aimed at showing that double multiplicative Poisson vertex algebras provide a convenient framework to study non-commutative differential-difference equations. Several examples are presented in Section 6.4.

$$\begin{array}{ccc}
(\mathcal{V}, \{\!\{-, -\}\!\}, S) & \xleftarrow{\text{Prop. 3.14}} & (\mathcal{V}, \{\!\{-_\lambda -\}\!\}, S) \\
\downarrow (1.1) & & \downarrow (5.1) \\
(\mathcal{V}_N, \{-, -\}, S) & \xleftarrow{\text{Prop. 2.3}} & (\mathcal{V}_N, \{-_\lambda -\}, S)
\end{array}$$

FIGURE 1.

6.1. The trace map and connection to Lie algebras. Let \mathcal{V} be a unital associative algebra endowed with an automorphism S of infinite order. We denote by $m : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$ the multiplication map. We denote also by $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ the commutator subspace. For $f \in \mathcal{V}$, we let $\text{tr}(f) \in \mathcal{V}/[\mathcal{V}, \mathcal{V}]$ be the corresponding coset. Furthermore, for $f \in \mathcal{V}$, we also let $\int f$ be the coset of $f \in \mathcal{V}$ in the quotient space

$$\mathcal{F} := \mathcal{V}/([\mathcal{V}, \mathcal{V}] + (S - 1)\mathcal{V}).$$

Given a double multiplicative λ -bracket $\{\!\{-_\lambda -\}\!\}$ on \mathcal{V} we define the following map $\{-_\lambda -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda, \lambda^{-1}]$ by

$$\{a_\lambda b\} = m\{\!\{a_\lambda b\}\!\} \quad (6.1)$$

(the multiplication map on $\mathcal{V} \otimes \mathcal{V}$ is extended to a multiplication map $m : (\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}] \rightarrow \mathcal{V}[\lambda, \lambda^{-1}]$ in the obvious way), and we also define the map $\{-, -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}$ by

$$\{a, b\} = m\{\!\{a_\lambda b\}\!\}|_{\lambda=1}. \quad (6.2)$$

Lemma 6.1. *If $\{\!\{-_\lambda -\}\!\}$ is a skewsymmetric double multiplicative λ -bracket on \mathcal{V} , then the following identity holds in $\mathcal{V}^{\otimes 2}[\lambda^{\pm 1}, \mu^{\pm 1}]$ ($a, b, c \in \mathcal{V}$):*

$$\{a_\lambda \{\!\{b_\mu c\}\!\}\} - \{\!\{b_\mu \{a_\lambda c\}\}\!\} - \{\!\{\{a_\lambda b\}_{\lambda\mu} c\}\!\} = (m \otimes 1)\{\!\{a_\lambda b_\mu c\}\!\} - (1 \otimes m)\{\!\{b_\mu a_\lambda c\}\!\}. \quad (6.3)$$

where we set $\{a_\lambda b \otimes c\} = \{a_\lambda b\} \otimes c + b \otimes \{a_\lambda c\}$.

Proof. Let us compute the three terms on the LHS of (6.3). We get after a straightforward computation

$$\begin{aligned}
\{a_\lambda \{\!\{b_\mu c\}\!\}\} &= (m \otimes 1)\{\!\{a_\lambda \{\!\{b_\mu c\}\!\}\}\!\}_L + (1 \otimes m)\{\!\{a_\lambda \{\!\{b_\mu c\}\!\}\}\!\}_R, \\
\{\!\{b_\mu \{a_\lambda c\}\}\!\} &= (1 \otimes m)\{\!\{b_\mu \{\!\{a_\lambda c\}\!\}\}\!\}_L + (m \otimes 1)\{\!\{b_\mu \{\!\{a_\lambda c\}\!\}\}\!\}_R, \\
\{\!\{\{a_\lambda b\}_{\lambda\mu} c\}\!\} &= (m \otimes 1)\{\!\{\{\!\{a_\lambda b\}\!\}_{\lambda\mu} c\}\!\}_L + (1 \otimes m)\{\!\{\{\!\{a_\lambda b\}\!\}_{\lambda\mu} c\}\!\}_R.
\end{aligned}$$

By skewsymmetry and Lemma 3.11, we can replace the last term in the RHS of the third equation by $-(1 \otimes m)\{\!\{\{\!\{b_\mu a\}\!\}_{\lambda\mu} c\}\!\}_L$. Combining the three equations above, we then get (6.3). \square

Theorem 6.2. *Let \mathcal{V} be a unital associative algebra with an automorphism S , endowed with a double multiplicative λ -bracket $\{\!\{-_\lambda -\}\!\}$. Let $\{-_\lambda -\}$ and $\{-, -\}$ be defined as in (6.1) and (6.2).*

- (a) $S[\mathcal{V}, \mathcal{V}] \subset [\mathcal{V}, \mathcal{V}]$. Hence, we have a well defined induced map (denoted, by abuse of notation, by the same symbol) $S : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V}/[\mathcal{V}, \mathcal{V}]$, given by $S(\text{tr } f) = \text{tr}(Sf)$.
- (b) $\{[\mathcal{V}, \mathcal{V}]_\lambda \mathcal{V}\} = 0$, and $\{\mathcal{V}_\lambda [\mathcal{V}, \mathcal{V}]\} \subset [\mathcal{V}, \mathcal{V}] \otimes \mathbb{k}[\lambda^{\pm 1}]$. Hence, we have well defined induced maps (denoted, by abuse of notation, by the same symbol)

$$\{-_\lambda -\} : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \times \mathcal{V} \rightarrow \mathcal{V}[\lambda^{\pm 1}],$$

and

$$\{-_\lambda -\} : \mathcal{V}/[\mathcal{V}, \mathcal{V}] \times \mathcal{V}/[\mathcal{V}, \mathcal{V}] \rightarrow \mathcal{V}/[\mathcal{V}, \mathcal{V}][\lambda^{\pm 1}],$$

given, respectively, by

$$\{\text{tr}(f)_\lambda g\} = m\{\!\{f_\lambda g\}\!\}, \quad (6.4)$$

and

$$\{\text{tr}(f)_\lambda \text{tr}(g)\} = \text{tr}(m\{\!\{f_\lambda g\}\!\}). \quad (6.5)$$

(c) If the double multiplicative λ -bracket is skewsymmetric, then so is the λ -bracket (6.5):

$$\{\mathrm{tr}(f)_\lambda \mathrm{tr}(g)\} = -\{\mathrm{tr}(g)_{\lambda^{-1}S^{-1}} \mathrm{tr}(f)\}.$$

(d) If the double multiplicative λ -bracket defines a structure of a double multiplicative Poisson vertex algebra on \mathcal{V} , then the multiplicative λ -bracket (6.5) satisfies the Jacobi identity ($f, g, h \in \mathcal{V}$)

$$\{\mathrm{tr}(f)_\lambda \{\mathrm{tr}(g)_\mu \mathrm{tr}(h)\}\} - \{\mathrm{tr}(g)_\mu \{\mathrm{tr}(f)_\lambda \mathrm{tr}(h)\}\} = \{\{\mathrm{tr}(f)_\lambda \mathrm{tr}(g)\}_{\lambda\mu} \mathrm{tr}(h)\}.$$

Hence, (6.5) endows $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ with a structure of multiplicative Lie conformal algebra (cf. Remark 2.4). Furthermore, the λ -action (6.4) of $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ on \mathcal{V} defines a representation of the multiplicative Lie conformal algebra $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ given by conformal derivations of \mathcal{V} .

(e) $\{[\mathcal{V}, \mathcal{V}] + (S-1)\mathcal{V}, \mathcal{V}\} = 0$, and $\{\mathcal{V}, [\mathcal{V}, \mathcal{V}] + (S-1)\mathcal{V}\} \subset ([\mathcal{V}, \mathcal{V}] + (S-1)\mathcal{V}) \otimes \mathbb{k}[\lambda^{\pm 1}]$. Thus, we have well defined induced brackets (denoted, by abuse of notation, by the same symbol)

$$\{-, -\} : \mathcal{F} \times \mathcal{V} \rightarrow \mathcal{V},$$

and

$$\{-, -\} : \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F},$$

given, respectively, by

$$\{\int f, g\} := m\{\{f_\lambda g\}\}_{\lambda=1}, \quad (6.6)$$

and

$$\{\int f, \int g\} := \int m\{\{f_\lambda g\}\}_{\lambda=1}. \quad (6.7)$$

(f) If the double multiplicative λ -bracket is skewsymmetric, then so is the bracket (6.7).

(g) If the double multiplicative λ -bracket defines a structure of a double multiplicative Poisson vertex algebra on \mathcal{V} , then the bracket (6.7) defines a structure of a Lie algebra on \mathcal{F} . Furthermore, the action of \mathcal{F} on \mathcal{V} , given by (6.6), defines a representation of the Lie algebra \mathcal{F} by derivations of \mathcal{V} commuting with S .

Proof. Part (a) is straightforward, since S is an automorphism of \mathcal{V} . Using the right Leibniz rule we have, for $a, b, c \in \mathcal{V}$,

$$\{ab_\lambda c\} = m\{\{ab_\lambda c\}\} = m\left((|_{x=Sa}) \otimes_1 \{\{b_\lambda x c\}\} + \{\{a_\lambda x c\}\} \otimes_1 (|_{x= Sb})\right).$$

The expression in parenthesis in the RHS above is unchanged if we switch a and b (since $x \otimes_1 (y' \otimes y'') = y' \otimes x \otimes y'' = (y' \otimes y'') \otimes_1 x$), so that we have $\{ab_\lambda c\} = \{ba_\lambda c\}$. Furthermore, we can compute that

$$\{a_\lambda bc\} = m\{\{a_\lambda bc\}\} = m(\{\{a_\lambda b\}\}c + b\{\{a_\lambda c\}\}) = \{a_\lambda b\}c + b\{a_\lambda c\}.$$

Namely, the λ -action $\{a_\lambda -\}$ is by derivations of the associative product of \mathcal{V} . This yields

$$\{a_\lambda bc - cb\} = [b, \{a_\lambda c\}] + [\{a_\lambda b\}, c] \in [\mathcal{V}, \mathcal{V}] \otimes \mathbb{k}[\lambda^{\pm 1}],$$

which finishes the proof of part (b). Part (c) is immediate. Part (d) is a direct consequence of Lemma 6.1. Finally, parts (e), (f) and (g) can be proven in the same way. Alternatively, one can use the standard construction that associates with a multiplicative Lie conformal algebra V the corresponding Lie algebra $V/(S-1)V$ and its representation on V . \square

Remark 6.3. Lemma 6.1 and Theorem 6.2 are multiplicative versions of [DSKV, Lemma 3.5] and [DSKV, Theorem 3.6], respectively.

6.2. Evolution differential-difference equations and Hamiltonian differential-difference equations. Let \mathcal{V} be an algebra of difference functions in the variable u_i , $i \in I = \{1, \dots, \ell\}$, see Definition 4.2. An *evolution differential-difference equation* over \mathcal{V} has the form

$$\frac{du_i}{dt} = P_i \in \mathcal{V}, \quad i \in I. \quad (6.8)$$

Assuming that time derivative commutes with the automorphism S , we have $\frac{du_{i,n}}{dt} = S^n(P_i)$, and, by the chain rule, a function $f \in \mathcal{V}$ evolves according to

$$\frac{df}{dt} = \sum_{(i,n) \in I \times \mathbb{Z}} m \left((S^n P_i) *_1 \frac{\partial f}{\partial u_{i,n}} \right) = X_P(f). \quad (6.9)$$

We call $\mathcal{F} = \mathcal{V}/([\mathcal{V}, \mathcal{V}] + (S - 1)\mathcal{V})$ the space of local functionals. An *integral of motion* is a local functional $\int f \in \mathcal{F}$ constant in time:

$$\frac{d \int f}{dt} = \int \sum_{(i,n) \in I \times \mathbb{Z}} m \left((S^n P_i) *_1 \frac{\partial f}{\partial u_{i,n}} \right) = 0. \quad (6.10)$$

We call *vector field* on \mathcal{V} any derivation $X : \mathcal{V} \rightarrow \mathcal{V}$ of the form

$$X(f) = \sum_{(i,n) \in I \times \mathbb{Z}} m \left(P_{i,n} *_1 \frac{\partial f}{\partial u_{i,n}} \right), \quad (6.11)$$

where $P_{i,n} \in \mathcal{V}$ for all i, n . Note that the RHS of (6.11) is a finite sum because $\frac{\partial f}{\partial u_{i,n}} = 0$ for all but finitely many choices of indices (i, n) , as \mathcal{V} is an algebra of difference functions.

An *evolutionary vector field* is a vector field commuting with the automorphism S . Gathering (4.9) and (6.9), it must have the form

$$X_P(f) = \sum_{(i,n) \in I \times \mathbb{Z}} m \left((S^n P_i) *_1 \frac{\partial f}{\partial u_{i,n}} \right), \quad (6.12)$$

for $P = (P_i)_{i=1}^\ell \in \mathcal{V}^\ell$, called the *characteristics* of the evolutionary vector field X_P .

Vector fields form a Lie algebra, and evolutionary vector fields form a Lie subalgebra, which we denote respectively by $\text{Vect}(\mathcal{V})$ and $\text{Vect}^S(\mathcal{V})$. The Lie bracket of two evolutionary vector fields $X_P, X_Q \in \text{Vect}^S(\mathcal{V})$ takes the usual form

$$[X_P, X_Q] = X_{[P, Q]}, \quad \text{where } [P, Q]_i = X_P(Q_i) - X_Q(P_i), i \in I.$$

Equation (6.8) is called *compatible* with another evolution differential-difference equation $\frac{du_i}{d\tau} = Q_i$, $i \in I$, if the corresponding evolutionary vector fields commute.

More generally, let \mathcal{V} be a double multiplicative Poisson vertex algebra. The *Hamiltonian equation* on \mathcal{V} , associated with the *Hamiltonian functional* $\int h \in \mathcal{F}$ is

$$\frac{du}{dt} = \{\int h, u\}, \quad (6.13)$$

for any $u \in \mathcal{V}$. An *integral of motion* for the Hamiltonian equation (6.13) is a local functional $\int f \in \mathcal{F}$ such that $\{\int h, \int f\} = 0$. In this case, by Theorem 6.2(g), the evolutionary vector fields $X^h = \{\int h, -\}$ and $X^f = \{\int f, -\}$ (called Hamiltonian vector fields) commute, hence equations $\frac{du}{dt} = \{\int f, u\}$ and (6.13) are compatible. If we are given a family of independent local functionals $\{\int f_k \mid k \in \mathbb{Z}_+\}$ whose evolutionary Hamiltonian vector fields $\{X^{f_k} \mid k \in \mathbb{Z}_+\}$ are pairwise commuting, we will say that the corresponding Hamiltonian equations defined through (6.13) form an *integrable hierarchy* of Hamiltonian differential-difference equations.

Let us now assume that \mathcal{V} is an algebra of difference functions in the variables u_i , $i \in I$, and that the double multiplicative λ -bracket $\{\!\![-\lambda-]\!\!$ on \mathcal{V} is given by a Poisson structure $H(S)$ via (4.10), where $\{\!\! [u_i \lambda u_j] \!\!\} = H_{ji}(\lambda)$. The Hamiltonian equation (6.13) becomes the following evolution equation

$$\frac{du_i}{dt} = m \sum_{j=1}^\ell H_{ij}(S) \bullet \left(\frac{\delta h}{\delta u_j} \right)^\sigma, \quad (6.14)$$

where we introduce the *difference variational derivative* $\frac{\delta h}{\delta u_j} \in \mathcal{V} \otimes \mathcal{V}$ of h by

$$\frac{\delta h}{\delta u_j} = \sum_{n \in \mathbb{Z}_+} S^{-n} \frac{\partial h}{\partial u_{j,n}}. \quad (6.15)$$

In equation (6.14) S is moved to the right of the \bullet product, acting on $\frac{\delta h}{\delta u_j}$. Moreover, the Lie bracket $\{-, -\}$ on \mathcal{F} defined by (6.7), becomes such that for all $f, g \in \mathcal{V}$:

$$\{f, g\} = \int \sum_{i,j \in I} m \left(\frac{\delta g}{\delta u_j} \right)^\sigma m \left(H_{ji}(S) *_1 m \left(\frac{\delta f}{\delta u_i} \right)^\sigma \right). \quad (6.16)$$

Then, the notions of compatibility and of integrals of motion are consistent with those for general evolution differential-difference equations, due to Theorem 4.3.

Remark 6.4. For $H(S) \in \text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[S]$ and $F \in \mathcal{V}^{\oplus \ell}$, let $H(S)F \in \mathcal{V}^\ell$ be defined by

$$(H(S)F)_i = \sum_{j \in I} m(H_{ij}(S) *_1 F_j) = \sum_{j \in I, n \in \mathbb{Z}} H'_{ij,n}(S^n F_j) H''_{ij,n}. \quad (6.17)$$

(Here, we used the notation $H_{ij}(S) = \sum_{n \in \mathbb{Z}} (H'_{ij,n} \otimes H''_{ij,n}) S^n$.) Then, formula (6.16) can be written in the more traditional form

$$\{f, g\} = \int \delta g \cdot (H(S) \delta f),$$

where $(\delta f)_i = m \left(\frac{\delta f}{\delta u_i} \right)^\sigma$ and \cdot denotes the usual dot product of vectors. The latter notation is compatible with the theory of the variational complex developed in §6.3, cf. (6.25).

Remark 6.5. Many of the results derived so far are compatible with their commutative analogues [DSKVW1, DSKVW2] when we go from \mathcal{V} to the N -th representation algebra \mathcal{V}_N , $N \geq 1$ (see Section 6). The role of $\mathcal{V}/[\mathcal{V}, \mathcal{V}]$ is played by

$$\mathcal{V}_N^{\text{tr}} := \{\text{tr } \mathcal{X}(a) = \sum_{1 \leq i \leq N} a_{ii} \mid a \in \mathcal{V}\}$$

where we denote by $\mathcal{X}(a) = (a_{ij})$ the matrix-valued function representing the element $a \in \mathcal{V}$. Similarly, we have to replace \mathcal{F} by

$$\mathcal{F}_N := \mathcal{V}_N^{\text{tr}} / ((S - 1)\mathcal{V}_N^{\text{tr}}).$$

We note two important such results (assuming that \mathcal{V} is a double multiplicative Poisson vertex algebra). First, (6.6) induces a representation of the Lie algebra \mathcal{F}_N on \mathcal{V} by derivations commuting with S through

$$\left\{ \int \text{tr } \mathcal{X}(f), \mathcal{X}(g) \right\} = \mathcal{X}(m \{f_\lambda g\} |_{\lambda=1}).$$

(The Lie bracket on \mathcal{F}_N is obtained by projecting this identity to $\mathcal{F}_N \times \mathcal{V}_N^{\text{tr}}$ then $\mathcal{F}_N \times \mathcal{F}_N$, in agreement with (6.7).) Second, a Hamiltonian functional $\int h \in \mathcal{F}$ gives rise to such a functional $\int \text{tr } \mathcal{X}(h) \in \mathcal{F}_N$, and (6.13) induces the following Hamiltonian equation at the level of the representation algebra \mathcal{V}_N :

$$\frac{du_{ij}}{dt} = \left\{ \int \text{tr } \mathcal{X}(h), u_{ij} \right\} = (m \{h_\lambda u\} |_{\lambda=1})_{ij},$$

for all $u \in \mathcal{V}$ and $1 \leq i, j \leq N$. In particular, an integral of motion $\int f \in \mathcal{F}$ for $\int h$ induces the integral of motion $\int \text{tr } \mathcal{X}(f) \in \mathcal{F}_N$ for $\int \text{tr } \mathcal{X}(h)$, and a (“non-commutative”) integrable hierarchy on \mathcal{V} as defined above induces a non-abelian (i.e. matrix-valued) integrable hierarchy on \mathcal{V}_N in the usual sense. Applying this point of view to the different examples gathered in §6.4 gives non-abelian integrable hierarchies of differential-difference equations.

6.3. de Rham complex over an algebra of difference functions. Let \mathcal{V} be an algebra of difference functions. The *de Rham complex* $\tilde{\Omega}(\mathcal{V})$ of \mathcal{V} is defined as the free product of the algebra \mathcal{V} and the algebra $\mathbb{k}\langle \delta u_{i,n} \mid i \in I = \{1, \dots, \ell\}, n \in \mathbb{Z} \rangle$ of non-commutative polynomials in the variables $\delta u_{i,n}$. The action of the automorphism S is extended from \mathcal{V} to $\tilde{\Omega}(\mathcal{V})$ by letting $S(\delta u_{i,n}) = \delta u_{i,n+1}$ for all $(i, n) \in I \times \mathbb{Z}$.

The algebra $\tilde{\Omega}(\mathcal{V})$ has a \mathbb{Z}_+ -grading, denoted by p , such that $f \in \mathcal{V}$ has degree $p(f) = 0$ and the $\delta u_{i,n}$ ’s have degree $p(\delta u_{i,n}) = 1$. We consider $\tilde{\Omega}(\mathcal{V})$ as a superalgebra, with superstructure

compatible with the \mathbb{Z}_+ -grading. Then, the subspace of elements of degree k , denoted $\tilde{\Omega}^k(\mathcal{V})$, consists of linear combinations of terms of the form

$$\tilde{\omega} = f_1 \delta u_{i_1, m_1} f_2 \delta u_{i_2, m_2} \cdots f_k \delta u_{i_k, m_k} f_{k+1}, \text{ where } f_1, \dots, f_{k+1} \in \mathcal{V}. \quad (6.18)$$

Note that $\tilde{\Omega}^0(\mathcal{V}) = \mathcal{V}$ and $\tilde{\Omega}^1(\mathcal{V}) \simeq \oplus_{(i,n) \in I \times \mathbb{Z}} \mathcal{V} \delta u_{i,n} \mathcal{V}$.

We turn $\tilde{\Omega}(\mathcal{V})$ into a differential algebra by considering the *de Rham differential* δ on $\tilde{\Omega}(\mathcal{V})$ defined as the odd derivation of degree 1 on the superalgebra $\tilde{\Omega}(\mathcal{V})$ satisfying

$$\delta f = \sum_{(i,n) \in I \times \mathbb{Z}} \left(\frac{\partial f}{\partial u_{i,n}} \right)' \delta u_{i,n} \left(\frac{\partial f}{\partial u_{i,n}} \right)'' \in \tilde{\Omega}^1(\mathcal{V}) \text{ for } f \in \mathcal{V}, \text{ and } \delta(\delta u_{i,n}) = 0. \quad (6.19)$$

The proof that δ is a differential, i.e. $\delta^2 = 0$, is a direct computation (it is the same as for the algebra of differential functions, cf. [DSKV, §2.7]). Therefore we can consider the corresponding cohomology complex $(\tilde{\Omega}(\mathcal{V}), \delta)$.

Given a vector field $X_P = \sum_{(i,n) \in I \times \mathbb{Z}} m \circ \left(P_{i,n} * 1 \frac{\partial}{\partial u_{i,n}} \right) \in \text{Vect}(\mathcal{V})$ (cf. (6.11)), we define the associated *Lie derivative* $L_P : \tilde{\Omega}(\mathcal{V}) \rightarrow \tilde{\Omega}(\mathcal{V})$ as the even derivation of degree 0 which extends X_P from \mathcal{V} , in such a way that $L_P(\delta u_{i,n}) = \delta P_{i,n}$, $i \in I$, $n \in \mathbb{Z}$. We can also define the associated *contraction operator* $\iota_P : \tilde{\Omega}(\mathcal{V}) \rightarrow \tilde{\Omega}(\mathcal{V})$ as the odd derivation of degree -1 given on generators by $\iota_P(f) = 0$, for $f \in \mathcal{V}$, and $\iota_P(\delta u_{i,n}) = P_{i,n}$. In analogy with [DSKV, Proposition 2.17], we remark that $\tilde{\Omega}(\mathcal{V})$ is a $\text{Vect}(\mathcal{V})$ -complex, which means that the following results hold in the multiplicative setting as well.

Proposition 6.6. *Fix $P, Q \in \mathcal{V}^{I \times \mathbb{Z}}$. Under the identifications $P \leftrightarrow X_P$ and $Q \leftrightarrow X_Q$, we have:*

- (a) $[\iota_P, \iota_Q] = 0$;
- (b) $[L_P, \iota_Q] = \iota_{[P, Q]}$;
- (c) $[L_P, L_Q] = L_{[P, Q]}$;
- (d) $L_P = \iota_P \delta + \delta \iota_P$ (Cartan's formula).

Proof. These are equalities of derivations of the superalgebra $\tilde{\Omega}(\mathcal{V})$, so they only need to be checked on generators. For example, let us establish (c). We start by noting that $L_P \delta - \delta L_P = 0$ because it is an equality of derivations that is easily checked on the generators $u_{i,n}, \delta u_{i,n}$. Thus, using the identification from the statement, the left-hand side of (c) satisfies

$$\begin{aligned} [L_P, L_Q](f) &= L_P(X_Q(f)) - L_Q(X_P(f)) = [X_P, X_Q](f), \quad \forall f \in \mathcal{V}, \\ [L_P, L_Q](\delta u_{i,n}) &= \delta([L_P, L_Q](u_{i,n})) = \delta([X_P, X_Q](u_{i,n})). \end{aligned}$$

Meanwhile, we get for the right-hand side

$$\begin{aligned} L_{[P, Q]}(f) &= [X_P, X_Q](f), \quad \forall f \in \mathcal{V}, \\ L_{[P, Q]}(\delta u_{i,n}) &= \delta(L_{[P, Q]}(u_{i,n})) = \delta([X_P, X_Q](u_{i,n})). \end{aligned} \quad \square$$

Our next step is to construct a reduction of the complex $(\tilde{\Omega}(\mathcal{V}), \delta)$.

Proposition 6.7. *In the de Rham complex $(\tilde{\Omega}(\mathcal{V}), \delta)$ we have:*

- (a) *The commutator subspace $[\tilde{\Omega}(\mathcal{V}), \tilde{\Omega}(\mathcal{V})]$ is compatible with the \mathbb{Z}_+ -grading and is preserved by δ .*
- (b) *δ and S commute, therefore $(S-1)\tilde{\Omega}(\mathcal{V})$ is compatible with the \mathbb{Z}_+ -grading and is preserved by δ .*
- (c) *Given an evolutionary vector field X_P of characteristics $P = (P_i)_{i=1}^\ell$ (cf. (6.12)), The associated Lie derivative L_P and contraction operator ι_P commute with the action of S on $\tilde{\Omega}(\mathcal{V})$.*

Proof. The proof is analog to the proof of Proposition 3.15 in [DSKV]. Part (a) follows immediately since δ is a derivation of the associative product on $\tilde{\Omega}(\mathcal{V})$. Part (b) is proven if we can show that $\delta(S\tilde{\omega}) = S(\delta\tilde{\omega})$ for every $\tilde{\omega} \in \tilde{\Omega}(\mathcal{V})$. Given $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\Omega}(\mathcal{V})$, it is easy to check that

$$[\delta, S](\tilde{\omega}_1 \tilde{\omega}_2) = [\delta, S](\tilde{\omega}_1) S(\tilde{\omega}_2) + (-1)^{p(\tilde{\omega}_1)} S(\tilde{\omega}_1) [\delta, S](\tilde{\omega}_2).$$

Hence, to prove the claim it suffices to check that $[\delta, S]$ is zero on $\delta u_{i,n}$ and $f \in \mathcal{V}$. The identity $[\delta, S](\delta u_{i,n}) = 0$ is obvious from the second equation in (6.19) and the action of S on $\tilde{\Omega}(\mathcal{V})$. On the other hand, using the first identity in (6.19) we have

$$S(\delta f) = \sum_{(i,n) \in I \times \mathbb{Z}} \left(S \frac{\partial f}{\partial u_{i,n}} \right)' \delta u_{i,n+1} \left(S \frac{\partial f}{\partial u_{i,n}} \right)''$$

and

$$\delta(Sf) = \sum_{(i,n) \in I \times \mathbb{Z}} \left(\frac{\partial(Sf)}{\partial u_{i,n}} \right)' \delta u_{i,n} \left(\frac{\partial(Sf)}{\partial u_{i,n}} \right)''.$$

Hence, $[\delta, S](f) = 0$ by (4.9). Part (c) can be proven similarly. \square

Thanks to Proposition 6.7(a-b), we can form the \mathbb{Z}_+ -graded *variational complex*

$$\Omega(\mathcal{V}) = \tilde{\Omega}(\mathcal{V}) / ((S-1)\tilde{\Omega}(\mathcal{V}) + [\tilde{\Omega}(\mathcal{V}), \tilde{\Omega}(\mathcal{V})]) = \oplus_{n \in \mathbb{Z}_+} \Omega^n(\mathcal{V}), \quad (6.20)$$

which is equipped with a differential induced by δ . Using Proposition 6.7(c), the Lie derivatives L_P and contraction operators ι_P , associated with the evolutionary vector field X_P of characteristics $P \in \mathcal{V}^\ell$, descend to well defined maps on the variational complex $\Omega(\mathcal{V})$.

Example 6.8. For \mathcal{R}_ℓ as in §4.2, the total degree vector field X_Δ , with characteristics $\Delta = (u_i)_{i=1}^\ell$, is an evolutionary vector field on \mathcal{R}_ℓ . By adapting [DSKV, Theorem 2.18], we can show that the contraction operator ι_Δ associated with X_Δ is a homotopy operator for the complex $(\tilde{\Omega}(\mathcal{R}_\ell), \delta)$, hence it is acyclic: $H^n(\tilde{\Omega}(\mathcal{R}_\ell), \delta) = \delta_{n,0}\mathbb{k}$. In the exact same way, we can see that the complex $(\Omega(\mathcal{R}_\ell), \delta)$ is acyclic as well, i.e.

$$H^k(\Omega(\mathcal{R}_\ell), \delta) = \delta_{k,0}\mathbb{k}. \quad (6.21)$$

Next, we give an explicit description of the complex $(\Omega(\mathcal{V}), \delta)$ by adapting [DSK1, DSK2, DSKV]. It is clear that $\Omega^0(\mathcal{V}) = \mathcal{F}$, the space of local functionals. For $k \geq 1$, let us introduce the space $\Sigma^k(\mathcal{V})$ of arrays $(A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell$ with entries

$$A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}) \in \mathcal{V}^{\otimes k}[\lambda_1^{\pm 1}, \dots, \lambda_{k-1}^{\pm 1}],$$

satisfying the following skewadjointness condition ($i_1, \dots, i_k \in I$):

$$A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}) = -(-1)^k |_{x=S} (A_{i_2 \dots i_k i_1}(\lambda_2, \dots, \lambda_{k-1}, (\lambda_1 \dots \lambda_{k-1} x)^{-1}))^\sigma, \quad (6.22)$$

where σ denotes the action of the cyclic permutation on $\mathcal{V}^{\otimes k}$ as in (2.6), and we are using the same notation as in (3.5). We claim that there is an isomorphism $\Omega^k(\mathcal{V}) \simeq \Sigma^k(\mathcal{V})$, which we prove by writing explicitly the maps in both directions.

Fix a coset $\omega = [\tilde{\omega}] \in \Omega^k(\mathcal{V})$, where $\tilde{\omega}$ is as in (6.18). We map ω to the array $A = (A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell \in \Sigma^k(\mathcal{V})$, with entries $A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}) = 0$ unless (j_1, \dots, j_k) is a cyclic permutation of (i_1, \dots, i_k) , and

$$\begin{aligned} A_{j_1 \dots j_k}(\lambda_1, \dots, \lambda_{k-1}) &= \frac{1}{k} (-1)^{s(k-s)} \lambda_1^{n_{s+1}} \dots \lambda_{k-s}^{n_k} \lambda_{k-s+1}^{n_1} \dots \lambda_{k-1}^{n_{s-1}} \\ &(\lambda_1 \dots \lambda_{k-1} S)^{-n_s} (f_{s+1} \otimes \dots \otimes f_k \otimes f_{k+1} f_1 \otimes f_2 \otimes \dots \otimes f_s), \end{aligned} \quad (6.23)$$

for $(j_1, \dots, j_k) = (i_{\sigma^s(1)}, \dots, i_{\sigma^s(k)})$. The inverse map $\Sigma^k(\mathcal{V}) \rightarrow \Omega^k(\mathcal{V})$ is given by

$$\begin{aligned} \left(\sum_{n_1, \dots, n_{k-1} \in \mathbb{Z}} A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}} \lambda_1^{n_1} \dots \lambda_{k-1}^{n_{k-1}} \right)_{i_1, \dots, i_k=1}^\ell &\mapsto \\ \sum_{\substack{i_1, \dots, i_k \in I \\ n_1, \dots, n_{k-1} \in \mathbb{Z}}} [(A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})' \delta u_{i_1, n_1} \dots (A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})^{\overbrace{\dots}^{k-1}}' \delta u_{i_{k-1}, n_{k-1}} (A_{i_1 \dots i_k}^{n_1 \dots n_{k-1}})^{\overbrace{\dots}^k} \delta u_{i_k}]. \end{aligned} \quad (6.24)$$

(Here, we use Sweedler's notation.) It is not hard to verify that the maps (6.23) and (6.24) are well defined and inverse to each other. Hence the space of degree k elements in the variational complex $\Omega^k(\mathcal{V})$ and the space of arrays $\Sigma^k(\mathcal{V})$ can be identified using these maps.

We can explicitly translate the differential δ of the variational complex $\Omega(\mathcal{V})$ to a differential $\delta : \Sigma^k(\mathcal{V}) \rightarrow \Sigma^{k+1}(\mathcal{V})$ under the above identification. For $k = 0$, we have

$$\delta(f) = \left(\sum_{n \in \mathbb{Z}} S^{-n} m \left(\frac{\partial f}{\partial u_{i,n}} \right)^\sigma \right)_{i=1}^\ell = \left(m \left(\frac{\delta f}{\delta u_i} \right)^\sigma \right)_{i=1}^\ell, \quad (6.25)$$

where in the second identity we used equation (6.15). More generally, if $k \geq 1$ and $A = (A_{i_1 \dots i_k}(\lambda_1, \dots, \lambda_{k-1}))_{i_1, \dots, i_k=1}^\ell \in \Sigma^k(\mathcal{V})$, we have that

$$\begin{aligned} & (\delta A)_{i_1 \dots i_{k+1}}(\lambda_1, \dots, \lambda_k) \\ &= \frac{k}{k+1} \sum_{n \in \mathbb{Z}} \left(\sum_{s=1}^k (-1)^{s+1} \left(\frac{\partial}{\partial u_{i_s, n}} \right)_{(s)} A_{i_1 \dots \overset{s}{i_{k+1}}}(\lambda_1, \dots, \overset{s}{\lambda_k}) \lambda_s^n \right. \\ & \quad \left. + (-1)^k (\lambda_1 \dots \lambda_k S)^{-n} \left(\left(\frac{\partial}{\partial u_{i_{k+1}, n}} \right)_{(1)} A_{i_1, \dots, i_k}(\lambda_1, \dots, \lambda_{k-1}) \right)^{\sigma^k} \right). \end{aligned} \quad (6.26)$$

The notation $\overset{t}{\dots}$ means that we skip the object in position t , σ denotes the action of the cyclic permutation in (2.6), and we use the extended derivations $(\partial/\partial u_{i,n})_{(s)} : \mathcal{V}^{\otimes k} \mapsto \mathcal{V}^{\otimes(k+1)}$ defined through (2.11).

Equation (6.26), for $k = 1$ and $F = (F_j)_{j=1}^\ell \in \mathcal{V}^\ell = \Sigma^1(\mathcal{V})$, gives

$$(\delta F)_{ij}(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\frac{\partial F_j}{\partial u_{i,n}} \lambda^n - (\lambda S)^{-n} \left(\frac{\partial F_i}{\partial u_{j,n}} \right)^\sigma \right). \quad (6.27)$$

For $F \in \mathcal{V}^{\oplus \ell} = \Sigma^1(\mathcal{V})$, define the corresponding *Frechet derivative*

$$D_F(\lambda) = \left(\sum_{n \in \mathbb{Z}} \frac{\partial F_i}{\partial u_{j,n}} \lambda^n \right)_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}].$$

From (6.27) we see that $\delta F = 0$ if and only if $D_F(S)$ is a selfadjoint non-commutative difference operator.

For $k = 2$, let $A = (A_{ij}(\lambda))_{i,j=1}^\ell \in \Sigma^2(\mathcal{V})$, i.e. the entries $A_{ij}(\lambda) \in \mathcal{V}^{\otimes 2}[\lambda, \lambda^{-1}]$ satisfy $(|_{x=S} A_{ji}(\lambda^{-1} S^{-1}))^\sigma = -A_{ij}(\lambda)$. Equation (6.26) gives

$$\begin{aligned} (\delta A)_{ijk}(\lambda, \mu) &= \frac{2}{3} \sum_{n \in \mathbb{Z}} \left(\left(\frac{\partial}{\partial u_{i,n}} \right)_L A_{jk}(\mu) \lambda^n \right. \\ & \quad \left. - \left(\frac{\partial}{\partial u_{j,n}} \right)_R A_{ik}(\lambda) \mu^n + (\lambda \mu S)^{-n} \left(\left(\frac{\partial}{\partial u_{k,n}} \right)_L A_{ij}(\lambda) \right)^{\sigma^2} \right). \end{aligned} \quad (6.28)$$

As an application of (6.21), we get the following result which is a multiplicative version of [DSKV, Corollary 3.17].

Corollary 6.9. (a) A 0-form $\int f \in \Omega^0(\mathcal{R}_\ell)$ is closed if and only if $f \in \mathbb{k} + [\mathcal{R}_\ell, \mathcal{R}_\ell] + (S-1)\mathcal{R}_\ell$.
 (b) A 1-form $F = (F_i)_{i=1}^\ell \in \mathcal{R}_\ell^{\oplus \ell} = \Sigma^1(\mathcal{R}_\ell)$ is closed if and only if there exists a local functional $\int f \in \mathcal{R}_\ell / ([\mathcal{R}_\ell, \mathcal{R}_\ell] + (S-1)\mathcal{R}_\ell)$ such that $F_i = m \left(\frac{\delta f}{\delta u_i} \right)^\sigma$ for every $i = 1, \dots, \ell$.
 (c) A 2-form $\alpha = (A_{ij}(\lambda))_{i,j=1}^\ell \in \Sigma^2(\mathcal{R}_\ell)$ is closed if and only if there exists $F = (F_i)_{i=1}^\ell \in \mathcal{R}_\ell^{\oplus \ell}$ such that

$$A_{ij}(\lambda) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\frac{\partial F_j}{\partial u_{i,n}} \lambda^n - (\lambda S)^{-n} \left(\frac{\partial F_i}{\partial u_{j,n}} \right)^\sigma \right),$$

for every $i, j = 1, \dots, \ell$.

6.4. Examples.

6.4.1. *Integrable hierarchies on \mathcal{R}_1 .* Recall from Example 3.16 that $\mathcal{R}_1 = \mathbb{k}\langle u_i \mid i \in \mathbb{Z} \rangle$ is a double multiplicative Poisson vertex algebra for

$$\{ \{ u_i \lambda u_j \} \} = \lambda^{j-i} (u_j \otimes 1 - 1 \otimes u_j).$$

It is easy to check that the local functionals $\{ \int u^k \mid k \in \mathbb{Z}_+ \}$ satisfy $\{ \int u^k, \int u^l \} = 0$ for any $k, l \in \mathbb{Z}_+$, therefore the Hamiltonian vector fields that they define are pairwise commuting. Note however that

$$\frac{du}{dt_k} := \{ \int u^k, u \} = 0, \quad k \in \mathbb{Z}_+,$$

so these Hamiltonian vector fields do not yield non-trivial differential-difference equations. The same observation can be made if we use the double multiplicative λ -bracket obtained by combining Proposition 3.14 with any double Poisson bracket from Example 3.4 (the case above corresponds to $\alpha = 1, \beta = \gamma = 0$). We have been unable to construct an integrable hierarchy on \mathcal{R}_1 starting with one of the cases from the classification given in Proposition 4.8. To get non-trivial examples on \mathcal{R}_1 , it seems necessary to use non-local double multiplicative Poisson vertex algebras as defined in Section 7, see [CW1, CW2].

6.4.2. *Integrable hierarchies on \mathcal{R}_2 .* As part of Proposition 4.11, we have constructed double multiplicative Poisson vertex algebra structures on $\mathcal{R}_2 = \mathbb{k}\langle u_i, v_i \mid i \in \mathbb{Z} \rangle$ such that

$$\{ \{ u \lambda u \} \} = 0, \quad \{ \{ v \lambda v \} \} = 0.$$

In particular, the local functionals $\{ \int u^k \mid k \in \mathbb{Z}_+ \}$ satisfy $\{ \int u^k, \int u^l \} = 0$ trivially, hence they yield commuting Hamiltonian vector fields on \mathcal{R}_2 . A similar result holds for the set of functionals $\{ \int v^k \mid k \in \mathbb{Z}_+ \}$ by symmetry.

Example 6.10. Consider $\{ \{ u \lambda v \} \} = 1 \otimes 1$, which corresponds to case (i) of Theorem 4.14. We easily compute that for the vector field $d/dt_k := \frac{1}{k} \{ \int u^k, - \}$, $k \geq 1$, we have

$$\frac{dv}{dt_k} = \frac{1}{k} \{ \int u^k, v \} = \frac{1}{k} m \left\{ \left\{ u^k \lambda v \right\} \right\} \big|_{\lambda=1} = u^{k-1}, \quad \frac{du}{dt_k} = 0. \quad (6.29)$$

Since d/dt_k commutes with S by part (g) of Theorem 6.2, note that (6.29) is equivalent to the Hamiltonian differential-difference equations

$$\frac{dv_i}{dt_k} = u_i^{k-1}, \quad \frac{du_i}{dt_k} = 0, \quad i \in \mathbb{Z}.$$

Since the vector fields $(d/dt_k)_{k \geq 1}$ are pairwise commuting for different $k \in \mathbb{Z}_+$ due to $\{ \int u^k, \int u^l \} = 0$, we get in this way an integrable hierarchy of differential-difference equations. The solution to the k -th system of equations is simply given by $u_i(t_k) = \alpha_i$, $v_i(t_k) = \beta_i + t_k \alpha_i^{k-1}$ for $i \in \mathbb{Z}$, where $\alpha_i, \beta_i \in \mathbb{k}$. Compatibility of the solution with S implies that $\alpha_i = \alpha_0$, $\beta_i = \beta_0$ for each $i \in \mathbb{Z}$.

Remark 6.11. While we observed in Remark 6.5 that differential-difference equations on an associative algebra \mathcal{V} induce such equations on the representation algebra \mathcal{V}_N , $N \geq 1$, solving the equation on \mathcal{V} does not provide all the solutions on \mathcal{V}_N . Combining Remark 6.5 and Example 6.10, we see that (6.29) induces the non-abelian equation

$$\frac{d\mathcal{X}(v)}{dt_k} = \mathcal{X}(u)^{k-1}, \quad \frac{d\mathcal{X}(u)}{dt_k} = 0, \quad (6.30)$$

while its solution $u(t_k) = \alpha_0$, $v(t_k) = \beta_0 + t_k \alpha_0^{k-1}$, leads to

$$\mathcal{X}(u)(t_k) = \alpha_0 \text{Id}_N, \quad \mathcal{X}(v)(t_k) = \beta_0 \text{Id}_N + t_k \alpha_0^{k-1} \text{Id}_N.$$

However, an arbitrary solution of (6.30) is of the form $\mathcal{X}(u)(t_k) = A_0$, $\mathcal{X}(v)(t_k) = B_0 + t_k A_0^{k-1}$ for $A_0, B_0 \in \text{Mat}_{n \times n}(\mathbb{k})$.

Example 6.12. Fix $r \in \mathbb{Z}$, $\alpha \in \mathbb{k}^\times$ and take

$$\llbracket u_\lambda v \rrbracket = (\alpha v u_r \otimes u_r v + v u_r \otimes 1 + 1 \otimes u_r v + \alpha^{-1} 1 \otimes 1) \lambda^r,$$

corresponding to case (v) of Theorem 4.14. The Hamiltonian vector fields $\frac{d}{dt_k} = \frac{1}{k} \{\int u_k, -\}$ are commuting and they define the following differential-difference equations:

$$\frac{dv}{dt_k} = \alpha v u_r^{k+1} v + v u_r^k + u_r^k v + \alpha^{-1} u_r^{k-1}, \quad \frac{du}{dt_k} = 0. \quad (6.31)$$

Example 6.13. Fix $r \in \mathbb{Z}$, $\alpha \in \mathbb{k}^\times$ and take as in Example 4.13

$$\llbracket u_\lambda v \rrbracket = (\alpha v \otimes v + v \otimes u_r + u_r \otimes v + \alpha^{-1} u_r \otimes u_r) \lambda^r,$$

that corresponds to case (iv) of Theorem 4.14. The Hamiltonian vector fields $\frac{d}{dt_k} = \frac{1}{k} \{\int u_k, -\}$ are commuting and they define the following differential-difference equations:

$$\frac{dv}{dt_k} = \alpha v u_r^{k-1} v + (v u_r^k + u_r^k v) + \alpha^{-1} u_r^{k+1}, \quad \frac{du}{dt_k} = 0. \quad (6.32)$$

Note that if we allow each u_i to be invertible by working in $\mathbb{k}\langle u_i^{\pm 1}, v_i \mid i \in \mathbb{Z} \rangle$, we have commuting Hamiltonian vector fields $\frac{d}{dt_k}$ for any $k \in \mathbb{Z}$ (not only for $k \in \mathbb{Z}_+$). In particular, remark that the differential-difference equation (6.32) defined for $\frac{d}{dt_k}$ can be transformed into (6.31) if we relabel $u_r \leftrightarrow u_r^{-1}$.

6.4.3. *Integrable hierarchies using a weak version of Jacobi identity.* As a slight generalisation of (6.32), it can be checked that the vector fields defining the differential-difference equations

$$\frac{dv}{dt_k} = \alpha v u_r^{k-1} v + (v u_r^k + u_r^k v) + \beta u_r^{k+1}, \quad \frac{du}{dt_k} = 0, \quad k \in \mathbb{Z}_+, \quad (6.33)$$

commute for any fixed $r \in \mathbb{Z}$ and $\alpha, \beta \in \mathbb{k}$. By considering the equations associated with the local functionals $\{\int u^k \mid k \in \mathbb{Z}_+\}$ for all the cases from Theorem 4.14, we can see that (6.33) can not always be obtained from a double multiplicative Poisson vertex algebra structure on \mathcal{R}_2 . It can, nevertheless, be obtained from a double multiplicative λ -bracket (see Example 6.16) using the framework that we introduce in this paragraph.

From now on, we consider a skewsymmetric double multiplicative λ -bracket $\llbracket -_\lambda - \rrbracket$ on a unital associative algebra \mathcal{V} with an automorphism S . Recall that $\{-_\lambda -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V}[\lambda^{\pm 1}]$ denotes the associated map (6.1) obtained from $\llbracket -_\lambda - \rrbracket$ by multiplication of the two factors. In the same way, if $\llbracket -_\lambda -_\mu - \rrbracket$ is the map (3.10) defined from $\llbracket -_\lambda - \rrbracket$, we introduce

$$\{-_\lambda -_\mu -\} : \mathcal{V}^{\otimes 3} \rightarrow \mathcal{V}[\lambda^{\pm 1}, \mu^{\pm 1}], \quad \{a_\lambda b_\mu c\} = m \circ (m \otimes 1) \llbracket a_\lambda b_\mu c \rrbracket. \quad (6.34)$$

Lemma 6.14. For any $a, b, c \in \mathcal{V}$,

$$\{a_\lambda \{b_\mu c\}\} - \{b_\mu \{a_\lambda c\}\} - \{\{a_\lambda b\}_\mu c\} = \{a_\lambda b_\mu c\} - \{b_\mu a_\lambda c\}. \quad (6.35)$$

Proof. It follows from (6.3) by applying the multiplication map m . \square

Recall from §6.2, that for the local functional $\int f \in \mathcal{F}$, we denote by X^f its associated Hamiltonian vector field. Recall also that \mathcal{F} is a Lie algebra with Lie bracket (6.7) satisfying $\{\int f, \int g\} = \int \{f, g\}$, where $f, g \in \mathcal{V}$, and $\{f, g\}$ is given by (6.2).

Lemma 6.15. Let $\int f, \int h \in \mathcal{F}$. Then $[X^f, X^h] = X^{\{f, h\}}$ if and only if the derivation

$$D_{f, h} := \{f_\lambda h_\mu -\} \big|_{\lambda=\mu=1} - \{h_\mu f_\lambda -\} \big|_{\lambda=\mu=1} \quad (6.36)$$

vanishes identically.

Proof. By (6.35), we have for any $c \in \mathcal{V}$ that

$$\begin{aligned} [X^f, X^h](c) &= (\{f_\lambda \{h_\mu c\}\} - \{h_\mu \{f_\lambda c\}\}) \big|_{\lambda=\mu=1} \\ &= (\{\{f_\lambda h\}_\mu c\} + \{f_\lambda h_\mu -\} - \{h_\mu f_\lambda -\}) \big|_{\lambda=\mu=1} \\ &= \{\int \{f_\lambda h\} \big|_{\lambda=1}, c\} + D_{f, h}(c) = X^{\{f, h\}}(c) + D_{f, h}(c). \end{aligned}$$

Thus $[X^f, X^h](c) = X^{\{f,h\}}(c)$ if and only if $D_{f,h}(c)$, and we can conclude as $c \in \mathcal{V}$ is arbitrary. The fact that $D_{f,h}$ is a derivation easily follows from (3.13). \square

As a consequence of Lemma 6.15, the two Hamiltonian vector fields X^f, X^h commute whenever $\int \{f, h\} = \{\int f, \int h\} = 0$ and $D_{f,h} = 0$ for $D_{f,h}$ defined through (6.36). In the case of a double multiplicative Poisson vertex algebra, the operation $\{\{-\lambda - \mu -\}$ (3.10) is identically vanishing, so for any $f, h \in \mathcal{V}$ we have $D_{f,h} = 0$. Therefore we have $[X^f, X^h] = 0$ whenever $\int \{f, h\} = 0$, as we already observed in §6.2.

Building on the observation that we have just made, we can seek to construct commuting families of vector fields in the presence of a skewsymmetric double multiplicative λ -bracket that has some failure to satisfy Jacobi identity, i.e. when $\{\{-\lambda - \mu -\}$ is non-zero. This weaker notion has been identified in [CW2] on $\mathcal{V} = \mathbb{R}\langle u_{i,n} \mid i \in I, n \in \mathbb{Z} \rangle$ by studying bivector fields, see [CW2, §4.4]. The main examples outside the class of double multiplicative Poisson vertex algebras that they investigated [CW2, Sect.6] are the double *quasi-Poisson* brackets due to Van den Bergh [VdB1]. Such double brackets have the property that $\{\{-\lambda - \mu -\} \neq 0$ has a particular form that entails the vanishing² of the associated map $\{-\lambda - \mu -\}$. Note that these are very special double multiplicative λ -brackets, because they have image in $\mathcal{V}^{\otimes 2}$ and not in $\mathcal{V}^{\otimes 2}[\lambda^{\pm 1}]$. Below, we provide a new non-trivial example which is *not* independent of λ .

Example 6.16. Fix $\alpha, \beta \in \mathbb{k}$ and $r \in \mathbb{Z}$. Consider the skewsymmetric double multiplicative λ -bracket on $\mathcal{R}_2 = \mathbb{k}\langle u_i, v_i \mid i \in \mathbb{Z} \rangle$ given by

$$\{\{u_\lambda u\}\} = 0 = \{\{v_\lambda v\}\}, \quad \{\{u_\lambda v\}\} = (v \otimes u_r + u_r \otimes v + \alpha v \otimes v + \beta u_r \otimes u_r) \lambda^r.$$

This operation does not satisfy Jacobi identity when $\alpha\beta \neq 1$ because

$$\{\{u_\lambda u_\mu v\}\} = (1 - \alpha\beta)(v \otimes u_r \otimes u_r - u_r \otimes u_r \otimes v) \lambda^r \mu^r.$$

Using this identity and (3.15)-(3.16), we get that for any $M, N \geq 1$,

$$\begin{aligned} & \{\{u_\lambda^M u_\mu^N v\}\} \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(\left|_{y=S} u^m \right\rangle *_1 \left(\left|_{x=S} u^n \right\rangle *_2 \{\{u_\lambda u_\mu v\}\} *_1 \left(\left|_{x=S} u^{N-n-1} \right\rangle *_2 \left(\left|_{y=S} u^{M-m-1} \right\rangle \right) \right) \right) \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (1 - \alpha\beta) [v u_r^{M-m-1} \otimes u_r^{N-n+m} \otimes u_r^{n+1} - u_r^{M-m} \otimes u_r^{N-n+m} \otimes u_r^n] \lambda^r \mu^r. \end{aligned}$$

Therefore, from (6.34) we have $\{u_\lambda^M u_\mu^N v\}|_{\lambda=\mu=1} = (1 - \alpha\beta)MN(vu_r^{M+N} - u_r^{M+N}v)$. In particular, this implies that for any $k, l \in \mathbb{Z}_+$ the derivation $D_{u^k, u^l} : \mathcal{R}_2 \rightarrow \mathcal{R}_2$ defined through (6.36) with $f = u^k, h = u^l$ vanishes on v . We trivially have $D_{u^k, u^l}(u) = 0$ as $\{u_\lambda u\} = 0$, so that $D_{u^k, u^l} = 0$ identically. As a consequence of Lemma 6.15, we get that

$$\left[\frac{d}{dt_k}, \frac{d}{dt_l} \right] = \frac{1}{kl} \{ \int \{u^k, u^l\}, - \}, \quad \text{where } \frac{d}{dt_k} := \frac{1}{k} \{ \int u^k, - \}.$$

As the local functionals $\{\int \frac{1}{k} u^k \mid k \in \mathbb{Z}_+\}$ are such that $\{\int u^k, \int u^l\} = 0$, we thus obtain that their Hamiltonian vector fields $\frac{d}{dt_k}$ pairwise commute. The associated differential-difference equations are given by (6.33).

7. NON-LOCAL AND RATIONAL DOUBLE MULTIPLICATIVE POISSON VERTEX ALGEBRAS

In this section we formalize the theory of non-local and rational double multiplicative Poisson vertex algebras. They play a crucial role in the context of non-commutative Hamiltonian differential-difference equations, see [CW1, CW2]. The exposition follows [DSKVVW1] where the commutative case is treated.

²This property was already known by Van den Bergh, see [VdB1, Proposition 5.1.2].

7.1. Non-local double multiplicative Poisson vertex algebras. Let \mathcal{V} be a unital associative algebra with an automorphism S . We denote by $(\mathcal{V} \otimes \mathcal{V})[[\lambda, \lambda^{-1}]]$ the space of bilateral series $\sum_{n \in \mathbb{Z}} a_n \lambda^n$, where $a_n \in \mathcal{V} \otimes \mathcal{V}$ for all $n \in \mathbb{Z}$.

Non-local double multiplicative λ -brackets differ from local ones just in replacing in Definition 3.7 the algebra $(\mathcal{V} \otimes \mathcal{V})[\lambda, \lambda^{-1}]$ by the vector space $(\mathcal{V} \otimes \mathcal{V})[[\lambda, \lambda^{-1}]]$. Note that in the non-local case, despite the fact that $(\mathcal{V} \otimes \mathcal{V})[[\lambda, \lambda^{-1}]]$ is not an algebra, all axioms (3.4a), (3.4b), (3.4c), (3.8a) and (3.8b) still make perfect sense. Hence we have the following definition.

Definition 7.1. A *non-local double multiplicative Poisson vertex algebra* is a unital associative algebra \mathcal{V} endowed with an automorphism $S : \mathcal{V} \rightarrow \mathcal{V}$ and a non-local double multiplicative λ -bracket, $\{\{-\lambda-\}\} : \mathcal{V} \otimes \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})[[\lambda, \lambda^{-1}]]$ satisfying sesquilinearity (3.4a), Leibniz rules (3.4b) and (3.4c), skewsymmetry (3.8a) and Jacobi identity (3.8b).

Let \mathcal{V} be an algebra of non-commutative difference functions in ℓ variables $u_i, i \in I$. We call the space $\text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[[S, S^{-1}]]$, the space of non-local difference operators. Note that this space is not an algebra with respect to the product (4.1), and its elements do not act on \mathcal{V}^ℓ (such an action would involve divergent sums, cf. (4.2)). However, if $H(S) = (H_{i,j}(S))_{i,j \in I} \in \text{Mat}_{\ell \times \ell}(\mathcal{V} \otimes \mathcal{V})[[S, S^{-1}]]$, then we can define a non-local double multiplicative λ -bracket on \mathcal{V} using the Master Formula (4.10) with $\{\{u_i \lambda u_j\}\} = H_{ji}(\lambda)$, which makes sense also for bilateral series. One can check that Theorem 4.3 still holds in the non-local case.

Theorem 7.2. Given an algebra of non-commutative difference functions \mathcal{V} in ℓ variables $u_i, i \in I$, and an $\ell \times \ell$ matrix $H(\lambda) = (H_{ij}(\lambda))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell} \mathcal{V}[[\lambda, \lambda^{-1}]]$, the double multiplicative λ -bracket (4.10) defines a structure of non-local double multiplicative Poisson vertex algebra on \mathcal{V} if and only if skewsymmetry and the Jacobi identity hold on the generators u_i . In this case we call the matrix H a non-local Poisson structure on \mathcal{V} .

Example 7.3. We can get examples of non-local double multiplicative Poisson vertex algebras on an algebra of difference function in one variable u , which generalize the λ -bracket given by (4.17)-(4.18). Indeed, note that in the proof of Proposition 4.7, equation (4.19), which is the Jacobi identity on generators, still holds if we assume $r(\lambda) = \sum_{n \in \mathbb{Z}} r_n \lambda^n \in \mathbb{k}[[\lambda, \lambda^{-1}]]$ and such that $r(\lambda) = -r(\lambda^{-1})$ in (4.17). Hence, for example, the non-local multiplicative λ -bracket defined on \mathcal{V} by letting

$$\{\{u_\lambda u\}\} = (u \otimes u) \bullet r(\lambda S)(u \otimes u) = \sum_{n \in \mathbb{Z}} r_n (u u_n \otimes u_n u) \lambda^n,$$

and extended to \mathcal{V} by the Master Formula (4.10), defines a non-local double multiplicative Poisson vertex algebra structure on \mathcal{V} . On the other hand, the proof of the “only if” part of Proposition 4.7 does not generalize to the non-local setting since it heavily relies on the fact that $r_n = 0$ for every $n > N$ for some $N \in \mathbb{Z}_{>0}$.

7.2. Pseudodifference operators. Let \mathcal{V} be a unital associative algebra with an automorphism S . The algebra of non-commutative difference operators $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]$ defined in §4.1 is \mathbb{Z} -graded by the powers of S and can be completed either in the positive or negative directions, giving rise to two algebras of non-commutative pseudodifference operators:

$$(\mathcal{V} \otimes \mathcal{V})((S)) = (\mathcal{V} \otimes \mathcal{V})[[S]][S^{-1}] \quad \text{and} \quad (\mathcal{V} \otimes \mathcal{V})((S^{-1})) = (\mathcal{V} \otimes \mathcal{V})[[S^{-1}]]S.$$

In the sequel we will use the notation $(\mathcal{V} \otimes \mathcal{V})((S^{\pm 1}))$ to denote $(\mathcal{V} \otimes \mathcal{V})((S))$ or $(\mathcal{V} \otimes \mathcal{V})((S^{-1}))$ respectively, and it should not be confused with $(\mathcal{V} \otimes \mathcal{V})((S, S^{-1}))$. Given a non-commutative pseudodifference operator $A(S) = \sum_n a_n S^n \in (\mathcal{V} \otimes \mathcal{V})((S^{\pm 1}))$, its formal adjoint is (cf. (4.3))

$$A^*(S) = \sum_n S^{-n} \bullet a_n^\sigma \in (\mathcal{V} \otimes \mathcal{V})((S^{\mp 1})), \quad (7.1)$$

and its symbol is (cf. (4.5))

$$A(z) = \sum_n a_n z^n \in (\mathcal{V} \otimes \mathcal{V})((z^{\pm 1})). \quad (7.2)$$

Formulas (4.6) still make sense for non-commutative pseudodifference operators.

7.3. Pseudodifference operators of rational type. Let

$$\mathbb{k}(z) = \left\{ \frac{p(z)}{q(z)} \mid p(z), q(z) \in \mathbb{k}[z], q(z) \neq 0 \right\}$$

denote the field of rational functions in the indeterminate z . It can be embedded in both spaces of Laurent series $\mathbb{k}((z))$ or $\mathbb{k}((z^{-1}))$. Indeed, if $q(z) = \sum_{n=M}^N q_n z^n \in \mathbb{k}[z]$, where $0 \leq M \leq N$, is non-zero, then we can factor it as

$$q(z) = q_M z^M \left(1 + \sum_{n=M+1}^N \frac{q_n}{q_M} z^{n-M} \right)$$

and expand $\frac{1}{q(z)}$, via geometric series expansion, as an element of $\mathbb{k}((z))$, or we can factor

$$q(z) = q_N z^N \left(1 + \sum_{n=M}^{N-1} \frac{q_n}{q_N} z^{n-N} \right)$$

and expand $\frac{1}{q(z)}$, via geometric series expansion, as an element of $\mathbb{k}((z^{-1}))$. We denote by ι_{\pm} the resulting embedding of the field of rational functions into the space of Laurent series

$$\iota_{\pm} : \mathbb{k}(z) \hookrightarrow \mathbb{k}((z^{\pm 1})). \quad (7.3)$$

For example, we have

$$\iota_+ \frac{1}{1-z} = \sum_{n \geq 0} z^n \in \mathbb{k}((z)), \quad \iota_- \frac{1}{1-z} = - \sum_{n \geq 1} z^{-n} \in \mathbb{k}((z^{-1})). \quad (7.4)$$

From now on we will work with the algebra of pseudodifference operators $(\mathcal{V} \otimes \mathcal{V})((S))$ but all the definitions and results still hold if we replace it by $(\mathcal{V} \otimes \mathcal{V})((S^{-1}))$ and use ι_- in place of ι_+ . Let $f_1, \dots, f_{n+1} \in \mathcal{V} \otimes \mathcal{V}$ and $r_1(z), \dots, r_n(z) \in \mathbb{k}(z)$, using (4.1) and the embedding ι_+ defined in (7.3) we define the following non-commutative pseudodifference operator

$$f_1 \iota_+ r_1(S) \bullet f_2 \iota_+ r_2(S) \bullet \dots \bullet f_n \iota_+ r_n(S) \bullet f_{n+1} \in (\mathcal{V} \otimes \mathcal{V})((S)). \quad (7.5)$$

For example, for $f, g \in \mathcal{V} \otimes \mathcal{V}$, we have, using (7.4),

$$f \iota_+ \frac{1}{1-S} \bullet g = \sum_{n \geq 0} (f \bullet S^n(g)) S^n \in (\mathcal{V} \otimes \mathcal{V})((S)).$$

Definition 7.4. A non-commutative pseudodifference operator of rational type with values in \mathcal{V} is a linear combination of non-commutative pseudodifference operators of the form (7.5). We denote by $\mathcal{Q}(\mathcal{V}) \subset (\mathcal{V} \otimes \mathcal{V})((S))$ the space of non-commutative pseudodifference operators of rational type.

It is clear from (7.5) and Definition 7.4 that $\mathcal{Q}(\mathcal{V})$ is an algebra with respect to the product (4.1). Given a non-commutative pseudodifference operator of rational type

$$A(S) = \sum f_1 \iota_+ r_1(S) \bullet f_2 \iota_+ r_2(S) \bullet \dots \bullet f_n \iota_+ r_n(S) \bullet f_{n+1}$$

we define its adjoint $A^*(S)$ by

$$A^*(S) = \sum f_{n+1}^{\sigma} \iota_+ r_n(S^{-1}) \bullet f_n^{\sigma} \iota_+ r_{n-1}(S^{-1}) \bullet \dots \bullet f_2^{\sigma} \iota_+ r_1(S^{-1}) \bullet f_1^{\sigma} \in (\mathcal{V} \otimes \mathcal{V})((S)). \quad (7.6)$$

Note that (7.6) is an element of $\mathcal{Q}(\mathcal{V})$ and does not coincide with the formal adjoint in the space $(\mathcal{V} \otimes \mathcal{V})((S))$ defined in §7.2 even though, by an abuse of notation, we are denoting it with the same symbol. In fact, the adjoint of a pseudodifference operator in $(\mathcal{V} \otimes \mathcal{V})((S))$ is an element of $(\mathcal{V} \otimes \mathcal{V})((S^{-1}))$, see (7.1).

Remark 7.5. Pseudodifference operators of rational type may not be rewritten as the ratio of two difference operators (that is, an expression of the form $A(S) \bullet B(S)^{-1}$, $A(S), B(S) \in (\mathcal{V} \otimes \mathcal{V})[S]$). Indeed, let us assume that \mathcal{V} is a division ring. In general $\mathcal{V} \otimes \mathcal{V}$ is not a division ring and the (non-commutative) field of fractions of $(\mathcal{V} \otimes \mathcal{V})[S]$ may not exist.

7.4. Rational double multiplicative Poisson vertex algebras. Let \mathcal{V} be a unital associative algebra with an automorphism S . By a rational double multiplicative λ -bracket on \mathcal{V} , we mean a double multiplicative λ -bracket as in Definition 3.7 with the only difference that we assume

$$\{a_\lambda b\} = A_{ab}(\lambda) \in (\mathcal{V} \otimes \mathcal{V})((\lambda)) \quad (7.7)$$

being the symbol of a pseudodifference operator of rational type $A_{ab}(S) \in \mathcal{Q}(\mathcal{V})$, for every $a, b \in \mathcal{V}$.

7.4.1. Definition. In analogy with the vector space $(\mathcal{V} \otimes \mathcal{V})((\lambda))$, we introduce for $k \geq 1$

$$\mathcal{V}^{\otimes k}((\lambda, \mu)) = \left\{ \sum_{m \geq M, n \geq N} a_{m,n} \lambda^m \mu^n \mid a_{m,n} \in \mathcal{V}^{\otimes k}, M, N \in \mathbb{Z} \right\}.$$

Let us remark that

$$\mathcal{V}^{\otimes k}((\lambda, \mu)) = \mathcal{V}^{\otimes k}((\lambda))((\mu)) \cap \mathcal{V}^{\otimes k}((\mu))((\lambda)).$$

The following results will be used throughout this section.

Lemma 7.6. *Let T_1, T_2 be automorphisms of \mathcal{V} , and let $A(\lambda, \mu) \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$ and $B(\lambda, \mu) \in (\mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$. Then,*

$$A(\lambda T_1, \mu T_2) \bullet_i B(\lambda, \mu), B(\lambda T_1, \mu T_2) \bullet_i A(\lambda, \mu) \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu)).$$

Proof. Straightforward. □

Lemma 7.7. *Let $A(S) \in \mathcal{Q}(\mathcal{V})$ be a pseudodifference operator of rational type. Then,*

$$\{a_\lambda A(\mu)\}_L, \{a_\lambda A(\mu)\}_R, \{A(\lambda)_{\lambda\mu} a\}_L \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu)),$$

for every $a \in \mathcal{V}$.

Proof. Recall from Definition 7.4 that $A(S)$ is a finite linear combination of pseudodifference operators as in (7.5). Hence, it suffices to prove the claim for

$$A(S) = A_1(S) \bullet \cdots \bullet A_n(S),$$

where $A_i = f_i \iota_+(r(S))$, $f_i \in \mathcal{V} \otimes \mathcal{V}$, $r_i(S) \in \mathbb{k}(S)$, $i = 1, \dots, n$. We will prove that $\{a_\lambda A(\mu)\}_L \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$, by induction on n .

For $n = 1$ we have $\{a_\lambda A(\mu)\}_L = \{a_\lambda f_1\} \iota_+(r_1(\mu))$ which clearly lies in $(\mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$. Let us now assume that $\{a_\lambda A(\mu)\}_L \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$, and let $f \in \mathcal{V} \otimes \mathcal{V}$, $r(S) \in \mathbb{k}(S)$. Then, by Lemma 3.10 we have

$$\{a_\lambda f \iota_+(r(\mu S)) \bullet A(\mu)\}_L = \{a_\lambda f\} \iota_+(r(\mu S)) \bullet_1 A(\mu) + f \iota_+(r(\lambda \mu S)) \bullet_2 \{a_\lambda A(\mu)\}_L. \quad (7.8)$$

Note that, by the base case we have that $\{a_\lambda f\} \iota_+(r(\mu)) \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$. Moreover, $f \iota_+(r(\lambda \mu)) \in (\mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$. Hence, the RHS of equation (3.10) lies in $(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})((\lambda, \mu))$ by Lemma 7.6 and the inductive assumption. The other claims can be proven similarly. □

We say that a rational double multiplicative λ -bracket as in (7.7) is skewsymmetric if

$$A_{ab}(\lambda) = -A_{ba}^*(\lambda), \quad (7.9)$$

for every $a, b \in \mathcal{V}$ (cf. Remark 4.4). In (7.9) we are using the formal adjoint (7.6) in $\mathcal{Q}(\mathcal{V})$. Hence, (7.9) is equivalent to an identity in the space $\mathcal{Q}(\mathcal{V})$.

Definition 7.8. A rational double multiplicative Poisson vertex algebra is a unital associative algebra \mathcal{V} endowed with an automorphism S and a rational double multiplicative λ -bracket $\{-_\lambda -\} : \mathcal{V} \otimes \mathcal{V} \rightarrow (\mathcal{V} \otimes \mathcal{V})((\lambda))$ (i.e. for every $a, b \in \mathcal{V}$, $\{a_\lambda b\}$ is the symbol of a pseudodifference operator of rational type $A_{ab}(S) \in \mathcal{Q}(\mathcal{V})$, cf. (7.7)) satisfying skewsymmetry (7.9) and Jacobi identity (3.8b).

Note that, if $\{\{-\lambda-\}\}$ has values in $(\mathcal{V} \otimes \mathcal{V})(\lambda)$ then

$$\{\{a_\lambda \{\{b_\mu c\}\}\}_L \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})(\lambda)(\mu),$$

while

$$\{\{b_\mu \{\{a_\lambda c\}\}\}_R \in (\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})(\mu)(\lambda).$$

Since these two terms lie in different spaces, the Jacobi identity could not make sense. However, for a rational double multiplicative λ -bracket (7.7), by Lemma 7.7, all three terms of the Jacobi identity lie in $(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})(\lambda, \mu)$, hence it is well-defined.

Let \mathcal{V} be an algebra of non-commutative difference functions in ℓ variables u_i , $i \in I$. We call the space $\text{Mat}_{\ell \times \ell}(\mathcal{Q}(\mathcal{V}))$, the space of matrix pseudodifference operators of rational type. If $H(S) = (H_{i,j}(S))_{i,j \in I} \in \text{Mat}_{\ell \times \ell}(\mathcal{Q}(\mathcal{V}))$, then we can define a rational double multiplicative λ -bracket on \mathcal{V} using the Master Formula (4.10) with $\{\{u_{i\lambda} u_j\}\} = H_{ji}(\lambda)$ and get an analogue of Theorem 4.3.

Theorem 7.9. *Given an algebra of non-commutative difference functions \mathcal{V} in ℓ variables u_i , and an $\ell \times \ell$ matrix $H(\lambda) = (H_{ij}(\lambda))_{i,j=1}^\ell \in \text{Mat}_{\ell \times \ell}(\mathcal{Q}(\mathcal{V}))$, the double multiplicative λ -bracket (4.10) defines a structure of rational double multiplicative Poisson vertex algebra on \mathcal{V} if and only if skewsymmetry and the Jacobi identity hold on the generators u_i . In this case we call the matrix H a Poisson structure of rational type on \mathcal{V} .*

Proof. It suffices to show that $\{\{f_\lambda g\}\}$ defined by the RHS of (4.10) is the symbol of a pseudodifference operator of rational type, for every $f, g \in \mathcal{V}$. The rest is the same as in the proof of Theorem 4.3. The claim follows since the RHS of (4.10) is the symbol of a finite sum of products of elements in $\mathcal{Q}(\mathcal{V})$ and $(\mathcal{V} \otimes \mathcal{V})[S, S^{-1}] \subset \mathcal{Q}(\mathcal{V})$. \square

Example 7.10. We have the analogue of the λ -bracket (4.17) (see also Example 7.3) in the rational case. Let \mathcal{R} denote the algebra of non-commutative difference polynomials in one variable u and let $r(\lambda) \in \mathbb{k}(\lambda)$ be a rational function such that $r(\lambda) = -r(\lambda^{-1})$. Then, consider the double multiplicative λ -bracket on \mathcal{R} defined by

$$\{\{u_\lambda u\}\} = g_{\iota_+ r(\lambda S)} \bullet g^\sigma,$$

where $g = (\alpha u + \beta) \otimes (\alpha u + \beta)$, $\alpha, \beta \in \mathbb{k}$. Skewsymmetry and Jacobi identity hold on generators (same proof as for Proposition 4.7) and by Theorem 7.9 we get a rational double multiplicative Poisson vertex algebra structure on \mathcal{R} .

Remark 7.11. Let \mathcal{V} be an algebra of non-commutative difference functions in ℓ variables. We introduce the following sets

$$\text{Loc}(\mathcal{V}) := \{H(S) \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[S, S^{-1}]) \mid H(S) \text{ local Poisson structure on } \mathcal{V}\},$$

$$\text{NonLoc}(\mathcal{V}) := \{H(S) \in \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[[S, S^{-1}]]) \mid H(S) \text{ non-local Poisson structure on } \mathcal{V}\},$$

$$\text{Rat}(\mathcal{V}) := \{H(S) \in \text{Mat}_{\ell \times \ell}(\mathcal{Q}(\mathcal{V})) \mid H(S) \text{ Poisson structure of rational type on } \mathcal{V}\},$$

(cf. Definition 4.5 and Theorems 7.2 and 7.9). We point out that despite there is the obvious inclusion $\text{Mat}_{\ell \times \ell}(\mathcal{Q}(\mathcal{V})) \subset \text{Mat}_{\ell \times \ell}((\mathcal{V} \otimes \mathcal{V})[[S, S^{-1}]])$, we have

$$\text{Rat}(\mathcal{V}) \not\subset \text{NonLoc}(\mathcal{V})$$

in view of the difference in the skewsymmetry axiom in Definitions 7.1 and 7.8. As an example, let \mathcal{V} be an algebra of non-commutative difference functions in one variable u . By Example 7.10 we have that

$$\{\{u_\lambda u\}\} = \iota_+ \frac{1+\lambda}{1-\lambda} (1 \otimes 1) = 1 \otimes 1 + \sum_{n \geq 1} 2(1 \otimes 1) \lambda^n, \quad (7.10)$$

defines a rational double multiplicative Poisson vertex algebra structure on \mathcal{V} , since $r(\lambda) = \frac{1+\lambda}{1-\lambda}$ satisfies $r(\lambda) = -r(\lambda^{-1})$. The latter condition is equivalent to the skew-symmetry axiom (7.9). On the other hand, if we think of (7.10) as a non-local double multiplicative λ -bracket, we can compute

$$-|_{x=S} \{\{u_{\lambda^{-1}x^{-1}} u\}\} = -1 \otimes 1 - \sum_{n \geq 1} 2(1 \otimes 1) \lambda^{-n},$$

which is clearly different from $\{\!\{u_\lambda u\}\!\}$ so that the skew-symmetry axiom (3.8a) does not hold in the non-local case. Finally, it is immediate to check that

$$\text{Rat}(\mathcal{V}) \cap \text{NonLoc}(\mathcal{V}) = \text{Loc}(\mathcal{V}).$$

7.4.2. Poisson structure of rational type for the non-commutative Narita-Itoh-Bogoyavlensky hierarchy. Let $\mathcal{R} = \mathcal{R}_1$ be the algebra of non-commutative difference polynomials in one variable u . For $a(z), b(z), c(z), d(z) \in \mathbb{k}(z)$ we consider, using the symbol map (7.2), the following pseudodifference operator of rational type

$$\begin{aligned} H(S) = & (1 \otimes u) \iota_+ a(S) \bullet (1 \otimes u) + (1 \otimes u) \iota_+ b(S) \bullet (u \otimes 1) \\ & + (u \otimes 1) \iota_+ c(S) \bullet (1 \otimes u) + (u \otimes 1) \iota_+ d(S) \bullet (u \otimes 1) \in \mathcal{Q}(\mathcal{R}). \end{aligned} \quad (7.11)$$

By (7.6) we have

$$\begin{aligned} H^*(S) = & (u \otimes 1) \iota_+ a(S^{-1}) \bullet (u \otimes 1) + (1 \otimes u) \iota_+ b(S^{-1}) \bullet (u \otimes 1) \\ & + (u \otimes 1) \iota_+ c(S^{-1}) \bullet (1 \otimes u) + (1 \otimes u) \iota_+ d(S^{-1}) \bullet (1 \otimes u). \end{aligned} \quad (7.12)$$

Using the notation (3.5), the skewsymmetry condition $H(S) = -H^*(S)$ is equivalent to the identity

$$\begin{aligned} & [\iota_+ a(x) + \iota_+ d(x^{-1})] (1 \otimes (|_{x=Su})u) + [\iota_+ d(y) + \iota_+ a(y^{-1})] (u(|_{y=Su}) \otimes 1) \\ & + [\iota_+ b(x) + \iota_+ c(y) + \iota_+ b(x^{-1}) + \iota_+ c(y^{-1})] ((x=Su) \otimes (|_{y=Su})) = 0. \end{aligned} \quad (7.13)$$

Since $1 \otimes (|_{x=Su})u$, $u(|_{y=Su}) \otimes 1$ and $(x=Su) \otimes (|_{y=Su})$ are linearly independent, then the skewsymmetry of $H(S)$ is equivalent to the conditions

$$d(z) = -a(z^{-1}), \quad b(x) + b(x^{-1}) = -c(y) - c(y^{-1}). \quad (7.14)$$

Since the LHS of the second equation in (7.14) is independent of y and the RHS is independent of x , then they need to be both equal to a constant $2\alpha \in \mathbb{k}$. Hence,

$$b(z) = b_1(z) + \alpha, \quad b_1(z) = -b_1(z^{-1}), \quad c(z) = c_1(z) - \alpha, \quad c_1(z) = -c_1(z^{-1}). \quad (7.15)$$

Inserting the first condition in (7.14) and the conditions in (7.15) in the definition of $H(S)$ given in (7.11), we see that $H(S)$ is skewadjoint if and only if it has the form

$$\begin{aligned} H(S) = & (1 \otimes u) \iota_+ a(S) \bullet (1 \otimes u) + (1 \otimes u) \iota_+ b(S) \bullet (u \otimes 1) \\ & + (u \otimes 1) \iota_+ c(S) \bullet (1 \otimes u) - (u \otimes 1) \iota_+ a(S^{-1}) \bullet (u \otimes 1), \end{aligned} \quad (7.16)$$

where $b(z) = -b(z^{-1})$ and $c(z) = -c(z^{-1})$.

Theorem 7.12. *The pseudodifference operator $H(S)$ in (7.16) defines a rational double multiplicative Poisson vertex algebra structure on \mathcal{R} if and only if for some $k \geq 1$ and $p \in \mathbb{Z}$,*

$$\begin{aligned} a(z) = & z^p a_1(z^k), \quad a_1(z) := \alpha \frac{1}{1-z}, \\ b(z) = & c(z) = b_1(z^k), \quad b_1(z) = \beta \frac{1+z}{1-z}, \end{aligned} \quad (7.17)$$

where $\alpha, \beta \in \mathbb{k}$ are such that $\alpha(2\beta + \alpha) = 0$.

Before proving Theorem 7.12 we need some preliminary results. Let $H(S)$ be as in (7.16) and let us define a double multiplicative λ -bracket on \mathcal{R} by setting

$$\begin{aligned} \{\!\{u_\lambda u\}\!\} = & H(\lambda) = (1 \otimes u) \iota_+ a(\lambda S) \bullet (1 \otimes u) + (1 \otimes u) \iota_+ b(\lambda S) \bullet (u \otimes 1) \\ & + (u \otimes 1) \iota_+ c(\lambda S) \bullet (1 \otimes u) - (u \otimes 1) \iota_+ a(\lambda^{-1} S^{-1}) \bullet (u \otimes 1) \in (\mathcal{R} \otimes \mathcal{R})((\lambda)), \end{aligned} \quad (7.18)$$

and extending to \mathcal{R} by the Master Formula (4.10).

Proposition 7.13. *Jacobi identity on generators holds for the double multiplicative λ -bracket (7.18) if and only if the rational functions $a(z), b(z), c(z) \in \mathbb{K}(z)$, with $b(z^{-1}) = -b(z)$ and $c(z^{-1}) = -c(z)$, satisfy*

$$\begin{aligned} (\iota_+ b(z) + \iota_+ b(w)) \iota_+ b(zw) - \iota_+ b(z) \iota_+ b(w) &= \gamma, \\ (\iota_+ c(z) + \iota_+ c(w)) \iota_+ c(zw) - \iota_+ c(z) \iota_+ c(w) &= \gamma, \\ (\iota_+ b(z) + \iota_+ b(w)) \iota_+ a(zw) + \iota_+ a(z) \iota_+ a(w) &= 0, \\ (\iota_+ c(z) + \iota_+ c(w)) \iota_+ a(zw) + \iota_+ a(z) \iota_+ a(w) &= 0, \end{aligned} \quad (7.19)$$

for some $\gamma \in \mathbb{K}$.

Proof. For convenience in the computations, let us use notation (3.5) to rewrite (7.18) as

$$\begin{aligned} \{u_\lambda u\} &= \iota_+ a(\lambda x) (1 \otimes (|_{x=Su})u) + \iota_+ b(\lambda x) ((|_{x=Su}) \otimes u) \\ &\quad + \iota_+ c(\lambda y) (u \otimes (|_{y=Su})) - \iota_+ a(\lambda^{-1} y^{-1}) (u(|_{y=Su}) \otimes 1). \end{aligned} \quad (7.20)$$

We start by computing explicitly the three terms appearing in the Jacobi identity

$$\{u_\lambda \{u_\mu u\}\}_L - \{u_\mu \{u_\lambda u\}\}_R = \left\{ \{u_\lambda u\}_{\lambda\mu} u \right\}_L. \quad (7.21)$$

By a long but straightforward computation, using the first equation in (3.7a), sesquilinearity (3.4a), the left Leibniz rule (3.4b) and (7.20), we get

$$\begin{aligned} &\{u_\lambda \{u_\mu u\}\}_L \\ &= \iota_+ a(\lambda x) \iota_+ b(\lambda \mu xy) (1 \otimes (|_{x=Su})(|_{y=Su}) \otimes u) \end{aligned} \quad (7.22)$$

$$+ \iota_+ b(\lambda x) \iota_+ b(\lambda \mu xy) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \quad (7.23)$$

$$+ \iota_+ b(\lambda \mu xy) \iota_+ c(\lambda y) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \quad (7.24)$$

$$- \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ b(\lambda \mu xy) ((|_{x=Su})(|_{y=Su}) \otimes 1 \otimes u) \quad (7.25)$$

$$+ \iota_+ a(\lambda x) \iota_+ c(\mu z) (1 \otimes (|_{x=Su})u \otimes (|_{z=Su})) \quad (7.26)$$

$$+ \iota_+ b(\lambda x) \iota_+ c(\mu z) ((|_{x=Su}) \otimes u \otimes (|_{z=Su})) \quad (7.27)$$

$$+ \iota_+ c(\lambda y) \iota_+ c(\mu z) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})) \quad (7.28)$$

$$- \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ c(\mu z) (u(|_{y=Su}) \otimes 1 \otimes (|_{z=Su})) \quad (7.29)$$

$$- \iota_+ a(\lambda x) \iota_+ a(\mu^{-1} z^{-1}) (1 \otimes (|_{x=Su})u(|_{z=Su}) \otimes 1) \quad (7.30)$$

$$- \iota_+ a(\mu^{-1} z^{-1}) \iota_+ b(\lambda x) ((|_{x=Su}) \otimes u(|_{z=Su}) \otimes 1) \quad (7.31)$$

$$- \iota_+ a(\mu^{-1} z^{-1}) \iota_+ c(\lambda y) (u \otimes (|_{y=Su})(|_{z=Su}) \otimes 1) \quad (7.32)$$

$$+ \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ a(\mu^{-1} z^{-1}) (u(|_{y=Su}) \otimes (|_{z=Su}) \otimes 1) \quad (7.33)$$

$$- \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ a(\lambda y) (u \otimes (|_{y=Su})(|_{z=Su}) \otimes 1) \quad (7.34)$$

$$- \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ b(\lambda y) (u(|_{y=Su}) \otimes (|_{z=Su}) \otimes 1) \quad (7.35)$$

$$- \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ c(\lambda z) (u(|_{y=Su}) \otimes (|_{z=Su}) \otimes 1) \quad (7.36)$$

$$+ \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ a(\lambda^{-1} z^{-1}) (u(|_{y=Su})(|_{z=Su}) \otimes 1 \otimes 1). \quad (7.37)$$

Similarly, but using the second equation in (3.7a) instead, we get

$$\begin{aligned} & \{\!\!\{ u_\mu \{\!\!\{ u_\lambda u \}\!\!\} \}\!\!\}_R \\ &= \iota_+ a(\lambda \mu x y) \iota_+ a(\mu x) (1 \otimes 1 \otimes (|_{x=Su})(|_{y=Su})u) \end{aligned} \quad (7.38)$$

$$+ \iota_+ a(\lambda \mu x y) \iota_+ b(\mu x) (1 \otimes (|_{x=Su}) \otimes (|_{y=Su})u) \quad (7.39)$$

$$+ \iota_+ a(\lambda \mu x y) \iota_+ c(\mu y) (1 \otimes (|_{x=Su}) \otimes (|_{y=Su})u) \quad (7.40)$$

$$- \iota_+ a(\lambda \mu x y) \iota_+ a(\mu^{-1} y^{-1}) (1 \otimes (|_{x=Su})(|_{y=Su}) \otimes u) \quad (7.41)$$

$$+ \iota_+ a(\lambda x) \iota_+ a(\mu y) (1 \otimes (|_{x=Su}) \otimes (|_{y=Su})u) \quad (7.42)$$

$$+ \iota_+ a(\lambda x) \iota_+ b(\mu y) (1 \otimes (|_{x=Su})(|_{y=Su}) \otimes u) \quad (7.43)$$

$$+ \iota_+ a(\lambda x) \iota_+ c(\mu z) (1 \otimes (|_{x=Su})u \otimes (|_{z=Su})) \quad (7.44)$$

$$- \iota_+ a(\lambda x) \iota_+ a(\mu^{-1} z^{-1}) (1 \otimes (|_{x=Su})u(|_{z=Su}) \otimes 1) \quad (7.45)$$

$$+ \iota_+ a(\mu y) \iota_+ b(\lambda x) ((|_{x=Su}) \otimes 1 \otimes (|_{y=Su})u) \quad (7.46)$$

$$+ \iota_+ b(\lambda x) \iota_+ b(\mu y) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \quad (7.47)$$

$$+ \iota_+ b(\lambda x) \iota_+ c(\mu z) ((|_{x=Su}) \otimes u \otimes (|_{z=Su})) \quad (7.48)$$

$$- \iota_+ a(\mu^{-1} z^{-1}) \iota_+ b(\lambda x) ((|_{x=Su}) \otimes u(|_{z=Su}) \otimes 1) \quad (7.49)$$

$$+ \iota_+ a(\mu y) \iota_+ c(\lambda \mu y z) (u \otimes 1 \otimes (|_{y=Su})(|_{z=Su})) \quad (7.50)$$

$$+ \iota_+ b(\mu y) \iota_+ c(\lambda \mu y z) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})) \quad (7.51)$$

$$+ \iota_+ c(\lambda \mu y z) \iota_+ c(\mu z) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})) \quad (7.52)$$

$$- \iota_+ a(\mu^{-1} z^{-1}) \iota_+ c(\lambda \mu y z) (u \otimes (|_{y=Su})(|_{z=Su}) \otimes 1). \quad (7.53)$$

Finally, using (3.7b), sesquilinearity (3.4a), the right Leibniz rule (3.4c) and (7.20) we get

$$\begin{aligned} & \{\!\!\{ \{\!\!\{ u_\lambda u \}\!\!\}_{\lambda \mu} u \}\!\!\}_L \\ &= \iota_+ a(\lambda \mu x y) \iota_+ b(\mu^{-1} x^{-1}) (1 \otimes (|_{x=Su}) \otimes (|_{y=Su})u) \end{aligned} \quad (7.54)$$

$$+ \iota_+ b(\lambda \mu x y) \iota_+ b(\mu^{-1} y^{-1}) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \quad (7.55)$$

$$+ \iota_+ b(\mu^{-1} y^{-1}) \iota_+ c(\lambda \mu y z) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})) \quad (7.56)$$

$$- \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ b(\mu^{-1} z^{-1}) (u(|_{y=Su}) \otimes (|_{z=Su}) \otimes 1) \quad (7.57)$$

$$+ \iota_+ a(\lambda \mu x y) \iota_+ c(\lambda x) (1 \otimes (|_{x=Su}) \otimes (|_{y=Su})u) \quad (7.58)$$

$$+ \iota_+ b(\lambda \mu x y) \iota_+ c(\lambda y) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \quad (7.59)$$

$$+ \iota_+ c(\lambda \mu y z) \iota_+ c(\lambda y) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})) \quad (7.60)$$

$$- \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ c(\lambda z) (u(|_{y=Su}) \otimes (|_{z=Su}) \otimes 1) \quad (7.61)$$

$$- \iota_+ a(\lambda \mu x y) \iota_+ a(\lambda^{-1} x^{-1}) ((|_{x=Su}) \otimes 1 \otimes (|_{y=Su})u) \quad (7.62)$$

$$- \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ b(\lambda \mu x y) ((|_{x=Su})(|_{y=Su}) \otimes 1 \otimes u) \quad (7.63)$$

$$- \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ c(\lambda \mu y z) (u(|_{y=Su}) \otimes 1 \otimes (|_{z=Su})) \quad (7.64)$$

$$+ \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \iota_+ a(\lambda^{-1} z^{-1}) (u(|_{y=Su})(|_{z=Su}) \otimes 1 \otimes 1) \quad (7.65)$$

$$- \iota_+ a(\lambda \mu x y) \iota_+ a(\mu x) (1 \otimes 1 \otimes (|_{x=Su})(|_{y=Su})u) \quad (7.66)$$

$$- \iota_+ a(\mu y) \iota_+ b(\lambda \mu x y) ((|_{x=Su}) \otimes 1 \otimes (|_{y=Su})u) \quad (7.67)$$

$$- \iota_+ a(\mu y) \iota_+ c(\lambda \mu y z) (u \otimes 1 \otimes (|_{y=Su})(|_{z=Su})) \quad (7.68)$$

$$+ \iota_+ a(\mu z) \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) (u(|_{y=Su}) \otimes 1 \otimes (|_{z=Su})). \quad (7.69)$$

The following terms cancel in the Jacobi identity (7.21):

$$\begin{aligned} (7.24) - (7.59) &= 0, & (7.25) - (7.63) &= 0, & (7.26) - (7.44) &= 0, & (7.27) - (7.48) &= 0, \\ (7.30) - (7.45) &= 0, & (7.31) - (7.49) &= 0, & (7.36) - (7.61) &= 0, & (7.37) - (7.65) &= 0, \\ (7.38) + (7.66) &= 0, & (7.50) + (7.68) &= 0, \end{aligned}$$

and using the fact that $b(z) = -b(z^{-1})$ we have also the cancellation

$$(7.39) + (7.54) = 0, \quad (7.51) + (7.56) = 0.$$

Next, observe that equation (7.21) can be understood as an identity in the space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$ with coefficients in $\mathbb{k}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]((\lambda, \mu))$. Since the elements $u \otimes u \otimes u$, $1 \otimes u^2 \otimes u$, $u^2 \otimes 1 \otimes u$, $u \otimes u^2 \otimes 1$, $u^2 \otimes u \otimes 1$, $1 \otimes u \otimes u^2$ and $u \otimes 1 \otimes u^2$ are linearly independent, the Jacobi identity (7.21) holds if and only if each coefficient of these elements vanishes, leading to seven further identities that we want to prove being equivalent to the four conditions (7.19).

Collecting the terms (7.23), (7.28), (7.47), (7.52), (7.55) and (7.60) acting on $u \otimes u \otimes u$, and using the fact that $b(z) = -b(z^{-1})$ we then get the following identity

$$\begin{aligned} & \left(\iota_+ b(\lambda x) \iota_+ b(\lambda \mu x y) + \iota_+ b(\mu y) \iota_+ b(\lambda \mu x y) - \iota_+ b(\lambda x) \iota_+ b(\mu y) \right) ((|_{x=Su}) \otimes (|_{y=Su}) \otimes u) \\ &= \left(\iota_+ c(\lambda y) \iota_+ c(\lambda \mu y z) + \iota_+ c(\mu z) \iota_+ c(\lambda \mu y z) - \iota_+ c(\lambda y) \iota_+ c(\mu z) \right) (u \otimes (|_{y=Su}) \otimes (|_{z=Su})). \end{aligned} \quad (7.70)$$

Note that the LHS of (7.70) is independent of z , hence of μ , while the RHS is independent of x , hence of λ . This forces both sides to be a constant multiple of $u \otimes u \otimes u$. This condition is equivalent to the first two conditions in (7.19).

Next, collecting the terms (7.22), (7.41), and (7.43) acting on $1 \otimes u^2 \otimes u$ we get the identity

$$\left(\iota_+ a(\lambda x) \iota_+ b(\lambda \mu x y) + \iota_+ a(\lambda \mu x y) \iota_+ a(\mu^{-1} y^{-1}) - \iota_+ a(\lambda x) \iota_+ b(\mu y) \right) (1 \otimes (|_{x=Su}) (|_{y=Su}) \otimes u) = 0. \quad (7.71)$$

Using the fact that $b(z) = -b(z^{-1})$ the identity (7.71) is equivalent to the third condition in (7.19). We get the same condition looking at the coefficient of $u^2 \otimes u \otimes 1$ and $u \otimes 1 \otimes u^2$.

Finally, collecting the terms (7.29), (7.64), and (7.69) acting on $u^2 \otimes 1 \otimes u$ we get the identity

$$\begin{aligned} & \left(\iota_+ a(\lambda^{-1} y^{-1}) \iota_+ c(\mu z) - \iota_+ a(\lambda^{-1} y^{-1}) \iota_+ c(\lambda \mu y z) \right. \\ & \left. + \iota_+ a(\mu z) \iota_+ a(\lambda^{-1} \mu^{-1} y^{-1} z^{-1}) \right) (u (|_{y=Su}) \otimes 1 \otimes (|_{z=Su})) = 0. \end{aligned} \quad (7.72)$$

Using the fact that $c(z) = -c(z^{-1})$ the identity (7.72) is equivalent to the fourth condition in (7.19). We get the same condition by looking at the coefficient of $1 \otimes u \otimes u^2$ and $u \otimes u^2 \otimes 1$. This concludes the proof. \square

Lemma 7.14. *Let $R(z), Q(z) \in \mathbb{k}((z))$.*

(a) *Let $\gamma \in \mathbb{k}$. Then, $R(z)$ satisfies the equation (in $\mathbb{k}((z, w))$)*

$$(R(z) + R(w))R(zw) - R(z)R(w) = \gamma, \quad (7.73)$$

if and only if $R(z) = \beta$ or $R(z) = R_1(z^k)$ for $k \geq 1$ and

$$R_1(z) = \beta \left(1 + 2 \sum_{n \geq 1} z^n \right) = \beta \iota_+ \frac{1+z}{1-z}, \quad (7.74)$$

where $\beta^2 = \gamma$.

(b) *Let $R(z) = R_1(z^k)$ for some $k \geq 1$ and $R_1(z)$ as in (7.74). Then $Q(z)$, satisfies the equation (in $\mathbb{k}((z, w))$)*

$$(R(z) + R(w))Q(zw) + Q(z)Q(w) = 0, \quad (7.75)$$

if and only if

$$Q(z) = \alpha \sum_{n \geq 0} z^{nk+p} = \iota_+ \frac{\alpha z^p}{1-z^k}, \quad p \in \mathbb{Z}, \quad (7.76)$$

where $\alpha, \beta \in \mathbb{k}$ satisfy $\alpha(2\beta + \alpha) = 0$.

Proof. For part (a) it is straightforward to check that $R(z) = \beta$ or $R(z)$ as in (7.74), with $\beta^2 = \gamma$, solve (7.73). On the other hand, let $N \in \mathbb{Z}$ and let us write $R(z) = \sum_{n \geq N} r_n z^n$, with $r_N \neq 0$. Then, (7.73) becomes

$$\sum_{m \geq 2N, n \geq N} r_{m-n} r_n z^m w^n + \sum_{m \geq N, n \geq 2N} r_{n-m} r_m z^m w^n - \sum_{n, m \geq N} r_m r_n z^m w^n = \gamma. \quad (7.77)$$

If $N < 0$, then equating the coefficient of $z^{2N} w^N$ in both sides of (7.77) we get $r_N^2 = 0$, hence $r_N = 0$, which is a contradiction. If $N > 0$, equating the coefficient of $z^N w^N$ in both sides of (7.75) we get again $r_N^2 = 0$, which leads to a contradiction. Hence, it remains to consider the case $N = 0$. By equating the coefficient of $z^0 w^0$ in both sides of (7.75) we get $r_0^2 = \gamma$, which leads to $r_0 = \beta$ with $\beta^2 = \gamma$. By equating the coefficient of $z^k w^k$, $k > 0$, in both sides of (7.75) we get

$$(2\beta - r_k) r_k = 0.$$

Hence, $r_k = 2\beta$ or $r_k = 0$, for $k \geq 1$. Let $k \geq 1$ be the smallest integer such that $r_k = 2\beta \neq 0$ but $r_h = 0$ for all $1 \leq h < k$. We claim that for each $n \geq 1$, $r_{nk} = r_k$ while $r_{nk+h} = 0$ for $1 \leq h < k$. This is shown by induction. Looking at the coefficient of $z^k w^{nk}$ in (7.75), we get $r_k r_{(n-1)k} - r_k r_{nk} = 0$ which yields the first equality. For the second, we look at the coefficient of $z^k w^{nk+h}$ in (7.75) which gives $r_k r_{(n-1)k+h} - r_k r_{nk+h} = 0$.

Hence, either $r_n = 0$ for every $n \geq 1$, thus $R(z) = \beta$, or the non-zero terms are $r_{nk} = 2\beta$, for every $n \geq 1$ and some $k \geq 1$, which gives that $R(z) = R_1(z^k)$ for $R_1(z)$ as in (7.74). This proves part (a).

For part (b), it is straightforward to check that $Q(z)$ in (7.76) solves (7.75). Moreover, if $Q(z)$ is a solution to (7.75), then $z^p Q(z)$ is also a solution, for every $p \in \mathbb{Z}$. Hence, discarding the trivial solution $Q(z) = 0$, we are left to seek solutions of the form $Q(z) = \sum_{n \geq 0} q_n z^n$ with $q_0 \neq 0$. Using (7.74) to expand $R(z) = R_1(z^k)$, we rewrite (7.75) as

$$2\beta \left(1 + \sum_{m \geq 1} z^{mk} + \sum_{n \geq 1} w^{nk} \right) \sum_{\ell \geq 0} q_\ell z^\ell w^\ell + \sum_{n, m \geq 0} q_m q_n z^m w^n = 0. \quad (7.78)$$

Equating the coefficient of $z^\ell w^\ell$, $\ell \geq 0$, in both sides of (7.78) we get

$$2\beta q_\ell + q_\ell^2 = 0. \quad (7.79)$$

Next, looking at the coefficient of $z^{h+nk} w^0$, for $1 \leq h < k$ and $n \geq 1$, we find that $q_{h+nk} q_0 = 0$. Since $q_0 \neq 0$ by assumption, this yields

$$q_j = 0 \quad \text{if} \quad j \notin k\mathbb{Z}_{\geq 0}. \quad (7.80)$$

Finally, if we look at the coefficient of $z^{nk} w^0$, $n \geq 1$, we get $q_0(2\beta + q_{nk}) = 0$. Together with the condition (7.79), we get that $q_0(q_0 - q_{nk}) = 0$, hence $q_{nk} = q_0$. Combining this identity with (7.80), we see that $q_\ell = 0$ except if ℓ is a multiple of k , in which case $q_{nk} = q_0$. Due to (7.79), we see by adding the trivial solution $Q(z) = 0$ that we can write $q_0 = \alpha$, where $\alpha \in \mathbb{k}$ is such that $\alpha(2\beta + \alpha) = 0$. This concludes the proof of part (b). \square

Proof of Theorem 7.12. By Theorem 7.9 we need to show that the double multiplicative λ -bracket defined by (7.18) satisfies skewsymmetry and Jacobi identity on generators. Skewsymmetry holds since, by construction, $H(S) = -H^*(S)$. By Proposition 7.13 Jacobi identity holds on generators if and only if the four conditions in (7.19) are satisfied. By Lemma 7.14(a) and the fact that $b(z) = -b(z^{-1})$, $c(z) = -c(z^{-1})$, the first two conditions in (7.19) give that

$$b(z) = \beta \frac{1 + z^k}{1 - z^k}, \quad c(z) = \beta \frac{1 + z^{\tilde{k}}}{1 - z^{\tilde{k}}},$$

for some $\beta \in \mathbb{k}$ and $k, \tilde{k} \geq 1$. By Lemma 7.14(b), the third equation in (7.19) is satisfied if and only if

$$a(z) = \frac{\alpha z^p}{1 - z^k}, \quad p \in \mathbb{Z},$$

where $\alpha \in \mathbb{k}$ is such that $\alpha(2\beta + \alpha) = 0$. Similarly, the fourth equation in (7.19) is satisfied if and only if

$$a(z) = \frac{\tilde{\alpha} z^{\tilde{p}}}{1 - z^{\tilde{k}}}, \quad \tilde{p} \in \mathbb{Z},$$

where $\tilde{\alpha} \in \mathbb{k}$ is such that $\tilde{\alpha}(2\beta + \tilde{\alpha}) = 0$. Equating both forms for $a(z)$ yields that $\tilde{k} = k$, $\tilde{p} = p$, and $\tilde{\alpha} = \alpha$, which concludes the proof. \square

Remark 7.15. Let us introduce the following notation (motivated by (4.2)): $r_u = 1 \otimes u$, $l_u = u \otimes 1$, $c_u = l_u - r_u$ and $a_u = l_u + r_u$. By Theorem 7.12 with $\alpha = -1$, $\beta = 1/2$, $k = 1$ and $p = q + 1$ for $q \geq 1$, we have that the pseudodifference operator of rational type

$$\begin{aligned} H(S) &= -r_u \iota_+ \frac{S^{q+1}}{1-S} \bullet r_u + \frac{1}{2} r_u \iota_+ \frac{1+S}{1-S} \bullet l_u + \frac{1}{2} l_u \iota_+ \frac{1+S}{1-S} \bullet r_u - l_u \iota_+ \frac{S^{-q}}{1-S} \bullet l_u \\ &= \sum_{i=1}^q (r_u S^i \bullet r_u - l_u S^{-i} \bullet l_u) - \frac{1}{2} a_u \bullet c_u - \frac{1}{2} c_u \iota_+ \frac{1+S}{1-S} \bullet c_u, \end{aligned} \quad (7.81)$$

defines a Poisson structure of rational type on \mathcal{R} . The operator $H(S)$ in (7.81) appeared in [CW2] (without the use of the embedding ι_+) and it is called the non-local Poisson structure of the non-commutative Narita-Itoh-Bogoyavlensky lattice hierarchy. However, note that $H(S)$ does not define a non-local Poisson structure on \mathcal{R} in the sense of Theorem 7.2. Indeed, let us replace in (7.16), the Laurent series $\iota_+ a(z)$, $\iota_+ b(z)$ and $\iota_+ c(z)$ by bilateral series $A(z), B(z), C(z) \in \mathbb{k}[[z, z^{-1}]]$ such that $B(z) = -B(z^{-1})$, $C(z) = -C(z^{-1})$. The same computations as in the proof of Proposition 7.13 show that $H(S)$ is a non-local Poisson structure if and only if conditions (7.19), obtained by replacing $\iota_+ a(z)$, $\iota_+ b(z)$ and $\iota_+ c(z)$ with $A(z), B(z), C(z)$, hold. It is not hard to check that the only solution to those equations is then $A(z) = B(z) = C(z) = 0$.

Remark 7.16. Motivated by the works [EKV1, EKV2] on a modification of the commutative Narita-Itoh-Bogoyavlensky lattice hierarchy, it is natural to ask whether there exists a rational Poisson structure with constant coefficient $K(S) = \iota_+ r(S)(1 \otimes 1)$, $r(z) \in \mathbb{k}(z)$, compatible with the rational Poisson structure $H(S)$ from Theorem 7.12. Compatibility means that $H(S) + K(S)$ is a rational Poisson structure as well. By Example 7.10, $K(S)$ is a rational Poisson structure if and only if $r(z) = -r(z^{-1})$. Let $\{u_\lambda u\}_H = H(\lambda)$ be as in (7.20), and let us set $\{u_\lambda u\}_K = K(\lambda) = \iota_+ r(\lambda)(1 \otimes 1)$. Then, by Theorem 7.9, $H(S) + K(S)$ is a rational Poisson structure if and only if $\{u_\lambda u\} = \{u_\lambda u\}_H + \{u_\lambda u\}_K$ defines a rational double multiplicative Poisson vertex algebra. To check this, it suffices to verify that Jacobi identity (7.21) holds. Using the fact that $H(S)$ and $K(S)$ are rational Poisson structures this reduces to the condition

$$\{u_\lambda \{u_\mu u\}_H\}_{K,L} - \{u_\mu \{u_\lambda u\}_H\}_{K,R} = \left\{ \{u_\lambda u\}_{H\lambda\mu} u \right\}_{K,L}. \quad (7.82)$$

Equation (7.82) is equivalent to the following three equations for the rational function $r(z)$ (we omit the details of the computations):

$$\begin{aligned} (\iota_+ b(zx) - \iota_+ b(wx)) \iota_+ r(zwx) + \iota_+ a(zx) \iota_+ r(w) + \iota_+ a(w^{-1}x^{-1}) \iota_+ r(z) &= 0, \\ (\iota_+ b(zx) + \iota_+ b(zw)) \iota_+ r(w) - \iota_+ a(zw) \iota_+ r(z) + \iota_+ a(zx) \iota_+ r(zwx) &= 0, \\ (\iota_+ b(zx) + \iota_+ b(zw)) \iota_+ r(w) + \iota_+ a(z^{-1}w^{-1}) \iota_+ r(z) - \iota_+ a(z^{-1}x^{-1}) \iota_+ r(zwx) &= 0, \end{aligned} \quad (7.83)$$

where $a(z)$ and $b(z)$ are as in (7.17). Let us show that these conditions imply $r(z) = 0$, i.e. there is no compatibility between the two Poisson structures. (We explain the case when $a(z) \neq 0$, and we leave to the reader the case with $a(z) = 0$, $b(z) \neq 0$.) Subtracting the second and third equations in (7.83), and using (7.17) we get that $r(z)$ should satisfy the identity

$$(1 - (zw)^{k-2p}) \iota_+ a(zw) \iota_+ r(z) = (1 - (zx)^{k-2p}) \iota_+ a(zx) \iota_+ r(zwx). \quad (7.84)$$

Note that the LHS of (7.84) does not depend on x . Hence, we can set $x = w$ in the RHS of (7.84) and get that $r(z)$ satisfies $\iota_+ r(z) = \iota_+ r(zw^2)$ whenever $k \neq 2p$. This forces $r(z) = \gamma$, for

some $\gamma \in \mathbb{K}$. Since $r(z) = -r(z^{-1})$, we then have $r(z) = 0$. When $k = 2p$ (which is a positive even integer by Theorem 7.12), the first equation in (7.83) for $w = x$ becomes

$$\iota_+ \frac{\alpha((zx)^p - (zx)^{-p})}{1 - (zx)^{2p}} \iota_+ r(z) = 0,$$

after using the form of $a(z)$ given in (7.17). Since the first term does not vanish, we must have $r(z) = 0$.

REFERENCES

- [CW1] Casati M., Wang J.P.: *Recursion and Hamiltonian operators for integrable nonabelian difference equations*. Nonlinearity 34, no. 1, 205-236 (2021).
- [CW2] Casati M., Wang J.P.: *Hamiltonian structures for integrable nonabelian difference equations*. Comm. Math. Phys. 392, no. 1, 219-278 (2022).
- [DSK1] De Sole A., Kac V.G.: *Lie conformal algebra cohomology and the variational complex*. Comm. Math. Phys. 292, no. 3, 667-719 (2009).
- [DSK2] De Sole A., Kac V.G.: *The variational Poisson cohomology*. Jpn. J. Math. 8, no. 1, 1-145 (2013).
- [DSK3] De Sole A., Kac V.G.: *Non-local Poisson structures and applications to the theory of integrable systems*. Jpn. J. Math. 8, no. 2, 233-347 (2013).
- [DSKV] De Sole A., Kac V.G., Valeri D.: *Double Poisson vertex algebras and non-commutative Hamiltonian equations*. Adv. Math. 281, 1025-1099 (2015).
- [DSKVW1] De Sole A., Kac V.G., Valeri D., Wakimoto M.: *Local and non-local multiplicative Poisson vertex algebras and differential-difference equations*. Comm. Math. Phys. 370, no. 3, 1019-1068 (2019).
- [DSKVW2] De Sole A., Kac V.G., Valeri D.: *Poisson λ -brackets for differential-difference equations*. Int. Math. Res. Not. IMRN 13, 4144-4190 (2020).
- [EKV1] Evripidou C., Kassotakis P., Vanhaecke P.: *Integrable deformations of the Bogoyavlenskij-Itoh Lotka-Volterra systems*. Regul. Chaotic Dyn. 22, no. 6, 721-739 (2017).
- [EKV2] Evripidou C., Kassotakis P., Vanhaecke P.: *Integrable reductions of the dressing chain*. J. Comput. Dyn. 6, no. 2, 277-306 (2019).
- [FH] Fernández D., Heluani R.: *Noncommutative Poisson vertex algebras and Courant-Dorfman algebras*. Preprint, [arXiv:2106.00270 \[math.QA\]](https://arxiv.org/abs/2106.00270).
- [GKK] Golenishcheva-Kutuzova M., Kac V.G.: Γ -conformal algebras. J. Math. Phys. 39, no. 4, 2290-2305 (1998).
- [Ko] Kontsevich, M.: *Formal (non)-commutative symplectic geometry*. In: I.M. Gelfand, L. Corwin, J. Lepowsky (Eds.), The Gelfand Mathematical Seminars, 1990-1992, Birkhäuser, Boston, pp. 173-187 (1993).
- [KR] Kontsevich, M., Rosenberg, A.L.: *Noncommutative smooth spaces*. The Gelfand Mathematical Seminars, 1996-1999, Gelfand Math. Sem., Birkhäuser, Boston, pp. 85-108 (2000).
- [P] Powell G.: *On double Poisson structures on commutative algebras*. J. Geom. Phys. 110, 1-8 (2016).
- [S] Schedler T.: *Poisson algebras and Yang-Baxter equations*. In: Advances in quantum computation, Contemp. Math., vol. 482, Amer. Math. Soc., Providence, RI, 2009, pp. 91-106.
- [ORS] Odesskii A.V., Rubtsov V.N., Sokolov V.V.: *Parameter-dependent associative Yang-Baxter equations and Poisson brackets*. Int. J. Geom. Methods Mod. Phys. 11, no. 9, 1460036, 18 pages (2014).
- [OS] Olver P.J., Sokolov V.V.: *Integrable evolution equations on associative algebras*. Comm. Math. Phys. 193, no. 2, 245-268 (1998).
- [VdB1] Van den Bergh M.: *Double Poisson algebras*. Trans. Amer. Math. Soc., 360, no. 11, 5711-5769 (2008).
- [VdB2] Van den Bergh M.: *Non-commutative quasi-Hamiltonian spaces*. In: Poisson geometry in mathematics and physics, volume 450 of Contemp. Math., 273-299. Amer. Math. Soc., Providence, RI (2008).

(Maxime Fairon) SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, UNIVERSITY PLACE, GLASGOW G12 8QQ, UK

Email address: Maxime.Fairon@glasgow.ac.uk

(current address) DEPARTMENT OF MATHEMATICAL SCIENCES, SCHOFIELD BUILDING, LOUGHBOROUGH UNIVERSITY, EPINAL WAY, LOUGHBOROUGH LE11 3TU, UK

Email address: M.Fairon@lboro.ac.uk

(Daniele Valeri) DIPARTIMENTO DI MATEMATICA & INFN, SAPIENZA UNIVERSITÀ DI ROMA, P.LE ALDO MORO 5, 00185 ROMA, ITALY

Email address: daniele.valeri@uniroma1.it