Identifying Competency Demands in Calculus Textbook Examples: The Case of Integrals

Abstract

This study investigates how three widely used calculus textbooks realize integrals as a potential to prompt mathematical competencies; adapting the rating scheme used in Boesen et al. (2014), Pettersen and Braeken (2019), and Turner, Blum, and Niss (2015). For this purpose, the study analysed examples (n = 444) about integrals – specifically, to assess the extent to which solving those examples calls for the activation of a particular set of mathematical competencies: *Communication*; *Devising Strategies*; *Mathematising*; *Representation*; *Using Symbols, Operations, and Formal Language* [*Symbols and Formalism*]; *Reasoning and Argument*. The competency demand of the examples was also identified on a scale from 0 (lowest demand) to 3 (highest demand) for each of six mathematical competencies. The findings revealed substantial similarities among the three calculus textbooks with regard to the level of competency demands: high level of *Communication* and *Symbols and Formalism*, and low level of *Devising Strategies*, *Representation, Reasoning and Argument,* and *Mathematising*. Relationships between these findings, implementations, and future research directions are also discussed.

Keywords: Calculus; calculus textbooks; integrals; mathematical competency demands; textbook analysis

Introduction

In calculus, the integral is a difficult concept for students to learn and understand due to the complexity of its definition, representation, and interpretation (Sealey, 2014). The majority of students enrolling in calculus courses at secondary (Kouropatov & Dreyfus, 2014; Swidan & Yerushalmy, 2014) and tertiary (Orton, 1983) levels face cognitive obstacles. More specifically, undergraduate students encounter difficulties with computing the single-variable definite integrals

(Jones, 2015b), multiple integrals (McGee & Martinez-Planell, 2014), carrying out various techniques of integration (Orton, 1983), understanding integration theorems (e.g., Fundamental Theorem of Calculus) that include antiderivative (i.e., accumulation) functions (Kouropatov & Dreyfus, 2014; Thompson, 1994), building links between Riemann sum and the Riemann integral (Jones, Lim, & Chandler, 2017), and making graph-based interpretations of definite integrals (Jones, 2018). In tertiary education, it is likely that undergraduate students work individually more frequently than they did at secondary grades and that some of their individual work may rely on the use of the textbook (Randahl, 2012). It is not surprising, then, to assume that many of the difficulties students have in understanding the integral concept and the integration process may be a result of how they perceive, approach, and use the calculus textbook.

Textbooks are the primary resource for teaching as they provide teachers/professors with the opportunity to prepare classes and to develop tests/exams (Kajander & Lovric, 2009) and thus, play an important role in teachers' classroom instruction (Tall, Smith, & Piez, 2008). Textbooks are also a fundamental resource for learning as they provide students with the opportunity to study for classes/exams, to do homework, and to develop projects (van Zanten & van den Heuvel-Panhuizen, 2018). When considering how these experiences are formed on the part of students, a central aspect is how students work with their textbooks and examples/exercises. In recent years, examples and exemplification play a central role in prompting students' mathematical learning, understanding, and thinking (Lithner, 2003). Sun (2011) indicates that when students experience an example in a textbook, they become aware of both essential and non-essential aspects of that example. However, essential aspects that would ideally capture a variety of mathematical competencies should be the focus of students' awareness. Thus, if mathematical competencies are important for the learning and understanding of mathematics, we must investigate the opportunities that students get acquainted with them. As the mathematics education community works to incorporate mathematical competencies throughout the international assessments (e.g., Programme for International Student Assessment (PISA), Trends in International Mathematics and Science Study (TIMSS)) conducted at the primary, middle, and secondary grades, it is important to not only look forward to the domains where mathematical competencies might be added at the undergraduate level but also to look back and reflect carefully on the extent and nature of mathematical competencies that already exist in tasks or assessment items. The present study promotes such integration and reflection by identifying the extent and nature of mathematical competencies in the context of integral examples in university calculus textbooks.

Against this background of the literature, our study set out to investigate the mathematical competency demands present in integral examples in a sample of three widely used calculus textbooks, focusing on the following research questions:

1. What are the mathematical competency demands of integral examples in calculus textbooks?
2. What is the competency level that best fit the demand of the integral examples in calculus textbooks?

Theoretical Background

Mathematical Competencies: A framework for Textbook Analysis

Recently, the concept of competence has gained an increased attention in mathematics education (e.g., Kilpatrick, 2014; Turner, 2012) and this focus has influenced curriculum design (Westera, 2001). *Competence* can be defined in a variety of ways such as 'the ability to do something successfully or efficiently'; or 'the state or quality of being adequately or well qualified'

(Kilpatrick, 2014, p. 85). It seems to possess a set of descriptions including ability, capability, efficiency, proficiency, or mastery (Pikkarainen, 2014) and covering a richer view of mastering mathematics, knowing mathematics, and understanding mathematics (Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016). In a related vein, *mathematical competency* (or *competencies* in plural) is seen as associated with one or a set of the constituent parts of mathematical competence (Niss, 2003, p. 6; Pettersen & Braeken, 2019, p. 406). That is, any one competency can be acquired or mastered in alignment with the other competencies. Viewed together, several mathematical competency frameworks, which refer to the structural plan for organizing the cognitive skills and abilities used in learning and doing mathematics (Kilpatrick, 2014), have emerged. These frameworks described different cognitive skills and abilities that constitute mathematical competence (Kilpatrick, 2014), which in common, divide mathematical competence into a set of mental processes with particular emphasis on the fact that doing mathematics requires a host of competencies, not limited to rote memorization (i.e., factual knowledge) or automatized calculation (i.e., procedural knowledge) (Niss et al., 2016). In this regard, one framework particularly had a considerable impact on the reform movements regarding the mathematics curricula and assessments in several European countries (e.g., Denmark, Sweden, Germany): the KOM (in Danish: Competencies and the Learning of Mathematics) framework (for a detailed review see Niss, 2015), putting forth eight mathematical competencies that overlap to a certain degree and have to be activated jointly while solving mathematical problems (Kilpatrick, 2014; Niss & Højgaard, 2011). When the aim of mathematics education in general, and tertiary education in particular, is mathematical competence, the textbook – as a primary teaching/learning material – must provide students with opportunities to develop the necessary prerequisites or essential activities for engaging in successful problem solving (Turner, Dossey,Blum, & Niss, 2013).

Drawing on these assertions the present study builds on the consecutive works by Pettersen and Nortvedt (2018) and Pettersen and Braeken (2019) who used the item analysis scheme developed to identify the competency demands of mathematical problems (Turner, Blum, & Niss, 2015). This item analysis scheme involved six mathematical competencies: *Communication*, *Devising Strategies*, *Mathematising*, *Representation*, *Using Symbols, Operations, and Formal Language* [*Symbols and Formalism*], and *Reasoning and Argument*. The item analysis scheme also included level descriptions of which four different rating values from 0 (lowest demand: a minimal degree of activation/no activation of the competency) to 3 (highest demand: advanced level of activation) to each item for each competency (Pettersen & Nortvedt, 2018). Table 1 presents the operational definitions of the six mathematical competencies and the four levels of demand for activation of each competency, which are adapted from the previous frameworks used by Boesen et al. (2014, p. 75-76), Pettersen and Braeken (2019, p. 408), and Turner et al. (2015, p. 85-115).

Table 1

Definitions of the mathematical competencies and level descriptions

Communication	
Reading, decoding, and **interpreting** statements, instructions, questions/tasks, examples/exercises, images/objects; **imagining** the situation to make sense of the information provided, understand the situation presented to make sense of the mathematical terms referred to, **presenting** and **explaining** one's mathematical work or reasoning.	
Level 0	Understand a short sentence or phrase relating to a single familiar concept that gives immediate access to the context: • all information is directly relevant to the task • the order of information matches the steps of thought required to make sense of the task • connection involves only presentation of a single word or numeric result
Level 1	Identify, select, and extract relevant elements of the information: • use links or connections within the text that are needed to understand the context/task • cycle within the text or between the text and other related representation(s) • connection required is simple, but beyond the presentation of a single numeric result (e.g., writing a short statement, doing a short calculation, expressing an interval or a range of values)

Level 2 Use repeated cycling to understand instructions, identify and select elements of the information to be combined, and decode/link multiple elements of the context/task:

- interpret conditional statements or instructions containing diverse elements; or actively communicate a constructed description or explanation
- connection includes giving a brief description, making a short explanation, or presenting a sequence of calculation steps

Level 3 Recognise and interpret logically complex relationships involving multiple elements, ideas and links

- create an economical, clear, coherent and complete description or explanation of a solution, process or argument
- connection is comprised of developing an argumentation that builds links within and among the multiple elements of the context/task

Devising Strategies

Selecting and **devising**, as well as **implementing**, a mathematical strategy to solve problems arising from the context/task; **monitoring** and **controlling** the implementation of the processes involved.

Level 0 Take direct actions, where the strategy needed is stated or obvious:

- activate a solution strategy which is explicitly stated

Level 1 Decide on a suitable strategy that uses the relevant given information to reach a conclusion:

- construct a straightforward strategy which includes a single step and combines the relevant elements of the context/task to reach a conclusion or a result

Level 2 Construct a strategy to transform given information to reach a conclusion:

- activate a straightforward strategy which includes multiple steps
- devise an identified multi-stage strategy repeatedly, where using the strategy requires controlled/targeted processing, in order to transform given information to reach a conclusion or a result

Level 3 Construct an elaborated strategy to find an exhaustive solution or a generalised conclusion; evaluate or compare strategies:

- devise a multi-stage strategy, where using the strategy involves substantial metacognitive in the implementation of the strategy towards a solution

Mathematising

Translating an extra-mathematical situation into a mathematical model (e.g., structuring, idealising, making assumptions, building a model), making use of a given or constructed model by **interpreting** or validating it in relation to the context.

Level 0 Either the situation is purely intra-mathematical, or the relationship between the real situation and the model is not needed in solving the problem

- ensuing issues within the mathematical structures of the problem

Level 1 Interpret and infer directly from a given model; translate directly from a situation into mathematics where the structure, variables, and relationships are given:

- conceptualise the situation in a relevant way
- identify and select relevant variables
- collect relevant measurements
- construct tables, diagrams, or figures

Level 2 Modify or use a given model to satisfy changed conditions or interpret inferred relationships:

- choose a familiar model within limited and clearly articulated constraints
- create a model where the required variables, relationships and constraints are explicit and clear

Level 3 Link, compare, evaluate or choose between different given models:

- create a model in a situation where the assumptions, variables, relationships and constraints are to be identified or defined
- check the created model whether it satisfies the requirements of the context/task

Representation

Interpreting, decoding, translating between, and making use of given representations in pursuit of a solution; **selecting** or **devising** representations to capture the situation or to present one's work. The representations are depictions of mathematical objects or relationships, which include equations, formulae, graphs, tables, diagrams, pictures, textual descriptions, concrete materials.

Level 0 Directly operate on a given representation, where minimal interpretation is required in relation to the context/task:

- go directly from text to numbers
- read a value directly from a diagram, graph or table

Level 1 Explore, select, use, and interpret one standard or familiar representation in relation to a context/task:

- compare data
- depict or interpret trends and relationships

Level 2 Translate between or use two or more different representations in relation to the context/task:

- modify a representation
- devise a simple representation of a situation
- construct representation that requires substantial decoding

Level 3 Understand and use a non-standard representation that requires substantial decoding and interpretation:

- devise a representation that captures the key aspects of a complex mathematical situation
- compare or evaluate multiple representations
- link representations of a variety of mathematical entities

Symbols and Formalism

Understanding and **making use** of definitions, symbols, and facts; **manipulating** symbolic expressions within a mathematical context (e.g., arithmetic expressions and operations or algorithms and procedures), governed by mathematical conventions and rules; understanding and **utilising constructs** based on definitions, rules and formal systems.

Level 0 No mathematical rules or symbolic expressions need to be activated beyond fundamental arithmetic calculations, operating with small or easily tractable numbers:

- activate only elementary mathematical definitions, symbols, and facts
- do few arithmetic calculations which involve only easily tractable numbers

Level 1 Make direct use of a simple functional relationship, either implicit or explicit:

- use familiar linear relationships
- use formal mathematical symbols by direct substitution or sustained arithmetic calculations involving fractions and decimals
- activate and directly use a formal mathematical definition, symbolic concept, convention, or rule

Level 2 Explicit use and manipulation of symbols:

- manipulate symbols by rearranging a formula algebraically

- activate mathematical rules, definitions, conventions, procedures or formulae using a combination of multiple relationships or symbolic concepts
- employ formally expressed mathematical relationships which include multiple components
- use repeated or sustained calculations which are comprised of simple functional relationships

Level 3 Multi-step application of formal mathematical procedures:

- work flexibly with functional or involved algebraic relationships
- use both mathematical technique and knowledge to produce results and draw conclusions
- apply multi-step formal mathematical procedures
- use repeated or sustained calculations which are comprised of multiple functional relationships

Reasoning and Argument

Logically rooted thought processes that explore and link problem elements so as to **make inferences** from them, or to **check a justification that is given** or **provide a justification** of statements; generate mathematical arguments about the plausibility of conclusions (e.g., why the conclusions are anchored in intrinsic properties of the mathematical components such as objects, concepts, and transformations).

Level 0 Make direct inferences from the information and instructions given:

- develop a simple mathematical argument

Level 1 Reflect to join information to make inferences:

- link separate components present in the problem
- use direct reasoning within one aspect of the context/task/problem
- activate direct reasoning within one aspect of the context/task/problem

Level 2 Analyse information to follow or create a multi-step argument:

- connect several mathematical elements (e.g., variables, objects, concepts, and transformations)
- reason from linked information sources

Level 3 Synthesize and evaluate, use or create chains of reasoning:

- justify inferences or make generalisations, drawing on and combining multiple elements of information in a sustained and directed way
- form, scrutinise or justify arguments and conclusions using multiple elements of the context/task/problem

Research on Students' Mathematical Competencies in Integral Concept

The concept of integral is considered to be central in the study of calculus and learning the integral concept is part of many high school mathematics curricula as well as university programs (Jones, 2015a). It is a key topic that deserves our attention for several reasons: (i) it is a significant component of coursework in both single- and multi-variable calculus series (Hughes-Hallet, Gleason, & McCallum, 2010; Stewart, 2010; Thomas, Weir, Hass, & Giordano, 2010); (ii) it is used in higher level mathematics, such as differential equations, complex analysis, and numerical analysis; (iii) it serves as the basis for many real world applications in physics (e.g., force, mass, impulse, circulation), engineering (e.g., energy, tension, kinematics), and economics (e.g., market prices, taxes) to define and compute natural phenomena (Lovell, 2004); and (iv) it is the most useful concept in underlying the idea of contemplation of the whole as the totality of its small parts, which enables drawing conclusions with regard to both the whole in its entirety and in its internal structure (Kouropatov & Dreyfus, 2014).

The literature base on student understanding of integrals includes work on understanding the symbolic integral notation (Jones, 2013; Rasslan & Tall, 2002), computing definite (McGee & Martinez-Planell, 2014; Orton, 1983; Rasslan & Tall, 2002) and indefinite (Swidan & Yerushalmy, 2014) integrals, understanding the Fundamental Theorem of Calculus theoretically and visually (Rasslan & Tall, 2002; Thompson, 1994), constructing knowledge about the notions of approximation and accumulation (Kouropatov & Dreyfus, 2014) as well as the accumulation process (Palha, Dekker, & Gravemeijer, 2015; Yerushalmy & Swidan, 2012), representing the definite integral as area under a curve (Rasslan & Tall, 2002), and comprehending Riemann Sums (Sealey, 2014). The most commonly reported issue in these studies is that even high achieving students fail to acquire comprehension regarding the integral concept and the integration process and that the majority of the students acquire no more than formal techniques and routine algorithms (*Symbols and Formalism competency*) for the solution of exercises or the execution of examples. Indeed, in his early study Orton (1983) investigated students' ability to carry out various integration techniques and found out that even the high ability students from his study had significant difficulties in understanding integration as the limit of a sum (*Communication competency*). Besides, although students were able to evaluate the definite integral or compute the area under a curve (*Representation competency*) and apply the Fundamental Theorem of Calculus to evaluate a definite integral (*Devising Strategies competency*), they were not aware of the reasons, for instance, why the area under a curve was represented by the definite integral (*Representation competency*). In other words, the best students failed to exhibit *Reasoning and Argument* because of not knowing *why* they were doing *what* they were doing.

Similarly, many researchers (Rasslan & Tall, 2002; Thompson, 1994) underlined that students' difficulties with proving the Fundamental Theorem of Calculus (*Reasoning and Argument competency*) stem from impoverished concepts of rates of change and poorly developed/coordinated images of functional covariation (*Communication competency*) or inappropriate connection between the area and the derivative on the graph of the corresponding "area collection" function (*Representation competency*). Along the same lines, researchers indicated that although the definite integral can be conceptualized in a variety of ways (Sealey, 2014), students had very limited *Representation competency*, especially in graphical depictions of the definite integral (Jones, 2015b; 2018). Data showed that the vast majority of students understood the expression $\int_a^b f(x)dx$, to represent a graphical entity in which the closed shape is formed by the two vertical lines at a and b, and the horizontal axis and that the area of this shape is the value of the integral and thus hardly see definite integrals as representing the limit of Riemann sums or as a method of approximating the area under a curve. Research has similarly shown that students' analysis of graphs (*Representation competency*) concerned with indefinite integrals (Swidan & Yerushalmy, 2014) and accumulation function (Yerushalmy & Swidan, 2012) was limited to visual considerations that did not lead them to awareness of the mathematical meaning of the connections they observed visually (e.g., consider Riemann sum rectangles are overlaid onto the region only as an approximate "filling in" of the area).

On an encouraging note, McGee and Martinez-Planell (2014) discussed that when different registers (i.e., geometric, numerical, and symbolic) which would trigger *Representation* and *Symbols and Formalism competency* were included in texts, students' comprehension of tracing the path from a numerical Riemann sum approximating the area under a curve to a definite integral representing the precise area (*Mathematising competency*) would improve. Indeed, researchers put forth the fact that the application of the definite integrals in extra-mathematical situations such as the Riemann sums would reflect a strong competency in mathematising that is translating the extra-mathematical situation of a numerical Riemann sum approximating the area under a curve into a mathematical model of a definite integral representing the precise area. Palha et al. (2015) also discussed that when calculation and proof tasks, which would trigger students' *Devising Strategies* and *Reasoning and Argument competency*, were designed to foster dynamic processes (e.g., total distance accumulation over time) students' would tend to connect and reflect on the different meanings of integral (e.g., explore the variation of lower and upper limits).

Methodology

Selection of Textbooks

We analyzed the examples from three calculus textbooks: *Calculus 7E 7th Edition* (Stewart, 2008–2012 https://www.stewartcalculus.com/), *Thomas' Calculus 12th Edition* (Thomas Jr., Weir, Hass, & Giordano, 1996 - 2019), and *Calculus Single & Multivariable 6th Edition* (Hughes-Hallett, McCallum, Gleason, Connally, Lovelock, Patterson et al., 1998 – 2013). In the following lines, these three textbooks were abbreviated as SC, TC, and HHC, respectively. The selection was based upon two standpoints: (1) information about the best-selling calculus textbooks from publishers (for details see https://www.amazon.com/Best-Sellers-Books-Calculus/zgbs/books/13905) and (2) reports of the Mathematical Association of America (MAA) from a survey (N = 700 instructors and N = 212 universities) about the most widely used calculus textbooks in the United States (Bressoud, Mesa, & Rasmussen, 2015) documenting that the majority of instructors preferred using in the Calculus courses SC (43%), HHC (19%), and TC (9%), whereas the other textbooks were used by less than 4%. The reason that we included these textbooks in our study was not only because these textbooks are purposely put in the market to promote calculus courses in which mathematical thinking and computation reinforce each other but also because they encourage students with various learning opportunities to expand their mathematical knowledge. For instance, HHC represents the mathematical ideas under the approach called the *Rule of Four* by using verbal, graphical, numeric, and symbolic representations of calculus concepts (Hughes-Hallett, 1991). SC has a formalistic approach to calculus in which harmonizes algebra and analytic geometry and motivates students to apply concepts rather than replicate the rules and facts behind the procedures and techniques (Stewart, 2010). On the other hand, TC introduces students to the intrinsic beauty of calculus and the power of its real-life applications with the goal of developing technical competence while furthering students' appreciation of the calculus subject (Thomas et al., 2010). Thus, for us it is interesting to investigate what additional opportunities to learn as well as to teach calculus these textbooks offer. As the unit of analysis, we used the examples referring to the unit that provides an explicit procedure of which the variables and relationships are required for solving the question and what kinds of strategies are involved in the solution process.

Data Analysis Procedures

The examples were coded based on the description of competency definitions and level descriptions as presented in Table 1. The classification of the examples was based on the judgement to what degree examples required high (Level 2 and 3) and low (Level 0 and 1) competency demand. When the examples (e.g., Example 1 (a) and (b)) were independent of each other, they were considered separately and counted as two examples. The number of examples reviewed was 444 in total: 140, 144, and 160 in TC, HHC, and SC, respectively. Analyses of data that involve computing descriptive (frequency and percentage distributions) and inferential (Kruskal-Wallis Test) statistics were performed with SPSS version 21.0 (IBM Corp, 2012). Kruskal-Wallis Test (Pallant, 2013), which allows to compare scores on some continuous variable for three or more groups, was conducted in order to determine whether there was a statistically significant difference among the percentage distribution of cognitive levels in the six mathematical competencies across the three textbooks, in general, and whether the percentage distributions of required levels of competency were significantly different for the three textbooks within each mathematical competency, in particular.

Reliability of Coding

Based on the mathematical competency framework, the two authors, fluent in English, independently coded each example in three textbooks. Next, a third rater, an independent expert in mathematics education, who was familiar with this study, performed a reliability check of the classification. For this purpose, a selection of examples (n = 50) was used from each of the three textbooks (N = 150) covering approximately a quarter of the total examples classified by the two authors. In this selection, all found appearances of the six mathematical competencies were included. Additionally, for this selection contained the level descriptions from 0 to 3 were included. When the two authors disagreed, those items were coded based on majority rule using the third expert's codes. There were only 10 items in which all three raters disagreed. The disagreement was due to the close relationship between the operational definition of the *Devising Strategies competency* and the *Symbols and Formalism competency*, both of which include an action (i.e., solution process). The percent agreement between the classifications of the two authors and the external rater was between 86% and 92%. After discussing the differences between the three classifications the percent agreement was between 96.5% and 98.2%. In all, 444 examples in the three calculus textbooks were analysed and coded.

Results

Table 2 displays the percentage of examples coded at each level of cognitive demand for each of the three textbooks.

Table 2
Percentages of distribution of examples

		Mathematical Competency Demand					
		C	DS	M	R	US	RA
TC	0	4.29	2.14	85.71	50	6.43	66.43
	1	34.29	42.86	9.29	37.86	39.29	20
	2	50.71	35.71	5	12.14	41.43	9.29
	3	10.71	19.29	0	0	12.86	4.29
SC	0	5	0.63	83.13	55.63	1.25	43.75
	1	41.88	46.88	4.38	34.38	30	35
	2	41.25	34.38	5.63	8.75	46.88	15.63
	3	11.88	18.13	6.88	1.25	21.88	5.63
HHC	0	5.56	7.64	70.83	59.72	2.78	52.08
	1	27.78	47.22	9.72	28.47	47.22	31.94
	2	52.08	34.72	13.89	8.33	40.28	10.42
	3	14.58	10.42	5.56	3.47	9.72	5.56

Note. HHC = Calculus Single and Multivariable 6th Edition (Hughes-Hallett et al., 2010), SC = Calculus 7E 7th Edition (Stewart, 2010), TC = Thomas' Calculus 12th Edition (Thomas et al., 2010), C = Communication, DS = Devising Strategies, M = Mathematising, R = Representation, US = Using Symbols, Operations, and Formal Language, RA = Reasoning and Argument.

From a general point of view, using the Kruskal-Wallis Test (Pallant, 2013), we determined that the percentage distribution of cognitive levels in the six mathematical competencies was not statistically significant across the three calculus textbooks (see Table 3). Taken together Table 2 and 3, findings implied that the three calculus textbooks included approximately equal numbers of integral examples at each levels of mathematical competency. From a specific point of view, we found that the distributions of required levels of mathematical competency were significantly different for the three calculus textbooks within a particular mathematical competency, with $p <$ .05 in each case, except *Mathematising* (see Table 4).

Table 3
Kruskal-Wallis Test results of the comparison of competency levels across three calculus textbooks

Level	Textbook	Mean Rank	χ^2	df	p
0	SC	8.17	.57	2	.75
	TC	10.00			
	HHC	10.33			
1	SC	9.83	.04	2	.97
	TC	9.17			
	HHC	9.50			
2	SC	9.17	.04	2	.97
	TC	9.50			
	HHC	9.83			
3	SC	11.17	.89	2	.64
	TC	8.50			
	HHC	8.83			

Note: $p > .05$

Table 4
Kruskal-Wallis Test results of the comparison of competency levels within mathematical competency demands

Competency	Level	Mean Rank	χ^2	df	p
Communication	0	2.00	9.97	3	.019*
	1	8.33			
	2	10.67			
	3	5.00			
Devising Strategies	0	2.00	10.38	3	.016*
	1	11.00			
	2	8.00			
	3	5.00			
Mathematising	0	11.00	6.84	3	.077
	1	5.67			
	2	5.67			
	3	3.67			
Representation	0	11.00	10.38	3	.016*
	1	8.00			
	2	5.00			
	3	2.00			

Symbols and Formalism	0	2.00	9.46	3	.024*
	1	9.00			
	2	10.00			
	3	5.00			
Reasoning and Argument	0	11.00	10.38	3	.016*
	1	8.00			
	2	5.00			
	3	2.00			

Note: $* p < .05$

As seen in Table 4, in *Communication* and *Symbols and Formalism* competencies, the mean rank for Level 2 was the highest (10.67 and 10.00, respectively), whereas the mean rank for Level 0 was the lowest (2.00 and 2.00, respectively), and the difference was significant ($\chi^2 = 9.97$ and $\chi^2 = 9.46$, respectively, $df = 3, p < .05$). The distribution of the mathematical competencies showed that the majority of examples in three calculus textbooks require high level of *Communication* and *Symbols and Formalism* competency (see Table 2). In particular, 41.2% of the examples in SC, 50.71% of them in TC, and 52.08% of them in HHC fit the Level 2 description for *Communication* competency. This showed that the examples required providing a brief explanation or presenting a sequence of calculation steps (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015). The receptive aspect of the examples involves understanding multiple elements that need to be linked in the solution steps. There is no simple and straightforward presentation of information. The constructive aspect involves providing an explanation or presenting a sequence of calculation steps. For *Symbols and Formalism* competency, 41.43% of the examples in TC, 46.88% of them in SC, and 40.28% of them in HHC were coded as Level 2, indicating that the examples require employing multiple rules, definitions, procedures, and formulas and using recurring calculations (see Table 2). An example from SC (p. 316-317):

Example 8 What is wrong with the following calculation?

$$\int_{-1}^{3}\frac{1}{x^2}dx=\left.\frac{x^{-1}}{-1}\right]_{-1}^{3}=-\frac{1}{3}-1=-\frac{4}{3}$$

Solution To start, we notice that this calculation must be wrong because the answer is negative but *f(x)* = $1/x^2 \geq 0$ and Property 6 of integrals says that $\int_a^b f(x)\,dx \geq 0$ when $f \geq 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because *f(x)* = $1/x^2$ is not continuous on $[-1, 3]$. In fact, f has infinite discontinuity at $x=0$, so $\int_{-1}^{3}\frac{1}{x^2}\,dx$ does not exist.

As seen in the example from SC, the explanation part of the example involved using the Property 6 (i.e., If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$), applying the Fundamental Theorem of Calculus, and then deciding whether *f(x)* has an infinite discontinuity at $x = 0$. Since the example required employing the definition of integral concept as well as to apply several rules, this example was associated with Level 2.

In *Devising Strategies* competency, as shown in Table 4, the mean rank for Level 1 was the highest (11.00), followed by Level 2 (8.00), Level 3 (5.00) and Level 0 (2.00), and the difference was significant ($\chi^2 = 10.38$, $df = 3$, $p < .05$). 42.86% of the examples in TC, 46.88% of examples in SC, and 47.22% of them in HHC correspond to the Level 1 description for *Devising Strategies* (see Table 2). This implied that the examples required constructing a strategy, which can be applied in a single step straightforwardly. For instance, in TC there is an example requiring high level of *Devising Strategies* competency: "The region bounded by the curve $\mathrm{y = x^2 + 1}$ and the line $\mathrm{y = -x + 3}$ is revolved about the $\mathrm{x-}$axis to generate a solid. Find the volume of the solid." (p. 314). In the explanation part of the example, a multi-stage strategy for calculating the volume of the solid are represented in this order: (1) drawing the region and sketching a line segment across it perpendicular to the axis of revolution, (2) finding the outer and inner radii of the washer that would swept out by the line segment, (3) finding the limits of integration by finding the x $-$coordinates of the intersection points of curve and line in represented in the figure, and (4) evaluating the volume integral. This example was associated with Level 2, since the solution steps require devising a straightforward strategy, which includes multiple steps (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015).

In *Representation* and *Reasoning and Argument* competencies (see Table 4), the mean rank for Level 0 was the highest (11.00), followed by Level 1 (8.00), Level 2 (5.00), and Level 3 (2.00), and the difference was significant ($\chi^2 = 10.38$, $df = 3$, $p < .05$). For the *Representation* competency, slightly more than half of the examples (i.e., 50% of them in TC, 55.63% of them in SC, and 59.72% of them HHC) were coded as Level 0 (see Table 2). This implied that the examples did not require translating between different representations, or comparing and evaluating representations. Furthermore, in each calculus textbook approximately 30% of the examples, which were coded as Level 1. The examples (coded as Level 1) require interpreting changes in two graphs representing the left-hand and right-hand sums under a curve, evaluating the definite integral as the limit of a sum of areas of rectangles, finding the area of a region, finding the integral given the volume or computing a volume or length using an integral, and finding the definite integral over an interval, from a graph showing the area of under the curve. For instance, in TC there are two kinds of examples requiring the *Representation* competency: (1) "Figure 5.21 shows the graph the function $f(x) = sinx$ between $x = 0$ and $x = 2\pi$. Compute (a) the definite integral of $f(x)$ over $[0, 2\pi]$, (b) the area between the graph of $f(x)$ and $x - axis$ over $[0, 2\pi]$", (2) "Find the area of the region between the $x - axis$ and the graph of $f(x) = x^3 - x^2 - 2x, -1 \leq x \leq 2$" (p. 281). In the first example, the figure represents the total area between $y = sinx$ and the $x - axis$ for $0 \leq x \leq 2\pi$. Students first need to read the graph of the sine function to interpret the area between the graph and the $x - axis$ over $[0, 2\pi]$, which is calculated by breaking up the domain of $sinx$ into two pieces (the interval $[0, \pi]$ over is nonnegative, the interval $[\pi, 2\pi]$ over is nonpositive), and then follow the computation of the area under the curves by adding the absolute values (*perceiving the transformation from graphical to algebra-symbolic representation*); whereas in the second, the graph is in the explanation part of the example. Students first need to realize how the zeros (-1, 0, and 2) of f subdivide the given integral into subintervals and integrate f over each subinterval, and then examine the graph to determine whether the features of the graph match the description of adding the absolute values of the calculated integrals (*perceiving the transformation from algebra-symbolic to graphical representation*). These instances reflect the Level 1 description of the *Representation* competency, which includes considerable interpretation of the transformation of the representations and using standard representations requiring minimal decoding (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015).

As shown in Table 2, 43.75% of the examples in SC, 52.08% of them in HHC, and 66.43% of them in TC correspond to the Level 0 description for *Reasoning and Argument*, which require students to draw direct inferences from the information and instructions given (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015). In TC and HHC, the examples, which demand lower levels of competency, were mostly related to the topics of integration by substitution, tables of integral, algebraic identities, trigonometric substitution, and trigonometric integrals. On the other hand, the examples, which demand higher levels of competency, were associated with the applications of definite integrals. An example for the high level of competency demands from HHC (p. 453) is presented as follows.

> **Example 6** It is reported that the Great Pyramid of Egypt was built in 20 years. If the stone making up the pyramid has a density of 200 pounds per cubic foot, find the total amount of work done in building the pyramid. The pyramid is 410 feet high and has a square base 755 feet by 755 feet. Estimate how many workers were needed to build the pyramid.

In this example, students need to realize that the stones are located at the approximate height of the construction site and calculate the total work done while building the pyramid. They should make inferences based on two assumptions: (1) “a laborer worked 10 hours a day, 300 days a year, for 20 years” to estimate the total number of workers needed and (2) “a typical worker carried ten blocks that weight 50 pounds along a 4 feet distance every hour” to perform each laborer works over a 20-year period (HHC, p. 453). This example is coded as Level 2 for *Reasoning and Argument* competency, indicating that it requires drawing inferences by connecting pieces of information from separate aspects of the example (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015).

In contrast, however, no significant difference was found between the competency levels in *Mathematising* competency (see Table 4). Accordingly, across the competency levels the highest mean rank was for Level 0 (11.00) followed by approximate mean ranks for Level 1, Level 2, and Level 3 (5.67, 5.67, and 3.67, respectively). As Table 2 presents, 85.71% of the examples in TC, 83.13% of them in SC, and 70.83% of them in HHC fit the Level 0 description for *Mathematising*. This indicated that the most examples were purely intra-mathematical, or the relationship between the real situation and the model is not needed in solving the problem (Boesen et al., 2014; Pettersen & Braeken, 2019; Turner et al., 2015). Significantly, the examples presenting the intra-mathematical situation were mostly emerged in the concept of integration by substitution and improper integrals in HHC, techniques of integration in TC, and the Fundamental Theorem of Calculus in SC. In contrast, less than 15% examples were related to the extra-mathematical situations across the three calculus textbooks. This implied that very few examples were aimed at connecting the integral concept to the other disciplines such as physics, economics, and biology, which required high levels of competency (i.e., Level 2 and 3) for *Mathematising*. Particularly, the examples, which were coded as Level 2 and 3, were related to the topic of density and center of mass, applications to physics, and application of definite integrals in the three calculus textbooks. For instance, in the following example from HHC (p. 441) students need to construct a population model where little guidance is provided regarding the variables and relationships, which must be defined and built by the student, respectively. This feature of the example reflects the Level 3 description of the *Mathematising* competency.

> **Example 4** The population density in Ringsburg is a function of the distance from the city center. At r miles from the center, the density is $P = f(r)$ people per square mile. Ringsburg is circular with radius 5 miles. Write a definite integral that expresses the total population of Ringsburg.

Discussion

The use of a rich variety of tasks comprising mathematical competencies has been shown to be important for students to engage successfully in problem solving and for teachers to provide opportunities to develop the competencies (Pettersen & Nortvedt, 2018; Turner et al., 2013). While prior research demonstrated the importance of the degree to which mathematical competencies to be designed and implemented with regard to the achievement level of students in educational settings (Niss et al., 2016), it offers little guidance for the written curriculum materials in general and for textbooks, in particular. In the present study, we investigated the mathematical competency demands of the examples about integrals in the three most commonly used calculus textbooks with respect to the six mathematical competencies. Since these competencies have a certain degree of overlap in the problem solving process (Turner et al., 2013) and they are not mutually distinct (Niss et al., 2016), we attempted at displaying the mathematical competency demands of integral examples in calculus textbooks but not establishing a sharp boundary among the mathematical competencies.

Regarding the different levels of demand for each competency, substantial similarities were documented across textbooks. Results revealed that the majority of the examples in each calculus textbook required high level of *Communication and Symbols and Formalism* competencies, whereas they required low level of *Devising Strategies, Representation, Reasoning and Argument,* and *Mathematising* competencies. Within the operational definitions of mathematical competencies primarily categorized by Niss (2003), some of the competencies (i.e., representation, symbols and formalism, communication, and aids and tools) were linked with the mathematical language and tools, whereas the others (i.e., mathematical thinking, problem handling, modeling, and reasoning) were associated with posing and answering questions in and with mathematics. Berger (2004) argued that a mathematical sign is used "as an object to communicate" and the meaning of the mathematical sign is related to the use of words for communication (p. 81). This might be also a plausible explanation about why the examples in calculus textbooks demanded a high level of *Communication* and *Symbols and Formalism* competencies. A possible reason for the examples requiring low level of *Devising Strategies, Representation, Reasoning and Argument,* and *Mathematising* competencies might be due to the fact that an overarching purpose of mathematical activity is "posing and answering questions in and by means of mathematics" (Niss & Højgaard, 2019, p.14), and that the nature of mathematical activity requires the ability to effectively engage in dealing with mathematical models and modelling (*Mathematising*), posing and solving mathematical problems (*Devising Strategies*), and undertaking mathematical reasoning (*Reasoning and Argument*).

It is noteworthy that the *Mathematising* competency was the only demand for which no significant differences were found among the distributions of required levels of mathematical competency across textbooks. One possible reason might be that, problem solving associated with mathematical competencies such as *Representation* or *Symbols and Formalism* are always brought on the scene as one of the main objectives of advanced mathematics (Tall et al., 2008). Given the strong focus on the role of problem solving particularly in integration process (Rasslan & Tall, 2002), it may be the case that *Mathematising* was overshadowed in the textbooks. Indeed, the examples, which displayed *Mathematising*, predominantly presented an intra-mathematical situation and were mostly related to the techniques of integration and the Fundamental Theorem of Calculus. To a lesser extent, examples presented an extra-mathematical situation in which relationships with other fields (e.g., physics, biology, economics etc.) were built. This deficiency in the textbooks was highlighted in McGee and Martinez-Planell (2014) with reference to the Riemann sums-definite integral association, which requires students show a strong mathematising competency where they formulate, employ, and interpret geometric, numerical, and symbolic representations. Viewed together, these findings implied that in general, calculus textbooks do not require students to identify the variables presented in the context and interpret a model in relation to a mathematical situation (Turner et al., 2013). This finding was consistent with previous research (Wijaya, van den Heuvel-Panhuizen, & Doorman, 2015) indicating that textbooks mostly provided tasks without a context, which would trigger students' mathematization and modeling skills (*Mathematising competency)*. Moreover, Randhal (2012) mentioned that there is not much attention paid to learning theories in the mathematics textbooks for tertiary level, which would have a particular impact on the treatment of mathematical definitions and procedures, and further students' previous learning experience and the curriculum objectives. Indeed, the treatment of the concepts in the textbooks becomes too difficult for students, and therefore they lose their interest in making sense of mathematics as well as learning calculus from the textbook. Besides, researchers (Garcia-Garcia & Dolores-Flores, 2018) underlined that while solving Calculus tasks, students need to be encouraged to make connections between the mathematical topics and prior knowledge, real life or other disciplines (extra-mathematical connection) and within various mathematical topics (intra-mathematical connection). Henceforth, the examples in the calculus textbooks concerning the extra-mathematical connections and consideration of problem contexts should play a prominent role in developing a high level of *Mathematising* competency.

In a similar vein, the inference about low level of *Reasoning and Argument* competency demand was supported by the high occurrence of Level 0 and 1 competency demands. Previous research revealed that the tasks in mathematics textbooks generally offered few instances of examples requiring mathematical reasoning (Dolev & Even, 2015; Stacey & Vincent, 2009). There is a limitation for the examples that need to be addressed before further discussing the finding. When we analyzed the examples, we considered the question stem and explanation part of the example. Particularly, the consideration of the explanation was an important restriction because the major purpose of the explanation was to introduce the formulas and rules for implementing them in the exercises, rather than to make a chain of inferences for the subsequent problems and exercises (Stacey & Vincent, 2009). In line with this assertion many researchers (e.g., Chang, Cromley, & Tran, 2016) indicated that students are not utilizing the mathematics textbooks efficiently, they fail to notice the important skills. Indeed, Lithner (2003) highlighted that students seem to accept the general solution strategies in the textbook exercises and apply the automatized procedure, but they do not attempt to learn the general ideas from a critical point of view. It should be kept in mind that the inclusion of problems and exercises within the investigation of *Reasoning and Argument* competency demands would provide a more comprehensive interpretation about the textbooks. Additionally, the content of the examples that required low level of *Reasoning and Argument* and *Mathematising* competencies, was reserved mostly to the applications of *definite* *integrals*. One possible reason might be that the context becomes the critical ingredient (diSessa, 2004) while solving the definite integral problems (Jones, 2015b) and that the majority of the examples, which activate the *Mathematising* competency, involve a mathematical context (e.g., Gravemeijer & Doorman, 1999).

On the other hand, the finding that examples required low level of *Representation* competency implied that students were prompted to use different representations and further grasp the transformation within (e.g., from geometric figures to geometric symbols) and among (e.g., from verbal explanations to algebraic expressions) these representations rather than engaging them to construct, evaluate, and/or modify various representations. In the case of functions, for instance, Duval (2006) indicated that reading the information from a function's algebraic expression together with its graph or constructing the graph of a function via the interpretation of its algebraic expression is not sufficient to recall the same function through these algebraic and graphical representations. Rather, it is only by *investigating* the representation variations and *realizing* what mathematically relevant in a representation. Obviously, the interpretation of the transformations within and among different representations is strictly related to the *Representation* competency possessed at all levels. Challenging examples prompting the acquisition of indepth knowledge are therefore necessary to have students investigate, evaluate, and construct transformations of representations and thus develop high level of *Representation* competency. In a related vein, the findings revealed that the graphical representations were mostly involved in the definite integral examples. Because the introduction and development of the definite integral concept are closely connected to the graph-based interpretation and representation in the calculus education community, it is not surprising to assume that the calculus textbooks present variety of graphical representations of the definite integral (Jones, 2018). Furthermore, Pettersen and Braeken (2019) found that *Representation* competency was not strong enough on their own to explain the mathematical competencies for solving the assessment items since decoding the symbolic representations was also linked to the *Symbols and Formalism* competency. In calculus, the use of symbols plays a fundamental role in developing the concepts, which could not exist independently from its presentation (de Almedia & da Silva, 2018).

As demonstrated in prior research, the textbook research has an evidence for providing affordances to learning mathematics (Fan, Zhu, & Miao, 2013). Still, there has been a strong need for making explicit the importance of the relationship between the role of calculus textbook and student learning. Furthermore, we limited the analyses to the examples in the integration chapters in three calculus textbooks. Future researchers might consider conducting studies on the examination of the mathematical competency demand in the expository text, end-of-chapter exercises, and problems. In doing so, an indepth understanding of the links between the various representations of the integral concept and the integration process and the mathematical competencies merits future research. For instance, by integrating additional textbooks into the investigation future researchers may consider implementing content analysis techniques for providing an exhaustive list of approaches to the integrals.

Similar patterns with corresponding inferences in other subject areas can be evident when investigating the mathematical competency demands of textbooks in other subject areas at the undergraduate level such as linear algebra, differential equations etc. In this sense, the big picture of the mathematical competencies designated in the present study offers descriptions specific for calculus as well as holds parallels and provides directions that can be specified to account for other subject areas in textbook analysis research.

References

Berger, M. (2004). The functional use of a mathematical sign. *Educational Studies in Mathematics*, *55*(1-3), 81-102.

Boesen, J., Helenius, O., Bergqvist, E., Bergqvist, T., Lithner, J., Palm, T., & Palmberg, B. (2014). Developing mathematical competence: From the intended to the enacted curriculum. *The Journal of Mathematical Behavior*, *33*, 72-87.

Bressoud, D., Mesa, V., & Rasmussen, C. (2015). *Insights and recommendations from the MAA national study of college calculus*. MAA Press.

Chang, B. L., Cromley, J. G., & Tran, N. (2016). Coordinating multiple representations in a reform calculus textbook. *International Journal of Science and Mathematics Education*, *14*(8), 1475-1497.

diSessa, A.A. (2004). Metarepresentation: Native competence and targets for instruction, *Cognition and Instruction*, *22*(3), 293-331.

de Almedia, L. M. W. & da Silva, K. A. P. (2018). A semiotic interpretation of the derivative concept in a textbook. *ZDM Mathematics Education*, *50*(5), 881-892.

Dolev, S. & Even, R. (2015). Justifications and explanations in Israeli 7th grade math textbooks. *International Journal of Science and Mathematics Education, 13*(2), 309-327.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*(1-2), 103-131.

García-García, J. & Dolores-Flores, C. (2018). Intra-mathematical connections made by high school students in performing Calculus tasks. *International Journal of Mathematical Education in Science and Technology*, *49*(2), 227-252.

Gravemeijer, K. & Doorman, M. (1999). Context problems in Realistic Mathematics Education: A Calculus course as an example. *Educational Studies in Mathematics*, *39*(1-3), 111-129.

Fan, L., Zhu, Y., & Miao, Z. (2013). Textbook research in mathematics education: development status and directions. *ZDM Mathematics Education*, *45*(5), 633-646.

Hughes-Hallett, D. (1991). Visualisation and calculus reform. In W. Zimmerman and S. Cunningham (Eds.), *Visualisation in teaching and learning mathematics* (pp. 121-126), MAA Notes 19.

Hughes-Hallet, D., Gleason, A. M., & McCallum, W. G. (2010). *Calculus: Single and multivariable*. New York: Wiley.

IBM Corp (2012). IBM SPSS Statistics for MAC, Version 21.0. Armonk, NY: IBM Corp.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141.

Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior, 38,* 9-28.

Jones, S. R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals. *International Journal of Mathematical Education in Science and Technology, 46*(5), 721-736.

Jones, S. R. (2018). Prototype images in mathematics education: the case of the graphical representation of the definite integral. *Educational Studies in Mathematics, 97*(3), 215-234.

Jones, S. R., Lim, Y., & Chandler, K. R. (2017). Teaching integration: How Certain instructional moves may undermine the potential conceptual value of the Riemann Sum and the Riemann Integral. *International Journal of Science and Mathematics Education, 15*(6), 1075-1095.

Kajander, A. & Lovric, M. (2009). Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology, 40*(2), 173-181.

Kilpatrick, J. (2014). Competency frameworks in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 85–87). Dordrecht: Springer.

Kouropatov, A., & Dreyfus, T. (2014). Learning the integral concept by constructing knowledge about accumulation. *ZDM Mathematics Education, 46*(4), 533-548.

Lithner, J. (2003). Students' mathematical reasoning in university textbook exercises. *Educational Studies in Mathematics, 52*(1), 29-55.

Lovell, M. C. (2004). *Economics with calculus*. Hackensack, NJ: World Scientific Publishing.

McGee, D. L. & Martinez-Planell, R. (2014). A study of semiotic registers in the development of the definite integral of functions of two and three variables. *International Journal of Science and Mathematics Education, 12*(4), 883-916.

Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. *Third Mediterranean conference on mathematics education* (pp. 115–124).

Niss, M. (2015). Mathematical Competencies and PISA. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy, Chapter 2* (pp. 35–56). Heidelberg, New York, Dordrecht: Springer.

Niss, M., Bruder, R., Planas, N., Turner, R. & Villa-Ochoa, J. A. (2016). Survey team on: Conceptualisation of the role of competencies, knowing and knowledge in mathematics education research. *ZDM Mathematics Education*, *48*(5), 611–632.

Niss, M., & Højgaard, T. (Eds.) (2011). *Competencies and mathematical learning—ideas and inspiration for the development of mathematical teaching and learning in Denmark*. English Edition. Tekster fra IMFUFA nr. 485. Roskilde: Roskilde University- IMFUFA [Translation of parts of Niss and Jensen (2002).].

Niss, M. & Højgaard, T. (2019). Mathematical competencies revisited. *Educational Studies in Mathematics, 102*(1), 9-28.

Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1-18.

Palha, S., Dekker, R., & Gravemeijer, K. (2015). The effect of shift-problem lessons in the mathematics classroom. *International Journal of Science and Mathematics Education, 13*(6), 1589-1623.

Pallant, J. (2013). *SPSS survival manual*. McGraw-Hill Education (UK).

Pettersen, A. & Braeken, J. (2019). Mathematical competency demands of assessment items: a search for empirical evidence. *International Journal of Science and Mathematics Education, 17*(2), 405-425.

Pettersen, A. & Nortvedt, G.A. (2018). Identifying competency demands in mathematical tasks: Recognising what matters. *International Journal of Science and Mathematics Education, 16*(5), 949-965.

Pikkarainen, E. (2014). Competence as a key concept of educational theory: A semiotic point of view. *Journal of Philosophy of Education, 48*(4), 621-636.

Randahl, M. (2012). Approach to mathematics in textbooks at tertiary level – exploring authors' views about their texts. *International Journal of Mathematical Education in Science and Technology, 43*(7), 881-896.

Rasslan, S. & Tall, D.O. (2002). *Definitions and images for the definite integral concept*. In Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, in A. D. Cockburn and E. Nardi (Eds.), Norwich, UK, Vol. 4, 89–96.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals, *The Journal of Mathematical Behavior, 33,* 230-245.

Stacey, K., & Vincent, J. (2009). Modes of reasoning in explanations in Australian eighth-grade mathematics textbooks. *Educational Studies in Mathematics*, *72*(3), 271.

Stewart, J. (2010). *Calculus*. Mason, OH: Brooks/Cole Cengage Learning.

Sun, X. (2011). "Variation problems" and their roles in the topic of fraction division in Chinese mathematics textbook examples. *Educational Studies in Mathematics*, *76*(1), 65-85.

Swidan, O. & Yerushalmy, M. (2014). Learning the indefinite integral in a dynamic and interactive technological environment. *ZDM Mathematics Education*, *46*(4), 517-531.

Tall, D., Smith, D., & Piez, C. (2008). Technology and calculus. In M. K. Heid & G. W. Blume (Eds.). *Research on technology and the teaching and learning of mathematics: Research syntheses* (Vol. 1, pp. 207–258). Charlotte, NC: Information Age.

Thomas, G. B., Weir, M. D., Hass, J., & Giordano, R. F. (2010). *Thomas' calculus: Early transcendentals*. Boston: Pearson Addison-Wesley.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274.

Turner, R. (2012). *Some drivers of test item difficulty in mathematics*. Paper presented at the Annual Meeting of the American Educational Research Association (AERA), Vancouver, 13-17 April 2012.

Turner, R., Blum, W., & Niss, M. (2015). Using competencies to explain mathematical item demand: A work in progress. In K. Stacey & R. Turner (Eds.), *Assessing mathematical literacy: The PISA experience* (pp. 85–115). New York: Springer.

Turner, R., Dossey, J., Blum, W., & Niss, M. (2013). Using mathematical competencies to predict item difficulty in PISA: A MEG study. In M. Prenzel, M. Kobarg, K. Schöps, & S. Rönnebeck (Eds.), *Research on PISA* (pp.23-37). Springer, Dordrecht.

van Zanten, M., & van den Heuvel-Panhuizen, M. (2018). Opportunity to learn problem solving in Dutch primary school mathematics textbooks. *ZDM Mathematics Education*, *50*(5), 827-838.

Westera, W. (2001). Competences in education: A confusion of tongues. *Journal of Curriculum Studies*, *33*(1), 75-88.

Wijaya, A., van den Heuvel-Panhuizen, M., & Doorman, M. (2015). Opportunity-to-learn context-based tasks provided by mathematics textbooks. *Educational Studies in Mathematics, 89*(1), 41-65.

Yerushalmy, M. & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representation environment. *Educational Studies in Mathematics, 80*(3), 287-306.