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Teaching Place Value Conceptually – Part I

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Place value plays a central role in the primary mathematics curriculum, as it is the basis for understanding arithmetic operations (Nuerk et al., 2015). However, it is a difficult concept to operate with (Hart, 2009) for a majority of pupils from primary through secondary school (Chan, Au, & Tang, 2014). A typical way to introduce place value in primary schools is through the use of physical or pictorial base-ten blocks. Such introduction is typically followed by labelling digit position (e.g., units, tens, hundreds) and operating on those labels for given arithmetic problems in primary and secondary schools. These experiences are mostly about procedural applications involving standard algorithms (Kamii, 2004).

Is such a teaching approach effective? Do pupils learn place value by manipulating base-ten blocks, labelling digit position and working with those labels? Are these experiences conducive to enabling pupils to analyse place value structure of numbers? Many teachers would answer "yes" to these questions. Nevertheless, teachers keep re-teaching place value from scratch, instead of building on previous work, almost every year throughout primary grades and revisit the concept in lower secondary grades. If pupils learn place value well, why do teachers need to re-teach this concept at almost every school level?

We investigate the concept of place value in a series of two articles. This article, Part I, focuses on the concept of place value by providing an effective method of teaching it at the primary level based on a theoretical foundation. Part II provides the particulars of this teaching through an analysis of it.

Meaning of Place Value

In all number systems, users organise numbers according to certain conventions. In the base-ten system, "372" represents a different quantity than "723" even though both numbers use the same three symbols 2, 3, 7. The symbol '7' in 372 represents 7 bundles of ten, while in 723 it represents 7 bundles of one-hundred. The bundle size (ten or one-hundred) is determined by the position of '7' within the number, and the quantity of bundles (how many) is given by the face value of '7'. Therefore,

> "in a place value system each digit provides information about the size and the quantity of the bundle it represents. While the quantity of the represented bundles is indicated by the face value of the digit, the bundles' size has to be derived from the position within the number" (Herzog, Ehlert, & Fritz, 2019, p.562).

In this sense, understanding place value requires making sense of the relationship between the position of a given symbol and the size of the bundle that position posits in representing numbers and of the hierarchical relation between consecutive bundles represented by their position – one bundle size is ten times as large as its predecessor from left to right.

We base our way of looking and interpreting place value knowledge as detailed above on Piaget's classification of knowledge types. Piaget (1971, cited in Kamii, 2004) describes three different kinds of knowledge: social knowledge, physical knowledge and logico-mathematical knowledge. ***Social knowledge*** is the knowledge of conventions, rules, habits, or agreed-upon information. The fact that "12" is read as "twelve" is an example of social knowledge. We do not question this convention on which mathematicians previously agreed, but we accept it as is. Therefore, social knowledge can be directly transmitted from generation to generation and can be taught directly in classes. ***Physical knowledge*** is the knowledge of observables or experiments, in other words, noticeable characteristics of an object. For example, one can look at a "thousand" block and describe it as a "cube", by focusing on the outer characteristics of it and ignoring the thousand-ness it represents. ***Logico-mathematical knowledge*** is the knowledge of relations between at least two objects established by individuals in their minds. For example, we can think about the number 43 as consisting of two partitions/columns. Each partition/column has two meanings involved in it. The column/partition that involves "4" means that we have 4 bundles but it also means that each bundle has 10 items in it. Hence, relating each column/partition to these two meanings can be considered as logico-mathematical knowledge. The relations made by individuals are unique to those individuals. Therefore, those relations cannot be directly transmitted: individuals should construct them.

Teachers must understand the distinction among these three knowledge types in order to decide on when to transmit knowledge to students directly and when to engage pupils in specific activities to help them construct the targeted mathematical relations. Treating logico-mathematical knowledge (e.g., the fact that the value of the place a digit occupies increases in powers of 10 from right to left) as social knowledge (e.g., the symbolic form '153') is problematic since it does not allow students to understand the relations embedded in the symbolic representation. In this article, we refer to the teaching that allows students to construct these relationships (or logico-mathematical knowledge) as "teaching conceptually".

How can we help pupils develop a solid understanding of place value considering the size and quantity of bundles as well as growth rate and trading (exchanging) of bundles? The following section will describe a set of activities that have a real potential to improve pupils' understanding of place value as logico-mathematical knowledge.

A Way to Teach Place Value Conceptually

We here present an instructional sequence developed by the first author that uses manipulatives in a non-conventional way (starting with sticks rather than the traditional blocks and exploring bases that are not 10) to help students develop logico-mathematical knowledge of the concept of place value that is built upon the interrelationships between: meaning of groups/bundles, size of groups/bundles, hierarchy of groups/bundles, trading (exchanging) groups/bundles. Please note that we use the terms groups and bundles interchangeably throughout the paper.

This sequence involves a series of sections that make pupils represent the same set of quantities (eight, thirty-four, forty-one, …) in different ways depending on the base they are working with (grouping in threes, fives, sevens, and so on). Figure 1 shows an example (grouping in threes). In a classroom setting, the teacher starts the activity (Section 1 of Activity 1 in Figure 1) by distributing several sticks (e.g., 29) to each pair or small group of pupils and asking them to group 'items' in threes. As pupils make several groups of three sticks, the teacher interacts with the pupils to make them notice that an 'item' can also be a group of three sticks, or a group of 'three groups of three' sticks, and so on, so that pupils not only focus on groups of 3 sticks but also higher-order groups. The teacher keeps reinforcing the language of 'groups' consistently throughout the activities.

The grouping activity continues with a more structured sequence (Section 2 of Activity 1 in Figure 1) where pupils not only work on grouping the sticks, but also record the numbers that correspond to "how many" of each group. For example, pupils group eight sticks in two bundles of threes resulting in 2 three-stick groups of 2 leftover sticks. Then, they record and/or draw these results in the first row of the table (Figure 1): '2' in the 'leftover' column and '2' in the 'Groups of first-order' column.

Once pupils have worked on 'eight' under the guidance of the teacher, they continue working on larger numbers like 'thirty-four' and 'forty-one'. The last row in the activity (Section 2 of Activity 1 in Figure 1) makes the pupils work on the reverse process: pupils are given "how many" of each group/leftover and have to work out the corresponding total number of individual sticks.

Activity #1 - Let's Group!

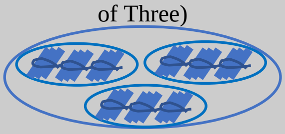
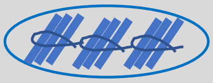
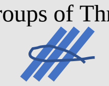
Section 1 – Warm-Up Activity – Forming Groups

You have a certain number of sticks or items at your tables. Please put those sticks into groups of **Three** as many times as possible. Make sure that you considered all types of groups of **Threes**!

Section 2 – Forming Number of Sticks in Groups of Three

Please use the given sticks to model each given number below (e.g., eight, thirty four, etc.) in groups of **Three**. Then fill in the table using the number of groups.

[***Note:*** *For the last row, the number of each type of group is given, and you are asked to find the amount of sticks that corresponds to these groups.*]

Number to be Modelled	***The Number Grouped in Threes***			
	Groups of Third Order (Three groups of Three groups of Three) 	***Groups of Second Order*** (Three groups of Three) 	***Groups of First Order*** (Groups of Three) 	***Leftovers*** (Ones) 
Eight			2	2
Thirty four				
Forty one				
Fifty one				
Sixty one				
Seventy nine				
?	**1**	**2**	**0**	**2**

1. Based on your work so far, what can be the **minimum** number of items in each column (representing '*Groups of First Order*', '*Group of Second Order*' and '*Groups of Third Order*')?

2. What can be the **maximum** number of items in each column (representing '*Groups of First Order*', '*Group of Second Order*' and '*Groups of Third Order*')? **Why?**

3.Explain how groups grow as you move from smaller to bigger groups (from rightmost column to left). **Why?**

Figure 1. Grouping/Bundling by 3 using sticks.

During the activity, as pupils work on grouping sticks and filling in the table, the teacher should circulate and pay attention to any issues arising within the pupil groups. After the pupils have completed the table in Figure 1, they are asked to look back and reflect on three key constructs for place value: minimum (0) and maximum (2) number of sticks a group can have and the growth rate from one group to the next (3). The teacher supports pupil reasoning with questions like, "Why is the number recorded in first-order no more than 2? Why can't it be 3? What happens when we have 3 groups that are the same? How does the number of sticks in a group grow from one column to the next?"

The same activity occurs for grouping by 5, 7 and 10. As the pupils are already familiar with the process of grouping, they progress through grouping in 5, 7 and 10 faster than the first one. The teacher still has pupils focus on grouping, the minimum and the maximum number of sticks in a group, and the growth rate from one group to the next.

After the pupils have completed the grouping activities for groups of 5, 7 and 10 sticks, the teacher takes the opportunity to make them reflect on all the previous work (Section 6 of Activity 1 in Figure 2) by making pupils think about what they have done until this point in terms of grouping, minimum/maximum number of items in a group and growth rate (in 3s, 5s, 7s and 10s) among groups. The purpose here is not to make pupils memorise tabulated answers but to help them reflect on what they have done up to this point. The final aim is to help them conceptualise the common patterns and aforesaid underlying ideas in all four types of grouping. During this whole class discussion, the teacher introduces the words "place value" alongside "grouping".

Section 6 – Let's Think A Little More Deeply!!!

1. Up until now you have modelled the given numbers by piling sticks in groups of **Three**, **Five**, **Seven**, and **Ten**. Based on this experience please fill in the below table.

Grouping by	What is the **maximum** number of items in '*Groups of First Order*', '*Group of Second Order*' and '*Groups of Third Order*'?	*Under what condition can we move from one group to a bigger one?*
Three		
Five		
Seven		
Ten		

CHALLENGE YOURSELF!

You only have **2 minutes**! Use sticks/cubes at your tables to make "***one hundred twenty-eight***" using one of the groupings given above.

Which grouping is more advantageous in organizing the given number? Why?

Figure 2. Reflection over sticks activities.

Activity #2 - Let's Continue Grouping in the World of Interlocking Units!

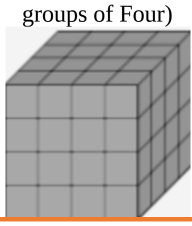
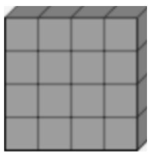
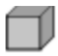
Section 1 – Warm-Up Activity – Forming Groups

You have a certain number of cubes on your tables. Please connect those cubes into groups of Four as much as possible. Make sure that you considered all groups of Fours!

Section 2 – Forming Numbers in Groups of Four with Singles, Longs, Flats and Cubes

Please use the given cubes to model each given number below (e.g., eight, thirty four, etc.) in groups of Four as much as possible. Then put the number of groups in each column.

[***Note:*** *For the last row, the number of each type of group is given, and you are asked to find the amount of cubes that corresponds to these groups.*]

Number to be Modelled	***The Number Grouped in Fours***			
	Groups of Third Order [Cubes] (Four groups of Four groups of Four) 	***Groups of Second Order*** [Flats] (Four groups of Four) 	***Groups of First Order*** [Longs] (Groups of Four) 	***Leftovers*** [Singles] 
Eight				
Thirty four				

…

Figure 3. Grouping/Bundling by 4 using interlocking cubes.

Pupils then move onto another set of grouping activities (for groups of 4, 6 and 10 for the same numbers given in Figure 1 in addition to hundred-thirty-four for groups of 10) using interlocking cubes instead of sticks (Activity 2 in Figure 3). Pupils are asked to change the manipulatives they use because, if their experience is limited to sticks, they might not be able to generalise the concept of grouping, as they might not see beyond the specific manipulatives used. Pupils need to apply grouping to different objects, so they will focus their attention on the nature of grouping and place value (logico-mathematical knowledge aspects) rather than the practical element of handling the objects in front of them (physical knowledge aspects). Grouping with cubes is also structurally different from grouping with sticks and pave the way to the use of base-ten (Dienes) blocks later on. Interlocking cubes allow pupils to build connected array models that represent groups of groups (row of four cubes for first order groups, 4x4 area model for second order groups, 4x4x4 cube models for third order groups), while sticks do not lend themselves to this organisation.

A follow-up article in a later issue will provide an analysis of this sequence.

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