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# Real options, risk aversion and markets: A corporate finance perspective

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ABSTRACT

We analyze how the presence of financial markets affects the optimal exercise of real options for a risk averse agent. Extending the results of Shackleton and Sodal (2005), we characterize the optimal exercise rule in terms of a benchmark portfolio, even for the case of an incomplete market, facilitating the minimal martingale measure. We unambiguously characterize the effect of idiosyncratic risk on the speed of exercise of the option. We further show that systematic risk can accelerate execution and reduce the value of a call-type option, in contrast with the standard view that both value and execution threshold are increasing in volatility.

## 1. Introduction

Real option theory originated mainly from the works of Myers (1977), McDonald and Siegel (1986), Myers and Majd (1990) as well as Dixit and Pindyck (1994) and has had a fundamental impact on how investment decisions are made, in practice as well as in theory. A large body of literature has evolved since, spanning across the research fields of finance, economics, management science and operations research. In corporate finance, real options methodology provided a new way of looking at the optimal capital structure of firms beyond the basic theory propounded by Modigliani and Miller (1958, 1963), including elements such as dynamic strategic default and re-organization. The paper by Strebulaev and Whited (2012) provides an excellent overview of these applications. Contributions to this literature include Leland (1994), Fischer et al. (1989), Goldstein et al. (2001), Lambrecht and Myers (2012, 2017) and Koussis et al. (2017). This literature has aligned the modeling more closely with empirical findings on optimal capital structure. Beyond this, real option theory has contributed to corporate finance in equity investment and IPOs (Perotti and Rossetto, 2007; Busaba, 2006), investment timing under a managerial structure (Hori and Osano, 2014; Ewald and Zhang, 2016), as well as convertible bonds (Zeidler et al., 2012; Finnerty, 2015).

All of the literature above relies in some way or another on the application of contingent claim analysis but it struggles in connecting the contingent claims approach to classical equilibrium approaches like the Capital Asset Pricing Model (CAPM), dealing with

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unspanned risks and the possibility of incomplete markets, capturing the effect of interaction of markets and the exercising of real options, as well as providing economic intuition.

Our article therefore focuses on four key issues in the context of real options:

- We clarify the relationship between the value and optimal exercise of real options and financial market equilibrium, specifically how the market portfolio determines the rate of return of the real option in line with the CAPM, as with other financial assets.
- We go beyond the the existing limits of the contingent claims approach to real option pricing, providing a rigorous basis for determining the value and optimal exercise of a real option in the case of possibly incomplete markets without the requirement of a spanning asset, via the *minimal martingale measure*, which enables the expression of returns in terms of the CAPM even in incomplete market settings.
- In the incomplete market setting, using the CAPM decomposition of returns enabled by the minimal martingale measure, we quantitatively characterize the specific roles of systematic and idiosyncratic risk in determining the exercise threshold, demonstrating that increased risk can decrease option value and hasten execution, in direct contradiction of conventional belief.
- We analyze how risk aversion affects the utility-based exercise and value of real options when risk is not properly connected to the market price of risk, quantitatively characterizing the resulting value loss.

As we will show, all four points are inherently interconnected, and only by recognizing the relationship between them will it be possible to correctly price real options, understand qualitative features like the volatility effect, and make investment decision that are optimal and in accordance with capital markets.

There is an abundance of literature where real option theory is applied and risk aversion is accounted for via appropriately specified utility functions, but with the existence of financial markets and the possibility of diversification largely ignored. There also seem to be some misperceptions about the concept of risk-neutral pricing developed in the derivatives pricing and mathematical finance literature. The risk-neutral approach does not ignore risk but, on the contrary, correctly accounts for risk via the risk aversion of the representative agent through an appropriate change of probability measure, but circumventing the use of explicit utility functions. It further accounts for hedging possibilities and diversification.

The role of capital markets in the context of real options has been recognized in the finance literature, perhaps most notably in Constantinides (1978), Dixit and Pindyck (1994), Henderson and Hobson (2002), Hugonnier and Morellec (2007), Henderson (2007) and Thijssen (2010). All these references take account of capital markets and some form of market portfolio which serves as a benchmark, at least implicitly. Constantinides (1978), Dixit and Pindyck (1994) and Thijssen (2010) only cover the one dimensional, and hence complete-market, case. They identify appropriate discount rates and pricing kernels which are compatible with the CAPM (or any arbitrage-free pricing rule in essence because of market completeness).

Henderson and Hobson (2002), Hugonnier and Morellec (2007) and Henderson (2007) also consider the multi-dimensional (in fact two-dimensional) case, where additional sources of risk, orthogonal to the market risk, affect the underlying of the real option. They apply a utility-based approach but allow for some level of diversification through trading of the market portfolio, with the remaining idiosyncratic risk affecting the valuation of the real option. Their pricing rules however do not originate from within an equilibrium framework and are not presented through a pricing kernel.

Our paper builds on the CAPM and was originally inspired by the paper of Shackleton and Sodal (2005), which also studies real options in the CAPM context. Shackleton and Sodal show that within a one-dimensional complete market setting, exercising is optimal when the rate of return of the real option value is the same as the return of a certain leveraged portfolio where sunk costs are financed through a loan at the riskless rate and investment is into a risky portfolio which has the same CAPM beta as the project. We are able to extend the Shackleton-Sodal framework to the multi-dimensional incomplete market case and identify the optimal exercise rule.

More generally, we provide a rigorous foundation for applying contingent claim analysis beyond the limits of perfect spanning assets and complete markets, while still connecting to CAPM equilibrium and providing economic intuition. In consequence our paper adds to the literature on real options under incomplete markets, such as Miao and Wang (2007) and Leippold and Stromberg (2017).

Following an idea presented by Brennan and Schwartz (1985), Dixit and Pindyck (1994, pages 114–120) use contingent claim analysis in order to price real options. In doing so, they also link the pricing problem to the CAPM and benchmarking against the market portfolio. Their assumptions are highly restrictive, however, and the mathematical justification for their approach has some loose ends. First, they require a perfect spanning asset, which is an asset that is driven by exactly the same random shocks as the real option underlying. This is a much stronger assumption than requiring that the spanning asset has the same risk characteristics as the real option underlying; for example it is not sufficient that it has the same return distribution and same correlation to the market portfolio, or the same CAPM beta.

Second, Dixit and Pindyck attempt to replicate the returns of the real option value by trading in the spanning asset and from this they derive the relevant pricing partial differential equation (PDE) and boundary conditions. The PDE and boundary conditions are correct, however it is well known in the option pricing literature that an American type option cannot be simply hedged with a self-financing trading strategy, as the American option price discounted at the riskless rate is in general only a supermartingale and not a local martingale. It therefore cannot be replicated in the way that is required for Dixit and Pindyck's (1994) argument to hold; we refer the reader to Jaillet et al. (1990) or Huang et al. (1996) for details.

Knudsen et al. (1999) take account of these issues and provide a more rigorous setup in which they demonstrate the equivalence of the contingent claim analysis approach and the dynamic programming approach. However they proceed in a very different manner, e.g. they do not use a hedge ratio, or riskless portfolio as would appear in a replication argument, but rather use super-replication and a no-arbitrage argument. Their results also depend on some rather restrictive assumptions, among these the existence of a perfect spanning asset as well. Insley and Wirjanto (2010) find a contrary result: the contingent claims approach, even in the presence of a perfect spanning asset, can lead to different results than the dynamic programming approach. Their result, however, depends on the use of a constant discount rate rather than a dynamic pricing kernel reflecting the appropriate measure. Our analysis demonstrates that the two approaches are in fact aligned when using the pricing kernel of the minimal martingale measure as the dynamic stochastic risk-adjusted discount factor.

The methodology developed in our paper provides what we think is the correct and direct implementation of the contingent claim analysis approach. Instead of assuming the existence of a perfect spanning asset that is priced in accordance with the CAPM as in Dixit and Pindyck (1994), we simply demand that the returns of the real option *value*, distinct from the stochastic process of the underlying, are aligned with the CAPM.[1] We do not require a spanning asset at all or replication in any way, nor is market completeness essential. This far less restrictive requirement will identify the minimal martingale measure as the appropriate pricing measure and produce the same real option prices as Dixit and Pindyck's (1994) contingent claim approach under their very restrictive condition of a perfect spanning asset.

Our approach is functional beyond these assumptions. We comprehensively investigate the role of the appropriate martingale measure (stochastic discount factor), which we identify as the minimal martingale measure, the only martingale measure that prices the real option consistent with the CAPM. This allows us to connect the utility-based approach as well as the contingent claims-based approach to the use of martingale measures (and therefore stochastic discount factors). As such, our article also links Thijssen (2010) with Shackleton and Sodal (2005), but for the more general case of multi-dimensional and incomplete market models.

A central question in the literature has also been the relation between risk and optimal exercise. The conventional belief is that increased risk delays exercise, or equivalently that higher volatility increases the exercise threshold.[2] This perspective explains why, in periods of high volatility, investment (which can be viewed as the exercise of a real option when a project is initiated) is depressed.

To explore this issue we develop a continuous-time example in which we compare the value of a company developing and selling (for commercial use) a patent for a vaccine under public (shareholder) versus private (not fully diversified) ownership. The example can be formulated as a conventional real option, and we can solve it in closed form. We then show that systematic risk and idiosyncratic risk have differing effects on the exercise threshold; and that the exercise threshold and the option value can in fact be *decreasing* with increasing total volatility when correctly evaluated in alignment with the CAPM. This establishes that the simple but somewhat stylized examples in Jagannathan (1984), Carr and Lee (2003) and Carr et al. (2008) are not just arcane in nature but that the effects observed there can and should appear in a standard CAPM-based setup.

The remainder of the article is organized as follows. In Section 2 we compare various utility-based approaches to real options and highlight contradictions that arise with standard approaches. In Sections 3 and 4 we consider complete markets and investigate the connection of the optimal exercise problem with the CAPM. In Section 5 we look at the case of incomplete markets, the absence of spanning assets, and the role of the minimal martingale measure in the valuation. We present numerical illustrations in Sections 6 and 7. Section 6 focuses on the issue of value loss from exercise rules arising from the incorrect use of risk aversion, while Section 7 looks more closely at the volatility effect. Section 8 concludes. We relegate two technical proofs to the Appendix.

## 2. Utility indifference pricing of real options

Consider an optimal stopping problem in which an agent receives a yield $B(X_t)$ during the time interval $[0, \tau]$ and a final payment of $H(X_\tau)$ at a stopping time $\tau$ chosen by the agent, where $X_t$ follows a diffusion process. In present value terms the payoff is

$$\int_0^\tau e^{-rt}B(X_t)\mathrm{d}t + e^{-r\tau}H(X_\tau), \tag{1}$$

with

$$\mathrm{d}X_t = \phi(X_t,t)\mathrm{d}t + \gamma(X_t,t)\mathrm{d}Z_t \tag{2}$$

and $(Z_t)$ a standard Brownian motion. We assume that the functions $\phi$ and $\gamma$ are sufficiently regular so as to allow for a unique solution of (2). The parameter $r$ denotes the risk free rate. We stress that the yield and final payoff are monetary values. The optimal stopping problem is equivalent to a real option problem, with the state $X_t$ reflecting the state of the project, accounting for irreversibility in the sense that once the project is stopped, it cannot be restarted again.

Often real option theory ignores risk aversion and aims to solve the problem

$$\sup_{\tau\in\mathcal{T}}\mathbb{E}\left(\int_0^\tau e^{-rt}B(X_t)\mathrm{d}t + e^{-r\tau}H(X_\tau)\right) \tag{3}$$

subject to (2), where $\mathcal{T}$ denotes the set of admissible stopping times. The value function

[1] McDonald and Siegel (1986) already hinted at this, but did not strictly follow this approach. Their results are tied to a very specific dynamic and they do not make use of the minimal martingale measure.

[2] See for example Dixit and Pindyck (1994) page 153. Lund and Nymoen (2018) provide an excellent discussion of this topic and the literature that addresses it.

$$V(x,s)=\sup_{\tau\geq s\in\mathcal{T}}\mathbb{E}\left(\int_s^\tau e^{-r(t-s)}B(X_t)\mathrm{d}t+e^{-r(\tau-s)}H(X_\tau)\middle|X_s=x\right),\tag{4}$$

is the present value in monetary terms of the project at time $t$ from the perspective of a risk neutral agent (representative) agent. The problem can be solved by using the fact that $V(x,s)$ needs to satisfy a corresponding Hamilton-Jacobi-Bellman (HJB) equation, at least under certain regularity conditions.

To take account of risk aversion in problem (3), much of the mainstream real option literature assumes the existence of two utility functions $U_1(x)$ and $U_2(x)$ and aims to solve

$$V(x,s)=\sup_{\tau\geq s\in\mathcal{T}}\mathbb{E}\left(\int_s^\tau e^{-\eta(t-s)}U_1(B(X_t))\mathrm{d}t+e^{-\eta(\tau-s)}U_2(H(X_\tau))\middle|X_s=x\right)\tag{5}$$

through setting up the corresponding HJB equation. Here $\eta$ is an appropriate subjective discount rate. This has a number of disadvantages. The right hand side of Eq. (5) adds up utilities and not monetary values. The subjective discount rate $\eta$ no longer corresponds to a market interest rate and the value function $V(x,s)$ does not express a market value in general. In addition, one of the key principles of corporate finance is that managers maximize the market value of the firm. Eq. (5) is simply incompatible with this guiding principle.

In both of the approaches above, by construction an explicit investment asset—risky or not—is missing. The only benefit of the continuous yield comes from consumption, and it is not possible to carry forward the benefits in time. In addition the agent receives utility from the terminal payoff. However the effective discount rates $r$ and $\eta$ above can (and should) account for marginal rates of intertemporal substitution, at least in a partial equilibrium setting, which makes this consistent with the approach discussed further below. Nevertheless, as the continuous yield and terminal payoffs present themselves in monetized form, the utility functions mainly capture the effect of risk aversion.

Miao and Wang (2007) and also Henderson (2007) present a setup that addresses some of these critiques. First, they include an investment asset into the setup and allow the agent to accumulate wealth and finance consumption until the option is exercised; the agent then receives utility from final wealth and the real option payoff. The value function derived is still measured in terms of utility, and therefore incompatible with most corporate finance models, although both also determine the value of the real option in terms of a certainty equivalent, which is in monetary terms. However, introducing a wealth process to the model comes at the cost of additional complexity, as it adds an additional state variable to the model, precluding an analytical solution.

Our approach circumvents this problem: the appropriate martingale measure implicitly takes account of agents trading optimally in available assets in addition to optimally exercising the real option. The marginal utilities of the optimal wealth process are encoded into the martingale measure, but without introducing an additional state variable. We are then able to solve the model explicitly.

In the complete market case, the martingale measure can be nicely introduced by what is best described as step-by-step utility indifference pricing. In the incomplete market case it reflects the choice of certain properties of the martingale measure. While utility indifference pricing has been discussed in the mathematical option pricing literature, mainly in the context of incomplete markets and European style options, it has received less consideration in the real options literature.

As Kasanen and Trigeorgis (1994) already demonstrated in a static one-period framework, in the presence of markets, utility indifference pricing is linked to certain pricing kernels that identify probability measures. These measures can then be used in pricing real options consistently with markets, while taking appropriate account of risk aversion.

To implement utility indifference pricing, we begin by examining a time discretized setup. We identify $dt=\Delta$ and consider a discretized version of (1). At each discretized point in time $t$ an agent would then make a decision about either stopping the project and receiving the final payoff $H(X_t)$ (given she has not stopped before) or continuing with the project and receiving a yield of $B(X_t)\cdot\Delta$ and the discounted future project value $e^{-r\Delta}V(X_{t+\Delta},t+\Delta)$ (given she cannot stop in between $t$ and $t+\Delta$). However the future project value is uncertain, as it depends on $X_{t+\Delta}$, which is not known at time $t$. A risk averse agent would evaluate the uncertain monetary payoffs of exercising or continuing through expected utility. Assuming that the agent's risk aversion is characterized through the utility function $U(x)$ the agent would stop the project if

$$U(H(X_t))\geq\mathbb{E}(U(B(X_t\cdot\Delta)+e^{-r\Delta}V(X_{t+\Delta},t+\Delta))|\mathcal{F}_t)\tag{6}$$

or at the first time she is indifferent (in terms of utility), i.e. at equality. Eq. (6) can also be expressed in terms of the certainty equivalent, and exercise occurs the first time the current payoff is larger or equal to the certainty equivalent of continuing. In conclusion we refer to this approach as utility indifference pricing, even though this specific dynamic adaptation can to the best of our knowledge not been found in the utility indifference pricing literature.[3]

[3] Note that Eq. (6) is structurally different from the stopping rule obtained

$$U_2(H(X_t))\geq U_1(B(X_t)\cdot\Delta)+\mathbb{E}(e^{-\eta\Delta}V(X_{t+\Delta},t+\Delta)|\mathcal{F}_t),\tag{7}$$

which is tied to the corresponding discretized version of (5). In (6) the value function appears as an argument in the utility function, in (7) it is added to the value of the utility function. However, with the choice of $U(x)=U_1(x)=U_2(x)=x$ one obtains the basic risk-neutral optimal stopping rule in both cases.

## 3. Project value and stopping in the presence of complete markets

In this section we assume that the financial market offers a complete set of securities which span the risk sources inherent in $(X_t)$ and solve (6). We also assume for simplicity that these securities do not pay a dividend. In reality these could be a set of commodity futures, index futures etc.[4] We denote the prices of these securities with

$$S_t^i, i \in \{0, 1, ...., n\}, \tag{8}$$

and assume that $S_t^0$ represents a riskless bond. Further, we denote with $Y_{t+\Delta}^{\varphi_t, x}$ the value at time $t+\Delta$ of a self-financing portfolio strategy $\varphi_t$ with initial investment $x$ at time $t$.

At time $t$ the project holder can now exercise the real option, i.e. stop, and invest the proceeds $H(X_t)$ into the market to obtain a portfolio value of $Y_{t+\Delta}^{\varphi_t, H(X_t)}$ or continue and invest the yield $B(X_t) \cdot \Delta$ into the market in addition to owning the market value of the project $V(X_{t+\Delta}, t+\Delta)$ at time $t+\Delta$. A rational agent will seek optimal investment and exercise if and only if

$$\sup_{\varphi_t \in \mathcal{A}} \mathbb{E}\left(U\left(Y_{t+\Delta}^{\varphi_t, H(X_t)}\right) \middle| \mathcal{F}_t\right) \geq \sup_{\psi_t \in \mathcal{A}} \mathbb{E}\left(U\left(Y_{t+\Delta}^{\psi_t, B(X_t)\cdot\Delta} + V(X_{t+\Delta}, t+\Delta)\right) \middle| \mathcal{F}_t\right), \tag{9}$$

where $\mathcal{A}$ denotes the set of admissible self-financing strategies. Note that in Eq. (9) discounting is no longer necessary, as payoffs occur at the same time $t+\Delta$.

Because of our assumption that the market for securities is arbitrage-free and complete, then as demonstrated in Pliska (1997), there exists a unique risk neutral measure $\mathbb{Q}$. In the following we denote expectations under the measure $\mathbb{Q}$ as $\mathbb{E}^{\mathbb{Q}}$ while expectations under the physical measure $\mathbb{P}$ will be denoted as $\mathbb{E}^{\mathbb{P}}$ or simply $\mathbb{E}$ if there is no risk of confusion.

**Proposition 3.1**. *Inequality (9) is equivalent to the risk-neutral criterion*

$$H(X_t) \geq B(X_t) \cdot \Delta + \mathbb{E}^{\mathbb{Q}}(e^{-r\Delta} V(X_{t+\Delta}, t+\Delta) | \mathcal{F}_t). \tag{10}$$

The proof is in the Appendix. Thus, solving the real option problem under risk aversion and the real world measure as expressed in (6) is equivalent to solving the problem without using utility functions but under a risk neutral measure that reflects an arbitrage free and complete market. Empirically the measure $\mathbb{Q}$ can be inferred from market data; see for example Ewald et al. (2017).

Eq. (10) can also be interpreted in terms of certainty equivalents at time $t$. The left hand side is the certain payoff from exercise at time $t$, the right hand side is the certainty equivalent of the value of continuation for one period. In the case of complete markets, it is possible to replicate the uncertain value of continuation through a one-period trading strategy, given that that the real option cannot be exercised in between periods. The value of the one-period replicating strategy coincides with the certainty equivalent, and can be determined as the expected present value under the risk-neutral measure $\mathbb{Q}$, which is unique here. As such Proposition 3.1 is not too surprising, however to the best of our knowledge it has not appeared in the real option literature. Note that our use of certainty equivalent here is very different from Miao and Wang (2007), where we use a step by step certainty equivalent whereas Miao and Wang (2007) consider the whole period from start until exercise.

## 4. CAPM and real options pricing: one dimensional case

In this section we assume that all risk is spanned by a market portfolio, which follows the dynamics

$$\mathrm{dS}_t = S_t(\mu^M \mathrm{dt} + \sigma^M dZ_t). \tag{11}$$

We also assume that the coefficient functions in the dynamics of the real option underlying $(X_t)$ in Eq. (2) do not explicitly depend on $t$, this is for simplicity only. Furthermore, we assume that $X_t > 0$ a.s. for all $t$, so that the dynamics of $X_t$ can be expressed in the following form

$$\mathrm{dX}_t = X_t(\mu^X(X_t)\mathrm{dt} + \sigma^X(X_t)\mathrm{dZ}_t), \tag{12}$$

i.e. $\phi(x, t) = \phi(x) = x \cdot \mu^X(x)$ and $\gamma(x, t) = \gamma(x) = x\sigma^X(x)$. In addition to the market portfolio, there exists a riskless bond paying the riskless rate $r$. We assume that the investment into the project occurs at a sunk cost of $K$ and pays a benefit of $X_\tau$ at the time of execution $\tau$.[5] We also assume that there are no other assets or sources of uncertainty and that in conclusion the market is complete.

In the following we extend the work of Shackleton and Sodal (2005) who studied an option on the market portfolio, while our work features an option on the more general underlying (12). As the market is complete, then as we noted in the previous section, there is a unique risk neutral measure $\mathbb{Q}$ and the only way to value our real option consistently with markets and agent's risk aversion is through using this risk-neutral measure, i.e. to compute

[4] At the extreme this could entail the entire universe of financial securities traded world wide.

[5] This can be easily generalized to a more general payoff of $H(X_\tau)$ discussed in Section 1, indeed, the Itô rule could be applied to $H(X_t)$ and then only the coefficient functions would need to be adapted.

$$V(x) = \sup_{\tau \geq t} \mathbb{E}^{\mathbb{Q}}(e^{-r(\tau-t)}(X_\tau - K)|X_t = x). \tag{13}$$

Note that due to our assumptions the problem is time-homogenous and that the value function does not depend explicitly on $t$, similarly with Shackleton and Sodal.

The risk neutral measure is identified through a Girsanov transformation, with the associated Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}_t(\theta) := exp\left( -\theta Z_t - \frac{1}{2}\theta^2 t \right), \tag{14}$$

where $\mathcal{E}_t(\theta)$ defines the stochastic exponential of the market price of risk (also Sharpe ratio)

$$\theta = \frac{\mu^M - r}{\sigma^M}. \tag{15}$$

We now demonstrate that the real option value $V(x)$ as defined in (13) will automatically satisfy the CAPM without requiring additional conditions. First note that the dynamics of $X_t$ under the measure $\mathbb{Q}$ is given by

$$\mathrm{d}X_t = X_t((\mu^X(X_t) - \sigma^X(X_t)\theta)\mathrm{d}t + \sigma^X(X_t)\mathrm{d}Z_t^*), \tag{16}$$

where $Z_t^* = Z_t + \theta t$ is a Brownian motion under $\mathbb{Q}$. Further note that $e^{-rt}V(X_t)$ is a super-martingale under $\mathbb{Q}$ and that in the continuation region $C$ (prior to execution) we have by Ito's lemma that

$$\mathbb{E}_t^{\mathbb{Q}}(\mathrm{dV}) = V_x x(\mu^X(x) - \sigma^X(x)\theta)\mathrm{d}t + \frac{1}{2}\sigma^X(x)^2 x^2 V_{\mathrm{xx}}\mathrm{d}t = \mathrm{r\ Vd}t, \tag{17}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the conditional expectation with respect to $\mathcal{F}_t$, $V_x$ and $V_{xx}$ the respective derivatives, and everything is evaluated at $X_t = x \in C$. Applying the Itô formula as before, but under $\mathbb{P}$ rather than $\mathbb{Q}$ and combining the result with Eq. (17) we obtain

$$\mathbb{E}_t^{\mathbb{P}}(\mathrm{dV}) = V_x x\mu^X(x)\mathrm{d}t + \frac{1}{2}\sigma^X(x)^2 x^2 V_{\mathrm{xx}}\mathrm{d}t = \mathrm{r\ Vd}t + V_x x\sigma^X(x)\theta \mathrm{d}t. \tag{18}$$

Combining these two equations, the expected return (under $\mathbb{P}$) of the real option then satisfies

$$\frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{dV}}{V}\right) = r + \frac{V_x X\sigma^X(x)}{V\sigma^M}(\mu^M - r). \tag{19}$$

The correlation of the option returns and the market returns can be computed as

$$\beta_V(x) = \frac{\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{dV}}{V}\frac{\mathrm{dS}}{S}\right)}{\frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left(\left(\frac{\mathrm{dS}}{S}\right)^2\right)} = \frac{\frac{V_x}{V}X\sigma^X(x)\sigma^M}{(\sigma^M)^2}, \tag{20}$$

and hence

$$\frac{1}{\mathrm{d}t}\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{dV}}{V}\right) = r + \beta_V(x)(\mu^M - r). \tag{21}$$

which is the classical expression of the CAPM. We note that Eq. (20) establishes that the CAPM $\beta$ is an elasticity, as do Shackleton and Sodal (2005), but for general underlying.

Before continuing we make three observations. First, if the real option were evaluated under a different measure without taking risk aversion appropriately into account, possibly through the measure $\mathbb{P}$ as it so often found in the literature, the expression of the CAPM in the form of (21) would in general be violated. Second, drawing from our results in Section 3, when appropriately accounting for risk aversion, the specifics of the utility function or even the level of risk aversion (as long as it is positive) do not really matter and it comes all down to CAPM pricing. Third, it is not necessary to assume that the real option underlying itself or a suitable spanning asset satisfy a CAPM condition, as do Dixit and Pindyck (1994, pages 115 and 148); in some cases it will not or will fail to exist. Nevertheless, the real option value will always meet the CAPM condition if correctly evaluated.

Let us now investigate the value-matching and smooth-pasting condition through the CAPM condition (21). Note that $\beta_V(x)$ depends on the current value $X_t = x$. It is well known that in this one dimensional time-homogenous setup, exercise of the real option occurs at a threshold $x^*$, i.e. the optimal stopping time is of the type[6]

[6] The threshold may take the value $x^* = \infty$ in which case it is optimal to never exercise the real option.

$$\tau^* = inf\{t > 0 | X_t = x^*\}. \tag{22}$$

As indicated earlier, we assume a payoff function of $H(x) = x - K$ in the following, i.e. investment is at a fixed cost $K$.[7] As has been shown in the classical real option literature, the threshold $x^*$, if finite, can then be determined through the value-matching and smooth-pasting conditions, i.e. $V(x^*) = x^* - K$ and $V_X(x^*) = 1$. Using the latter and evaluating (20) at $x^*$, we therefore obtain

$$\beta_V(x^*) = \frac{x^* \sigma^X(x^*)}{(x^* - K)\sigma^M} \tag{23}$$

and the expected return $r_V(x^*)$ of the real option at $x^*$ is given by

$$r_V(x^*) = r + \frac{x^* \sigma^X(x^*)}{(x^* - K)\sigma^M}(\mu^M - r) = \frac{x^* \left[\frac{\sigma^X(x^*)}{\sigma^M}\mu^M + \left(1 - \frac{\sigma^X(x^*)}{\sigma^M}\right) r\right] - \mathrm{rK}}{x^* - K}. \tag{24}$$

The expression in the square brackets corresponds to the return of a portfolio which invests a proportion $\frac{\sigma^X(x^*)}{\sigma^M}$ into the market portfolio and a proportion $\left(1 - \frac{\sigma^X(x^*)}{\sigma^M}\right)$ into the riskless asset. This portfolio has a CAPM beta of $\frac{\sigma^X(x^*)}{\sigma^M}$. Further, using the value-matching and smooth-pasting conditions once more, the correlation of the real option underlying with the market returns can easily be computed as

$$\beta_V(x) = \frac{\mathbb{E}_t^{\mathbb{P}}\left(\frac{\mathrm{dX}}{X}\frac{\mathrm{dS}}{S}\right)}{\frac{1}{\mathrm{dt}}\mathbb{E}_t^{\mathbb{P}}\left(\left(\frac{\mathrm{dS}}{S}\right)^2\right)} = \frac{\sigma^X(x^*)}{\sigma^M}. \tag{25}$$

Investment in the project therefore occurs exactly at the time when the return of the project is equal to the return of a leveraged portfolio, where sunk costs $K$ are financed through a loan at the riskless rate and investment is into a portfolio of assets which has the same beta as the project and by nature lies on the capital market line. This is a generalization of the result presented in equation (9) of Shackleton and Sodal (2005).

## 5. Real options in incomplete markets within the CAPM context

In this section we allow for non-spanned risk in the real option underlying. The market as such is therefore incomplete and a manifold of martingale measures $\mathbb{Q}$ and in consequence arbitrage free pricing rules exist. The choice of a single martingale measure $\mathbb{Q}$ is determined through the market, in practice often through the introduction and pricing of derivatives.

We assume that agents' attitude toward risk is in accord with the CAPM and that hence according to Ingersoll (1987) their individual utilities can be represented as quadratic functions. Further, for any agent a solution to the optimal portfolio allocation problem can be expressed as a suitable combination of investment into a riskless asset and the market portfolio. The model is set within a filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t))$ where the filtration $(\mathcal{F}_t)$ is generated by a multi dimensional Brownian motion. Merton (1990), section 15, provides the continuous time analogue of the results above. In this case the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ obtained from the marginal utilities is a function of the market portfolio only and hence the Girsanov transformation from $\mathbb{P}$ to $\mathbb{Q}$ leaves all random sources orthogonal to the market portfolio invariant, while the drift of the market portfolio is changed to the riskless rate.[8] In mathematical finance this measure is referred to as the minimal martingale measure—the corresponding change of measure is as "small" as possible.[9]

Cerny (1999) discusses some properties of the minimal martingale measure both in discrete and continuous time and shows how it is connected to CAPM through exploring the property of the minimal martingale measure to identify so called (locally) risk-minimizing hedges. In the following, we provide a more direct argument as to why the minimal martingale measure is attached to the CAPM. Our argument does not require that idiosyncratic risk is diversifiable. We assume that the dynamics of the market portfolio under $\mathbb{P}$ is given as

$$\mathrm{dS}_t = S_t(\mu_t^M \mathrm{dt} + \sigma_t^M \mathrm{dB}_t), \tag{26}$$

where $\mu_t^M$ and $\sigma_t^M$ are progressively measurable processes and $B_t$ is a Brownian motion under $\mathbb{P}$. Let $X_T > 0$ be an arbitrary $\mathcal{F}_T$-measurable contingent claim, which in general is not spanned through the assets in the market. Under not too stringent conditions

[7] This is a very common setup in the literature, however our analysis would also hold for more general $H(x)$, where in the smooth-pasting condition (28) $H'(x^*)$ would occur. The interpretation of some terms occurring in the analysis would require further analysis however.

[8] Orthogonal here means zero instantaneous correlation.

[9] Beyond its links to the CAPM, the minimal martingale measure has many other attractive properties that have been explored mainly in the mathematical finance literature. Its has evolved from results by Föllmer and Schweizer (1991). Schweizer (1991) and Schweizer (1995) on hedging in incomplete markets. Intuitively it provides optimal hedging strategies in the sense that the risk originating from the dynamic hedging/tracking error is minimized.

as set out in Clark (1970) and Ocone and Karatzas (1991), $X_T$ can be realized as an Itô process of type

$$dX_t = X_t\left(\mu_t^X dt + \sigma_t^X\left[\rho_t dB_t + \sqrt{1-\rho_t^2}\,dW_t\right]\right) \tag{27}$$

with $W_t$ a second Brownian motion under $\mathbb{P}$ such that $dB_t dW_t = 0$ and $\mu_t^X$, $\sigma_t^X$ and $\rho_t^X \in [-1,1]$ are progressively measurable processes. Further assume that $(S_t, X_t)$ represents a two dimensional Markov process.[10] For any martingale measure $\mathbb{Q}$ define the price process of the contingent claim $X_T$ determined through $\mathbb{Q}$ by[11]

$$V^{\mathbb{Q}}(t) = \mathbb{E}_t^{\mathbb{Q}}\left(e^{-r(T-t)}X_T\right). \tag{28}$$

In the context of incomplete markets there is a manifold of martingale measures, however as the following proposition shows, only one, the minimal martingale measure, leads to pricing that is consistent with CAPM. This result is little known; see Cerny (1999).

**Proposition 5.1**. *Assume that the two Brownian motions $B_t$ and $W_t$ span the uncertainty in the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the minimal martingale measure $\mathbb{Q}^{min}$ is the only martingale measure under which pricing is consistent with CAPM.*

The proof is in the Appendix. Our proof is very different from Cerny's proof and does not rely on discrete time approximation. It is also far more direct and tied to the agenda of our paper.

Proposition 5.1 can be rephrased as follows: if the real option is priced under any measure other than the minimal martingale measure, then the CAPM equilibrium condition is violated.[12]

Let us now reconsider the situation in Section 4. For simplicity we assume again that the coefficients $\mu_t^M$ and $\sigma_t^M$ appearing in the dynamics of the market portfolio (26) are constant, i.e. $\mu_t^M = \mu^M$ and $\sigma_t^M = \sigma^M$.[13] We assume that the coefficients of the real options underlying $X_t$ are deterministic functions of $X_t$, i.e.

$$dX_t = X_t\left[\mu^X(X_t)dt + \sigma^X(X_t)\left(\rho(X_t)dB_t + \sqrt{1-\rho(X_t)^2}\,dW_t\right)\right], \tag{29}$$

with $\mu_t^X = \mu^X(X_t)$ and $\sigma_t^X = \sigma^X(X_t)$ in the previous notation.

We know from Proposition 5.1 that the minimal martingale measure $\mathbb{Q}^{min}$ is the only measure leading to an evaluation of the real option that is consistent with the CAPM. We therefore fix $\mathbb{Q} = \mathbb{Q}^{min}$ in the following. We then know that the return of the option satisfies

$$r_{V^{\mathbb{Q}}} = r + \beta_{V^{\mathbb{Q}}}(\mu^M - r) \tag{30}$$

with $\beta_{V^{\mathbb{Q}}}$ given by Eq. (77) in the Appendix. We now make use of the assumption that the coefficients of the underlying are functions of $X_t$ only, and do not explicitly depend on time $t$ or the value of the market portfolio $S_t$.[14] This guarantees that the value function attached to the real option is a function of $X_t$ alone, i.e.

$$V^{\mathbb{Q}}(t, X_t, S_t) = V^{\mathbb{Q}}(X_t). \tag{31}$$

As in Section 4, if it is optimal to exercise the real option at finite time,[15] then optimal exercise occurs at a threshold $x^*$, where value matching and smooth pasting hold, i.e. $V^{\mathbb{Q}}(x^*) = x^* - K$ and $V_x^{\mathbb{Q}}(x^*) = 1$. Substituting these into (30) and using Eq. (77) from the Appendix yields

$$r_{V^{\mathbb{Q}}}^* = \frac{x^*\left(\frac{\sigma^X(x^*)}{\sigma^M}\rho(x^*)\mu^M + \left(1 - \frac{\sigma^X(x^*)}{\sigma^M}\rho(x^*)\right)r\right) - rK}{x^* - K}. \tag{32}$$

This generalizes our result (24) from Section 4, which in fact can be obtained from (32) for the case $\rho = 1$. We note that under any other measure than the minimal martingale measure, this extension of the Shackleton and Sodal (2005) rule would be violated.

It is also useful to examine the result for $\rho = 0$, i.e. the case where the real option underlying is uncorrelated with the market portfolio. In this case $r_{V^{\mathbb{Q}}} = r$, which means investment occurs when the option return is equal to the riskless rate $r$.

[10] These conditions are always satisfied within a multi-dimensional diffusion setting.

[11] We use the notation $V^{\mathbb{Q}}(t)$ for the stochastic process describing the value function rather than the notation $V_t^{\mathbb{Q}}$ with a sub-index, as the latter could be confused with the time derivative of the value function, which is also in use in the following.

[12] In fact, choosing $\theta_t^W \neq 0$ in Eq. (78) in the proof of Proposition 5.1 violates the CAPM equilibrium condition. On the other hand choosing $\theta_t^W = 0$ produces the CAPM equation, which means that the option is priced in alignment with the CAPM equilibrium condition. This choice reflects that no risk premium is paid on idiosyncratic risk, risk that is orthogonal to market risk.

[13] This can be generalized to the case where these coefficients are progressively measurable stochastic processes with respect to the filtration generated by the one-dimensional Brownian motion $(B_t)$.

[14] Note implicitly $X_t$ and $S_t$ are linked as they are generally correlated to some degree.

[15] There are cases, where it is optimal to never exercise the real option, such as those discussed in Section 6 Fig. 1 for $\rho < -0.5$.

From a CAPM point of view this makes sense, as in this case the real option does not carry any systematic market risk. From a CAPM point of view, it is perfectly reasonable that a higher correlation with the market requires a higher return in the real option value when exercising the real option. Depending on the coefficient functions determining the dynamics of the real option underlying, it is possible that higher returns occur at lower values of the underlying, hence higher correlation can reduce the exercise threshold. Fig. 1 in Section 6 illustrates this. This is in contrast to Henderson (2007) where higher correlation increases the threshold. Henderson's (2007) solution is also rather symmetric (but not perfectly symmetric) in the sign of $\rho$ and therefore less aligned with the CAPM. It follows from the proof of Proposition 5.1 and specifically from Eq. (71) in the Appendix, that under the minimal martingale measure and the assumptions above (and hence in alignment with CAPM), the drift of the underlying of the real option is reduced by the term $\rho(X_t)\theta\sigma^X(X_t)$. More specifically we obtain that under $\mathbb{Q}^{\min}$ we have

$$dX_t = X_t(r - \delta^X(X_t))dt + \sigma^X(X_t)dZ_t^*) \tag{33}$$

with

$$\delta^X(X_t) \equiv r - \mu^X(X_t) + \rho(X_t)\theta\sigma^X(X_t) \tag{34}$$

denoting the implied convenience yield.[16]

The correlation $\rho(X_t)$ and the volatility $\sigma^X(X_t)$ both have a direct impact on the implied convenience yield. Increasing the implied convenience reduces the option value and hence the exercise threshold. Therefore, as long as the market price of risk for the market portfolio is positive, a higher correlation will have the effect of a lower threshold. For positive correlation $\rho(X_t)$, increasing the volatility $\sigma^X(X_t)$ also increases the implied convenience yield, which has the same threshold-reducing effect. However here it is partly offset by the classical volatility effect due to convexity, which pushes the option value and hence the threshold upwards. It is not clear in general which of the two effects dominates. Dixit and Pindyck (1994) do not recognize the effect of the volatility on the implied convenience yield and therefore conclude that volatility unambiguously increases the exercise threshold, at least in a GBM based model. In Section 7 we provide an explicit counterexample demonstrating that the first effect can dominate the second effect, contrary to common belief; see Lund and Nymoen (2018).

The assumption that the coefficients of the underlying are functions of the underlying only is crucial here. If they were in fact functions of $X_t$ and $S_t$, the option value would in general depend on the values of $X_t$ and $S_t$ and instead of a single exercise threshold $x^*$ a free exercise boundary $x^*(S)$ would need to be identified. The value-matching and smooth-pasting conditions (in the multi dimensional context also referred to as high contact conditions) would become much more involved; see Brekke and Oksendal (1990). We leave this as an avenue for future research.

## 6. Does it matter? Value loss from non-market risk aversion

In this section we show that the loss in market value by following a stopping rule that is not aligned with the markets can be quite substantial. It can be helpful to have a real world example in mind, so we provide the following story that fits nicely with our technical analysis. However we would like to stress at this point that this example is more anecdotal, and not intended to provide any policy advice.

Assume that a governing authority wants to put in place a small biotech company that develops a patent for a vaccine against an infectious disease, with a fixed development cost $K$. This patent will eventually be sold to a larger pharmaceutical company for commercial use, at a time $\tau$ that is determined by the biotech company. The sale of the patent is irreversible and thus constitutes a real option, and the value of the company is comprised entirely of the value of this option.

The value of the patent, $X_t$, varies stochastically with the level of infection and other factors. In this case, it cannot realistically be assumed that the random shocks that are driving the value of the patent $X_t$ can be identically replicated with a perfect spanning asset, as in Dixit and Pindyck's (1994) contingent claims approach. Further, there is no direct reason for the value of the patent itself to be tied to the CAPM. However as we showed, this is not necessary in order to price the real option in accordance to the CAPM and hence determine the value of the company. In fact if the company that is due to develop the patent is established as a shareholder-owned publicly-listed company, then our only requirement is that the company value needs to meet the CAPM condition. The management of the company then aims at maximizing the shareholder value, which is the equivalent of solving problem (28) under the minimal martingale measure $\mathbb{Q}_{\min}$. This is a much more realistic assumption we believe.

As an alternative, rather than establishing the company as a publicly listed company, the governing authority could also put in place a privately-owned company. Private companies are less diversified and their management may be averse to idiosyncratic risk. Here patent development and timing would be determined through a utility based approach, as in (5). Nevertheless, company

[16] According to Casassus and Collin-Dufresne (2005), the convenience yield $\delta^X(t)$ is defined through the equation

$$r(t) - \delta(t) = \frac{d}{dt}\mathbb{E}_t^{\mathbb{Q}}\left(\frac{dX(t)}{X(t)}\right), \tag{35}$$

for the relevant risk-neutral measure $\mathbb{Q}$. That is, the risk-neutral expected returns of the underlying are equal to the interest rate $r(t)$ minus the convenience yield $\delta(t)$. In this way the convenience yield can be considered as an implicit net-dividend that takes account of costs like storage and benefits like liquidity for example. Costs increase the required return, benefits decrease the required return.

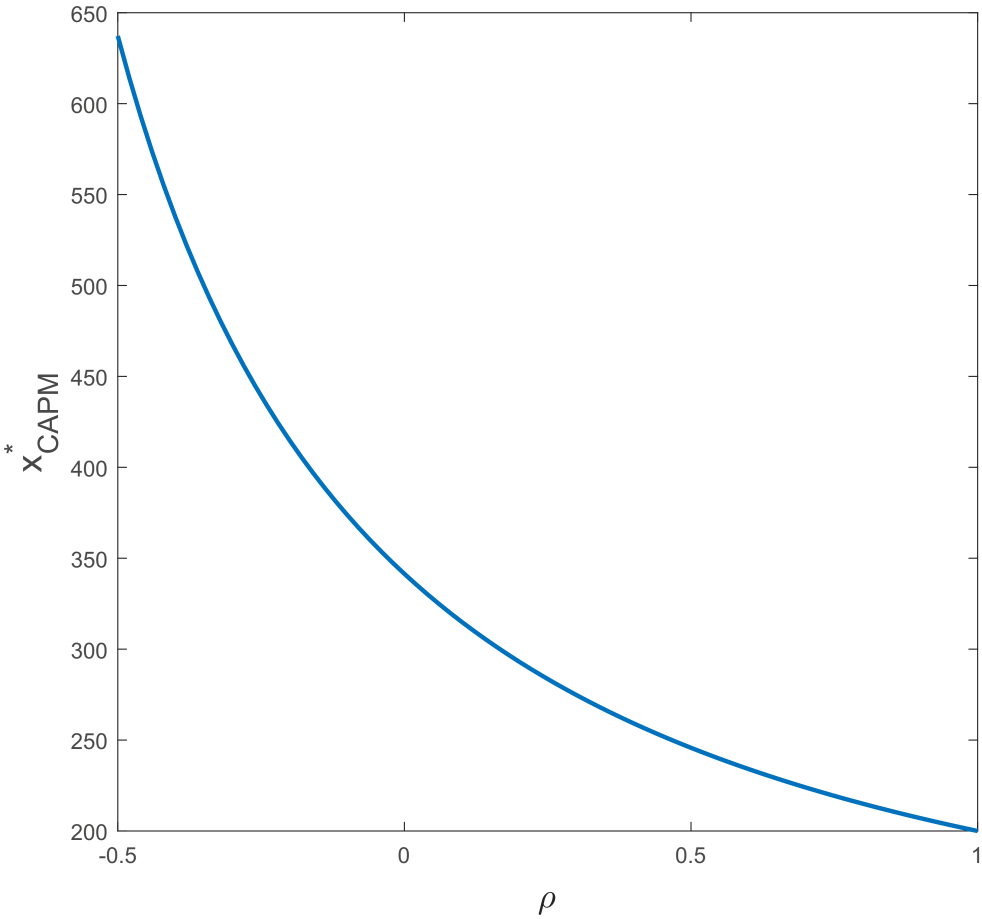


**Fig. 1.** Threshold $x^*_{\mathrm{CAPM}}$ as a function of $\rho$ for parameter choices of $r = 0.04$, $\mu^X = 0.02$, $\sigma^X = 0.2$, $K = 100$, $\theta = 0.1$.

valuation would still need to be aligned with CAPM, at least from the governing authority's (and welfare) perspective.

We evaluate the two scenarios within the methodology presented in the previous sections. We demonstrate that the second scenario creates financial losses in comparison to the first; which scenario causes the vaccine patent to be developed and released for commercial use more quickly, however, is ambiguous.

To formalize this we model the market portfolio $S_t$, and the value of the patent $X_t$, the real option underlying, as correlated geometric Brownian motions. Under $\mathbb{P}$:

$$\mathrm{dS}_t = \quad S_t(\mu^M \mathrm{dt} + \sigma^M \mathrm{dB}_t) \tag{36}$$

$$\mathrm{dX}_t = \quad X_t(\mu^X \mathrm{dt} + \sigma^X [\rho \mathrm{dB}_t + \sqrt{1-\rho^2}\mathrm{dW}_t]). \tag{37}$$

Using the results of the previous sections, the pricing and exercising of the real option consistent with the CAPM model is undertaken under the minimal martingale measure $\mathbb{Q}_{min}$. Under this measure, the dynamics of $X_t$ is given as

$$\mathrm{dX}_t = X_t([\mu^X - \rho\theta\sigma^X]\mathrm{dt} + \sigma^X[\rho \mathrm{dB}_t^{\mathbb{Q}_{min}} + \sqrt{1-\rho^2}\mathrm{dW}_t^{\mathbb{Q}_{min}}]), \tag{38}$$

where $\theta = \frac{\mu^M - r}{\sigma^M}$ is the Sharpe ratio of the market portfolio and $B_t^{\mathbb{Q}_{min}}$ as well as $W_t^{\mathbb{Q}_{min}}$ are Brownian motions under $\mathbb{Q}_{min}$.[17] As $X_t$ remains a geometric Brownian motion under $\mathbb{Q}_{min}$ the solution of the optimal stopping problem

[17] The dynamics (37) could also be expressed using the implied convenience yield $\delta_t^X$, and in fact this would simplify some of the expressions below, however it is dangerous to do so, as it may hide that both correlation and volatility affect the implied convenience yield.

$$V_{\text{CAPM}}(x) = \sup_{\tau \geq t} \mathbb{E}^{\mathbb{Q}_{min}}\left(e^{-r(\tau-t)}(X_\tau - K)|X_t = x\right) \tag{39}$$

can be obtained explicitly; see Oksendal (2007), page 223 for example. Denoting

$$\varepsilon_{\text{CAPM}} = (\sigma^X)^{-2}\left[\frac{1}{2}(\sigma^X)^2 - \mu^X + \rho\theta\sigma^X + \sqrt{\left(\frac{1}{2}(\sigma^X)^2 - \mu^X + \rho\theta\sigma^X\right)^2 + 2r(\sigma^X)^2}\right], \tag{40}$$

then if $\varepsilon_{CAPM} > 1$, i.e. $\mu^X - \rho\theta\sigma^X < r$[18], the optimal exercise time is whenever $X_t$ passes the threshold

$$x^*_{\text{CAPM}} = \frac{K\varepsilon_{\text{CAPM}}}{\varepsilon_{\text{CAPM}} - 1} = \frac{K}{1 - \frac{1}{\varepsilon_{\text{CAPM}}}} \tag{41}$$

for the first time and the value function is given by

$$V_{\text{CAPM}}(x) = (x^*_{\text{CAPM}} - K)\left(\frac{x}{x^*_{\text{CAPM}}}\right)^{\varepsilon_{\text{CAPM}}}. \tag{42}$$

For later use, we define $\varepsilon^{\rho=0}_{\text{CAPM}}$ as the corresponding value of (40) when $\rho = 0$ and similarly for $x^{*\rho=0}_{\text{CAPM}}$. The latter value represents the exercise threshold of an agent who is risk neutral. Fig. 1 shows this threshold as a function of the correlation between the market portfolio and the real option underlying.[19]

Fig. 1 can also be interpreted in the following way. At least for positive $\rho$, the more idio-syncratic the risk in the real option underlying holding total volatility constant, the higher the exercise threshold $x^*_{\text{CAPM}}$. This observation is in line with the conventional belief. However the effect is reversed for negative $\rho$. This effect is also aligned with our observation in Section 5, that $\rho$ increases the implied convenience yield, which under constant total volatility reduces the option value and therefore the exercise threshold.

Our setup allows us to find a closed form solution to the real option problem, even though we take full account of agent's optimally trading assets on a financial market. We are able to do this because we do not model the corresponding (sub-optimal) wealth processes explicitly, unlike Miao and Wang (2007). In our case, optimal wealth management is already encoded in the use of the minimal martingale measure. In Miao and Wang's case, the introduction of a second state variable, the wealth process, in fact prevents them from obtaining a closed form solution.

We now consider a possibly risk averse agent who evaluates the real option option through expected utility, while being ignorant of the market portfolio. To allow for explicit solutions, we assume HARA utility of the following form[20]

$$U(x) = \frac{1}{1-\gamma}(x+K)^{1-\gamma} - K. \tag{43}$$

where the parameter $\gamma$ measures the level of risk aversion and $\gamma = 0$ represents the case of a risk neutral agent. In general $\gamma \geq 0$, but for our specific application we need to restrict $\gamma \in [0, 1)$. The agent now maximizes

$$\sup_{\tau \geq t} \mathbb{E}\left(e^{-r(\tau-t)}U(X_\tau - K)|X_t = x\right) \tag{44}$$

Note that if $\gamma = 0$ and $\rho = 0$, then problem (44) coincides with problem (39), and hence the specific form of the utility function also enables us to calibrate the following around this case. The fact that

$$\begin{aligned} \frac{1}{1-\gamma}X_t^{1-\gamma} &= \frac{1}{1-\gamma}X_0^{1-\gamma}exp\left(\left(\widetilde{\mu}_X - \frac{1}{2}\widetilde{\sigma}_X^2\right)t + \widetilde{\sigma}_X Z_t\right) \\ \text{with} \quad \widetilde{\mu}_X &= (1-\gamma)\left(\mu^X - \frac{1}{2}(\sigma^X)^2\right) + \frac{1}{2}(1-\gamma)^2(\sigma^X)^2 \\ \text{and} \quad \widetilde{\sigma}_X &= (1-\gamma)\sigma^X \end{aligned} \tag{45}$$

is also a geometric Brownian motion enables us to find an explicit solution to problem (44). Once again following Oksendal, page 223, the solution has the same form as before, but with

$$\varepsilon_\gamma = \quad \frac{1}{1-\gamma}\varepsilon^{\rho=0}_{\text{CAPM}} \tag{46}$$

[18] If this condition is violated, then it is optimal to never exercise the option.

[19] Obviously, $\varepsilon_{CAPM}$ is an increasing function in $\rho$ and from this it can be rather easily concluded that $x^*_{\text{CAPM}}$ is indeed a decreasing function of $\rho$.

[20] The parameter $K$ in the utility function refers to the fixed investment cost, as used previously in this article. This very specific choice allows for closed form solutions and helps to illustrate the points we are making. Letting this parameter run freely, would not change any of the following conclusions, but the analysis would then need to rely on numerical simulation.

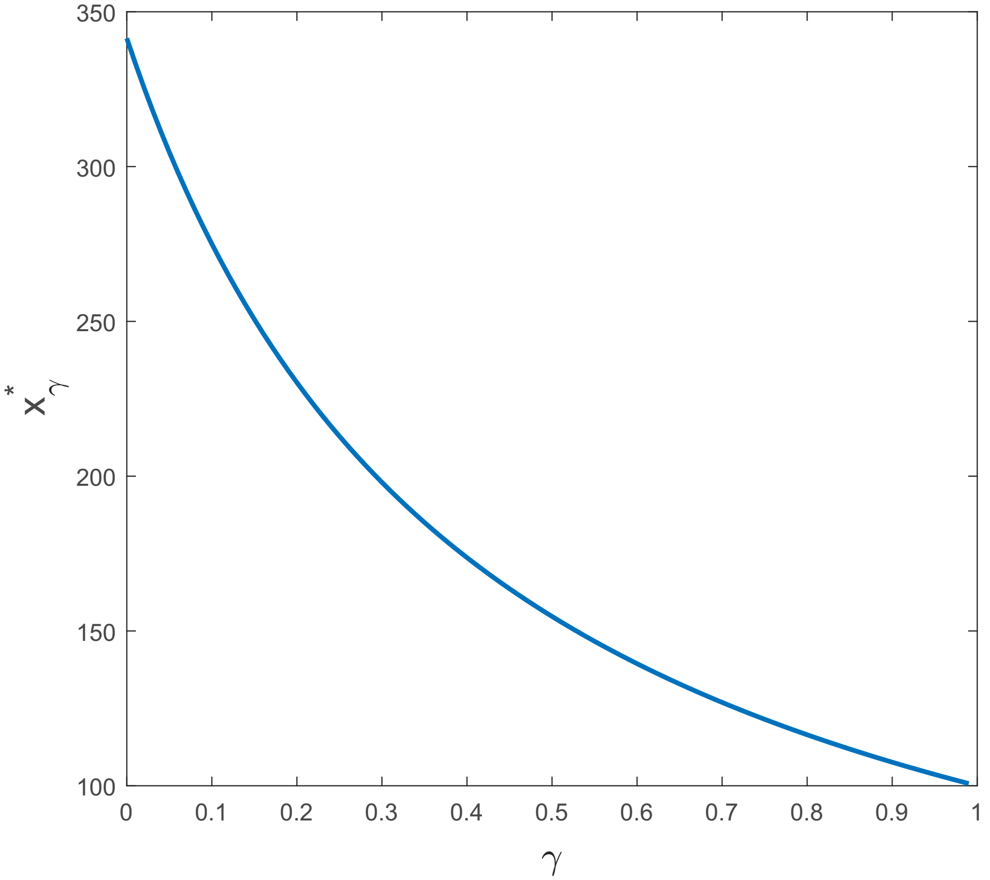


**Fig. 2.** Threshold $x_\gamma^*$ as a function of $\gamma$ for parameter choices of $r = 0.04$, $\mu^X = 0.02$, $\sigma^X = 0.2$, $K = 100$, $\theta = 0.1$.

and

$$x_\gamma^* = \frac{K\varepsilon_\gamma}{\varepsilon_\gamma - 1} = \frac{K\frac{1}{1-\gamma}\varepsilon_{\text{CAPM}}^{\rho=0}}{\frac{1}{1-\gamma}\varepsilon_{\text{CAPM}}^{\rho=0} - 1} = \frac{K}{1 - \frac{1-\gamma}{\varepsilon_{\text{CAPM}}^{\rho=0}}}. \tag{47}$$

Fig. 2 shows the exercise threshold depending on the level of risk aversion $\gamma$. Note that the intersection of the graphs and the y-axis in Figs. 1 and 2 coincide, and for the particular choice of $\rho = \gamma = 0$, the two models coincide, i.e. $x_{\text{CAPM}}^{*\rho=0} = x_{\gamma=0}^*$.

By comparing the two expressions in (41) and (47) we can explicitly identify the effect of risk aversion. We can now use the threshold $x_\gamma^*$ from (47) to compute the value of the investment. We assume that the company undertaking the investment is traded on the market. This is enough to justify the application of the CAPM, and as such

$$V_{\text{CAPM}}^\gamma(x) = \mathbb{E}^{\mathbb{Q}_{min}}\left(e^{-r(\tau_t^\gamma - t)}(X_{\tau_t^\gamma} - K)\Big|X_t = x\right), \tag{48}$$

with

$$\tau_t^\gamma = inf\{s \geq t | X_s = x_\gamma^* \,\text{given}\, X_t = x\}. \tag{49}$$

Once again, we conclude from Oksendal, page 223 that this value is given as

$$V_{\text{CAPM}}^\gamma(x) = (x_\gamma^* - K)\left(\frac{x}{x_\gamma^*}\right)^{\varepsilon_{\text{CAPM}}}. \tag{50}$$

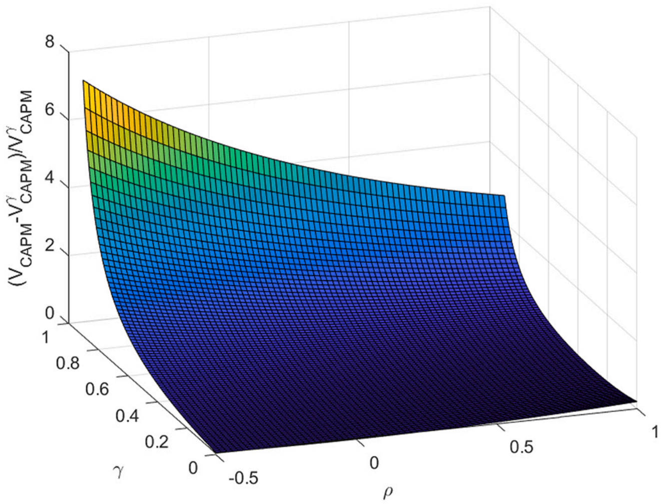


**Fig. 3.** Relative loss in value of the project, when following the stopping rule obtained from the risk averse utility based approach as compared to the market based approach as function of $\rho$ and $\gamma$. Parameter choices $r = 0.04$, $\mu^X = 0.02$, $\sigma^X = 0.2$, $K = 100$, $\theta = 0.1$.

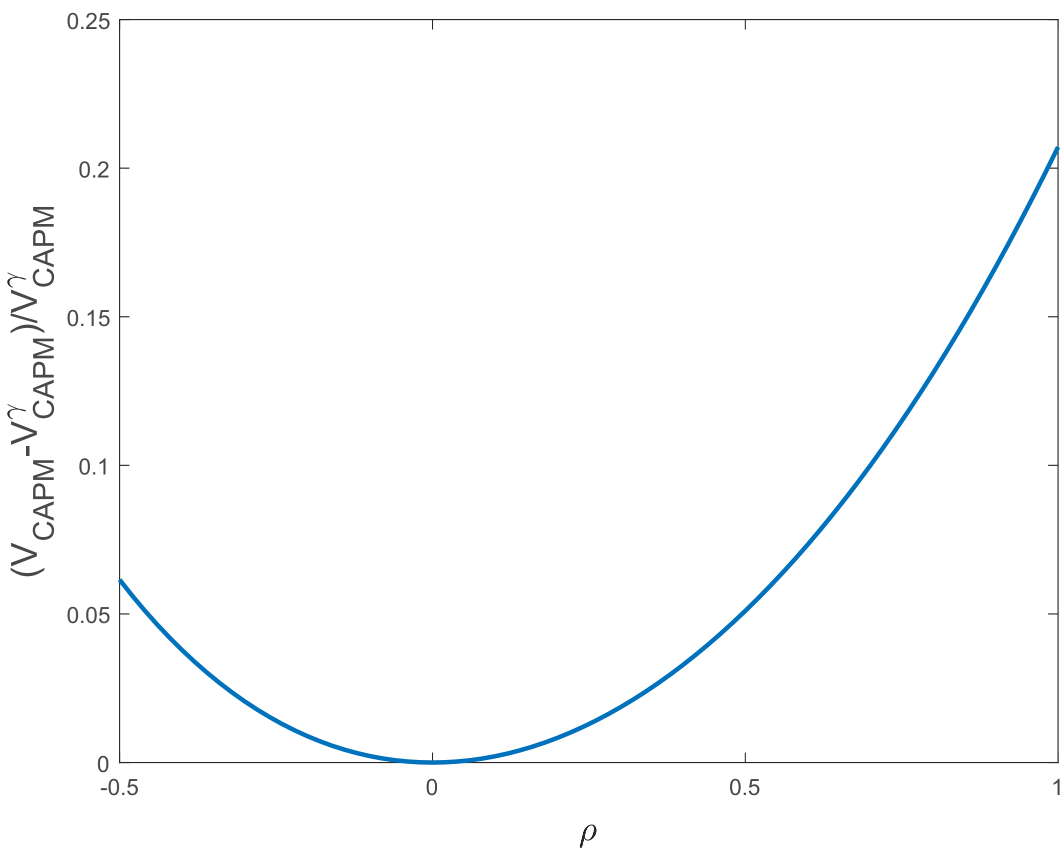


**Fig. 4.** Relative loss in value of the project, when following the stopping rule obtained from the risk averse utility based approach as compared to the market based approach as function of $\rho$ and $\gamma = 0$. Parameter choices $r = 0.04$, $\mu^X = 0.02$, $\sigma^X = 0.2$, $K = 100$, $\theta = 0.1$.

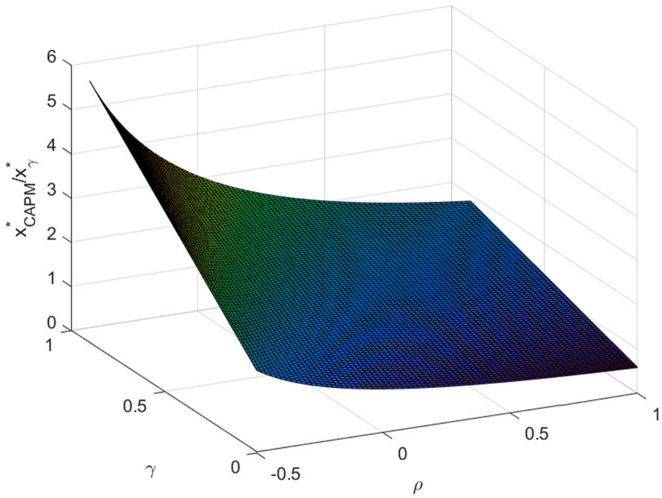


**Fig. 5.** Ratio of exercise thresholds, market based approach (public company) versus utility based approach (private company) as a function of $\rho$ and $\gamma$. Parameter choices $r = 0.04$, $\mu^X = 0.02$, $\sigma^X = 0.2$, $K = 100$, $\theta = 0.1$.

Note the use of $\varepsilon_{CAPM}$ in the exponent in (50) instead of $\varepsilon_\gamma$, which reflects the use of $\mathbb{Q}_{min}$ in the valuation.[21]

Fig. 3 shows the ratio

$$\frac{V_{\mathrm{CAPM}}(x) - V^\gamma_{\mathrm{CAPM}}(x)}{V^\gamma_{\mathrm{CAPM}}(x)} = \left(\frac{x^*_{\mathrm{CAPM}} - K}{x^*_\gamma - K}\right)\left(\frac{x^*_\gamma}{x^*_{\mathrm{CAPM}}}\right)^{\varepsilon_{\mathrm{CAPM}}} - 1 \tag{51}$$

$$\text{for} x \le min\{x^*_\gamma, x^*_{\mathrm{CAPM}}\},$$

which reflects the relative loss in value of the project, when following the stopping rule obtained from the risk averse utility based approach as compared to the market based approach. The expression in (51) does not depend on the state of the underlying anymore.

The losses can be quite dramatic. Comparing Figs. 1 and 2, one can see that for a significant fraction of the range of $\rho$, the value loss is due to earlier than optimal exercise, although this is not true in general.

It is obvious from Fig. 3 that risk aversion accounted for in terms of a utility function creates a very large loss, which overshadows the correlation effect. Fig. 4 below shows a cross section of Fig. 3 for $\gamma = 0$ from which we conclude that the loss due to not accounting for market correlation can still be very significant, up to 20%.

A different question is how the ownership structure of the company affects the exercise time of the real option, i.e. the time until the company sells the patent for the vaccine in order to facilitate commercial use through a larger pharmaceutical company. It may be in the interests of public welfare that this time is relatively short. Fig. 5 however shows that the answer to this question is ambiguous and depends on the parameters of the model, specifically the correlation $\rho$ between the value of the patent and the market portfolio and the risk-aversion $\gamma$ of the private management. A closer inspection of Fig. 5 shows that for positive correlation $\rho$ and reasonable levels of risk aversion $\gamma \le 0.5$ the shareholder owned company delivers the patent earlier than the privately owned company.

## 7. The volatility effect: a counterexample

In this section we provide an explicit example where, contrary to conventional belief, increasing risk reduces the exercise threshold of the real option when the option is evaluated in alignment with the CAPM. Our analysis emphasizes that it fundamentally matters what source of risk one considers (systematic or idiosyncratic) when doing comparative statics.

We use the same notation as before and assume that the dynamics of the underlying under $\mathbb{P}$ is given as

$$\mathrm{d}X_t = X_t(\mu^X \mathrm{d}t + \sigma_B \mathrm{d}B_t + \sigma_W \mathrm{d}W_t) = X_t\left(\mu^X \mathrm{d}t + \sigma^X \cdot (\rho \mathrm{d}B_t + \sqrt{1-\rho^2}\, \mathrm{d}W_t)\right), \tag{52}$$

with

$$(\sigma^X)^2 = \sigma_B^2 + \sigma_W^2 \text{ and } \rho = \frac{\sigma_B}{\sqrt{\sigma_B^2 + \sigma_W^2}}. \tag{53}$$

The parameters $\sigma_B$ and $\sigma_W$ therefore measure the exposure of the underlying toward systematic and idiosyncratic risk respectively. One

[21] $V^\gamma_{\mathrm{CAPM}}(x)$ represents the market value of the project, when following the stopping rule obtained from (5).

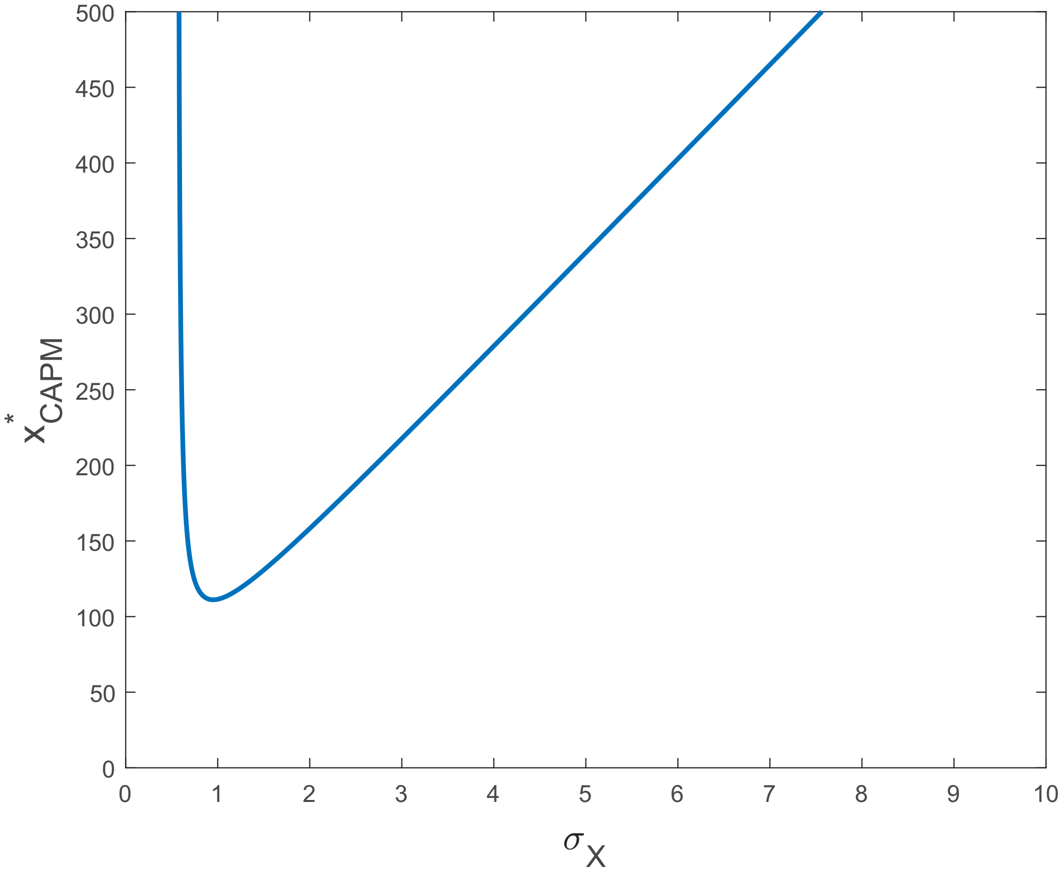


**Fig. 6.** Threshold $x^*_{\text{CAPM}}$ as a function of volatility $\sigma^X$ with $\sigma_W = 0.5$ fixed, $\mu^X = 0.02$, $\theta = 10$, $K = 10$ and $r = 0$.

can now increase or decrease these parameters independently from each other and observe different effects on total volatility and correlation, which will effect the option value and threshold. The following realization about the impact on the implied convenience yield is crucial: Combining (34) together with (53) yields

$$\delta^X = r - \mu^X + \rho\theta\sigma^X = r - \mu^X + \theta\sigma_B. \tag{54}$$

leading to the conclusion that increasing total volatility by increasing idiosyncratic volatility $\sigma_W$ has no effect on the implied convenience yield and therefore unambiguously increases the option value and threshold. However, increasing systematic volatility $\sigma_B$ increases the implied convenience yield, which reduces the option value and exercise threshold. This effect is countered to some degree by the increase in total volatility. Which of the effects dominates the other is unclear and in fact depends on the value of the state variable $X_t$. We will illustrate this trade-off in our numerical example.

We can clarify the relative influence of the two types of risk by computing the actual threshold for this example, where the real option is evaluated in alignment with the CAPM in the same way as before, i.e.

$$\varepsilon_{\text{CAPM}} = \frac{1}{\sigma_B^2 + \sigma_W^2} \cdot \left[ \frac{1}{2}(\sigma_B^2 + \sigma_W^2) - \mu^X + \sigma_B \cdot \theta + \sqrt{\left( \frac{1}{2}(\sigma_B^2 + \sigma_W^2) - \mu^X + \sigma_B \cdot \theta \right)^2 + 2r(\sigma_B^2 + \sigma_W^2)}. \right] \tag{55}$$

This can be simplified substantially if we assume that the riskless rate is equal to zero, i.e. $r = 0$. Then

$$\varepsilon_{\text{CAPM}} = 1 - \frac{2\mu^X}{\sigma_B^2 + \sigma_W^2} + \frac{2\sigma_B\theta}{\sigma_B^2 + \sigma_W^2} = 1 - \frac{2\mu^X}{\sigma_B^2 + \sigma_W^2} + \frac{2\beta\mu^M}{\sigma_B^2 + \sigma_W^2}, \tag{56}$$

where $\beta$ is the CAPM beta of the real option underlying. In order for a finite exercise threshold to exist, we need $\varepsilon_{CAPM} > 1$ which requires a sufficiently large $\sigma_B$ or $\theta$. However, it is then obvious from looking at the second and third term in (56) that for positive $\mu^X$ and $\mu^M$ these two terms pull in different directions and that the expression $\varepsilon_{CAPM}$ is therefore non-monotonic and has regions for $\sigma_B$ in which it either increases or decreases.

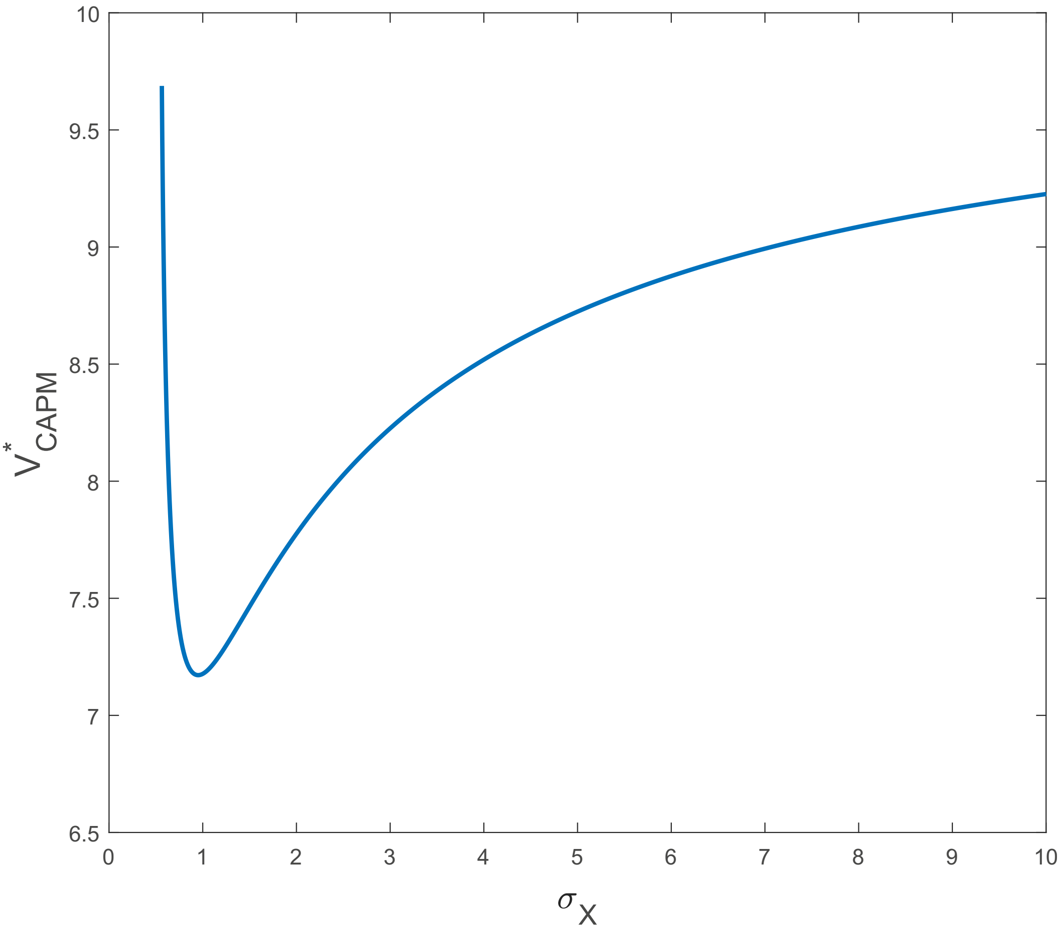


**Fig. 7.** Real option value $V_{CAPM}(x)$ as a function of volatility $\sigma^X$ at the money $x = K$ with $\sigma_W = 0.5$ fixed, $\mu^X = 0.02$, $\theta = 10$, $K = 10$ and $r = 0$.

More precisely, introducing the function $f(\sigma_B) := \varepsilon_{\text{CAPM}} - 1 = 2 \cdot \frac{\sigma_B\theta - \mu^X}{\sigma_B^2 + \sigma_W^2}$, defined for $\sigma_B \geq 0$, then $f(0) = -2\frac{\mu^X}{\sigma_W^2} < 0, f\left(\frac{\mu^X}{\theta}\right) = 0$ and $\lim_{\sigma_B \to \infty} f(\sigma_B) = 0$. Furthermore $f(\sigma_B)$ has a unique maximum at $\sigma_B^* := \frac{\mu^X}{\theta} + \sqrt{\left(\frac{\mu^X}{\theta}\right)^2 + \sigma_W^2}$. We conclude that on the interval

$$\left[\frac{\mu^X}{\theta}, \frac{\mu^X}{\theta} + \sqrt{\left(\frac{\mu^X}{\theta}\right)^2 + \sigma_W^2}\right] \tag{57}$$

the function $\varepsilon_{CAPM}(\sigma_B) = f(\sigma_B) + 1$ is larger than 1 and increasing, while it decreases to the right of this interval. Further and in consequence, the exercise threshold

$$x^*_{\text{CAPM}} = \frac{K\varepsilon_{\text{CAPM}}}{\varepsilon_{\text{CAPM}} - 1} \tag{58}$$

is finite and decreasing on the interval (57) and increasing for $\sigma_B > \frac{\mu^X}{\theta} + \sqrt{\left(\frac{\mu^X}{\theta}\right)^2 + \sigma_W^2}$.

Expression (56) allows to draw on some further intuition. The second term captures the real option underlying's drift, while the third term is determined through the real option underlying's beta as well as the market drift. If we hold total volatility $\sqrt{\sigma_B^2 + \sigma_W^2}$ constant and only increase the systematic volatility exposure identified through $\beta$, then $\varepsilon_{CAPM}$ increases and hence the threshold decreases. This confirms the effect observed above. It also corresponds to our previous observation in Fig. 1, as increasing $\sigma_B$ (or equivalently $\beta$) while holding $\sqrt{\sigma_B^2 + \sigma_W^2}$ constant means to increase $\rho$ via (52).

The opposing effects of implied convenience yield and volatility can also be highlighted by expressing (56) in terms of the implied convenience yield. We then get

$$\varepsilon_{\mathrm{CAPM}} = 1 + \frac{2\delta^X}{\sigma_X}, \tag{59}$$

and in consequence

$$x^*_{\mathrm{CAPM}} = \frac{K}{2}\frac{\sigma^X}{\delta^X} - 1. \tag{60}$$

Note that in our case, both $\sigma^X$ and $\delta^X$ are increasing functions of the systematic volatility $\sigma_B$ and the monotonicity of $x^*_{\mathrm{CAPM}}$ depends on which effect is stronger.

Fig. 6 shows initially an inverse relationship between the systematic risk measured in form of $\sigma_B$, or alternatively total volatility $\sigma^X$ as $\sigma_W$ is fixed, and the optimal threshold $x^*_{\mathrm{CAPM}}$ for a suitable choice of parameters in the more general case. Both observations are in contradiction to conventional belief that increasing total volatility always increases the exercise threshold, as stated for example in Dixit and Pindyck (1994), page 153.

Perhaps even more interesting is that the value of the real option as assessed in our CAPM based framework is also initially decreasing with systematic volatility $\sigma_B$, and in conclusion also with total volatility and risk, while holding idiosyncratic volatility $\sigma_W$ constant. Fig. 7 shows this relationship. In fact to the best of our knowledge this presents the first example in continuous time and under realistic modeling assumptions, where a plain-vanilla type option decreases in value with increasing volatility. Previous examples featuring this result are somewhat stylized; they are in discrete time over one or two periods and with finite probability space only; see Jagannathan (1984), Carr and Lee (2003, pages 20 and 21) and Carr et al. (2008). Our example shows that the effect observed in this literature is not just an artificial construct, but can and in fact should appear in a standard continuous time CAPM based setup.[22]

## 8. Conclusions

We have presented a framework in which real options can be priced appropriately by taking account of the effect of financial markets and risk aversion following an agenda motivated by utility indifference pricing and optimal hedging placed within the CAPM. Our pricing approach is aligned with fundamental principles in corporate finance and provides a rigorous justification of Dixit and Pindyck's (1994) contingent claim analysis approach, even without the existence of perfect spanning assets and complete markets. We also shed light on the important question how an increase in uncertainty effects investment. As we show, the answer to this question delicately depends on to which extent the uncertainty is systematic and to which extent it is idiosyncratic. While idiosyncratic risk always increases the exercise threshold, an increase in systematic risk can reduce the threshold. We then demonstrate that this effect can be so strong that even an increase in total volatility can decrease the exercise threshold, contradicting the conventional view.

Our work builds on and extends Shackleton and Sodal (2005), including in particular the incomplete market case. We demonstrated the importance of taking account of markets and risk aversion in real option evaluation by showing that not doing so can incur substantial financial losses, which we illustrated with an example investigating how the ownership structure of a biotech company (shareholder vs. private) affects its value and the time to develop and sell a patent for a vaccine to a pharmaceutical company for commercial use.

## Declaration of Competing Interest

None.

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## Appendix

**Proof of Proposition 3.1**. *Proof.* To simplify the proof, we assume that the suprema on both sides of (9) are actually achieved, i.e. there do exist maximizing strategies. This is without loss of generality, in fact the proof could be adapted in slightly modified form using an epsilon argument. Let $\psi^*_t$ be the strategy maximizing the right hand side of (9). Note that since $\psi^*_t$ is self-financing, we have that

[22] We thank Peter Carr for making us aware of the opportunity to provide such an example within our framework.

$$B(X_t)\cdot\Delta = \mathbb{E}^{\mathbb{Q}}\left(e^{-r\Delta}\left(Y_{t+\Delta}^{\psi_t^*,B(X_t)\cdot\Delta}\right)\Big|\mathcal{F}_t\right) \tag{61}$$

and in consequence the right hand side of (10) is equal to

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-r\Delta}\left(Y_{t+\Delta}^{\psi_t^*,B(X_t)\cdot\Delta} + V(X_{t+\Delta}, t+\Delta)\right)\Big|\mathcal{F}_t\right). \tag{62}$$

Let us assume now that (10) holds. Then there exist a self-financing strategy $\varphi_t$ s.t.

$$Y_{t+\Delta}^{\varphi_t,H(X_t)} \geq Y_{t+\Delta}^{\psi_t^*,B(X_t)\cdot\Delta} + V(X_{t+\Delta}, t+\Delta). \tag{63}$$

To see this, note that the amount of $\mathbb{E}^{\mathbb{Q}}(e^{-r\Delta}V(X_{t+\Delta}, t+\Delta)|\mathcal{F}_t)$ can be taken off $H(X_t)$ to perfectly replicate $V(X_{t+\Delta}, t+\Delta)$ and that the remainder

$$H(X_t) - \mathbb{E}^{\mathbb{Q}}(e^{-r\Delta}V(X_{t+\Delta}, t+\Delta)|\mathcal{F}_t) \geq B(X_t)\cdot\Delta \tag{64}$$

can be used to generate a larger portfolio value than $Y_{t+\Delta}^{\psi_t^*,B(X_t)\cdot\Delta}$. As the utility function $U(x)$ is monotonic increasing (9) follows from (63). We have therefore shown that (10) implies (9). To show that (9) implies (10) we assume that (10) does not hold, i.e.

$$H(X_t) < B(X_t)\cdot\Delta + \mathbb{E}^{\mathbb{Q}}(e^{-r\Delta}V(X_{t+\Delta}, t+\Delta)|\mathcal{F}_t). \tag{65}$$

Let $\alpha_t$ be a self-financing strategy which perfectly replicates $V(X_{t+\Delta}, t+\Delta)$. As before, the investment required for this is $\mathbb{E}^{\mathbb{Q}}(e^{-r\Delta}V(X_{t+\Delta}, t+\Delta)|\mathcal{F}_t)$. In addition, let $\beta$ be the strategy which invests

$$m := B(X_t)\cdot\Delta - (H(X_t) - \mathbb{E}^{\mathbb{Q}}(e^{-r\Delta}V(X_{t+\Delta}, t+\Delta)|\mathcal{F}_t))\rangle 0 \tag{66}$$

into the bank account. Let $\varphi^*$ be the self-financing strategy maximizing the left hand side of (9). The self-financing strategy $\psi = \varphi^* - \alpha + \beta$ then requires an investment of $B(X_t)\cdot\Delta$ and satisfies

$$Y_{t+\Delta}^{\psi,B(X_t)\cdot\Delta} + V(X_{t+\Delta}, t+\Delta) = Y_{t+\Delta}^{\varphi^*,H(X_t))} + m(1+r\Delta), \tag{67}$$

but this contradicts (9) and therefore (10) needs to hold. □ **Proof of Proposition 5.1**. *Proof.* The density of any equivalent martingale measure $\mathbb{Q}$ with respect to $\mathbb{P}$ relative to $\mathcal{F}_t$ can be expressed as

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} = exp\left(-\left[\int_0^t \theta_s^B \mathrm{dB}_s + \int_0^t \theta_s^W \mathrm{dW}_s\right] - \frac{1}{2}\int_0^t\left((\theta_s^B)^2 + (\theta_s^W)^2\right)\mathrm{ds}\right) \tag{68}$$

with a progressively measurable two dimensional process $(\theta_s^B, \theta_s^W)$ such that

$$\left[B_t^{\mathbb{Q}}\ W_t^{\mathbb{Q}}\right] = \begin{bmatrix} B_t + \int_0^t \theta_s^B \mathrm{ds} \\ W_t + \int_0^t \theta_s^W \mathrm{ds}\end{bmatrix} \tag{69}$$

is a Brownian motion under $\mathbb{Q}$, with $\mathrm{dB}_t^{\mathbb{Q}}\cdot \mathrm{dW}_t^{\mathbb{Q}} = 0$. As the discounted market portfolio $e^{-rt}S_t$ needs to be a local martingale under $\mathbb{Q}$, we must have

$$\theta_s^B = \frac{\mu_s^M - r}{\sigma_s^M} \tag{70}$$

from applying a Girsanov transformation and hence the dynamics of $X_t$ under $\mathbb{Q}$ can be written as

$$\mathrm{dX}_t = X_t\left(\left[\mu_t^X - \left(\rho_t\frac{\mu_t^M - r}{\sigma_t^M}\sigma_t^X + \sqrt{1-\rho_t^2}\,\theta_t^W\sigma_t^X\right)\right]\mathrm{dt} + \sigma_t^X\left(\rho_t \mathrm{dB}_t^{\mathbb{Q}} + \sqrt{1-\rho_t^2}\,\mathrm{dW}_t^{\mathbb{Q}}\right)\right). \tag{71}$$

We also recall that by construction under any martingale measure $\mathbb{Q}(t)$ the market portfolio process obeys

$$\mathrm{dS}_t = S_t(\mathrm{r\ dt} + \sigma_t^S \mathrm{dB}_t) \tag{72}$$

Under our assumptions, the process $V^{\mathbb{Q}}(t)$ can be written as a function of $S_t$ and $X_t$, i.e.

$$V^{\mathbb{Q}} = V^{\mathbb{Q}}(t, X_t, S_t), \tag{73}$$

and from Itô's lemma we have[23]

$$\mathrm{d}V^{\mathbb{Q}} = \left[ V_t^{\mathbb{Q}} + V_x^{\mathbb{Q}} \cdot X_t \left\{ \mu_t^X - \left( \rho_t \frac{\mu_t^M - r}{\sigma_t^M} \sigma_t^X + \sqrt{1-\rho_t^2}\, \theta_t^W \sigma_t^X \right) \right\} + V_S^{\mathbb{Q}} S_t r \right. \tag{74}$$
$$\left. + \frac{1}{2} V_{xx}^{\mathbb{Q}} X_t^2 (\sigma_t^X)^2 + \frac{1}{2} V_{SS}^{\mathbb{Q}} S_t^2 (\sigma_t^M)^2 + V_{xS}^{\mathbb{Q}} X_t S_t \sigma_t^X \sigma_t^M \rho_t \right] \mathrm{d}t + (\ldots)\mathrm{d}B_t^{\mathbb{Q}} + (\ldots)\mathrm{d}W_t^{\mathbb{Q}},$$

where the terms in the round brackets in front of $\mathrm{d}B_t^{\mathbb{Q}}$ and $\mathrm{d}W_t^{\mathbb{Q}}$ do not need to be determined at the moment. For arbitrage reasons however, the long expression in the square brackets needs to be equal to $rV^{\mathbb{Q}}$. Similar to the argument leading to (18), this then gives

$$\mathbb{E}_t^{\mathbb{P}}(\mathrm{d}V^{\mathbb{Q}}) = rV^{\mathbb{Q}}\mathrm{d}t + V_S^{\mathbb{Q}} S_t (\mu_t^M - r)\mathrm{d}t \tag{75}$$
$$+ V_x^{\mathbb{Q}} X_t \left( \rho_t \frac{\mu_t^M - r}{\sigma_t^M} \sigma_t^X + \sqrt{1-\rho_t^2}\, \theta_t^W \sigma_t^X \right) \mathrm{d}t.$$

Hence, for the expected return of $V^{\mathbb{Q}}$ under $\mathbb{P}$ this gives

$$r_{V^{\mathbb{Q}}} = \frac{1}{\mathrm{d}t} \mathbb{E}_t^{\mathbb{P}} \left( \frac{\mathrm{d}V^{\mathbb{Q}}}{V^{\mathbb{Q}}} \right) = r + \frac{V_S^{\mathbb{Q}}}{V^{\mathbb{Q}}} S_t (\mu_t^M - r) + \frac{V_X^{\mathbb{Q}}}{V^{\mathbb{Q}}} X_t \left( \rho_t \frac{\mu_t^M - r}{\sigma_t^M} \sigma_t^X + \sqrt{1-\rho_t^2}\, \theta_t^W \sigma_t^X \right). \tag{76}$$

On the other hand the correlation of the options return with the market portfolio's return is given by

$$\beta_{V^{\mathbb{Q}}} = \frac{\mathbb{E}_t^{\mathbb{P}} \left( \frac{\mathrm{d}V^{\mathbb{Q}}}{V^{\mathbb{Q}}} \frac{\mathrm{d}S}{S} \right)}{\mathbb{E}_t^{\mathbb{P}} \left( \left( \frac{\mathrm{d}S}{S} \right)^2 \right)} = \frac{\frac{V_X^{\mathbb{Q}}}{V^{\mathbb{Q}}} X_t \sigma_t^X}{\sigma_t^M} \rho_t + \frac{V_S^{\mathbb{Q}}}{V^{\mathbb{Q}}} S_t, \tag{77}$$

These correlations reflect the "(…)" terms in (74). Substituting into (76),

$$r_{V^{\mathbb{Q}}} = r + \beta_{V^{\mathbb{Q}}} (\mu_t^M - r) + \frac{V_X^{\mathbb{Q}}}{V^{\mathbb{Q}}} X_t \sqrt{1-\rho_t^2}\, \theta_t^W \sigma_t^X. \tag{78}$$

This is consistent with CAPM if and only if $\theta_t^W = 0$, or $\rho_t = \pm 1$; in which case all risk is spanned by the market portfolio and we are back in the situation considered in the previous section. It then follows from El Karoui et al. (1997, Remark 1.2) that $\mathbb{Q}$ must be the minimal martingale measure. □

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[23] The expressions $V_t$, $V_x$, $V_S$, $V_{xx}$, $V_{SS}$ and $V_{xS}$ refer to derivatives of the value-function evaluated at $(t, X_t, S_t)$ while $X_t$ and $S_t$ denote the values of the two stochastic processes at time $t$ respectively.

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