Binary Darboux Transformation for the Gerdjikov-Ivanov equation

Halis Yilmaz

School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8SQ, UK

Department of Mathematics, University of Dicle, 21280 Diyarbakir, Turkey

Abstract

We present a standard binary Darboux transformation for the Gerdjikov-Ivanov equation and then construct its quasigrammian solutions. Further, the multi- soliton, breather and rogue wave solutions of the Gerdjikov-Ivanov equation are given as particular examples.

Keywords: Gerdjikov-Ivanov equation, Derivative nonlinear Schrödinger equation, Binary Darboux transformation, Quasideterminants, Quasigrammians

1. Introduction

The nonlinear Schrödinger (NLS) equation [49] is a physically significant model and has many physical applications, such as deep water waves [4, 47], plasma physics [48], nonlinear optical fibers [20, 21], magnetostatic spin waves [55] and so on. The high-order nonlinear forms of the NLS equation have been studied by various scholars. Among them, there are three well known equations with derivative–type nonlinearities which are called the Kaup-Newell equation (DNLSI) [27], the Chen–Lee–Liu equation (DNLSII) [5] and the Gerdjikov–Ivanov equation (DNLSIII) [15]

$$iq_t + q_{xx} + \frac{1}{2}q|q|^4 + iq^2 q_x^* = 0, \tag{1}$$

where the asterisk * denotes complex conjugation and the dependent variable $q(x, t)$ represents a complex-valued wave profile with x and t being the independent spatial and temporal variables respectively. The classical nonlinear Schrödinger equation [49] with the derivative-type NLS (DNLS) equations [5, 15, 27] are completely integrable and play an important role in the study of mathematical physics [3, 7, 23, 26, 28, 40].

Recently, the explicit solutions of the Gerdjikov-Ivanov (GI) equation are studied in [9, 17, 42] via *Darboux-type* transformations. These solutions are often expressed in terms of determinants with a complicated structure. On the other hand, in [46], the exact solutions of the GI equation are constructed in terms of *quasideterminants* in compact forms by a standard Darboux transformation (DT). Quasideterminant was introduced first by Gelfand and Retakh [10] in 1991 as a replacement for the determinant over non-commutative rings R. They play crucial roles in constructing explicit solutions of integrable systems [12, 13, 19, 22, 24, 38, 29].

In this paper, we present for the first time a standard binary Darboux transformation (BDT) for the GI equation (1) and then construct its quasigrammian solutions. The DT is known as one of the most powerful methods for finding the explicit solutions of the integrable systems. For the sake of clarity we emphasize that the method we employ in the present article is based on Darboux's [8] and Matveev's original ideas [35, 36]. In addition to this, our solutions are expressed in terms of quasigrammians [11] rather than determinants. Therefore, our approach should be considered on its own merits.

Email address: Halis.Yilmaz@glasgow.ac.uk (Halis Yilmaz)

The structure of this paper is as follows. In Section 2, we construct a 2×2 eigenfunction and corresponding constant 2×2 square matrix for the eigenvalue problem of the GI equation (1) using two symmetries of the Lax pair of the GI equation. In Section 3, we state a standart BDT [8, 35, 36] for the GI equation in $(2+1)$ dimensions. Then, by a separation of variables technique, we present the reduced BDT, which can be considered as a dimensional reduction from $(2+1)$ to $(1+1)$ dimensions. The reader is referred to the article [37] for a more detailed treatment. In Section 4, via the reduced BDT, we construct the exact quasigrammian solutions of the GI equation. We then show that these explicit solutions can be written in terms of quasideterminants. In Section 5, as applications of the BDT, we present the explicit solutions of the Gerdjikov-Ivanov equation from zero and non-zero seed solutions. These particular solutions include the multi-soliton, breather and rogue-wave solutions. A brief summary and discussion are given in the last Section 6.

2. Gerdjikov-Ivanov equation

Let us start with the coupled Gerdjikov-Ivanov equations:

$$\begin{aligned} &iq_t+q_{xx}+iq^2r_x+\frac{1}{2}q^3r^2=0,\\ &ir_t-r_{xx}+ir^2q_x-\frac{1}{2}q^2r^3=0, \end{aligned} \tag{2}$$

which are reduced to the GI equation (1) when $r=q^*$. The Lax pair [15] for the system (2) is given by

$$L = \partial_x + J\lambda^2 - R\lambda + \frac{1}{2}qrJ, \tag{3}$$

$$M = \partial_t + 2J\lambda^4 - 2R\lambda^3 + qrJ\lambda^2 + U\lambda + W, \tag{4}$$

where J, R, U and W are 2×2 matrices

$$J=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad R=\begin{pmatrix} 0 & q\\ r & 0\end{pmatrix},\quad U=\begin{pmatrix} 0 & -iq_x\\ ir_x & 0\end{pmatrix} \tag{5}$$

and

$$W=\begin{pmatrix} -\frac{1}{2}\left(rq_x-qr_x\right)-\frac{1}{4}iq^2r^2 & 0\\ 0 & \frac{1}{2}\left(rq_x-qr_x\right)+\frac{1}{4}iq^2r^2\end{pmatrix}. \tag{6}$$

Here the spectral parameter λ is an arbitrary complex number. We can easily see that the potential matrix R in (5) has two symmetry properties. The first one is that it is Hermitian: $R^\dagger=R$. The second one is that $SRS^{-1}=-R$, where

$$S=S^{-1}=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{7}$$

Let $\phi=(\varphi,\psi)^T$ be a vector eigenfunction for the Lax operators (3)-(4) corresponding an eigenvalue λ. By using the second symmetry, we can see that $S\phi=(\varphi,-\psi)^T$ is another eigenfunction for eigenvalue $-\lambda$. By using these vector eigenfunctions we can construct a square 2×2 matrix eigenfunction θ with 2×2 eigenvalue Λ

$$\theta=\begin{pmatrix} \varphi & \varphi\\ \psi & -\psi\end{pmatrix},\quad \Lambda=\begin{pmatrix} \lambda & 0\\ 0 & -\lambda\end{pmatrix}, \tag{8}$$

satisfying

$$\theta_x + J\theta\Lambda^2 - R\theta\Lambda + \frac{1}{2}qrJ\theta = 0, \tag{9}$$

$$\theta_t + 2J\theta\Lambda^4 - 2R\theta\Lambda^3 + qrJ\theta\Lambda^2 + U\theta\Lambda + W\theta = 0. \tag{10}$$

3. Darboux transformations and Dimensional reductions

3.1. Darboux transformation

Let us begin by considering the following linear operators:

$$L = \partial_x + \sum_{i=0}^{N} u_i \partial_y^i, \qquad M = \partial_t + \sum_{i=0}^{N} v_i \partial_y^i, \tag{11}$$

where u_i, v_i are $m \times m$ are matrices. Let $L(\phi) = M(\phi) = 0$, where ϕ is any eigenfunction of L and M. Let us introduce a gauge operator $G_\theta = \theta \partial_y \theta^{-1}$, where θ is a non-singular matrix solution of the same linear system $L(\phi) = M(\phi) = 0$. Then the Darboux transformation $\phi \to \tilde{\phi} = G_\theta(\phi)$ keeps the linear system $L(\phi) = M(\phi) = 0$ invariant:

$$\tilde{L}(\tilde{\phi}) = \tilde{M}(\tilde{\phi}) = 0, \tag{12}$$

where $\tilde{\phi} = G_\theta(\phi)$ is an eigenfunction of $\tilde{L}$ and $\tilde{M}$ in which the linear operators $\tilde{L} = G_\theta L G_\theta^{-1}$ and $\tilde{M} = G_\theta M G_\theta^{-1}$ have the same form as L and M in (11) but with different coefficients:

$$\tilde{L} = \partial_x + \sum_{i=0}^{N} \tilde{u}_i \partial_y^i, \qquad \tilde{M} = \partial_t + \sum_{i=0}^{N} \tilde{v}_i \partial_y^i. \tag{13}$$

3.2. Binary Darboux transformation

Suppose G_θ and $G_{\hat{\theta}}$ be two standard Darboux transformations map two linear operators L and $\hat{L}$ onto a common linear operator $\tilde{L}$ such that

$$\tilde{L} = G_\theta L G_\theta^{-1} = G_{\hat{\theta}} \hat{L} G_{\hat{\theta}}^{-1}. \tag{14}$$

Then one may define a binary Darboux transformation $B_{\theta,\hat{\theta}} = G_{\hat{\theta}}^{-1} G_\theta$ such that $\hat{L} = B_{\theta,\hat{\theta}} L B_{\theta,\hat{\theta}}^{-1}$. In order to define $G_{\hat{\theta}}$ one needs an eigenfunction of $\hat{L}$. This problem can be got around by using the formal adjoint operator $L^\dagger$ constructed according to the rule $\left(a\partial_y^i\right)^\dagger = (-\partial_y)^i a^\dagger$, where † denotes the Hermitian conjugate. If θ and ρ be two $m \times k$ matrix solutions of the linear system $L(\phi) = M(\phi) = 0$ and its adjoint system $L^\dagger(\psi) = M^\dagger(\psi) = 0$ respectively, we then derive the eigenfunction $\hat{\theta}$ as

$$\hat{\theta} = \theta \Omega(\theta, \rho)^{-1}, \tag{15}$$

where the eigenfunction potential Ω is defined as $\Omega(\theta,\rho)_y = \rho^\dagger \theta$. We may now construct the binary Darboux transformation explicitly as

$$B_{\theta,\rho} = I - \theta \Omega(\theta,\rho)^{-1} \partial_y^{-1} \rho^\dagger \tag{16}$$

with its adjoint

$$B_{\theta,\rho}^{-\dagger} = I - \rho \Omega(\theta,\rho)^{-\dagger} \partial_y^{-1} \theta^\dagger. \tag{17}$$

Let $\phi_{[1]} = \phi$ and $\psi_{[1]} = \psi$ be two general eigenfunctions for the operators $L_{[1]} = L$, $M_{[1]} = M$ and the adjoint Lax operators $L_{[1]}^\dagger = L^\dagger$, $M_{[1]}^\dagger = M^\dagger$ respectively, where L and M are given by (11). We then define the binary Darboux transformations of the eigenfunctions ϕ and ψ as

$$\begin{aligned}
\phi_{[2]} &= B_{\theta,\rho}\left(\phi_{[1]}\right) = \phi_{[1]} - \theta_{[1]} \Omega\left(\theta_{[1]}, \rho_{[1]}\right)^{-1} \Omega\left(\phi_{[1]}, \rho_{[1]}\right), \\
\psi_{[2]} &= B_{\theta,\rho}^{-\dagger}\left(\psi_{[1]}\right) = \psi_{[1]} - \rho_{[1]} \Omega\left(\theta_{[1]}, \rho_{[1]}\right)^{-\dagger} \Omega\left(\theta_{[1]}, \psi_{[1]}\right)^\dagger,
\end{aligned}$$

with

$$\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}, \qquad \rho_{[2]} = \psi_{[2]}|_{\psi\to\rho_2}.$$

After $N \geqslant 1$ iterations, the N-fold BDTs are given below as

$$\begin{aligned} \phi_{[N+1]} &= B_{\theta,\rho}\left(\phi_{[N]}\right) = \phi_{[N]} - \theta_{[N]}\Omega\left(\theta_{[N]},\rho_{[N]}\right)^{-1}\Omega\left(\phi_{[N]},\rho_{[N]}\right), \\ \psi_{[N+1]} &= B^{-\dagger}_{\theta,\rho}\left(\psi_{[N]}\right) = \psi_{[N]} - \rho_{[N]}\Omega\left(\theta_{[N]},\rho_{[N]}\right)^{-\dagger}\Omega\left(\theta_{[N]},\psi_{[N]}\right)^{\dagger}, \end{aligned} \tag{18}$$

with

$$\theta_{[N]} = \phi_{[N]}|_{\phi\to\theta_N}, \qquad \rho_{[N]} = \psi_{[N]}|_{\psi\to\rho_N}, \tag{19}$$

where $L(\theta_i) = M(\theta_i) = 0$ and $L^{\dagger}(\rho_i) = M^{\dagger}(\rho_i) = 0$ such that $i = 1, \ldots, N$. By introducing the notations $\Theta = (\theta_1, \ldots, \theta_N)$ and $P = (\rho_1, \ldots, \rho_N)$, these eigenfunctions can be expressed in terms of quasigrammians

$$\phi_{[N+1]} = \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Theta & \boxed{\phi} \end{vmatrix}, \quad \psi_{[N+1]} = \begin{vmatrix} \Omega(\Theta,P)^{\dagger} & \Omega(\Theta,\psi)^{\dagger} \\ P & \boxed{\psi} \end{vmatrix}, \tag{20}$$

and

$$\Omega(\phi_{[N+1]},\psi_{[N+1]}) = \begin{vmatrix} \Omega(\Theta,P) & \Omega(\phi,P) \\ \Omega(\Theta,\psi) & \boxed{\Omega(\phi,\psi)} \end{vmatrix}. \tag{21}$$

3.3. Dimensional reductions of the binary Darboux transformation

In this subsection, we describe a dimensional reduction of the BDT from $(2+1)$ to $(1+1)$ dimensions by a separation of variables:

$$\begin{aligned} \phi(x,y,t) &= \phi^r(x,t)e^{\lambda y}, \qquad \theta(x,y,t) = \theta^r(x,t)e^{\Lambda y}, \\ \psi(x,y,t) &= \psi^r(x,t)e^{\mu y}, \qquad \rho(x,y,t) = \rho^r(x,t)e^{\Pi y}, \end{aligned} \tag{22}$$

where λ, μ are two constant scalars and Λ, Π are two $k \times k$ constant square matrices. Here, the superscript ^{r} denotes reduced functions. Then, the differential operators L and M in (11) become

$$L^r \quad = \quad \partial_x + \sum_{i=0}^{N} u_i\lambda^i, \qquad M^r = \partial_t + \sum_{i=0}^{N} v_i\lambda^i, \tag{23}$$

and the reduced eigenfunction θ^r satisfying

$$\theta^r_x + \sum_{i=0}^{N} u_i\theta^r\Lambda^i = 0, \qquad \theta^r_t + \sum_{i=0}^{N} v_i\theta^r\Lambda^i = 0. \tag{24}$$

Similarly we can write down the reduced adjoint eigenfunction ρ^r. It follows that the $y-$dependence of the potential Ω can also be made explicit by letting

$$\begin{aligned} \Omega(\theta,\rho) &= e^{\Pi^{\dagger}y}\Omega^r\left(\theta^r,\rho^r\right)e^{\Lambda y}, \\ \Omega(\phi,\rho) &= e^{\left(\Pi^{\dagger}+\lambda I\right)y}\Omega^r\left(\phi^r,\rho^r\right), \\ \Omega(\theta,\psi) &= \Omega^r\left(\theta^r,\psi^r\right)e^{\left(\Lambda+\mu^{\dagger}I\right)y}, \end{aligned} \tag{25}$$

where the reduced potential Ω^r satisfying the following conditions

$$\begin{aligned} &\Pi^{\dagger}\Omega^r\left(\theta^r,\rho^r\right) + \Omega^r\left(\theta^r,\rho^r\right)\Lambda = \rho^{r\dagger}\theta^r, \\ &\left(\Pi^{\dagger}+\lambda I\right)\Omega^r\left(\phi^r,\rho^r\right) = \rho^{r\dagger}\phi^r, \\ &\Omega^r\left(\theta^r,\psi^r\right)\left(\Lambda+\mu^{\dagger}I\right) = \psi^{r\dagger}\theta^r. \end{aligned} \tag{26}$$

Then the dimensionally reduced binary Darboux transformations are written as

$$\begin{aligned} B_{\theta^r,\rho^r} &= I - \theta^r \Omega^r \left(\theta^r, \rho^r\right)^{-1} \left(\Pi^\dagger + \lambda I\right)^{-1} \rho^{r\dagger}, \\ B^{-\dagger}_{\theta^r,\rho^r} &= I - \rho^r \Omega^r \left(\theta^r, \rho^r\right)^{-\dagger} \left(\Lambda^\dagger + \mu I\right)^{-1} \theta^{r\dagger}. \end{aligned} \tag{27}$$

The transformed operators

$$\begin{aligned} L^r_{[N+1]} &= B_{\theta^r,\rho^r} L^r_{[N]} B^{-1}_{\theta^r,\rho^r}, \\ M^r_{[N+1]} &= B_{\theta^r,\rho^r} M^r_{[N]} B^{-1}_{\theta^r,\rho^r} \end{aligned} \tag{28}$$

have generic eigenfunctions and adjoint eigenfunctions

$$\begin{aligned} \phi^r_{[N+1]} &= \phi^r_{[N]} - \theta^r_{[N]} \Omega^r \left(\theta^r_{[N]}, \rho^r_{[N]}\right)^{-1} \Omega^r \left(\phi^r_{[N]}, \rho^r_{[N]}\right), \\ \psi^r_{[N+1]} &= \psi^r_{[N]} - \rho^r_{[N]} \Omega^r \left(\theta^r_{[N]}, \rho^r_{[N]}\right)^{-\dagger} \Omega^r \left(\theta^r_{[N]}, \psi^r_{[N]}\right)^{\dagger}, \end{aligned} \tag{29}$$

with

$$\theta^r_{[N]} = \phi^r_{[N]}|_{\phi^r \to \theta_N}, \qquad \rho^r_{[N]} = \psi^r_{[N]}|_{\psi^r \to \rho^r_N}. \tag{30}$$

From now on, for notational simplicity, we omit the superscript ^{r} and consider only the reduced BDT.

4. Constructing quasigrammian solutions of the GI equation

In this section, we find out the effect of the BDT on the differential operator L given by (3). It can be easily checked that the corresponding results also hold for the operator M given by (4). We observe that the Lax operators L and M are both anti-Hermitian, i.e., $L^\dagger = -L$ and $M^\dagger = -M$. Due to this property, we choose the adjoint eigenfunction $\rho = \theta$ and the constant matrix $\Pi = \Lambda$. Furthermore, under the BDT, the operator L is transformed to a new operator $\hat{L}$:

$$L \to \hat{L} = B_\theta L B_\theta^{-1}, \tag{31}$$

where

$$B_\theta = I - \theta \Omega(\theta, \theta)^{-1} \left(\Lambda^\dagger + \lambda I\right)^{-1} \theta^\dagger. \tag{32}$$

It follows that

$$\hat{R} = R + \left[J, \theta \Omega(\theta, \theta)^{-1} \theta^\dagger\right], \tag{33}$$

where the potential $\Omega(\theta, \theta)$ satisfies

$$\Omega(\theta, \theta)\Lambda + \Lambda^\dagger \Omega(\theta, \theta) = \theta^\dagger \theta, \tag{34}$$

in which the eigenfunction θ and the diagonal constant matrix Λ are given by (8). For notational convenience let us introduce a 2×2 matrix Q such that $R = [J, Q]$, of the form

$$Q = \frac{1}{2i} \begin{pmatrix} & q \\ -r & \end{pmatrix}, \tag{35}$$

where the entries left blank are arbitrary and do not contribute to R. From (33), it follows that $\hat{R} = [J, \hat{Q}]$ where

$$\hat{Q} = Q - \theta \Omega(\theta, \theta)^{-1} \theta^\dagger \tag{36}$$

which can be written in a quasigrammian form as

$$\hat{Q} = Q + \begin{vmatrix} \Omega(\theta, \theta) & \theta^\dagger \\ \theta & \boxed{0_2} \end{vmatrix}. \tag{37}$$

4.1. Iteration of the binary Darboux transformation

Let $Q_{[1]} = Q$, $Q_{[2]} = \hat{Q}$, $\theta_{[1]} = \theta_1 = \theta$ and $\Lambda_1 = \Lambda$ such that $\Lambda_1 = \mathrm{diag}(\lambda_1, -\lambda_1)$. Then, the solution (36) can be written as

$$Q_{[2]} = Q_{[1]} - \theta_{[1]}\Omega(\theta_{[1]}, \theta_{[1]})^{-1}\theta_{[1]}^{\dagger}. \tag{38}$$

After N repeated applications of the reduced BDT (32), we have

$$Q_{[N+1]} = Q_{[N]} - \theta_{[N]}\Omega(\theta_{[N]}, \theta_{[N]})^{-1}\theta_{[N]}^{\dagger} \tag{39}$$

with $\theta_{[N]} = \phi_{[N]}|_{\phi\to\theta_N}$, where the general eigenfunction $\phi_{[N]}$ is given by (29) as

$$\phi_{[N+1]} = \phi_{[N]} - \theta_{[N]}\Omega(\theta_{[N]}, \theta_{[N]})^{-1}\Omega(\phi_{[N]}, \theta_{[N]}). \tag{40}$$

Let $\phi_1, \ldots, \phi_N$ be a particular set of eigenfunctions of the linear operators L, M given by (3)−(4), and define $\Theta = (\theta_1, \ldots, \theta_N)$ for the 2×2 matrices θ_i $(i = 1, \ldots, N)$ such that

$$\theta_i = \begin{pmatrix} \varphi_i & \varphi_i \\ \psi_i & -\psi_i \end{pmatrix}. \tag{41}$$

We may now express $Q_{[N+1]}$ and $\phi_{[N+1]}$ in quasigrammian form as

$$Q_{[N+1]} = Q + \begin{vmatrix} \Omega(\Theta,\Theta) & \Theta^{\dagger} \\ \Theta & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} \end{vmatrix}, \quad \phi_{[N+1]} = \begin{vmatrix} \Omega(\Theta,\Theta) & \Omega(\phi,\Theta) \\ \Theta & \boxed{\phi} \end{vmatrix}. \tag{42}$$

Let us define the $2\times 2N$ matrix Θ as

$$\Theta = \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{43}$$

where ϕ and ψ denote the row vectors

$$\phi = (\varphi_1, \varphi_1, \ldots, \varphi_N, \varphi_N), \tag{44}$$
$$\psi = (\psi_1, -\psi_1, \ldots, \psi_N, -\psi_N). \tag{45}$$

Then it follows from (42)

$$Q_{[N+1]} = Q + \begin{pmatrix} \begin{vmatrix} \Omega(\Theta,\Theta) & \phi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta,\Theta) & \psi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix} \\ \begin{vmatrix} \Omega(\Theta,\Theta) & \phi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix} & \begin{vmatrix} \Omega(\Theta,\Theta) & \psi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix} \end{pmatrix}. \tag{46}$$

By substituting (35) into (46), we obtain quasigrammian expressions for q and r, namely

$$q_{[N+1]} = q + 2i\begin{vmatrix} \Omega(\Theta,\Theta) & \psi^{\dagger} \\ \phi & \boxed{0} \end{vmatrix}, \tag{47}$$

$$r_{[N+1]} = r - 2i\begin{vmatrix} \Omega(\Theta,\Theta) & \phi^{\dagger} \\ \psi & \boxed{0} \end{vmatrix}. \tag{48}$$

We have constructed a quasigrammian solution $q_{[N+1]}$ of the GI equation (1), along with its complex conjugate $r_{[N+1]}$. It is necessary to show that these two expressions are consistent. That is, that the pair are indeed complex conjugate to each other. The proof of this is given in Section 4.2.

4.2. Proof of consistency

In the expressions (47)-(48), the potential $\Omega(\Theta,\Theta)$ is $2N\times 2N$ matrix satisfying the relation

$$\Omega(\Theta,\Theta)\Lambda+\Lambda^{\dagger}\Omega(\Theta,\Theta)=\Theta^{\dagger}\Theta, \tag{49}$$

where Λ is $2N\times 2N$ constant matrix such that $\Lambda=\operatorname{diag}(\Lambda_1,\dots,\Lambda_N)$. Solving this relation for $\Omega(\Theta,\Theta)$, we obtain the explicit expression

$$\Omega(\Theta,\Theta)=\begin{pmatrix}\Omega(\theta_1,\theta_1) & \Omega(\theta_2,\theta_1) & \dots & \Omega(\theta_N,\theta_1)\\ \Omega(\theta_1,\theta_2) & \Omega(\theta_2,\theta_2) & \dots & \Omega(\theta_N,\theta_2)\\ \vdots & \vdots & & \vdots\\ \Omega(\theta_1,\theta_N) & \Omega(\theta_2,\theta_N) & \dots & \Omega(\theta_N,\theta_N)\end{pmatrix}, \tag{50}$$

where $\Omega(\theta_i,\theta_j)$ is 2×2 potential satisfying the relation

$$\Omega(\theta_i,\theta_j)\Lambda_i+\Lambda_j^{\dagger}\Omega(\theta_i,\theta_j)=\theta_j^{\dagger}\theta_i, \tag{51}$$

in which $\Lambda_k=\operatorname{diag}(\lambda_k,-\lambda_k)$ and $i,j,k\in\{1,2,\dots,N\}$. It follows from this relation that the potential Ω is written explicitly as

$$\Omega(\theta_i,\theta_j)=\begin{pmatrix}f_{ij} & -g_{ij}\\ g_{ij} & -f_{ij}\end{pmatrix}, \tag{52}$$

where f_{ij} and g_{ij} are two scalar functions such that

$$\begin{aligned} f_{ij} &= \frac{1}{\lambda_i+\lambda_j^*}\left(\varphi_i\varphi_j^*+\psi_i\psi_j^*\right),\\ g_{ij} &= \frac{1}{\lambda_i-\lambda_j^*}\left(\varphi_i\varphi_j^*-\psi_i\psi_j^*\right).\end{aligned}$$

Here we observe that the functions f_{ij} and g_{ij} hold the relations $f_{ij}^*=f_{ji}$ and $g_{ij}^*=-g_{ji}$, where $i,j=1,\dots,N$. Then the 2×2 potential $\Omega(\theta_i,\theta_j)$ satisfies the symmetry condition

$$\Omega(\theta_i,\theta_j)^{\dagger}=\Omega(\theta_j,\theta_i), \tag{53}$$

and the $2N\times 2N$ matrix potential $\Omega(\Theta,\Theta)$, as given by (50), is self-adjoint,

$$\Omega(\Theta,\Theta)^{\dagger}=\Omega(\Theta,\Theta). \tag{54}$$

By using the symmetry property (54) of the potential Ω, it is easily seen that the expressions given in (47) and (48) are complex conjugate. This completes the proof.

4.3. Explicit quasideterminant solutions

In this subsection, we consider the quasigrammian solution (47) of the GI equation (1), in which we derive the following quasideterminant solution

$$q_{[N+1]}=q+2i\begin{vmatrix}F_{11} & F_{12} & \dots & F_{1n} & \varphi_1\\ F_{21} & F_{22} & \dots & F_{2n} & \varphi_2\\ \vdots & \vdots & & \vdots & \vdots\\ F_{n1} & F_{n2} & \dots & F_{nn} & \varphi_n\\ \psi_1^* & \psi_2^* & \dots & \psi_n^* & \boxed{0}\end{vmatrix}, \tag{55}$$

where

$$F_{ij} \;=\; \frac{1}{\Lambda_{ij}}\left(\lambda_i\psi_i\psi_j^* - \lambda_j^*\varphi_i\varphi_j^*\right), \tag{56}$$

in which $\Lambda_{ij} = \lambda_i^2 - \lambda_j^{*2}$. For one-fold BDT ($N = 1$), the solution (55) yields

$$q_{[2]} \;=\; q + 2i\begin{vmatrix} F_{11} & \varphi_1 \\ \psi_1^* & \boxed{0} \end{vmatrix}, \tag{57}$$

where

$$F_{11} \;=\; \frac{1}{\Lambda_{11}}\left(\lambda_1|\psi_1|^2 - \lambda_1^*|\varphi_1|^2\right),$$

in which $\Lambda_{11} = \lambda_1^2 - \lambda_1^{*2}$ such that $\Lambda_{11} \in i\mathbb{R}$. Thus, we obtain a new explicit solution for the GI equation (1), namely

$$q_{[2]} \;=\; q - 2i\Lambda_{11}\frac{\varphi_1\psi_1^*}{\lambda_1|\psi_1|^2 - \lambda_1^*|\varphi_1|^2}, \tag{58}$$

where $\Phi_1 = (\varphi_1, \psi_1)^T$ is a solution of the eigenvalue problems $L(\Phi_1) = M(\Phi_1) = 0$, in which L and M are given by (3)-(4). For two-fold BDT ($N = 2$), the solution (55) gives us

$$q_{[3]} \;=\; q + 2i\begin{vmatrix} F_{11} & F_{12} & \varphi_1 \\ F_{21} & F_{22} & \varphi_2 \\ \psi_1^* & \psi_2^* & \boxed{0} \end{vmatrix}, \tag{59}$$

where the scalar functions F_{ij} are given by (56) for $i, j = 1, 2$. This solution can be written as

$$q_{[3]} \;=\; q - \frac{2i}{|F|}\left(\varphi_1\psi_1^*F_{22} + \varphi_2\psi_2^*F_{11} - \varphi_1\psi_2^*F_{21} - \varphi_2\psi_1^*F_{12}\right), \tag{60}$$

where $|F| = F_{11}F_{22} - F_{12}F_{21}$. Here (60) is the explicit solution of the GI equation (1) which yields

$$q_{[3]} = q - 2i\frac{\Lambda_{11}\Lambda_{22}\left(\Lambda_{12}\varphi_1\psi_2^*h_{21} - \Lambda_{12}^*\varphi_2\psi_1^*h_{12}\right) + |\Lambda_{12}|^2\left(\Lambda_{11}\varphi_1\psi_1^*h_{22} + \Lambda_{22}\varphi_2\psi_2^*h_{11}\right)}{\Lambda_{11}\Lambda_{22}h_{12}h_{21} + |\Lambda_{12}|^2 h_{11}h_{22}}, \tag{61}$$

where $\Lambda_{11} = \lambda_1^2 - \lambda_1^{*2}$, $\Lambda_{22} = \lambda_2^2 - \lambda_2^{*2}$, $\Lambda_{12} = \lambda_1^2 - \lambda_2^{*2}$ such that $\Lambda_{11}, \Lambda_{22} \in i\mathbb{R}$ and

$$\begin{aligned} h_{11} &= \lambda_1|\psi_1|^2 - \lambda_1^*|\varphi_1|^2, \qquad h_{12} = \lambda_1\psi_1\psi_2^* - \lambda_2^*\varphi_1\varphi_2^*, \\ h_{21} &= \lambda_2\psi_1^*\psi_2 - \lambda_1^*\varphi_1^*\varphi_2, \qquad h_{22} = \lambda_2|\psi_2|^2 - \lambda_2^*|\varphi_2|^2, \end{aligned}$$

in which $\Phi_1 = (\varphi_1, \psi_1)^T$ and $\Phi_2 = (\varphi_2, \psi_2)^T$ are two solutions of the eigenvalue problems $L(\Phi_1) = M(\Phi_1) = 0$ and $L(\Phi_2) = M(\Phi_2) = 0$ respectively, where L and M are given by (3)-(4).

5. Particular solutions

Let us consider the spectral problem $L(\Phi_j) = M(\Phi_j) = 0$ with eigenvalue λ_j, where $\Phi_j = (\varphi_j, \psi_j)^T$ and L, M are given by (3)-(4) so that

$$\begin{aligned} {\Phi_j}_x + J\Phi_j\lambda_j^2 - R\Phi_j\lambda_j + \frac{1}{2}qrJ\Phi_j &= 0, \\ {\Phi_j}_t + 2J\Phi_j\lambda_j^4 - 2R\Phi_j\lambda_j^3 + qrJ\Phi_j\lambda_j^2 + U\Phi_j\lambda_j + W\Phi_j &= 0, \end{aligned} \tag{62}$$

where $j = 1, \ldots, N$.

5.1. Solutions for zero seed

For $q = r = 0$, the first-order linear system (62) becomes

$$\begin{aligned} \Phi_{j_x} + J\Phi_j\lambda_j^2 &= 0, \\ \Phi_{j_t} + 2J\Phi_j\lambda_j^4 &= 0, \end{aligned} \tag{63}$$

which is solved for the eigenfunction $\Phi_j = (\varphi_j, \psi_j)^T$ as

$$\Phi_j = \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix} = \begin{pmatrix} e^{-i\lambda_j^2\left(x+2\lambda_j^2 t\right)} \\ e^{i\lambda_j^2\left(x+2\lambda_j^2 t\right)} \end{pmatrix} \tag{64}$$

where $j = 1, \ldots, N$.

Case 1 ($N = 1$): One-soliton solution

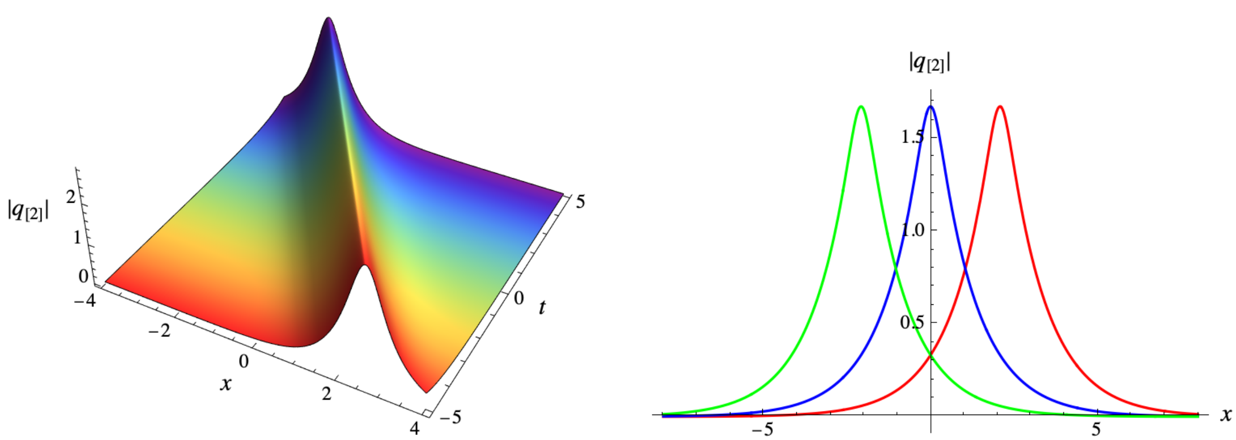
Substituting the eigenfunction $(\varphi_1, \psi_1)^T$ given by (64) into the solution (58) with the choice $\lambda_1 = \xi + i\eta$, leads to the one-soliton solution of the GI equation (1)

$$q_{[2]} = -4i\xi\eta \frac{e^{-2i\left[\left(\xi^2-\eta^2\right)x+2\left(\xi^4-6\xi^2\eta^2+\eta^4\right)t\right]}}{\eta\cosh\left(4\xi\eta\left[x+4\left(\xi^2-\eta^2\right)t\right]\right)+i\xi\sinh\left(4\xi\eta\left[x+4\left(\xi^2-\eta^2\right)t\right]\right)}, \tag{65}$$

which yields

$$\left|q_{[2]}\right|^2 = 16\frac{\xi^2\eta^2}{\eta^2\cosh^2\left(4\xi\eta\left[x+4\left(\xi^2-\eta^2\right)t\right]\right)+\xi^2\sinh^2\left(4\xi\eta\left[x+4\left(\xi^2-\eta^2\right)t\right]\right)}. \tag{66}$$

This solution is plotted in Fig. 1.



(*a*) (*b*)

Fig. 1. (Color online) One soliton solution $|q_{[2]}|$ of the GI equation (1) when $\xi = 0.7, \eta = 0.6$. Figure (*a*) describes its surface and (*b*) gives its profiles at different times $t = -4$ (red), $t = 0$ (blue), $t = 4$ (green).

Case 2 ($N = 2$): Two-soliton solution

Let $\lambda_2 = i\lambda_1$ with the choice $\lambda_1 = \xi + i\eta$ under the condition $\xi\eta \neq 0$. By substituting the eigenfunctions $(\varphi_1, \psi_1)^T$ and $(\varphi_2, \psi_2)^T$ given by (64) into (61), we obtain the two-soliton solution of the GI equation (1) as follows:

$$\left|q_{[3]}\right|^2 = K^2\,\frac{8\xi^2\eta^2\left(\xi^2-\eta^2\right)+\left(\xi^2+\eta^2\right)^3\cosh\alpha\cosh\gamma-\left(\xi^2+\eta^2\right)H_1\sin\beta+2\xi\eta H_2\cos\beta}{\left[\left(\xi^2-\eta^2\right)^2\sinh\alpha+\left(\xi^4-\eta^4\right)\sinh\gamma\right]^2+4\xi^2\eta^2\left[\left(\xi^2+\eta^2\right)\cosh\gamma+2\xi\eta\cos\beta\right]^2}, \tag{67}$$

where H_1 and H_2 are hyperbolic functions such that

$$
\begin{aligned}
H_1 &= \left(\xi^4-\eta^4\right)\sinh\alpha+\left(\xi^4-6\xi^2\eta^2+\eta^4\right)\sinh\gamma, \\
H_2 &= \left(\xi^2+\eta^2\right)^2\cosh\alpha+2\left(\xi^4-\eta^4\right)\cosh\gamma,
\end{aligned}
$$

in which

$$
\begin{aligned}
\alpha &= 8\xi\eta x, \\
\beta &= 4\left(\xi^2-\eta^2\right)x, \\
\gamma &= 32\xi\eta\left(\xi^2-\eta^2\right)t, \\
K &= 8\frac{\xi\eta\left(\xi^2-\eta^2\right)}{\xi^2+\eta^2}.
\end{aligned}
$$

By choosing appropriate parameters, the two soliton solution of the GI equation (1) is plotted in Fig. 2.

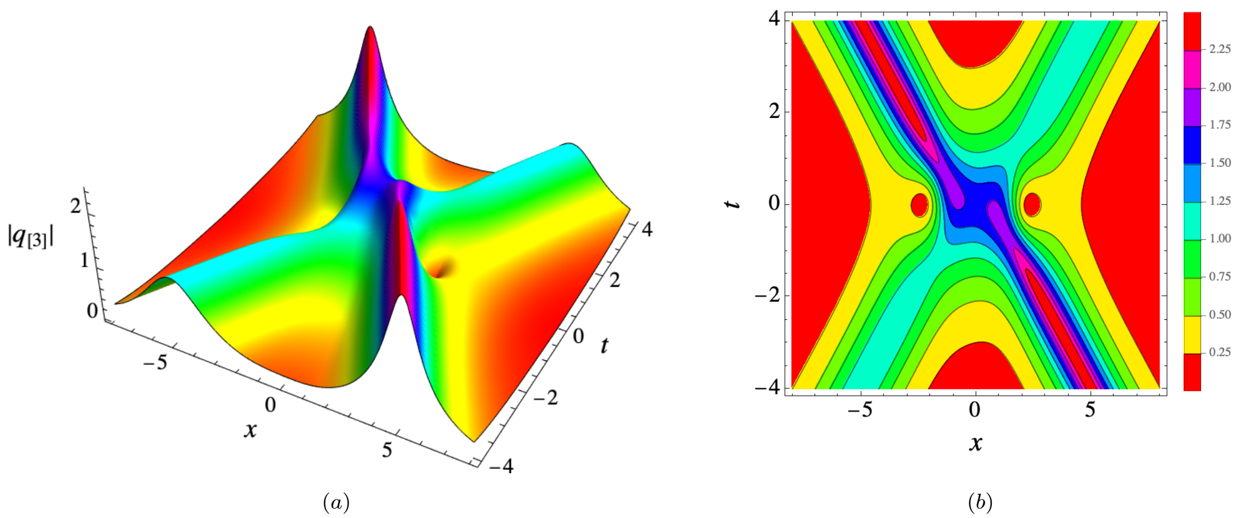


Fig. 2. (Color online) Two soliton solution $|q_{[3]}|$ of the GI equation (1) when $\xi = 0.6$ and $\eta = 0.3$. (a) Surface diagram. (b) Contour diagram.

5.2. Solutions for non-zero seed

For $q, r \neq 0$ with $r = q^*$,

$$
q = ke^{-i\mu} \tag{68}
$$

is a periodic solution of the GI equation (1), where $\mu = ax + bt$ in which $a, k \in \mathbb{R}$ and $b = a^2 + ak^2 - \frac{1}{2}k^4$. We use (68) as a seed solution for the application of the binary Darboux transformation. Substituting (68) into the linear system (62) and solving for the eigenfunction $\Phi_j = (\varphi_j, \psi_j)^T$, we obtain

$$
\begin{aligned}
\varphi_j(x,t,\lambda_j) = &\quad e^{-\frac{1}{2}i\mu}\left(c_1 e^{\frac{1}{2}i\gamma_j}+c_2 e^{-\frac{1}{2}i\gamma_j}\right), \\
\psi_j(x,t,\lambda_j) = &\quad \frac{ie^{\frac{1}{2}i\mu}}{2k\lambda_j}\left(\widetilde{c_1} e^{\frac{1}{2}i\gamma_j}+\widetilde{c_2} e^{-\frac{1}{2}i\gamma_j}\right),
\end{aligned} \tag{69}
$$

where $\gamma_j = s_j[x + (a + 2\lambda_j^2)t]$, $\widetilde{c_1} = c_1(k^2 - a + 2\lambda_j^2 + s_j)$ and $\widetilde{c_2} = c_2(k^2 - a + 2\lambda_j^2 - s_j)$ in which $s_j = \sqrt{k^4 - 2ak^2 + a^2 + 4\lambda_j^4 - 4a\lambda_j^2}$ and c_1, c_2 are integration constants ($j = 1, \ldots, N$).

In the case $N=1$, inserting the eigenfunction $\Phi_1=(\varphi_1,\psi_1)^T$ (69) into (58) with the choice $\lambda_1=\xi+i\eta$ $(\xi\eta\neq 0)$ yields

$$q_{[2]}=ke^{-i\left[ax+(a^2+ak^2-\frac{1}{2}k^4)t\right]}\left(1-16i\xi\eta\,\frac{a_1e^{\frac{1}{2}i\gamma}+a_2e^{-\frac{1}{2}i\gamma}+a_3e^{\frac{1}{2}i\widetilde{\gamma}}+a_4e^{-\frac{1}{2}i\widetilde{\gamma}}}{b_1e^{\frac{1}{2}i\gamma}+b_2e^{-\frac{1}{2}i\gamma}+b_3e^{\frac{1}{2}i\widetilde{\gamma}}+b_4e^{-\frac{1}{2}i\widetilde{\gamma}}}\right),\tag{70}$$

where

$$\begin{aligned}\gamma(x,t)&=(s-s^*)x+[(s-s^*)a+2(\lambda^2s-\lambda^{*2}s^*)]t,\\ \widetilde{\gamma}(x,t)&=(s+s^*)x+[(s+s^*)a+2(\lambda^2s+\lambda^{*2}s^*)]t,\end{aligned}$$

and

$$\begin{aligned}&a_1=|c_1|^2e_1^*,\ a_2=|c_2|^2e_2^*,\ a_3=c_1c_2^*e_2^*,\ a_4=c_1^*c_2e_1^*,\ b_1=|c_1|^2\left(|e_1|^2-4k^2\lambda^{*2}\right),\\ &b_2=|c_2|^2\left(|e_2|^2-4k^2\lambda^{*2}\right),\ b_3=c_1c_2^*\left(e_1e_2^*-4k^2\lambda^{*2}\right),\ b_4=c_1^*c_2\left(e_1^*e_2-4k^2\lambda^{*2}\right),\end{aligned}$$

in which $s=\sqrt{k^4-2ak^2+a^2+4\lambda^4-4a\lambda^2}$, $e_1=k^2-a+2\lambda^2+s$, $e_2=k^2-a+2\lambda^2-s$.

Case 3 ($N=1$): Breather solution

For the choices $k^2=2a$ and $c_1=c_2$ in (70), we have the breather solution of the GI equation (1) as

$$q_{[2]}=\sqrt{2a}\ e^{-i(ax+a^2t)}\ \frac{m_1\cosh\alpha+m_2\sinh\alpha+m_3(\sin\beta-i\cos\beta)}{n_1\cosh\alpha+n_2\sinh\alpha+8\xi\eta(a\sin\beta+i\cos\beta)},\tag{71}$$

where $\alpha(x,t)=4\xi\eta\left[x+4\left(\xi^2-\eta^2\right)t\right]$, $\beta(x,t)=\left[a+2\eta^2-2\xi^2\right]x+\left[a^2-4\left(\xi^4-6\xi^2\eta^2+\eta^4\right)\right]t$ and

$$\begin{aligned}m_1&=a^2-4a\left(\xi^2-\eta^2\right)+4\left(\xi^4-6\xi^2\eta^2+\eta^4\right)-4i\left(\xi^2-\eta^2\right),\\ m_2&=a^2-4\left(\xi^4-6\xi^2\eta^2+\eta^4\right)-2i\left(a+2\eta^2-2\xi^2\right),\\ m_3&=16\xi\eta\left(\xi^2-\eta^2-2i\xi\eta\right),\\ n_1&=a^2-4a\left(\xi^2-\eta^2\right)+4\left(\xi^2+\eta^2\right)^2+8i\xi\eta,\\ n_2&=a^2-4\left(\xi^2+\eta^2\right)^2.\end{aligned}$$

Fig. 3 shows the dynamical evolution of the breather solution of the GI equation (1).

Case 4 ($N=1$): Breather and Rogue wave solutions

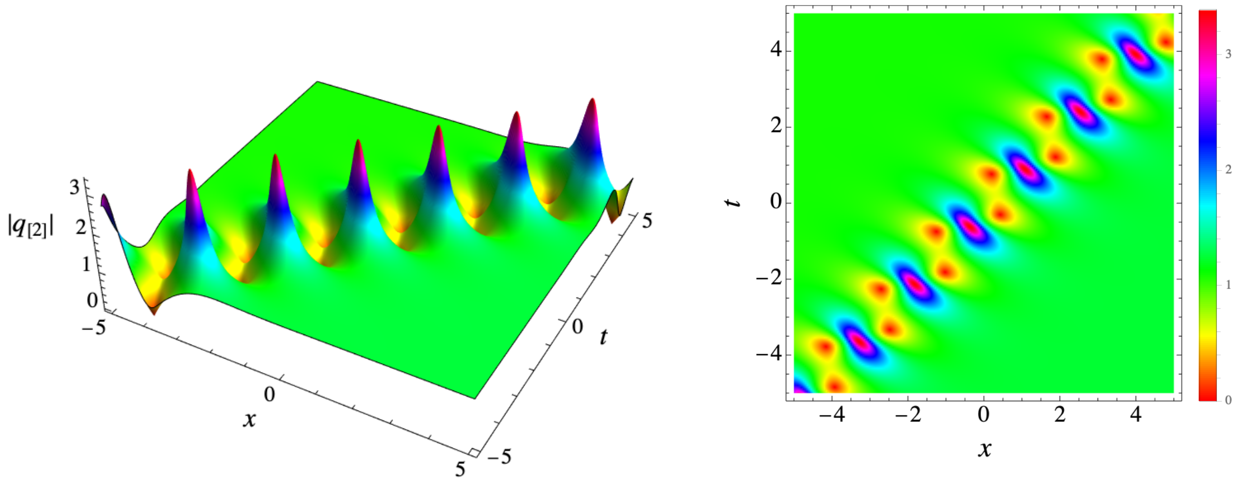
For simplicity, let $c_2=-c_1$ and $a=2\left(\xi^2-\eta^2\right)$ so that Im $(k^4-2ak^2+a^2+4\lambda^4-4a\lambda^2)=0$ in (70). Then we obtain the breather solution of the GI equation (1) in the following form

$$q_{[2]}=ke^{-2i\tilde{\mu}}\left[1+\frac{8\xi\eta\left[\left(4\xi\eta+ik^2\right)(\cosh\alpha-\cos\beta)-s(\sin\beta+i\sinh\alpha)\right]}{k^2\left[s\sinh\alpha-(k^2-2a+4i\xi\eta)\cosh\alpha\right]+4\xi\eta\left[s\sin\beta+(4\xi\eta+ik^2)\cos\beta\right]}\right],\tag{72}$$

where

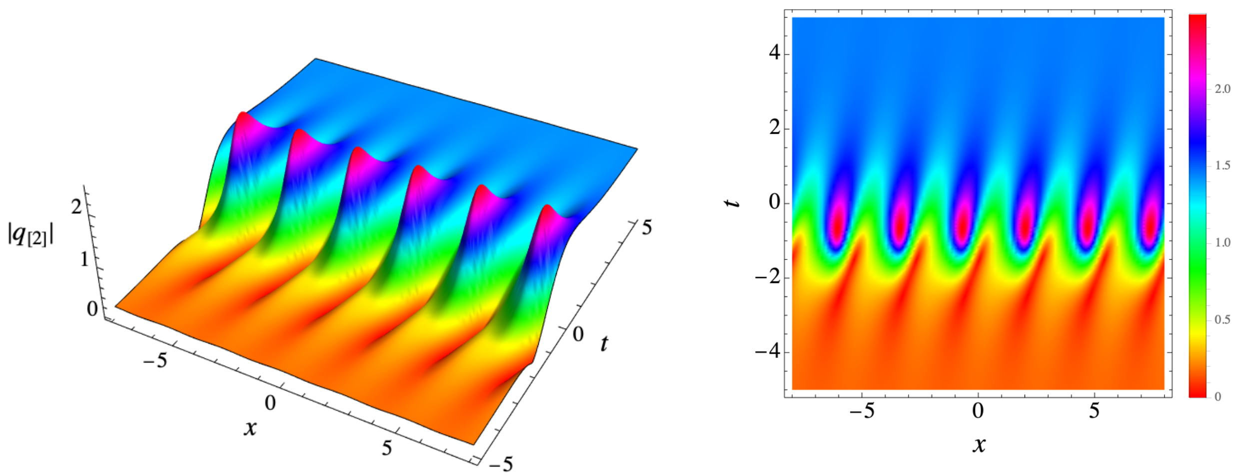
$$\begin{aligned}\tilde{\mu}(x,t)&=\left(\xi^2-\eta^2\right)x+\left[2\left(\xi^2-\eta^2\right)^2+\left(\xi^2-\eta^2\right)k^2-\frac{1}{4}k^4\right]t,\\ \alpha(t)&=4\xi\eta st,\\ \beta(x,t)&=s\left[x+4\left(\xi^2-\eta^2\right)t\right],\end{aligned}$$

in which $s=\sqrt{k^4+4\left(\eta^2-\xi^2\right)k^2-16\xi^2\eta^2}$. By choosing appropriate parameters, the breather solution of the GI equation (1) is plotted in Fig. 4.



(a) (b)

Fig. 3. (Color online) Breather solution $|q_{[2]}|$ of the GI equation (1) when $a = 1$, $\xi = 0.5$ and $\eta = 0.7$. (a) Surface diagram. (b) Density diagram.



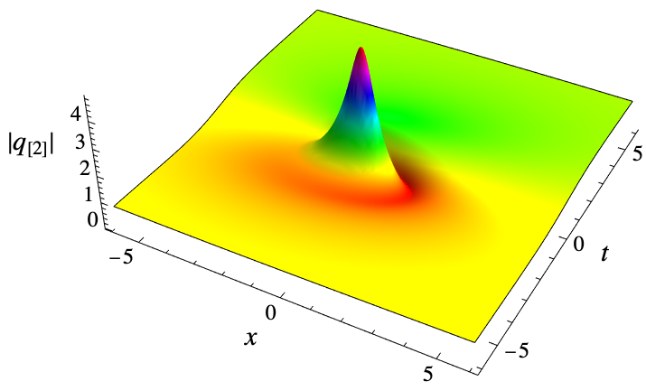
(a) (b)

Fig. 4. (Color online) Breather solution $|q_{[2]}|$ of the GI equation (1) when $k = 1.5$, $\xi = 0.3$ and $\eta = -0.4$. (a) Surface diagram. (b) Density diagram.

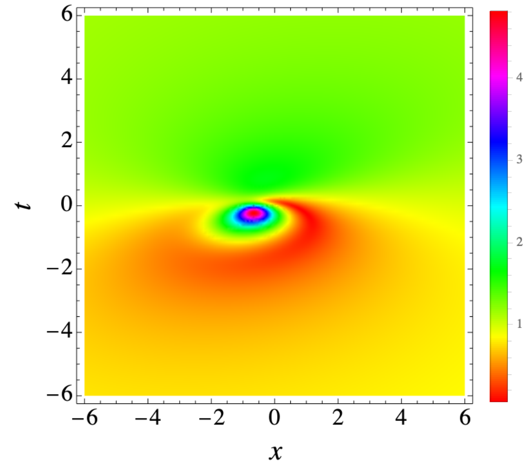
To derive a rogue wave solution of the Gerdjikov-Ivanov equation (1), we shall use the breather solution of the GI equation given by (72). Here we shall apply the Taylor expansion approach in order to construct the rogue wave solution of the GI equation. The Taylor expansion of the breather solution (72) with limit $k \to -2\xi$ gives us the first-order rogue wave solution of the GI equation as follows

$$q_{[2]} = 2\xi e^{-2i\tilde{\mu}} \left(1 + \frac{32\xi^2\eta\,(\xi - i\eta)\,t - 2}{1 + 8\xi^2\eta(\eta + i\xi)\,[x^2 + 8\zeta xt + 16\,(\xi^2\zeta + \eta^4)\,t^2] - 4\xi\eta\,[x + 4\,(\xi^2 + \zeta)\,t]}\right), \tag{73}$$

where $\tilde{\mu} = \left(\xi^2 - \eta^2\right)x + 2\left(\xi^4 - 4\xi^2\eta^2 + \eta^4\right)t$ and $\zeta = \xi^2 - \eta^2$. This first-order rogue wave solution of the GI equation is shown in Fig. 5.



(a)



(b)

Fig. 5. (Color online) The first-order rogue wave solution $|q_{[2]}|$ of the GI equation (1) when $\xi = -0.5$ and $\eta = 0.6$. (a) Surface diagram. (b) Density diagram.

6. Conclusion and discussion

In this article, the exact quasigrammian solutions of the GI equation have been constructed using the BDT method. Then we have showed that these quasigrammian solutions are written in terms of quasideterminants. As particular examples, we have presented the multi-soliton and breather solutions of the GI equation for zero and non-zero seeds. In addition to these particular solutions, the first-order rogue wave solution of the GI equation has been constructed explicitly from the breather solution given by (72). All these particular solutions are plotted in the figures $1-5$ with the chosen parameters.

We should emphasise that the method presented in this paper allows us to construct explicit solutions of various other integrable equations such as [14, 39, 52, 51]. Morever, we should also point out that we have constructed the first-order rogue wave solution of the GI equation via the Taylor expansion method. In general, the DT cannot be directly used to construct rational solutions for evolution equations. In [16], Gue *et al* proposed a simple method (the modified DT) which can be applied to the GI equation for constructing the higher-order rogue wave solutions [17, 32].

In the current work, we have presented the first-order rogue wave solution of the GI equation. In our future work, we would like to construct the multi-rogue wave solutions for integrable equations. In addition to this, we would also like to study *integrable nonlocal equations*. In 2013, by using the symmetry reduction method, Ablowitz and Musslimani [1] proposed a nonlocal nonlinear Schrödinger equation

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,t) = 0.$$

The authors proved that this equation is integrable. From then on, integrable nonlocal systems have become one of the most popular research topics in mathematical physics. Many researchers have made significant contributions to the study of this research area. These scholars have constructed exact solutions of many nonlocal equations by using different techniques such as Inverse Scattering transformation [2, 25, 33, 34, 53], Darboux transformation [43, 44, 45, 50, 54] and Hirota bilinear method [6, 18, 30, 31, 41]. We believe that the idea presented in our paper can be used to obtain explicit solutions to integrable nonlocal equations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The author would like to thank the anonymous reviewers and the editor of this journal for their valuable comments and suggestions which improved the quality of this manuscript.

References

[1] M.J. Ablowitz and Z.H. Musslimani, Integrable nonlocal nonlinear Schrödinger equation, Phys. Rev. Lett. **110** (2013), 064105.

[2] M.J. Ablowitz, Z.H. Musslimani, Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation, Nonlinearity **29** (2016) 915-946.

[3] G.P. Agrawal, Nonlinear Fibers Optics (Academic, New York, 2001).

[4] D. J. Benney and G. J. Roskes, Wave instabilities, Stud. Appl. Math. **48** (1969), 377–385.

[5] H.H. Chen, Y.C. Lee and C.S. Liu, Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method, Phys. Scr. **20** (1979), 490–492.

[6] J.C. Chen, Q.X. Yan, H. Zhang, Multiple bright soliton solutions of a reverse-space nonlocal nonlinear Schrödinger equation, Appl. Math. Lett. **106** (2020), 106375.

[7] P.A. Clarkson and J.A. Tuszynski, Exact solutions of the multidimensional derivative nonlinear Schrödinger equation for many-body systems near criticality, J. Phys. A: Math. Gen. **23** (1990), 4269–4288.

[8] G. Darboux, Comptes Rendus de l'Acadmie des Sciences **94** (1882), 1456–1459.

[9] E. Fan, Darboux transformation and soliton-like solutions for the Gerdjikov–Ivanov equation, J. Phys.A: Math. Gen. **33** (2000), 6925–6933.

[10] I. Gelfand and V. Retakh, Determinants of the matrices over noncomutative rings, Funct. Anal. App. **25** (1991), 91–102.

[11] I. Gelfand, S. Gelfand, V. Retakh, and R. L. Wilson, Quasideterminants, Adv.Math. **193** (2005), 56–141.

[12] C.R. Gilson and J.J.C. Nimmo, On a direct approach to quasideterminant solutions of a noncommutative KP equation, J. Phys. A: Math. Theor. **40** (2007), 3839–3850.

[13] C.R. Gilson, M. Hamanaka and J.C. Nimmo, Bäcklund transformations and the Atiyah-Ward ansatz for non-commutative anti-self-dual Yang-Mills equations, Proc. R. Soc. A **465** (2009), 2613–2632.

[14] C.R. Gilson and S.R. Macfarlane, Dromion solutions of noncommutative Davey-Stewartson equations, J. Phys. A: Math. Theor. **42** (2009), 235202.

[15] V.S. Gerdjikov and M.I. Ivanov, A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures, Bulgar. J. Phys. **10** (1983), 130–143.

[16] B. Guo, L. M. Ling, and Q. P. Liu, Nonlinear Schrödinger Equation: Generalized Darboux Transformation and Rogue Wave Solutions, Phys. Rev. E **85** (2012), 026607.

[17] L. Guo, Y. Zhang, S. Xu, W. Zhiwei and J. He, The higher order rogue wave solutions of the Gerdjikov-Ivanov equation, Phys. Scr. **89** (2014), 035501.

[18] M. Gürses, A. Pekcan, Nonlocal nonlinear Schrödinger equations and their soliton solutions, J. Math. Phys. **59** (2018), 051501.

[19] M. Hamanaka, Noncommutative solitons and quasideterminants, *Phys. Scr.* **89** (2014), 038006.

[20] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers I. Anomalous dispersion, Appl. Phy. Lett. **23** (1973), 142.

[21] A. Hasegawa and F. Tappert, Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers II. Normal dispersion, Appl. Phy. Lett. **23** (1973) , 171.

[22] M. Hassan, Darboux transformation of the generalized coupled dispersionless integrable system, J.Phys.A: Math. Theor. **42** (2009), 065203.

[23] Y.H. Ichikawa, K. Konno, M. Wadati and H. Sanuki, Spiky soliton in circular polarized Alfv*é*n wave, J. Phys.Soc. Japan **48** (1980), 279–286.

[24] M. Irfan, Lax pair representation and Darboux transformation of noncommutative Painlevé's second equation, J. Geom. Phys., **62** (2012), 1575–1582.

[25] J.L. Ji, Z.N. Zhu, Soliton solutions of an integrable nonlocal modified Korteweg-de Vries equation through inverse scattering transform, J. Math. Anal. Appl. **453** (2017) 973-984.

[26] R.S. Johnson, On the modulation of water waves in the neighbourhood of $kh \approx 1.363$, Proc. Roy.Soc.London Ser.A **357** (1977), 131–141.

[27] D.J. Kaup and A.C. Newell, An exact solution for derivative nonlinear Schrödinger equation, J.Math.Phys. **19** (1978), 798–801.

[28] Y. Kodama, Optical solitons in a monomode fiber, J. Statist. Phys. **39** (1985), 597–614.

[29] K. Kondo, Sato-theoretic construction of solutions to noncommutative integrable systems, Phys. Lett. A, **375** (2011), 488–492.

[30] L. Li, F.J. Yu and C.N. Duan, A generalized nonlocal Gross-Pitaevskii (NGP) equation with an arbitrary time-dependent linear potential, Appl. Math. Lett. **110** (2020), 106584.

[31] S.Z. Liu, J. Wang and D.J. Zhang,The Fokas-Lenells equations: Bilinear approach, Stud. Appl. Math. (2021), 12454.

[32] Y. Lou, Y. Zhang, R. Ye and M. Li, Modulation instability, higher–order rogue waves and dynamics of the Gerdjikov-Ivanov equation, Wave Motion **106** (2021): 102795.

[33] W.X. Ma, Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations, Appl. Math. Lett. **102** (2020) 106161.

[34] X. Ma and Y. Kuang, Inverse scattering transform for a nonlocal derivative nonlinear Schrödinger equation. Theor Math Phys **210** (2022), 31–45.

[35] V.B. Matveev, Darboux transformation and explicit solutions of the Kadomtcev-Petviaschvily equation, depending on functional parameters. Lett. Math. Phys. **3** (1979), 213–216.

[36] V.B. Matveev and M.A. Salle, Darboux transformations and solitons (Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1991).

[37] J. J. C. Nimmo, C. R. Gilson and Y. Ohta, Applications of Darboux transformations to the self-dual Yang-Mills equations, Theor. Math. Phys. **122** (2000), 239–246.

[38] J.J.C. Nimmo and H. Yilmaz, On Darboux Transformations for the derivative nonlinear Schrödinger equation, Journal of Nonlinear Mathematical Physics **21** (2014), 278–293.

[39] J.J.C. Nimmo and H. Yilmaz, Binary Darboux transformation for the Sasa-Satsuma equation, J. Phys. A: Math. Theor. **48** (2015), 425202.

[40] Y. Ohta and J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, Proc. R. Soc. A **468** (2012), 1716–1740.

[41] A. Pekcan, Shifted nonlocal Kundu type equations: Soliton solutions, Partial Differential Equations in Applied Mathematics **5** (2022), 100292.

[42] S. Xu and J. He, The rogue wave and breather solution of the Gerdjikov-Ivanov equation, J. Math. Phys. **53** (2012), 6063507.

[43] B. Yang and Y. Chen, Reductions of Darboux transformations for the PT-symmetric nonlocal Davey-Stewartson equations, Appl. Math. Lett. **82** (2018) 43-49.

[44] J.K. Yang, General N-solitons and their dynamics in several nonlocal nonlinear Schrödinger equations, Phys. Lett. A **383** (2019) 328-337.

[45] Y. Yang, T. Suzuki, X. Cheng, Darboux transformations and exact solutions for the integrable nonlocal Lakshmanan-Porsezian-Daniel equation, Appl. Math. Lett. **99** (2020), 105998.

[46] H. Yilmaz, Exact solutions of the Gerdjikov-Ivanov equation using Darboux transformations, Journal of Nonlinear Mathematical Physics **22** (2015), 32–46.

[47] V. E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, Sov. Phys. J. Appl. Mech. Tech. Phys. **4** (1968), 190–194.

[48] V.E. Zakharov, Collapse of Langmuir waves, Sov. Phys. JETP **35** (1972), 908–914.

[49] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media, Sov. Phys. JETP **34** (1972), 62–69.

[50] X.E. Zhang, Y. Chen and Y. Zhang, Breather, lump and X soliton solutions to nonlocal KP equation, Comput. Math. Appl. **74** (2017), 2341-2347.

[51] H.Q. Zhang, Y. Wang and W.X. Ma, Binary Darboux transformation for the coupled Sasa–Satsuma equations, Chaos **27** (2017), 073102.

[52] Y. Zhang, R. Ye, and W. Ma, Binary Darboux transformation and soliton solutions for the coupled complex modified Korteweg-de Vries equations, Mathematical Methods in the Applied Sciences **43** (2020), 613-627.

[53] G.Q. Zhang, Z.Y. Yan, Inverse scattering transforms and soliton solutions of focusing and defocusing nonlocal mKdV equations with non-zero boundary conditions, Physica D **402** (2020) 132170.

[54] S. Zhou, J. Liu, S. Chen and Y. Yau, The nth-Darboux Transformation and Explicit Solutions of the PT-Symmetry Second-Type Derivative Nonlinear Schrödinger Equation. J Nonlinear Math Phys (2022).

[55] A. K. Zvezdin and A. F. Popkov, Contribution to the nonlinear theory of magnetostaticspin waves, Sov. Phys. JETP **2** (1983), 350.