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Brief paper

An iterative strategy for robust integration of fault estimation and fault-tolerant control☆

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ABSTRACT

This paper considers fault estimation (FE) and fault-tolerant control (FTC) for linear parameter varying systems with actuator and sensor faults, uncertainties, and disturbances. The inevitable coupling between the FE and FTC functions needs to be taken into account in the design to ensure the overall FE-based FTC closed-loop system performance and robustness. This paper proposes an iterative strategy to achieve the robust integration of FE and FTC by leveraging the concepts of Separation Principle and Small Gain Theorem. The iterative algorithm involves solving multi-objective linear matrix inequality optimisation problems at each iteration and has finite-step convergence guarantee. Efficacy of the proposed algorithm and its advantages over the existing works are illustrated through numerical simulations.

1. Introduction

Active fault-tolerant control (FTC) has been demonstrated effective in maintaining robustly acceptable system performance in the presence of faults. The most studied active FTC framework is reconfigurable control using fault detection and isolation (FDI), see the celebrated book (Blanke et al., 2006), the rich works reviewed in Zhang and Jiang (2008) and some recent results in Ding (2021) and Zhang et al. (2020). In the past decade, great attention has been paid to another FTC framework, which involves (i) a fault estimation (FE) observer to estimate the system state and/or faults, and (ii) an FTC controller that combines a state feedback action to guarantee non-faulty system stability with a feedforward action to compensate the faults, see e.g. Abdullah and Zribi (2012), Bouarar et al. (2013), Chen et al. (2019), Do et al. (2020), Gao and Ding (2007), Lan and Patton (2016a, 2016b, 2016c), Liu et al. (2017) and Shi and Patton (2015). The FE-based FTC framework can avoid the complex design procedure of FDI and thus is the focus of this paper.

Most existing works cast the FE design as a state estimation problem by regarding the faults as auxiliary state variables (Chen et al., 2019; Do et al., 2020; Gao & Ding, 2007; Hashemi & Tan, 2020; Jiang et al., 2006; Lan & Patton, 2016c; Shi & Patton, 2015; Xu et al., 2021). However, it is shown in Lan and Patton (2016c) that there exist bidirectional robustness interactions between the FE and FTC designs: On the one side, the FE performance is affected by system control robustness because the FE observer leverages the system model where uncertainties inevitably exist; On the other side, the FTC performance is in turn influenced by the estimation robustness due to the use of the estimated state and faults. These bidirectional robustness interactions break down the Separation Principle (Luenberger, 1971), leading to the loss of guarantee in closed-loop stability by using the traditional *Separated strategy* (Jiang et al., 2006; Liu et al., 2017) to design the FTC controller and FE observer, where their coupling is ignored. This gives rise to the necessity of robust integration of FE and FTC with consideration of the coupling effects.

The robust integration of FE and FTC can be formulated as an observer-based robust control problem to be solved using the well-established robust control theory. Based on this formulation, several robust integration strategies have been developed in the literature using the linear matrix inequality (LMI) technique. The *Two-step strategy* is proposed in Shi and Patton (2015) for linear parameter varying (LPV) descriptor systems with actuator faults. The *Two-step strategy* designs the FTC controller first and then use it to determine the FE observer. However, it considers only the effects of FTC on FE, i.e. the unidirectional robustness interaction, and relies on a heuristic method to ensure feasibility of the FE observer design and balance the performance of FE and FTC. Building on the *Two-step strategy*, the *Iterative two-step strategy*

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is developed in Chen et al. (2019) for LPV systems with sensor faults. It adopts an iterative algorithm to systematically refine the FTC controller performance, which is less conservative than the *Two-step strategy* but still considers just the unidirectional robustness interaction. To account for the bidirectional robustness interactions, the *Integrated strategy* is first developed in Lan and Patton (2016c) for linear systems by solving the FE observer and FTC controller from a single LMI optimisation problem. The strategy is further extended to Lipschitz nonlinear systems (Hashemi & Tan, 2020), T-S fuzzy systems (Bouarar et al., 2013; Lan & Patton, 2016b), LPV systems (Do et al., 2020), input saturated systems (Hashemi et al., 2019) and large-scale interconnected systems (Lan & Patton, 2016a). Although the *Integrated strategy* gives a more optimal design than the *Two-step strategy* and *Iterative two-step strategy*, it has higher complexity in formulating and computing the LMI optimisation problem, especially for complex systems such as LPV, T-S fuzzy and large-scale interconnected systems.

This paper aims to develop an innovative robust integration strategy that can handle the bidirectional robustness interactions and has reduced complexity in the LMI formulation and computation. The key idea is to approximately recover the Separation Principle by ensuring that the pure feedback interconnection of the FTC system and FE estimation error system is small-gain stable (Isidori, 2017). This enables us to establish a stable closed-loop system by assembling the separately determined FTC controller and FE observer. Different from the *Separated strategy* where the coupling is ignored (Jiang et al., 2006; Liu et al., 2017), the present research "*approximately*" recovers the Separation Principle via attenuating the coupling effects as much as possible, but they still exist. The idea of using the Small Gain Theorem to approximately recover the Separation Principle has been adopted in Peaucelle et al. (2017), but it is not in the area of FE and FTC. Moreover, a heuristic design procedure is used in Peaucelle et al. (2017) to establish a stable closed-loop system, which is undesirable for real implementation. This paper will develop an efficient and easily implemented robust integration strategy. The main contributions are summarised as follows:

(1) An iterative strategy is proposed to achieve robust integration of FE and FTC for uncertain LPV systems with actuator faults, sensor faults and disturbances. The proposed strategy addresses the bidirectional robustness interactions rather than unidirectional robustness interaction as in the *Two-step strategy* (Shi & Patton, 2015) and *Iterative two-step strategy* (Chen et al., 2019).
(2) The multi-objective LMI optimisation problems solved at each iteration have feasibility guarantee under the given assumptions. The iterative algorithm is proved to have finite-step convergence.
(3) Extensive theoretic and numerical analysis show that the proposed strategy is less computationally complex than the *Iterative two-step strategy* and *Integrated strategy* (Lan & Patton, 2016c), and achieves better robust closed-loop performance than the *Separated strategy* (Jiang et al., 2006; Liu et al., 2017), *Two-step strategy* and *Iterative two-step strategy*.

The rest of this paper is organised as follows. Section 2 describes the system model and the robust integration problem. Section 3 presents the iterative strategy, followed by the analysis of convergence and computational complexity in Section 4. Section 5 provides a simulation example. Section 6 draws the conclusion.

Notations: $\mathbb{R}$ is the set of real numbers. $\|\cdot\|$ and $\|\cdot\|_\infty$ are the 2-norm and $\infty-$norm in the Euclidean space, respectively. $l_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$. I and $\mathbf{0}$ are identity and zero matrices of appropriate dimensions, respectively. s.p.d. and s.t. are short for symmetric positive definite and subject to, respectively. The superscript $\dagger$ denotes the pseudo-inverse of a matrix. $\star$ induces symmetry in a block matrix. diag$(\cdot)$ denotes a block diagonal matrix. $Z \succ (\prec) 0$ indicates that the matrix Z is positive (negative) definite. $\mathrm{Herm}(Z) = Z + Z^\top$. trace$(Z)$ is the sum of the diagonal elements of matrix Z. $Z^{(k)}$ denotes the value of matrix Z that is computed at iteration k.

2. Problem statement and preliminaries

Consider the uncertain LPV system

$$
\begin{aligned}
\dot{x} &= (A(\theta) + \Delta A(\theta))x + B(\theta)u + F(\theta)f_a + D(\theta)d \\
y &= Cx + Ef_s
\end{aligned} \tag{1}
$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f_a \in \mathbb{R}^q$, $d \in \mathbb{R}^l$, $y \in \mathbb{R}^p$, and $f_s \in \mathbb{R}^r$ are the vectors of state, control input, actuator fault, external disturbance, measured output, and sensor fault, respectively. $\Delta A(\theta)$ represents the system uncertainty. The matrices C and E are constant, while $A(\theta)$, $B(\theta)$, $F(\theta)$ and $D(\theta)$ depend on the known time-varying scheduling parameter $\theta \in \mathbb{R}^g$, which can be system state, inputs, outputs, or some exogenous signals. The system is assumed to have a polytopic representation with $[A(\theta) \mid B(\theta) \mid F(\theta) \mid D(\theta)] = \sum_{i=1}^{\ell} \phi_i [A_i \mid B_i \mid F_i \mid D_i]$, where ϕ_i, $i \in [1,\ell]$, are non-negative scalar functions of θ satisfying $\sum_{i=1}^{\ell} \phi_i = 1$.

Assumption 2.1. The matrix $\Delta A(\theta)$ satisfies $\Delta A(\theta) = \mathcal{M}(\theta)\mathcal{F}(t)\mathcal{N}$, where $\mathcal{M}(\theta) = \sum_{i=1}^{\ell} \phi_i \mathcal{M}_i$, $\mathcal{M}_i$ and $\mathcal{N}$ are known matrices of appropriate dimensions, and $\mathcal{F}(t)$ is an unknown matrix satisfying $\mathcal{F}^\top(t)\mathcal{F}(t) \preceq I$.

Assumption 2.2. The pair $(A(\theta) + \Delta A(\theta), B(\theta))$ is stabilisable for all admissible uncertainty. The quadruple $(A(\theta)+\Delta A(\theta), F(\theta), C, E)$ has no invariant zeros in the open right-half complex plane. The actuator fault f_a is matched, i.e. $\mathrm{rank}([B_i\ F_i]) = \mathrm{rank}(B_i) = m$.

Assumption 2.3. The disturbance d, the faults f_a and f_s, and $\ddot{f}_a$ and $\ddot{f}_s$ all belong to $l_2[0,\infty)$.

Assumption 2.1 is a standard assumption on system uncertainty in the robust control theory (Boyd et al., 1994). Assumption 2.2 is standard for designing FE-based FTC for LPV systems (see e.g. Abdullah and Zribi (2012) and Shi and Patton (2015)), where the first part ensures system stabilisability and the second part ensures that the system state, actuator faults and sensor faults can be fully estimated. For Assumption 2.3, it is realistic to consider disturbance and faults belonging to $l_2[0,\infty)$. The existence of $l_2[0,\infty)$ second derivatives is also not unrealistic for faults such as sinusoidal signals (Gao & Ding, 2007) and continuous time functions (Lan & Patton, 2016b).

This paper aims to design an FE observer to estimate the state and faults, and an FTC controller to compensate the faults and stabilise the state. The main purpose is illustrating the key ideas of the proposed strategy for robust integration of FE and FTC. Hence, for simplicity and clarity, a state feedback FTC controller and a Luenberger-type FE observer are used. This does not exclude the applicability of other forms of controllers and observers, e.g. the ones in Gao and Ding (2007), Lan and Patton (2016c) and Shi and Patton (2015).

To estimate the state and faults, the signals f_a, $\dot{f}_a$, f_s and $\dot{f}_s$ are regarded as auxiliary state variables. Define $\bar{x} = [x^\top\ f_a^\top\ f_s^\top\ \dot{f}_a^\top\ \dot{f}_s^\top]^\top$, then it follows from (1) that

$$
\begin{aligned}
\dot{\bar{x}} &= \bar{A}(\theta)\bar{x} + \Delta\bar{A}(\theta)x + \bar{B}(\theta)u + \bar{D}(\theta)\bar{d} \\
y &= \bar{C}\bar{x}
\end{aligned} \tag{2}
$$

where $\bar{d} = [d^\top\ \ddot{f}_a^\top\ \ddot{f}_s^\top]^\top$, $\bar{C} = [C\ \mathbf{0}\ E\ \mathbf{0}\ \mathbf{0}]$ and

$$\bar{A}(\theta) = \begin{bmatrix} A(\theta) & F(\theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \Delta\bar{A}(\theta) = \begin{bmatrix} \Delta A(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\bar{B}(\theta) = \begin{bmatrix} B(\theta) \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \quad \bar{D}(\theta) = \begin{bmatrix} D(\theta) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I \end{bmatrix}.$$

By regarding both f_a and f_s and their derivatives as auxiliary state, each fault is assumed to have second-order dynamics. It thus enables the proposed observer to estimate a wider range of actuator and sensor faults. To estimate an even more general class of faults, high-order dynamics can be used to characterise the faults (Gao & Ding, 2007; Lan & Patton, 2016b), but leading to an increased observer dimension.

Observability of the augmented system (2) is guaranteed under Assumptions 2.1–2.3 (Lan & Patton, 2016b). Hence, the state $\bar{x}$ can be estimated by the augmented Luenberger state observer

$$\begin{aligned} \dot{\hat{\bar{x}}} &= \bar{A}(\theta)\hat{\bar{x}} + \bar{B}(\theta)u + L(\theta)(y - \hat{y}) \\ \hat{y} &= \bar{C}\hat{\bar{x}} \end{aligned} \tag{3}$$

where $\hat{\bar{x}}$ and $\hat{y}$ are the estimates of $\bar{x}$ and y, respectively. The gain $L(\theta) = \sum_{i=1}^{\ell} \phi_i L_i$ is to be determined. Once $\hat{\bar{x}}$ is obtained, the estimated state and fault are $\hat{x} = [I_n\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\hat{\bar{x}}$, $\hat{f}_a = [\mathbf{0}\ I_q\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]\hat{\bar{x}}$ and $\hat{f}_s = [\mathbf{0}\ \mathbf{0}\ I_r\ \mathbf{0}\ \mathbf{0}]\hat{\bar{x}}$.

Let $e = \bar{x} - \hat{\bar{x}}$. Subtracting (3) from (2) yields the estimation error system

$$\dot{e} = \bar{A}_c(\theta)e + \Delta\bar{A}(\theta)x + \bar{D}(\theta)\bar{d} \tag{4}$$

where $\bar{A}_c(\theta) = \bar{A}(\theta) - L(\theta)\bar{C}$.

Design the FTC controller as

$$u = K(\theta)\hat{x} - H(\theta)\hat{f}_a \tag{5}$$

where $[K(\theta) \mid H(\theta)] = \sum_{i=1}^{\ell} \phi_i[K_i \mid H_i]$, with the design gains K_i and H_i, $i \in [1, \ell]$. Since f_a is matched, we design $H_i = B_i^\dagger F_i$ such that f_a is fully compensated if it is accurately estimated. The use of fault feedforward may cause input saturation and consequent control performance degradation. A way to handle this is referred to Hashemi et al. (2019) and is not the focus of this work.

Applying (5) to (1) gives the FTC closed-loop system

$$\dot{x} = (A_c(\theta) + \Delta A(\theta))x + B(\theta)G(\theta)e + D(\theta)d \tag{6}$$

with $A_c(\theta) = A(\theta) + B(\theta)K(\theta)$ and $G(\theta) = [-K(\theta)\ H(\theta)\ \mathbf{0}]$.

Combining (4) and (6) gives the FE-based FTC system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A_c(\theta) + \Delta A(\theta) & B(\theta)G(\theta) \\ \Delta\bar{A}(\theta) & \bar{A}_c(\theta) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D(\theta)d \\ \bar{D}(\theta)\bar{d} \end{bmatrix}. \tag{7}$$

This closed-loop system is perturbed by d (external disturbance) and $\bar{d}$ (including d and fault modelling errors $\ddot{f}_a$ and $\ddot{f}_s$). The off-diagonal elements $B(\theta)G(\theta)$ and $\Delta\bar{A}(\theta)$ in the system matrix clearly evidence the existence of bidirectional robustness interactions between the FTC and FE observer systems. Therefore, the gains $K(\theta)$ and $L(\theta)$ need to be designed to ensure that the system (7) is stable against the disturbances (d and $\bar{d}$) and the coupling effects ($B(\theta)G(\theta)e$ and $\Delta\bar{A}(\theta)x$).

The mutual coupling breaks down the Separation Principle, and thus closed-loop system stability cannot be guaranteed by the independent choices of stabilising gains $K(\theta)$ and $L(\theta)$ using the *Separated strategy* (Jiang et al., 2006; Liu et al., 2017). By using the *Two-step strategy* (Shi & Patton, 2015) and *Iterative two-step strategy* (Chen et al., 2019), suboptimally stable solutions may be attained but they are too restrictive. The *Integrated strategy* (Lan & Patton, 2016c) is considered to be an ideal strategy, but suffering from difficulty in solving the bilinear matrix inequality problem. Although a convex LMI problem can be obtained using equality constraint or Young inequality, its formulating and computing are complex, especially for systems such as T-S fuzzy, LPV and large-scale interconnected systems. A way around this difficulty is to recover the Separation Principle approximately by minimising the coupling effects $B(\theta)G(\theta)e$ and $\Delta\bar{A}(\theta)x$. In such case, the FTC controller and FE observer can be designed separately with reduced complexity in formulating and computing the LMI problems. This motivates the proposal of a decoupling strategy in this paper, which can minimise the coupling effects and guarantee closed-loop stability by assembling the separately designed FTC controller and FE observer.

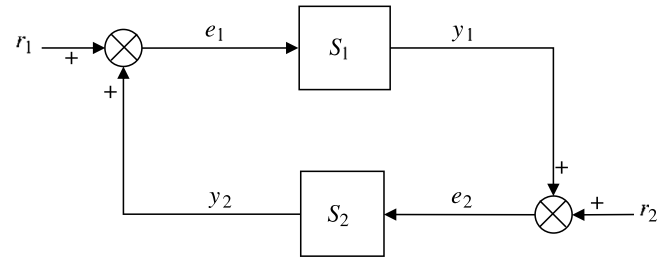

Fig. 1. A generic closed-loop system with two elements.

Before proceeding to present the proposed strategy, the Small Gain Theorem (Glad & Ljung, 2000; Isidori, 2017) and a lemma for robust stability of uncertain LPV systems (Montagner et al., 2005) are introduced.

Theorem 2.1 (*Small Gain Theorem*)**.** *Consider a feedback loop depicted in Fig. 1 composed of two stable systems S_1 and S_2, with inputs r_1 and r_2 and outputs e_1, e_2, y_1 and y_2. If $\|G_{y_1e_1}\|_\infty \cdot \|G_{y_2e_2}\|_\infty < 1$, then the closed-loop system is input-to-output stable. If in addition $r_1 = 0$ and $r_2 = 0$, then the closed-loop system is asymptotically stable.*

Lemma 2.1. *Consider the uncertain LPV system in the form of (1) but without faults (i.e. $F(\theta)f_a = \mathbf{0}$ and $Ef_s = \mathbf{0}$). The closed-loop system with the controller $u = \sum_{i=1}^{\ell} \phi_i K_i x$ is stable and satisfies the H_∞ performance $\|G_{yd}\|_\infty < \gamma$ if there is a s.p.d. matrix $\mathcal{P}$ such that*

$$\begin{bmatrix} \tilde{\Pi}_{ij} & \mathcal{P}D_i & \mathcal{P}\mathcal{M}_i & \mathcal{N}^\top & C^\top \\ \star & -\gamma^2 I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -I & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -I & \mathbf{0} \\ \star & \star & \star & \star & -I \end{bmatrix} \prec \mathbf{0}, \quad i, j \in [1, \ell]$$

where $\tilde{\Pi}_{ij} = \mathrm{Herm}(\mathcal{P}(A_i + B_iK_j))$.

The proposed iterative strategy will build on Lemma 2.1 that adopts the common Lyapunov method (with a single s.p.d. matrix $\mathcal{P}$). This is known to be conservative, but it can keep the presentation concise and clear to illustrate the key ideas of the proposed strategy. To reduce the conservativeness, one can directly replace the common Lyapunov function method by the parameter-dependent Lyapunov function method in de Oliveira et al. (2004) to have multiple s.p.d. matrices $\mathcal{P}_i$, $i \in [1, \ell]$.

3. Iterative robust integration of FE and FTC

3.1. Overview of the proposed strategy

A conceptual diagram of the FE-based FTC closed-loop system (7) is depicted in Fig. 2, where $\tilde{x}$ is the performance output

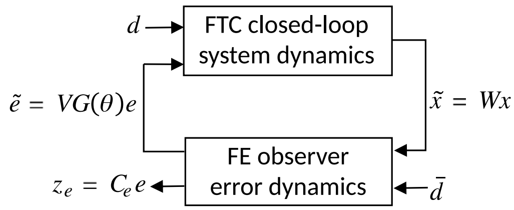

Fig. 2. Conceptual diagram of the FE-based FTC system (7).

associated with d and the coupling effect $\tilde{e}$; z_e and $\tilde{e}$ are the performance outputs associated with $\bar{d}$ and the coupling effect $\tilde{x}$, respectively. The weight C_e is given, while $W \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are tunable positive definite matrices. For design purpose, the outputs $\tilde{x}$ and $\tilde{e}$ are defined to reduce the coupling effects, while the real coupling terms within the closed-loop system remain as $\Delta\bar{A}(\theta)x$ and $B(\theta)G(\theta)e$ because $\Delta\bar{A}(\theta)x = \Delta\bar{A}(\theta)W^{-1}\tilde{x}$ and $B(\theta)G(\theta)e = B(\theta)V^{-1}\tilde{e}$.

One design objective is ensuring robust stability of (7) against d and $\bar{d}$. This is achieved by designing the observer and controller gains to minimise the H_∞ performance metrics $\|G_{\tilde{x}d}\|_\infty$ and $\|G_{z_e\bar{d}}\|_\infty$. Another objective is guaranteeing closed-loop stability when assembling the FTC controller and FE observer. Fig. 2 shows that without d and $\bar{d}$, the closed-loop system is a pure feedback interconnection of two systems with inputs (outputs) $\tilde{e}$ and $\tilde{x}$. According to Theorem 2.1, the closed-loop system is asymptotically stable if $\|G_{\tilde{x}\tilde{e}}\|_\infty \cdot \|G_{\tilde{e}\tilde{x}}\|_\infty < 1$. Moreover, it is desirable to attenuate the coupling effects as much as possible to enhance the closed-loop robustness. This can be realised through maximising the weights W and V, as delineated below: Let $\|G_{\tilde{x}\tilde{e}}\|_\infty \le \tilde{\gamma}_1$ and $\|G_{\tilde{e}\tilde{x}}\|_\infty \le \tilde{\gamma}_2$, where $\tilde{x} = Wx$ and $\tilde{e} = VG(\theta)e$. According to Isidori (2017), these ∞-norm inequalities are equivalent to the 2-norm inequalities $\|Wx\| \le \tilde{\gamma}_1\|\tilde{e}\|$ and $\|VG(\theta)e\| \le \tilde{\gamma}\|\tilde{x}\|$. Suppose W_1 and W_2 satisfy $W_1 \prec W_2$ and $\|W_1x\| = \|W_2x\| \le \tilde{\gamma}_1\|\tilde{e}\|$. Then it is clear that the effect of $\tilde{e}$ on the state x is attenuated more by using the weight W_2. Hence, a bigger weight W is desirable. A similar story applies for the weight V. In conclusion, maximising the weights W and V can attenuate the coupling effects as much as possible.

According to the above analysis, the proposed iterative strategy will consist of three parts:

(1) **FTC controller design:** Design the gains K_i, $i \in [1, \ell]$, and the weight W such that the FTC closed-loop system (6) is stable and satisfies $\|G_{\tilde{x}d}\|_\infty < \gamma_1$ and $\|G_{\tilde{x}\tilde{e}}\|_\infty < 1$, where $\gamma_1 > 0$. The details are given in Section 3.2.
(2) **FE observer design:** Design the gains L_i, $i \in [1, \ell]$, and the weight V such that the estimation error system (4) is stable and satisfies $\|G_{z_e\bar{d}}\|_\infty < \gamma_2$ and $\|G_{\tilde{e}\tilde{x}}\|_\infty < 1$, where $\gamma_2 > 0$. The details are given in Section 3.2.
(3) **Iterative refinement:** An iterative strategy to realise simultaneous maximisation of the weights W and V and refine the gains K_i and L_i, $i \in [1, \ell]$. The details are given in Section 3.3.

Remark 3.1. Two outputs z_e and $\tilde{e}$ are used to characterise the estimation performance against $\bar{d}$ and $\tilde{x}$, respectively. The reason of considering $\tilde{e}$ is that only the affine mapping $B(\theta)G(\theta)e$ of the estimation error e intrudes the FTC closed-loop system, and thus it is desirable to directly attenuate $B(\theta)G(\theta)e$ rather than e. This can reduce the design conservativeness of the existing strategies where a single output z_e is used.

3.2. FTC controller and FE observer designs

To design the FTC controller, rewriting (6) as

$$\begin{aligned}\dot{x} &= (A_c(\theta) + \Delta A(\theta))x + D(\theta)d + B(\theta)V^{-1}\tilde{e}\\ \tilde{x} &= Wx\end{aligned} \tag{8}$$

where $\tilde{e}$ is given in Fig. 2, V is known from the FE observer design, and W is to be determined. The FTC controller is designed using Theorem 3.1.

Theorem 3.1. *Under Assumptions 2.1–2.3, the FTC system (8) is stable with $\|G_{\tilde{x}d}\|_\infty < \gamma_1$ and $\|G_{\tilde{x}\tilde{e}}\|_\infty < 1$, if the following optimisation problem is feasible:*

$$\min_{X_i, i\in[1,\ell], P, S, \bar{\gamma}_1} \alpha_1\bar{\gamma}_1 + \alpha_2 \mathrm{trace}(S)$$

$$\begin{bmatrix} \check{\Pi}_{ij} & D_i & \mathcal{M}_i & P\mathcal{N}^\top & P\\ \star & -\bar{\gamma}_1 I & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \star & -I & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & -I & \mathbf{0}\\ \star & \star & \star & \star & -S \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{9a}$$

$$\begin{bmatrix} \check{\Pi}_{ij} & B_i & \mathcal{M}_i & P\mathcal{N}^\top & P\\ \star & -R^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \star & -I & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & -I & \mathbf{0}\\ \star & \star & \star & \star & -S \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{9b}$$

$$P = P^\top \succ \mathbf{0},\ S = S^\top \succ \mathbf{0},\ \bar{\gamma}_1 > 0 \tag{9c}$$

where $\check{\Pi}_{ij} = \mathrm{Herm}(A_iP + B_iX_j)$, $R = (V^\top V)^{-1}$, and α_1 and α_2 are prescribed positive scalars. The gains are obtained as: $K_i = X_iP^{-1}$, $i \in [1,\ell]$, $W = \sqrt{S^{-1}}$, $\gamma_1 = \sqrt{\bar{\gamma}_1}$.

Proof. Let $\tilde{e} = \mathbf{0}$. Consider the Lyapunov function $V_x = x^\top \mathcal{P}x$ with a s.p.d. matrix $\mathcal{P}$. By using Lemma 2.1, the system (8) is stable with $\|G_{\tilde{x}d}\|_\infty < \gamma_1$ if

$$\begin{bmatrix} \Pi_{ij} & \mathcal{P}D_i & \mathcal{P}\mathcal{M}_i & \mathcal{N}^\top & I\\ \star & -\gamma_1^2 & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \star & \star & -I & \mathbf{0} & \mathbf{0}\\ \star & \star & \star & -I & \mathbf{0}\\ \star & \star & \star & \star & -S \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{10}$$

where $\Pi_{ij} = \mathrm{Herm}(\mathcal{P}(A_i + B_iK_j))$ and $S = (W^\top W)^{-1}$. Let $P = \mathcal{P}^{-1}$ and $X_i = K_iP$. Multiplying both sides of (10) with $\mathrm{diag}(P, I, I, I, I)$ and its transpose, then it yields (9a). Let $d = \mathbf{0}$ and $R = (V^\top V)^{-1}$. Under Lemma 2.1, the system (8) is stable with $\|G_{\tilde{x}\tilde{e}}\|_\infty < 1$ if (9b) holds. To minimise γ_1 and maximise W (i.e. minimise S), the design of K_i and W is formulated as the multi-objective optimisation problem (9), where $\bar{\gamma}_1 = \gamma_1^2$. □

The FE error system (4) can be rewritten as

$$\begin{aligned}\dot{e} &= \bar{A}_c(\theta)e + \bar{D}(\theta)\bar{d} + \Delta\bar{A}(\theta)W^{-1}\tilde{x}\\ z_e &= C_e e\\ \tilde{e} &= VG(\theta)e\end{aligned} \tag{11}$$

where $\tilde{x}$ is given in Fig. 2, W is known from the FTC controller design, and V is to be determined. The FE observer is designed using Lemma 3.1.

Lemma 3.1. *Under Assumptions 2.1–2.3, the FE system (11) is stable with $\|G_{z_e\bar{d}}\|_\infty < \gamma_2$ and $\|G_{\tilde{e}\tilde{x}}\|_\infty < 1$, if the following optimisation problem is feasible:*

$$\min_{Y_i, i\in[1,\ell], Q, R, \bar{\gamma}_2} \beta_1\bar{\gamma}_2 + \beta_2 \mathrm{trace}(R)$$

$$\begin{bmatrix} \Omega_{ij} & Q\bar{D}_i & C_e^\top\\ \star & -\bar{\gamma}_2 I & \mathbf{0}\\ \star & \star & -I \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{12a}$$

$$\begin{bmatrix} \Omega_{ij} & Q\bar{\mathcal{M}}_i & G_i^\top & \mathbf{0} \\ \star & -I & \mathbf{0} & \mathbf{0} \\ \star & \star & -R & \mathbf{0} \\ \star & \star & \star & \mathcal{N}^\top\mathcal{N} - S^{-1} \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{12b}$$

$$Q = Q^\top \succ \mathbf{0},\ R = R^\top \succ \mathbf{0},\ \bar{\gamma}_2 > 0 \tag{12c}$$

where $\Omega_{ij} = \mathrm{Herm}(Q\bar{A}_i - Y_j\bar{C})$, $\bar{\mathcal{M}}_i = [\mathcal{M}_i^\top\ \mathbf{0}]^\top$, *and* β_1 *and* β_2 *are prescribed positive scalars. The gains are obtained as:* $L_i = Q^{-1}Y_i, i \in [1,\ell], V = \sqrt{R^{-1}}, \gamma_2 = \sqrt{\bar{\gamma}_2}$.

Proof. Consider the Lyapunov function $V_e = e^\top Qe$ with a s.p.d. matrix Q. The proof follows from Lemma 2.1 with $Y_i = QL_i$, $R = (V^\top V)^{-1}$, $S = (W^\top W)^{-1}$. □

A necessary condition to ensure feasibility of (12b) is

$$\mathcal{N}^\top\mathcal{N} - S^{-1} \prec \mathbf{0}. \tag{13}$$

Since S is the decision variable of the optimisation problem (9), the condition (13) can be satisfied by imposing the constraint $\lambda_{\max}(S) < \lambda_{\min}((\mathcal{N}^\top\mathcal{N})^{-1})$ given a nonsingular $\mathcal{N}^\top\mathcal{N}$. However, solving (9) with such an eigenvalue constraint is difficult. This paper proposes a method to ensure (13) without imposing any extra constraint. It is based on the observation that for any uncertainty matrix $\Delta A(\theta)$, the matrices $\mathcal{M}(\theta)$ and $\mathcal{N}$ can be freely chosen, as long as they abide by Assumption 2.1. The proposed method is described below: (i) When solving the optimisation problem (9), define $\mathcal{N} = \delta_0 \times I_n$ with $\delta_0 \in (0,1]$, then select $\mathcal{M}(\theta)$ and $\mathcal{F}(t)$ to satisfy Assumption 2.2. (ii) When solving the optimisation problem (12), re-scale $\mathcal{N}$ and $\mathcal{M}(\theta)$ as $\hat{\mathcal{N}} = \delta\mathcal{N}$ and $\hat{\mathcal{M}}(\theta) = \delta^{-1}\mathcal{M}(\theta)$, where $\delta = \sqrt{\lambda_{\min}(S^{-1})} - \epsilon_0$ and ϵ_0 is a small positive scalar. Using this method gives $\hat{\mathcal{N}}$ and $\hat{\mathcal{M}}(\theta)$ that always satisfy Assumption 2.1 and (13). Hence, Lemma 3.1 is reformulated as Theorem 3.2.

Theorem 3.2. *Under Assumptions 2.1–2.3, the FE system (11) is stable with* $\|G_{\tilde{e}\bar{d}}\|_\infty < \gamma_2$ *and* $\|G_{\tilde{e}\tilde{x}}\|_\infty < 1$, *if the following optimisation problem is feasible:*

$$\min_{Y_i, i\in[1,\ell], Q, R, \bar{\gamma}_2} \beta_1\bar{\gamma}_2 + \beta_2 \mathrm{trace}(R)$$

$$\begin{bmatrix} \Omega_{ij} & Q\bar{D}_i & G_i^\top \\ \star & -\bar{\gamma}_2 I & \mathbf{0} \\ \star & \star & -R \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{14a}$$

$$\begin{bmatrix} \Omega_{ij} & Q\hat{\bar{\mathcal{M}}}_i & G_i^\top \\ \star & -I & \mathbf{0} \\ \star & \star & -R \end{bmatrix} \prec \mathbf{0},\ i,j \in [1,\ell] \tag{14b}$$

$$Q = Q^\top \succ \mathbf{0},\ R = R^\top \succ \mathbf{0},\ \bar{\gamma}_2 > 0 \tag{14c}$$

where $\Omega_{ij} = \mathrm{Herm}(Q\bar{A}_i - Y_j\bar{C})$, $\hat{\bar{\mathcal{M}}}_i = [\hat{\mathcal{M}}_i^\top\ \mathbf{0}]^\top$, $\hat{\mathcal{M}}_i = \delta^{-1}\mathcal{M}_i$, $\delta = \sqrt{\lambda_{\min}(S^{-1})} - \epsilon_0$. ϵ_0, β_1 *and* β_2 *are all given positive scalar but* ϵ_0 *is required to be small. The gains are obtained as:* $L_i = Q^{-1}Y_i, i \in [1,\ell], V = \sqrt{R^{-1}}, \gamma_2 = \sqrt{\bar{\gamma}_2}$.

The iterative strategy to be detailed in Section 3.3 will be based on the optimisation problems (9) and (14). Hence, it is necessary to analyse their feasibility.

Proposition 3.1. *The optimisation problem (9) is feasible under Assumption 2.2 with an appropriate matrix R, while (14) is feasible under Assumption 2.2.*

Proof. By using Schur complement (Boyd et al., 1994), the LMIs (9a) and (9b) are equivalent to

$$\check{\Pi}_{ij} + D_i\bar{\gamma}_1^{-1}D_i^\top + \mathcal{M}_i\mathcal{M}_i^\top + P\mathcal{N}^\top\mathcal{N}P + PS^{-1}P \prec \mathbf{0},$$
$$\check{\Pi}_{ij} + B_iRB_i^\top + \mathcal{M}_i\mathcal{M}_i^\top + P\mathcal{N}^\top\mathcal{N}P + PS^{-1}P \prec \mathbf{0},$$

where $i,j \in [1,\ell]$. There is always a large enough $\bar{\gamma}_1$ and S satisfying the above inequalities, if it holds that

$$\check{\Pi}_{ij} + B_iRB_i^\top + \mathcal{M}_i\mathcal{M}_i^\top + P\mathcal{N}^\top\mathcal{N}P \prec \mathbf{0},\ i,j \in [1,\ell].$$

This means that feasibility of the optimisation problem (9) relies on an existing P to stabilise the uncertain system (1) and the matrix R that is obtained in the observer design. Therefore, the feasibility is guaranteed under Assumption 2.2 with an appropriate value of matrix R.

The LMIs (14a) and (14b) are equivalent to

$$\Omega_{ij} + (Q\bar{D}_i)\bar{\gamma}_2^{-1}(Q\bar{D}_i)^\top + G_i^\top R^{-1}G_i \prec \mathbf{0},\ i,j \in [1,\ell].$$
$$\Omega_{ij} + Q\hat{\bar{\mathcal{M}}}_i(Q\hat{\bar{\mathcal{M}}}_i)^\top + G_i^\top R^{-1}G_i \prec \mathbf{0},\ i,j \in [1,\ell].$$

One can always find a large enough scalar $\bar{\gamma}_2$ and a s.p.d. matrix R to satisfy the above inequalities, if there exists a s.p.d. matrix Q such that $\Omega_{ij} + Q\hat{\bar{\mathcal{M}}}_i(Q\hat{\bar{\mathcal{M}}}_i)^\top \prec \mathbf{0}$. This means that the feasibility of optimisation problem (14) relies on observability of the uncertain system (1), which is guaranteed by the second part of Assumption 2.2. Therefore, feasibility of the optimisation problem (14) is guaranteed under Assumption 2.2. □

3.3. Iterative algorithm

This section presents Algorithm 1 to determine the gains K_i and L_i, $i \in [1,\ell]$, based on Theorems 3.1 and 3.2. At the initialisation step, the optimisation problem (9) is solved to get the initial feasible FTC controller gains $K_i^{(0)}$, $i \in [1,\ell]$, and the weight $W^{(0)}$ based on the given matrix $R_{\mathrm{known}}^{(0)}$ (i.e. $V^{(0)}$). The iteration loop involves two steps: FE observer design and FTC controller refinement. At Step 1, the FE observer gains $L_i^{(k)}$, $i \in [1,\ell]$, and the matrix $R_{\mathrm{known}}^{(k)}$ are solved from the optimisation problem P1 based on $K_i^{(k-1)}$, $i \in [1,\ell]$, and $W^{(k-1)}$ that are obtained in the previous iteration. At Step 2, the FTC controller gains $K_i^{(k-1)}$, $i \in [1,\ell]$, are refined by solving optimisation problem P2 based on the matrix $R_{\mathrm{known}}^{(k)}$ updated at Step 1. The iteration continues until the relative change of the cost function $J_c^{(k)}$ of problem P2 is smaller than the prescribed tolerance ϵ.

The definition of the matrix $R_{\mathrm{known}}^{(k)}$ and how it is used in optimisation problem P2 are detailed below. Let $\bar{V} = V^\top V$. At iteration k, $\bar{V}^{(k)}$ is designed as

$$\bar{V}^{(k)} = \left(\bar{V}^{(0)} + \sum_{s=1}^{k} \Delta\bar{V}^{(s)}\right) \tag{15}$$

where $\bar{V}^{(0)}$ is the initial value of $\bar{V}$ and $\Delta\bar{V}^{(s)}$ are s.p.d. matrices determined at iterations $s \in [1,k]$. Hence, $\bar{V}^{(k)} = \bar{V}^{(k-1)} + \Delta\bar{V}^{(k)}$. Corresponding to (15), define

$$R_{\mathrm{known}}^{(k-1)} = (\bar{V}^{(k-1)})^{-1},\ \Delta R^{(k)} = (\Delta\bar{V}^{(k)})^{-1} \tag{16}$$

where $R_{\mathrm{known}}^{(k-1)}$ is known at iteration k while $\Delta R^{(k)}$ is the decision variable. Applying (16) to (14b) gives

$$\begin{bmatrix} \Omega_{ij}^{(k)} & Q^{(k)}\hat{\bar{\mathcal{M}}}_i & \Pi_1 \\ \star & -I & \mathbf{0} \\ \star & \star & \Pi_2 \end{bmatrix} \prec \mathbf{0}, i,j \in [1,\ell] \tag{17}$$

where $\Omega_{ij}^{(k)} = \mathrm{Herm}(Q^{(k)}\bar{A}_i - Y_j^{(k)}\bar{C})$, $\Pi_1 = [G_i^\top\ G_i^\top]$, and $\Pi_2 = \mathrm{diag}(-R_{\mathrm{known}}^{(k-1)}, -\Delta R^{(k)})$.

Note that using a too small $V^{(0)}$ leads to the need of large numbers of iterations to approach a good attenuation of the coupling, while using a too big value may cause early termination of the iteration. In this work, $V^{(0)}$ is chosen close to an identity matrix. The gain $\bar{\gamma}_1$ is optimised at the initialisation step and fixed at Step 2 to optimise $S^{(k)}$ only. Hence, the part $\alpha_1\bar{\gamma}_1$ is not included in the cost function of optimisation problem P2.

Algorithm 1 Iterative robust integration strategy

Input: A_i, B_i, F_i, D_i, $\bar{A}_i$, $\bar{D}_i$, $\mathcal{M}_i$, $i \in [1, \ell]$; C, $\mathcal{N}$, $\bar{C}$, C_e, $V^{(0)}$, α_1, α_2, β_1, β_2, ϵ_0, ϵ.
Initialisation: Set $K_i^{(0)} \leftarrow X_i^{(0)}(P^{(0)})^{-1}$, $i \in [1, \ell]$, $W^{(0)} \leftarrow \sqrt{(S^{(0)})^{-1}}$, $J_c^{(0)} \leftarrow \alpha_2 \mathrm{trace}(S^{(0)})$, where $P^{(0)}$ and $S^{(0)}$ are solutions to the problem (9) with $R = R_{\mathrm{known}}^{(0)} = (\bar{V}^{(0)})^{-1}$ and $\bar{V}^{(0)} = (V^{(0)})^\top V^{(0)}$.
for $k = 1, 2, \ldots$ **do**
  Step 1: FE observer design
  Set $L_i^{(k)} \leftarrow (Q^{(k)})^{-1} Y_i^{(k)}$, $i \in [1, \ell]$, $\Delta\bar{V}^{(k)} \leftarrow (\Delta R^{(k)})^{-1}$, $R_{\mathrm{known}}^{(k)} \leftarrow (\bar{V}^{(k-1)} + \Delta\bar{V}^{(k)})^{-1}$, where $Q^{(k)}$ and $\Delta R^{(k)}$ are solutions to problem P1:

$$J_o^{(k)} =: \min\ \beta_1 \bar{\gamma}_2^{(k)} + \beta_2 \mathrm{trace}(\Delta R^{(k)})$$
$$\text{s.t. (14a), (14c), (17)},\ \ \delta = \sqrt{\lambda_{\min}((S^{(k-1)})^{-1})} - \epsilon_0,$$
$$\hat{\bar{\mathcal{M}}}_i = [\delta^{-1}\mathcal{M}_i^\top\ 0]^\top,\ \ G_i = [-K_i^{(k-1)}\ B_i^\dagger F_i\ 0],$$

  with Y_i, Q, $\bar{\gamma}_2$ and R in (14a) and (14c) being replaced by $Y_i^{(k)}$, $Q^{(k)}$, $\bar{\gamma}_2^{(k)}$ and $\Delta R^{(k)}$, respectively.
  Step 2: FTC controller refinement
  Set $K_i^{(k)} \leftarrow X_i^{(k)}(P^{(k)})^{-1}$, $i \in [1, \ell]$, $W^{(k)} \leftarrow \sqrt{(S^{(k)})^{-1}}$, $J_c^{(k)} \leftarrow \alpha_2 \mathrm{trace}(S^{(k)})$, where $X_i^{(k)}$, $P^{(k)}$ and $S^{(k)}$ are solutions to problem P2:

$$J_c^{(k)} =: \min\ \alpha_2 \mathrm{trace}(S^{(k)})$$
$$\text{s.t. (9a), (9b), (9c)},\ \ R = R_{\mathrm{known}}^{(k)},$$

  with X_i, P and S in (9a), (9b) and (9c) being replaced by $X_i^{(k)}$, $P^{(k)}$ and $S^{(k)}$, respectively.
  if $|J_c^{(k)} - J_c^{(k-1)}| < \epsilon J_c^{(k-1)}$ **then**
    Set $K_i \leftarrow K_i^{(k)}$, $L_i \leftarrow L_i^{(k)}$, $i \in [1, \ell]$, and stop.
  end if
end for
Output: K_i, L_i, $i \in [1, \ell]$.

4. Convergence and computational complexity

4.1. Convergence analysis

Before analysing the convergence of Algorithm 1, it is necessary to discuss feasibility of the three optimisation problems that it involves. The initialisation step requires solving the optimisation problem (9), which is feasible under Assumption 2.2 with an appropriately selected matrix $R_{\mathrm{known}}^{(0)}$ (i.e. $V^{(0)}$), as shown in Proposition 3.1. The optimisation problem P1 solved at Step 1 is slightly different from the optimisation problem (14), with only the LMI (14b) being replaced by (17) to include $R_{\mathrm{known}}^{(k-1)}$. Hence, the proof of Proposition 3.1 can be directly adapted to show that problem P1 is feasible under Assumption 2.2 and a large enough matrix $R_{\mathrm{known}}^{(k-1)}$. Since $R_{\mathrm{known}}^{(k)} \prec R_{\mathrm{known}}^{(k-1)}$ by construction, the value of $R_{\mathrm{known}}^{(k)}$ decreases with iterations. Therefore, the problem P1 is feasible through the iterations until $R_{\mathrm{known}}^{(k-1)}$ reaches a certain small value, i.e. when the weight V is maximised to some value. The feasibility proof of the optimisation problem P2 needs to use Proposition 4.1 .

Proposition 4.1. *If the optimisation problem P2 is feasible at iteration $k-1$, then it is feasible at iteration k.*

Proof. Let $\mathcal{L}_{c1}(P^{(k)}, X_j^{(k)}, S^{(k)})$ be the left-hand side term of (9a), and $\mathcal{L}_{c2}(P^{(k)}, X_j^{(k)}, S^{(k)})$ be the left-hand side term of (9b) with deletion of the second row and second column. By using Schur complement, the LMIs (9a) and (9b) are then rewritten into the compact forms:

$$\mathcal{L}_{c1}(P^{(k)}, X_j^{(k)}, S^{(k)}) \prec \mathbf{0},\ i, j \in [1, \ell].$$
$$B_i^\dagger \mathcal{L}_{c2}(P^{(k)}, X_j^{(k)}, S^{(k)})(B_i^\dagger)^\top + R_{\mathrm{known}}^{(k)} \prec \mathbf{0},\ i, j \in [1, \ell].$$

By construction, $\mathbf{0} \prec R_{\mathrm{known}}^{(k)} \prec R_{\mathrm{known}}^{(k-1)}$. Hence, by induction, the solution to problem P2 at iteration $k-1$ is always feasible to this problem at iteration k by setting $X_j^{(k)} = X_j^{(k-1)}$, $P^{(k)} = P^{(k-1)}$ and $S^{(k)} = S^{(k-1)}$. □

Under Proposition 4.1, the optimisation problem P2 is always feasible if it is feasible under $R_{\mathrm{known}}^{(0)}$, which reverts to the same optimisation problem solved at the initialisation step. Therefore, problem P2 is always feasible under Assumption 2.2 with an appropriately selected $R_{\mathrm{known}}^{(0)}$ (i.e. $V^{(0)}$). Furthermore, the cost function of problem P2 is shown to converge in Lemma 4.1.

Lemma 4.1. *The cost function sequence $\{J_c^{(k)}\}_{k=0}^{\infty}$ of problem P2 converges to a (local) minimum J_c^*.*

Proof. By construction, Algorithm 1 generates a series of positive scalars $J_c^{(k)}$. Under Proposition 4.1, $J_c^{(k+1)} \leq J_c^{(k)}$. Hence, the sequence $\{J_c^{(k)}\}_{k=0}^{\infty}$ is non-increasing and bounded below by zero. Let J_c^* be the greatest lower bound of the sequence, then $J_c^{(k)} \geq J_c^*, \forall k \in [0, \infty)$. Moreover, there is an integer n_0 such that $J_c^{n_0} \leq J_c^* + \varepsilon,\ \forall \varepsilon > 0$, otherwise J_c^* is not the greatest lower bound. Hence, $J_c^* - \varepsilon \leq J_c^{(k)} \leq J_c^* + \varepsilon,\ \forall k \geq n_0$, meaning that $\{J_c^{(k)}\}_{k=0}^{\infty}$ converges to J_c^* and it is a Cauchy sequence (Rudin, 1964). Therefore, $|J_c^{(k)} - J_c^{(k-1)}| < \varepsilon, \forall k > n_0$. This confirms that Algorithm 1 terminates in finite iterations and gives an arbitrarily close approximation to the true local minimum J_c^*. □

Convergence of the cost function of the optimisation problem P1 is also analysed below.

Proposition 4.2. *The cost function sequence $\{J_o^{(k)}\}_{k=1}^{\infty}$ of problem P1 does not necessarily converge.*

Proof. Let $\mathcal{L}_{o1}(Q^{(k)}, Y_j^{(k)}, \bar{\gamma}_2^{(k)}, K_i^{(k-1)}, R_{\mathrm{known}}^{(k-1)}, \Delta R^{(k)})$ and $\mathcal{L}_{o2}(Q^{(k)}, Y_j^{(k)}, K_i^{(k-1)}, R_{\mathrm{known}}^{(k-1)}, \Delta R^{(k)})$ be the left-hand side terms of the LMIs (14a) and (14b), respectively. Then (14a) and (14b) are rewritten compactly as

$$\mathcal{L}_{o1}(Q^{(k)}, Y_j^{(k)}, \bar{\gamma}_2^{(k)}, K_i^{(k-1)}, R_{\mathrm{known}}^{(k-1)}, \Delta R^{(k)}) \prec \mathbf{0},$$
$$\mathcal{L}_{o2}(Q^{(k)}, Y_j^{(k)}, K_i^{(k-1)}, R_{\mathrm{known}}^{(k-1)}, \Delta R^{(k)}) \prec \mathbf{0},\ i, j \in [1, \ell].$$

As seen from (17), both the above two inequalities include the nonlinear term $\Pi_1^\top(-\Pi_2^{-1})\Pi_1$ that are dependent on the controller gain $K_i^{(k-1)}$. Hence, their solutions at iteration $k-1$ are not necessarily feasible at iteration k. This implies that the solution to the optimisation problem P1 at iteration $k-1$ is not necessarily a solution to it at iteration k. Therefore, convergence of the cost function sequence $\{J_o^{(k)}\}_{k=1}^{\infty}$ cannot be established. □

By using Lemma 4.1 and Proposition 4.2, the properties of Algorithm 1 are summarised in Theorem 4.1.

Theorem 4.1. *If the optimisation problem solved at the initialisation step is feasible, then Algorithm 1 terminates in finite iterations under the stopping criterion. The obtained gains K_i and L_i, $i \in [1, \ell]$, minimise the mutual coupling between the FE and FTC designs, guaranteeing robust stability of the FE-based FTC closed-loop system.*

Proof. The convergence follows from Proposition 4.1 and Lemma 4.1. Proposition 4.2 implies that the cost function of problem P1 does not necessarily decrease. However, since $\Delta R^{(k)} \succ \mathbf{0}$, $\forall k \geq 1$, the iteration gradually increases $\bar{V}$ and equivalently the weight V. Under Proposition 4.1, this leads to a reduction in $J_c^{(k)}$ and a consequent increase in W. Hence, the iterative algorithm maximises W and V, resulting in maximal attenuation of the coupling effects between FE and FTC designs. Under Theorems 3.1 and 3.2, the gains K_i and L_i, $i \in [1, \ell]$, generated by Algorithm 1 also guarantee robust stability of the FE-based FTC closed-loop system in the presence of uncertainties and disturbances. □

Due to the nonlinear nature of iteration, the sequence $\{J_c^{(k)}\}_{k=0}^{\infty}$ normally converges to a local minimum and gives suboptimal solutions to the integration of FE and FTC. However, this is not too conservative. Generally, suboptimal solutions are also obtained by the *Separated strategy* (Jiang et al., 2006; Liu et al., 2017) and *Two-step strategy* (Shi & Patton, 2015) due to the separate design procedure, by the *Iterative two-step strategy* (Chen et al., 2019) due to the iteration, and even by the *Integrated strategy* (Lan & Patton, 2016c) due to the linearisation needed to formulate the LMIs.

4.2. Computational complexity

This section analyses computational complexity of the LMI optimisation problems in Algorithm 1 with comparison to the existing strategies. To establish a fair comparison, the existing strategies are modified to have the same FE observer (3) and FTC controller (5). Moreover, the complexity of solving individual optimisation problem, rather than all the optimisation problems involved in a strategy, is investigated. The computational complexity of an LMI is estimated by Gahinet et al. (1995): $\mathcal{C} = \mathcal{R}\mathcal{S}^3$, where $\mathcal{R}$ is the LMI row size and $\mathcal{S}$ is the total number of scalar decision variables. Hence, computational complexity of different designs is derived below:

(1) *Separated strategy*: Complexity of solving the FTC controller is $\mathcal{C}_{s,1} = \mathcal{R}_{s,1}\mathcal{S}_{s,1}^3$ with $\mathcal{R}_{s,1} = (4n + l) \times 2^\ell + 2n + 1$ and $\mathcal{S}_{s,1} = 0.5n(n + 2m\ell + 1) + 1$; Complexity of solving the FE observer is $\mathcal{C}_{s,2} = \mathcal{R}_{s,2}\mathcal{S}_{s,2}^3$ with $\mathcal{R}_{s,2} = (2n + 5q + 5r + l) \times 2^\ell + 2(n + 2q + 2r) + 1$ and $\mathcal{S}_{s,2} = 0.5(n + 2q + 2r)(n + 2q + 2r + 4p\ell + 1) + 1$.
(2) *Two-step strategy*: Complexity of solving the FTC controller is $\mathcal{C}_{t,1} = \mathcal{R}_{t,1}\mathcal{S}_{t,1}^3$ with $\mathcal{R}_{t,1} = \mathcal{R}_{s,1}$ and $\mathcal{S}_{t,1} = \mathcal{S}_{s,1}$; Complexity of solving the FE observer is $\mathcal{C}_{t,2} = \mathcal{R}_{t,2}\mathcal{S}_{t,2}^3$ with $\mathcal{R}_{t,2} = \mathcal{R}_{s,2} + (4n + l) \times 2^\ell + 2n$ and $\mathcal{S}_{t,2} = \mathcal{S}_{s,2} + 0.5n(n + 1)$.
(3) *Iterative two-step strategy*: Complexity of solving the initial FTC controller is $\mathcal{C}_{it,0} = \mathcal{R}_{it,0}\mathcal{S}_{it,0}^3$ with $\mathcal{R}_{it,0} = \mathcal{R}_{t,1}$ and $\mathcal{S}_{it,0} = \mathcal{S}_{t,1}$; Complexity of solving the FE observer at each iteration is $\mathcal{C}_{it,1} = \mathcal{R}_{it,1}\mathcal{S}_{it,1}^3$ with $\mathcal{R}_{it,1} = \mathcal{R}_{t,2}$ and $\mathcal{S}_{it,1} = \mathcal{S}_{t,2}$; Complexity of refining the FTC controller at each iteration is $\mathcal{C}_{it,2} = \mathcal{R}_{it,2}\mathcal{S}_{it,2}^3$ with $\mathcal{R}_{it,2} = \mathcal{R}_{t,2}$ and $\mathcal{S}_{it,2} = \mathcal{S}_{t,2} + mn\ell$.
(4) *Integrated strategy*: The complexity is $\mathcal{C}_{in} = \mathcal{R}_{in}\mathcal{S}_{in}^3$ with $\mathcal{R}_{in} = \mathcal{R}_{t,2}$ and $\mathcal{S}_{in} = \mathcal{S}_{t,2} + mn\ell$.
(5) *Proposed strategy*: Complexity of solving the initial FTC controller is $\mathcal{C}_{p,0} = \mathcal{R}_{p,0}\mathcal{S}_{p,0}^3$ with $\mathcal{R}_{p,0} = 2\mathcal{R}_{t,1} + (m - l) \times 2^\ell - 1$ and $\mathcal{S}_{p,0} = \mathcal{S}_{t,1} + 0.5n(n + 1)$; Complexity of solving the FE observer at each iteration is $\mathcal{C}_{p,1} = \mathcal{R}_{p,1}\mathcal{S}_{p,1}^3$ with $\mathcal{R}_{p,1} = 2\mathcal{R}_{s,2} + (2m - l - 3q - 3r) \times 2^\ell + 2(m - n - 2q - 2r) - 1$ and $\mathcal{S}_{p,1} = \mathcal{S}_{s,2} + 0.5m(m + 1)$; Complexity of solving the refined FTC controller at each iteration is $\mathcal{C}_{p,2} = \mathcal{R}_{p,2}\mathcal{S}_{p,2}^3$ with $\mathcal{R}_{p,2} = 2\mathcal{R}_{t,1} + (m - l) \times 2^\ell - 1$ and $\mathcal{S}_{p,2} = \mathcal{S}_{t,1} + 0.5n(n + 1) - 1$.

It is observed that solving the FTC controllers in the *Separated strategy* and *Two-step strategy* have the same complexity, but the later has a more complex FE design. For the *Iterative two-step strategy*, computing the initial FTC controller and FE observer are as complex as the *Two-step strategy*. However, refining the FTC controller is much more complex than solving the FTC controller in the *Two-step strategy*. Complexity of the *Integrated strategy* is higher than that of the FE observer, but same as the FTC controller refinement, of the *Iterative two-step strategy*. Each single LMI optimisation problem of the *Proposed strategy* is less computationally complex than that of the *Integrated strategy*. Compared to the *Iterative two-step strategy*, the *Proposed strategy* solves less complex LMIs during iterations, but a more complex initialisation problem due to the multi-objective setting. It is also observed that the computational complexity of all designs mainly depends on the system dimension n and the number of scheduling parameters ℓ.

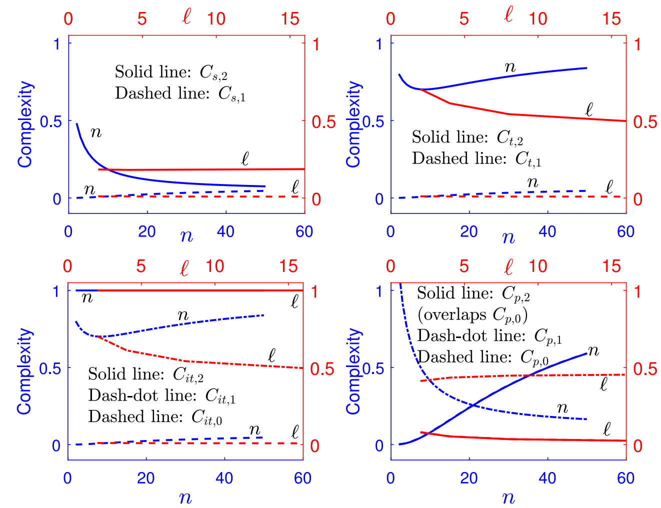

Fig. 3. Comparative computational complexity under various system dimension n and number of scheduling parameters ℓ. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

The above observations are illustrated by an example system with $m = 2$, $q = 1$, $l = 1$, $p = 2$ and $r = 1$. The values of n and ℓ are varied to show evolution of the complexity. In Fig. 3, corresponding to the blue axes are results of the first case with $\ell = 2$ and $n \in [2, 50]$, while corresponding to the red axes are results of the second case with $n = 10$ and $\ell = 2, 4, 8, 16$. The obtained results are normalised against the *Integrated strategy* complexity $\mathcal{C}_{in}$, which is within [1.6e8, 6.4e13] in the first case and within [1.6e10, 1.7e16] in the second case. The results confirm the analysis above and conclude that the *Proposed strategy* has reduced computational complexity in solving the LMI problems, compared with the existing *Integrated strategy* and *Iterative two-step strategy*.

5. Simulation example

Consider a mass–spring–damper system (De Caigny et al., 2010) in the form of (1) with the matrices:

$$A(\theta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\theta}{m_1} & \frac{\theta}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{\theta}{m_2} & -\frac{\theta+k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix},$$

$$\Delta A = \sigma_{c2} A_p, \quad A_p = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{m_1} & \frac{1}{m_1} \\ 0 & 0 & \frac{1}{m_2} & -\frac{1}{m_2} \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix},$$

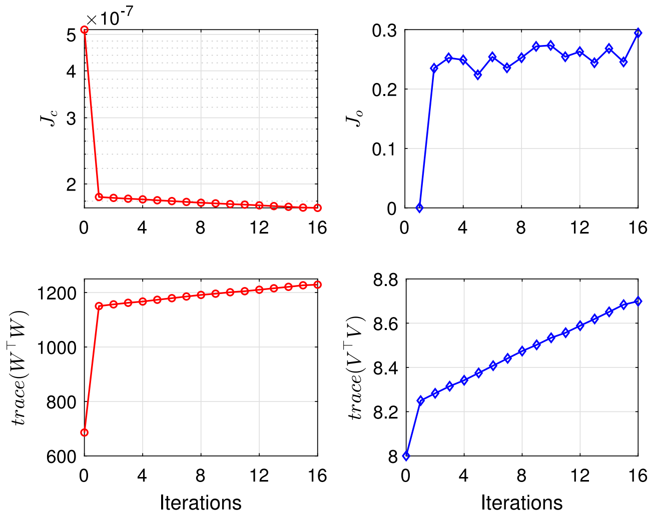

Fig. 4. Evolution of J_c, J_o, trace$(W^\top W)$ and trace$(V^\top V)$.

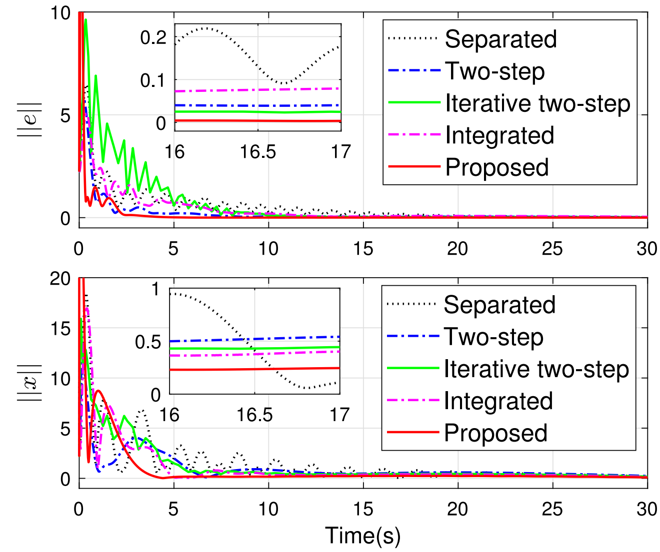

Fig. 5. FE and FTC performance under different strategies.

$$D = \begin{bmatrix} 0 & 0 & \frac{1}{m_1} & \frac{1}{m_2} \end{bmatrix}^\top, \; C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; E = [0 \; 1]^\top,$$

where c_1 and c_2 are the viscous frictions of the masses; θ and k_2 are the stiffnesses of the first and second masses, respectively; σ_{c2} is the variation of c_2 from its nominal value. The simulation uses: $m_1 = 10$ kg, $m_2 = 1$ kg, $k_2 = 5$ N/m, $c_1 = 0.2$ N s/m, $c_2 = 0.3$ N s/m, $\theta(t) = 13 - 3\cos(0.86\pi t/18)$ N/m, $\sigma_{c2} = 0.01\sin(0.1\pi t)$, $d = 0.01\sin(\pi t)$, $f_a = 10\sin(0.1t)$, $f_s = 0.1\cos(0.2t)$.

Since $\theta(t) \in [10, 16]$, designing $\phi_1 = (\theta(t) - 10)/6$ and $\phi_2 = (16 - \theta(t))/6$. It is verified that the example system satisfies Assumptions 2.1–2.3 with $\mathcal{M} = 0.01 \times A_p$, $\mathcal{F}(t) = \sin(0.1\pi t) \times I_4$, $\mathcal{N} = I_4$. The parameters used to run Algorithm 1 are $C_e = [\mathbf{0} \; 0.1 \times I_4]$, $V^{(0)} = 4 \times I_2$, $\alpha_1 = 1.0\text{e}{-5}$, $\alpha_2 = 1.0\text{e}{-5}$, $\beta_1 = 1.0\text{e}{-5}$, $\beta_2 = 2.0\text{e}{-6}$, $\epsilon = 2.0\text{e}{-3}$ and $\epsilon_0 = 2.2204\text{e}{-16}$. The algorithm terminates in 16 iterations. The evolution of J_c, J_o, trace$(W^\top W)$ and trace$(V^\top V)$ are shown in Fig. 4. The FTC controller cost J_c monotonically decreases and converges to the local minimum 1.7271e−07, while trace$(W^\top W)$ and trace$(V^\top V)$ monotonically increase. However, the FE observer cost J_o does not converge. These results coincide exactly with the convergence analysis in Section 4.1.

Comparative closed-loop simulations are performed using the observer (3) and controller (5) with the gains L_i and K_i, $i = 1, 2$, obtained from the *Separated strategy* (Liu et al., 2017), *Two-step strategy* (Shi & Patton, 2015), *Iterative two-step strategy* (Chen et al., 2019), *Integrated strategy* (Lan & Patton, 2016c), and *Proposed strategy*. The initial conditions are $x(0) = [1 \; 2 \; 0 \; 0]^\top$ and $\hat{\bar{x}}(0) = \mathbf{0}$. The estimation errors and state responses under different designs are shown in Fig. 5. All the strategies, except the *Separated strategy*, can achieve accurate estimation and state stabilisation at steady-state. This is because the mutual coupling between FE and FTC is ignored by the *Separated strategy*. Compared to the *Iterative two-step strategy*, the *Two-step strategy* has better estimation performance but worse state stabilisation. This is because the effects of estimation errors on the control performance are considered by the former but not by the later. Since the *Integrated strategy* considers the mutual coupling, it achieves better estimation and stabilisation than the *Iterative two-step strategy*. For this example, the *Proposed strategy* achieves even (slightly) better performance than the *Integrated strategy*, because it minimises the coupling effects and the separation of FE and FTC designs offers more freedom to obtain better solutions. The above results show that the *Proposed strategy* has the advantage in achieving better estimation and stabilisation than the *Integrated strategy*, by using an LMI formulation with less computational complexity.

6. Conclusion

An iterative strategy is developed for achieving robust integration of FE and FTC for uncertain LPV systems, via tuning two weighting matrices to minimise the coupling effects and refine the design gains. The iteration converges to a (local) minimum in finite steps under the stopping criterion. Compared to the existing strategies, the proposed strategy has reduced complexity in formulating and computing the LMI optimisation problems. Simulation results demonstrated efficacy of the proposed strategy and its advantages over the existing strategies. The strategy is directly applicable to the robust integration of FE and FTC for many other systems such as linear time-invariant, Lipschitz nonlinear and T-S fuzzy systems. Future research will be extending the iterative strategy for large-scale interconnected systems.

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