GAUSS-MANIN FROM SCRATCH: THEME, VARIATIONS AND FANTASIA.

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ABSTRACT. We discuss the explicit construction of Gauß-Manin connections on the cohomology of families of low genus Riemann surfaces represented as curves with branch points in general position. The approach is distinguished from others by the use of an equivariant basis of differential forms.

1. INTRODUCTION

This paper deals with the calculation of Gauß-Manin connections on the cohomologies of families of Riemann surfaces represented as curves.

The paradigm is the Picard-Fuchs equation for a one parameter family of elliptic curves but we wish to repeat the treatment for curves in general position and of greater genus. By this we mean that the curve equation will be of higher degree and have a maximal set of parameters. The motivation for this arises from the desire to describe the behaviour of the multivariate σ function as a function of parameters [5, 14, 15]. The second logarithmic derivatives of σ are the generalised Weierstraß $\wp$ functions defined on the Jacobian varieties of said curves. The study originates in work of Frobenius and Stickelberger [19].

The current paper therefore belongs to the classical problem of describing periods of Abelian integrals as functions of parameters [7, 6].

One consequence of the Picard-Fuchs equations is the result [7] that the periods of elliptic integrals are single valued, holomorphic functions of the J-invariant. Another application is that critical points of quotients of periods are related to the existence of limit cycles for certain planar vector fields [8, 11].

Abelian integrals are ubiquitous in the Algebro-Geometric theory of Integrable Systems. For a recent general collection of articles, see [12] and for more specific material, [20, 21]. The periods of such integrals are important in the physics of such systems, in particular the manner in which the periods vary with respect to physical parameters and conserved quantities such as energy.

We start with a recap of the Picard-Fuchs equation for the Legendre family of genus one curves, describing its Hamiltonian structure and connection with the heat equation for the σ-function first described by Weierstraß, before introducing a property of equivariance that pertains to objects associated with the maximal parameter curve. Polynomial functions of the parameters that are invariant under an $\mathfrak{sl}_2(\mathbb{C})$ action are the classical polynomial invariants of binary forms described in representation theory. We wish to use the representation theory [23, 29] to clarify the structure of the Gauß-Manin connection. A crucial component is the introduction of an equivariant basis for cohomology rather than the basis traditionally employed. We develop

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the calculation of the connection form for the generic genus one curve in stages successively simplifying the calculation. Finally we apply the simplified approach to the generic genus two curve subject to a conjecture about the action of invariant vector fields on discriminants.

Our approach differs from the usual one, where the curve is taken to be in canonical form, and can be regarded as a significant modification of that in [4, 6].

All calculations presented here are appropriate to the hyperelliptic class.

2. *Theme* PICARD-FUCHS

The treatment of the Picard-Fuchs equation in this section draws on [7, 10].

The Legendre family is a one (complex) parameter family of elliptic curves in $\mathbb{C}^2$ of the form,

$$y^2 = x(x-1)(x-\lambda). \tag{1}$$

As a function of x, y is multivalued with branch points for $x \in \{0, 1, \lambda, \infty\}$ and we assume $\lambda \neq 0, 1$ for the curve to be nonsingular. The discriminant of the curve is $(\lambda-1)^2\lambda^2$.

By examination in a local coordinate at any point of the curve it is verified that $\omega = \frac{dx}{y}$ is a globally defined holomorphic (first kind) differential on the curve and that $\eta = \frac{xdx}{y}$ is a meromorphic (second kind) differential, that is, it has vanishing residues at all poles.

For a closed curve γ on the Riemann surface of the curve, period integrals are defined as $\pi = \int_\gamma \xi$ where ξ is either ω or η. One verifies that for, say, ω the second order, Fuchsian equation,

$$2\lambda(\lambda-1)\frac{d^2\omega}{d\lambda^2} + 2(2\lambda-1)\frac{d\omega}{d\lambda} + \frac{1}{2}\omega = df, \tag{2}$$

holds where

$$f = -\frac{y}{(x-\lambda)^2} = -\frac{x(x-1)}{y}.$$

Since f is a rational function on the curve, $\int_\gamma df = 0$, and the period integrals themselves satisfy the second-order, hypergeometric equation

$$\lambda(\lambda-1)\pi'' + (2\lambda-1)\pi' + \frac{1}{4}\pi = 0. \tag{3}$$

For our later purposes it is better to rewrite the equation (2) as a first order system. Using $\{\omega, \eta\}$ as a basis for cohomology,

$$\begin{aligned}
\frac{d\omega}{d\lambda} &= \alpha_1\omega + \beta_1\eta + dA_1 \sim \alpha_1\omega + \beta_1\eta, \\
\frac{d\eta}{d\lambda} &= \alpha_2\omega + \beta_2\eta + dA_2 \sim \alpha_2\omega + \beta_2\eta,
\end{aligned}$$

where $\sim$ denotes equality modulo an exact differential of a rational function and

$$\begin{aligned}
\alpha_1 &= \frac{1}{2(1-\lambda)}, \quad \beta_1 = -\tfrac{1}{2\lambda(1-\lambda)}, \quad A_1 = \frac{x(x-1)}{\lambda(1-\lambda)}, \\
\alpha_2 &= \frac{1}{2(1-\lambda)}, \quad \beta_2 = -\tfrac{1}{2(1-\lambda)}, \quad A_2 = -\frac{x(x-1)}{1-\lambda}.
\end{aligned}$$

Equivalently we may write this as a connection form:

$$d\theta \sim \Gamma\theta, \quad \theta = \begin{bmatrix} \omega \\ \eta \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} d\lambda. \tag{4}$$

Note that we are abusing notation by using the symbol d to denote both the differential on the curve, $df = f_{,x}\,dx + f_{,y}\,dy$, and the differential on cohomology, $d\omega = \omega_{,\lambda}\,d\lambda$. It will be apparent which interpretation is intended in any particular context.

It's worth indicating explicitly how these expressions are obtained.

Firstly note that by differentiating the curve relation with respect to each of λ and x,

$$2y\frac{dy}{d\lambda} = -x(x-1),$$

and

$$2y\frac{dy}{dx} = 3x^2 - 2(1+\lambda)x + \lambda.$$

We wish to express the differential of ω with respect to λ by finding α_1 and β_1 as functions of that parameter only, i.e.

$$\frac{x(x-1)}{2y^3} = \alpha_1\frac{1}{y} + \beta_1\frac{x}{y} + \frac{d}{dx}(\frac{g}{y}),$$

g being a function of x, or equivalently

$$\frac{1}{2}x(x-1) = (\alpha_1 + \beta_1 x + \frac{dg}{dx})x(x-1)(x-\lambda) - \frac{1}{2}g(3x^2 - 2(1+\lambda)x + \lambda),$$

requiring that $g(x)$ be quadratic in x.

This relation is of degree four in x and so we obtain, writing $g(x) = g_2x^2 + g_1x + g_0$, five linear equations solvable for $\alpha_1, \beta_1, g_2, g_1$ and g_0. These give the expressions in the connection form (4) above.

Converting back to pairs of second order equations for ω and η we recover the Fuchsian equations (2),

$$\begin{aligned} \frac{d^2\omega}{d\lambda^2} &\sim -\frac{1-2\lambda}{\lambda(1-\lambda)}\frac{d\omega}{d\lambda} + \frac{1}{4\lambda(1-\lambda)}\omega, \\ \frac{d^2\eta}{d\lambda^2} &\sim \frac{1}{(1-\lambda)}\frac{d\eta}{d\lambda} - \frac{1}{4\lambda(1-\lambda)}\eta. \end{aligned}$$

The cohomology carries a symplectic form, $\Omega = \omega \wedge \eta$ which is λ invariant,

$$\begin{aligned} \frac{d\Omega}{d\lambda} &= \frac{d\omega}{d\lambda} \wedge \eta + \omega \wedge \frac{d\eta}{d\lambda} \\ &\sim (\alpha_1 + \beta_2)\Omega = 0, \end{aligned}$$

and hence the connection equations can be interpreted as Hamiltonian flows, λ playing the role of time, on the two dimensional cohomology space with λ dependent Hamiltonian,

$$H(q,p) = \frac{1}{4(1-\lambda)}(q^2 + 2qp - \frac{1}{\lambda}p^2),$$

writing the more conventional q, p for ω, η respectively.

3. *Variation 1* THE WEIERSTRASS FORM

Our first variation on the theme introduces a second parameter so that our curves are now described by the two classical invariants of binary forms of degree 4.

A generic form of an elliptic curve is

$$y^2 = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 \tag{5}$$

with four branch points in the complex plane, regular at ∞ for $a_0 \neq 0$. Three parameters can be removed using Möbius transformations,

$$\begin{aligned} x &= \frac{AX+B}{CX+D}, \\ y &= \frac{Y}{(CX+D)^2}, \end{aligned} \tag{6}$$

with $AD - BC = 1$. So a common choice for A, B, C and D is to set $a_0 = a_2 = 0$ and $a_1 = 4$ giving the Weierstraß form,

$$Y^2 = 4X^3 + \mu X + \nu.$$

The remaining parameters μ and ν are functions of the a_i, namely,

$$\mu = -a_0a_4 + 4a_1a_3 - 3a_2^2,$$

and

$$\nu = -a_0a_2a_4 + a_0a_3^2 + a_1^2a_4 - 2a_1a_2a_3 + a_2^3.$$

These are polynomial invariants in the parameters (coefficients) of the quartic form in (5) under the transformations (6), [23].

Let us repeat the previous calculation for this (X, Y) curve in the parameters, μ and ν with $\omega = \frac{dX}{Y}$, and $\eta = \frac{XdX}{Y}$. As before these are differentials of the first and second kinds on the Weierstraß curve.

Via a calculation a little more involved than for the Legendre family but the same in principle, we obtain the Gauß-Manin connection,

$$\Gamma = \begin{bmatrix} -\frac{1}{4}\frac{\mu^2}{\Delta} & \frac{9}{2}\frac{\nu}{\Delta} \\ \\ \frac{3}{8}\frac{\mu\nu}{\Delta} & \frac{1}{4}\frac{\mu^2}{\Delta} \end{bmatrix} d\mu + \begin{bmatrix} -\frac{9}{2}\frac{\nu}{\Delta} & -3\frac{\mu}{\Delta} \\ \\ -\frac{1}{4}\frac{\mu^2}{\Delta} & \frac{9}{2}\frac{\nu}{\Delta} \end{bmatrix} d\nu,$$

where $\Delta = \mu^3 + 27\nu^2$ is the discriminant of the quartic form.

The tracelessness of the connection matrices ensure that the form $\omega \wedge \eta$ is symplectic and the μ and ν flows Hamiltonian with Hamiltonian functions,

$$M(q,p) = \frac{1}{16\Delta}\left(36\nu p^2 - 4\mu^2 pq - 3\mu\nu q^2\right)$$

and

$$N(q,p) = \frac{1}{8\Delta}\left(-12\mu p^2 - 36\nu pq + \mu^2 q^2\right).$$

These Hamiltonians commute in the sense that (see [13])

$$\frac{\partial M}{\partial \nu} - \frac{\partial N}{\partial \mu} + \{M, N\}_{p,q} = 0.$$

We would like to address the question of describing the Gauß-Manin connection for the *generic* curve of genus g.

Here we restrict ourselves to the hyperelliptic cases,

$$y^2 = \sum_{r=0}^{2g+2} \binom{2g+2}{r} a_r x^{2g+2-r},$$

where g is the genus.

Transforming to the canonical form,

$$Y^2 = 4X^{2g+1} + \sum_{r=0}^{2g-1} \mu_i X^{2g+2-r},$$

using a Möbius transform

$$\begin{aligned} x &= \frac{AX+B}{CX+D}, \\ y &= \frac{Y}{(CX+D)^{g+1}}, \end{aligned} \tag{7}$$

we find that the μ_i are invariants but not in general *polynomial* invariants in the a_i.

This is already apparent for genus two. For example, the degree two invariant in that instance must satisfy

$$I_2 \equiv a_0 a_6 - 6a_1 a_5 + 15 a_2 a_4 - 10 a_3^2 = -\frac{2}{3}\mu_1 - \frac{1}{40}\mu_3^2,$$

and so the μ_i are generally roots of algebraic equations over the ring of polynomial invariants.

In the case of the Weierstraß form above the connection coefficients are rational in μ and λ and, as happens for this genus one representation, in the a_i also, but it is not clear if this is so for higher genus. A subsidiary question then is whether there is a representation for the Gauß-Manin connection at genus g which is rational or polynomial in the parameters of the generic curve.

4. The heat equation and the genus one curve

This section constitutes a brief *intermezzo* in which we explain the relation of the connection equations to those determining the σ function. Indeed, this is part of the motivation for the study.

The Hamiltonians M and N give rise to heat equations for the σ function via "quantization": $q \to u, p \to \partial_u$. Thus

$$\begin{aligned} \frac{\partial\sigma}{\partial\mu} = M(u,\partial_u)\sigma &= \frac{1}{16\Delta}(36\nu\partial_u^2\sigma - 2\mu^2(2u\partial_u\sigma + \sigma) - 3\mu\nu u^2\sigma), \\ \frac{\partial\sigma}{\partial\nu} = N(u,\partial_u)\sigma &= \frac{1}{8\Delta}(-12\mu\partial_u^2\sigma - 18\nu(2u\partial_u\sigma + \sigma) - \mu^2 u^2\sigma). \end{aligned}$$

If we replace the ∂_μ and ∂_ν vector fields with the combinations

$$\begin{aligned} \mathcal{D}_0 &= \mu\partial_\mu + \frac{3}{2}\nu\partial_\nu, \\ \mathcal{D}_1 &= \frac{3}{2}\nu\partial_\mu - \frac{1}{12}\mu^2\partial_\nu, \end{aligned} \tag{8}$$

then the connection equations and the Hamiltonian equations take on much simpler forms,

$$\begin{aligned}
\mathcal{D}_0 \begin{bmatrix} \omega \\ \eta \end{bmatrix} &= \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \omega \\ \eta \end{bmatrix}, \\
\mathcal{D}_1 \begin{bmatrix} \omega \\ \eta \end{bmatrix} &= \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{48}\mu & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \eta \end{bmatrix}, \\
\mathcal{D}_0\sigma &= \frac{1}{4}u\sigma_{,u} + \frac{1}{8}\sigma\,, \\
\mathcal{D}_1\sigma &= \frac{1}{8}\sigma_{,uu} - \frac{1}{96}\mu u^2\sigma\,,
\end{aligned} \tag{9}$$

in which the coefficients are *polynomial* in the invariants.

Unlike the derivatives $\frac{\partial}{\partial\mu}$ and $\frac{\partial}{\partial\nu}$, $\mathcal{D}_0$ and $\mathcal{D}_1$ are tangent to the discriminant variety, $\Delta = 0$:

$$\mathcal{D}_0\Delta = 3\Delta$$

and

$$\mathcal{D}_1\Delta = 0,$$

both the coefficients of the connection and of the differential operators being polynomial on the moduli space of invariants.

We will see that $\mathcal{D}_0$ and $\mathcal{D}_1$ arise quite naturally fromin terms of the invariant theory.

5. Equivariance

In this and the next section we wish to change to a different tonal philosophy by introducing new machinery. Usually one reduces the curve to its simplest form by a special choice of coordinates as in section 3. Alternatively, we can keep the curve in general position and think in terms of representation theory.

The role of equivariance in the theory arises from the transformations of the genus g hyperelliptic model:

$$y^2 = \sum_{i=0}^{2g+2} \binom{2g+2}{i} a_i x^{2g+2-i} \to Y^2 = \sum_{i=0}^{2g+2} \binom{2g+2}{i} A_i X^{2g+2-i},$$

where

$$y = \frac{Y}{(CX+D)^{g+1}}, \quad x = \frac{AX+B}{CX+D},$$

A, B, C and D being complex constants with $AD - BC = 1$.

The corresponding infinitesimal action is described by the $\mathfrak{sl}_2(\mathbb{C})$ generators

$$\mathbf{e} = \partial_x - \sum_{i=1}^{2g+2} i a_{i-1}\partial_{a_i},$$

$$\mathbf{f} = (g+1)xy\partial_y + x^2\partial_x + \sum_{i=0}^{2g+1} (2g+2-i)a_{i+1}\partial_{a_i},$$

and

$$\mathbf{h} = [\mathbf{e}, \mathbf{f}] = (g+1)y\partial_y + 2x\partial_x - \sum_{i=0}^{2g+2} (2g+2-2i)a_i\partial_{a_i},$$

under which the coefficients $\{a_0, a_1, \ldots, a_{2g+2}\}$ constitute a $2g + 3$ dimensional irreducible representation.

As an aside note that this action has proven useful [1] in discussions of the generalised $\wp_{ij}$-functions associated with the hyperelliptic curves. They are defined by rational expressions

$$y_i y - \mathbf{x}_i H_{g+2}(\wp_{ij})\mathbf{x}^t = 0, \quad \mathbf{x}_i = (x_i^{g+1}, x_i^g, \ldots, 1), \quad i = 1, \ldots g,$$

where (x_i, y_i) for $i = 1, \ldots g$ are arbitrary points on the genus g curve and H_{g+2} is a $(g+2) \times (g+2)$ matrix containing the $\wp_{ij}$ and the curve's coefficients.

The complicated singularity calculations that lead to the analogues of the classical (genus one) Weierstraß differential equation can be considerably abbreviated using the $\mathfrak{sl}_2$ action, leading to the simple form,

$$\mathbb{P}(\wp_{ijk}, \mathbf{l}_1, ..., \mathbf{l}_{g-1})\mathbb{P}(\wp_{ijk}, \mathbf{l}'_1, ..., \mathbf{l}'_{g-1}) = -\frac{1}{4}\begin{vmatrix} H_{g+2} & \mathbf{l}_1^t & \ldots & \mathbf{l}_{g-1}^t \\ \mathbf{l}'_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ \mathbf{l}'_{g-1} & 0 & \ldots & 0 \end{vmatrix},$$

where $\mathbf{l}_i$ and $\mathbf{l}'_i$ are $g+2$ dimensional vector parameters and the object $\mathbb{P}$ is formed from the dual pairing

$$\left(Symm \bigotimes^3 V_g\right) \times \left(\bigwedge^{g-1} V_{g+2}\right) \to \mathbb{C}.$$

In this pairing the symmetric arguments correspond to the $\wp_{ijk}$ (the analogues of $\wp'$ in the genus one Weierstraß case and the antisymmetric arguments to Plücker coordinates on the Grassmannian of $g-1$ planes in $\mathbb{C}^{g+2}$ spanned by the $2g-2$ vector parameters.

For details in the cases $g = 1$, 2 and 3 see [1].

Another incarnation of equivariance is described in [2] where the action is on divisor spaces on the curve.

6. Polynomial and Operator Invariants

The infinitesimal $\mathfrak{sl}_2$ actions of the Möbius group on the a_i and on the derivatives with respect to the a_i are given by

$$\begin{array}{ccccc} -(n-i)a_{i+1} & \stackrel{\mathbf{f}}{\leftarrow} & a_i & \stackrel{\mathbf{e}}{\rightarrow} & -ia_{i-1}, \\ (n-i+1)\partial_{a_{i-1}} & \stackrel{\mathbf{f}}{\leftarrow} & \partial_{a_i} & \stackrel{\mathbf{e}}{\rightarrow} & (i+1)\partial_{a_{i+1}}. \end{array}$$

Symmetric $\mathbb{C}$-tensor products of the $2g+3$ dimensional $\mathfrak{s}l_2$ representation with basis elements $\{a_0, \ldots, a_{2g+2}\}$ can be thought of as polynomial expressions in these variables. The occurrence of irreducible representations within these tensor products is ascertained from their formal characters [29].

Thus for the quartic (genus one) polynomial the $\{a_i\}$ form a five dimensional representation and the formal character of its symmetric tensor algebra is

$$\left[(1-\lambda^4 t_1^4)(1-\lambda^4 t_1^3 t_2)(1-\lambda^4 t_1^2 t_2^2)(1-\lambda^4 t_1 t_2^3)(1-\lambda^4 t_2^4)\right]^{-1}.$$

This decomposes at λ^{4n} into sums of expressions

$$t_1^{2n} t_2^{2n}$$

$$t_1^{2n-1}t_2^{2n-1}(t_1^2+t_1t_2+t_2^2)$$

$$t_1^{2n-2}t_2^{2n-2}(t_1^4+t_1^3t_2+t_1^2t_2^2+t_1t_2^3+t_2^4)$$

$$\vdots$$

and so on, each corresponding to irreducible polynomial representations of odd dimension.

By counting the occurrences of representations of dimension one (invariants) at each degree we sketch out a table as follows

$Weight$	0	1	2	3	4	5	6	7	8	9	...
$\#Invariants$	1	0	1	1	1	1	2	1	2	2	...
	1		I_2	I_3	I_2^2	I_2I_3	I_2^3	$I_2^2I_3$	I_2^4	$I_2^3I_3$	...
							I_3^2		$I_2I_3^2$	I_3^3	...

(By *weight* here is meant the value of n in the terms of the power series expansion.)

In this instance all polynomial invariants are generated using only those of weights 2 and 3 so the ring of polynomial invariants is $\mathbb{C}[I_2, I_3]$.

We can also tensor with the five dimensional dual representation with basis

$$\{\partial_{a_0}, \partial_{a_1}, \partial_{a_2}, \partial_{a_3}, \partial_{a_4}\},$$

and in this way create invariant differential operators using the bases of five dimensional polynomial representations. We define their degrees to be one less than the degree of the polynomial coefficients.

For succinctness we introduce the following notation:

$$\left[\bigotimes^p \mathbf{a}\right]_d \quad : \quad \text{degree } p \text{ irreducible representation of dimension } d\,;$$

$$\left\langle \left[\bigotimes^p \mathbf{a}\right]_d \otimes \nabla_{\mathbf{a}} \right\rangle_D \quad : \quad \begin{array}{c}\text{degree } p-1 \text{ differential operator,}\\ \text{an irreducible representation of dimension } D.\end{array}$$

Clearly the action of an invariant differential operator ($D=1$) of degree $p-1$ on an invariant ($d=1$) of degree q is to produce an invariant of degree $q+p-1$.

In the case of the sextic (the genus 2 hyperelliptic curve) the formal character of the symmetric tensor algebra is

$$\Pi_{i=0}^6(1-\lambda^6 t_1^{6-i}t_2^i)^{-1}.$$

It decomposes at λ^{6n} into sums of simple characters as before.

Counting the occurances of invariants we find generators I_2, I_4, I_6 and I_{10} at even degree and I_{15} at odd degree.

Because the Möbius transformations account for three degrees of freedom amongst the seven a_i, we expect four invariants instead of the five listed above. In fact, however, at degree 30 too many possible invariants can be generated and so there must be a relation:

$$I_{15}^2 \in \mathbb{C}[I_2, I_4, I_6, I_{10}].$$

The situation for higher genus curves becomes increasingly sporadic [24].

7. EQUIVARIANT FIRST AND SECOND KIND DIFFERENTIALS

In order to use the representation theory in constructing the connection coefficients we need to find bases of first and second kind differentials which transform as irreducible representations for $\mathfrak{sl}_2$. We call these *equivariant differentials* of the first and second kind and we must emphasise that they differ from the conventional differentials used elsewhere in the literature [3, 5].

For the genus g hyperelliptic curve

$$y^2 = a(x) = \sum_{i=0}^{2g+2} \binom{2g+2}{i} a_i x^{2g+2-i},$$

we define one-forms (differentials)

$$\omega_i = \frac{x^{i-1}dx}{y}, \quad i = 1, \dots g$$

and

$$\eta_1 = \frac{d^{g+1}a}{dx^{g+1}}\frac{dx}{y},$$
$$\eta_i = \mathbf{f}^{i-1}\eta_1, \quad i =, 2 \dots g.$$

Theorem 7.1. *Given the definitions of the ω_i and the iterative definitions of the η_i above:*

(i) $\{\omega_i\}$ *are first kind (holomorphic) and a g dimensional irreducible representation;*

(ii) $\{\eta_i\}$ *are second kind (zero residue, meromorphic) and a g dimensional irreducible representation. In particular* $\mathbf{f}\eta_g \sim 0$.

Proof. The first kind property is straightforward since the holomorphic property is classical [16]. Note that $\mathbf{e}(dx) = 0$ and $\mathbf{f}(dx) = 2x\,dx$. Then the action of $\mathfrak{sl}_2$ is

$$\mathbf{e}\omega_i = (i-1)\omega_{i-1}$$

and

$$\begin{aligned}\mathbf{f}\omega_i &= (i-1)\frac{x^i dx}{y} + 2x\frac{x^{i-1}dx}{y} - (g+1)xy\frac{x^{i-1}dx}{y^2} \\ &= (i-g)\omega_{i+1},\end{aligned}$$

vanishing when $i = g$.

In the case of the η_i we need to establish that the differentials have only poles with zero residues (are of the second kind) and we need to show that $\mathbf{e}\eta_1 = 0$ and $\mathbf{f}\eta_g \sim 0$.

The proof is more involved and is detailed in Appendix A.

8. *Variation 2*: GAUSS-MANIN, GENERIC GENUS ONE CURVE

Having assembled the orchestra we now make a first attempt at a performance by repeating the earlier programme for the generic curve but using the equivariant basis

$$\omega = \frac{dx}{y}, \quad \eta = \frac{a''(x)dx}{y},$$

and the quartic with five parameters $(a_0, \dots, a_4)$.

This will not be a great success. The simplicity for which we hoped will not emerge from the chaos.

We will abbreviate ∂_{a_i} to ∂_i henceforth.

It is enough to find (by quite a lengthy calculation analogous to those in the earlier part of the paper but best done in MAPLE) the relations belonging to the kernel of $\mathbf{f}$, that is of lowest weight in the Lie algebraic sense,

$$\begin{aligned}\partial_0\omega &\sim \frac{18(a_4a_2-a_3^2)I_3-a_4I_2^2}{4(I_2^3-27I_3^2)}\omega+\frac{6(a_4a_2-a_3^2)I_2-9a_4I_2^2}{8(I_2^3-27I_3^2)}\eta\,,\\ \partial_0\eta &\sim \frac{-2(a_4a_0-a_3^2)I_2^2+3a_4I_2I_3}{2(I_2^3-27I_3^2)}\omega+\frac{-18(a_4a_2-a_3^2)I_3+a_4I_2^2}{4(I_2^3-27I_3^2)}\eta\,,\end{aligned}$$

because we can then apply $\mathbf{e}$ successively to obtain the two sets of five Gauß-Manin connection equations corresponding to the flows $\{\partial_0,\partial_1,\partial_2,\partial_3,\partial_4\}$.

The invariants are given by the expressions

$$I_2 = a_0a_4 - 4a_1a_3 + 3a_2^2$$

and

$$I_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}.$$

Since in this case both ω and η are one-dimensional representations (invariant differential forms) it is only the coefficients, rational in the parameters that transform under $\mathbf{e}$. In this connection note that under the $\mathbf{e},\mathbf{f}$ action the quadratic terms appearing in the numerators are a basis for a five dimensional representation:

$$a_4a_2-a_3^3 \stackrel{\mathbf{e}}{\rightarrow} 2(a_3a_2-a_4a_1) \stackrel{\mathbf{e}/2}{\rightarrow} a_4a_0-3a_2^2+2a_3a_1 \stackrel{\mathbf{e}/3}{\rightarrow}$$

$$2(a_2a_1-a_3a_0) \stackrel{\mathbf{e}/4}{\rightarrow} a_2a_0-a_1^2 \rightarrow 0.$$

At this stage, given the complexity of these expressions, the equivariant approach confers no advantage over the earlier ones.

9. Variation 3: Gauss-Manin, invariant differential operators

We now repeat the treatment using invariant differential operators instead of the ∂_i. This effectively rearranges results of the last section into a compact form. The calculations dramatically simplify to the point where they are easy enough to do by hand!

Because the derivatives $\{\partial_0,\dots,\partial_4\}$ constitute a representation dual to the coefficients $\{a_0,\dots,a_4\}$, such differential operators correspond to covariants of the kind listed or described in [17, 29, 28] when monomials in x are replaced by derivations.

A $\mathbb{C}$-linearly independent set of five such operators are

$$\begin{aligned}\mathcal{D}_0 = \left\langle \left[\bigotimes^1 \mathbf{a}\right]_5 \otimes \nabla_{\mathbf{a}} \right\rangle_1 &= a_0\partial_0+a_1\partial_1+a_2\partial_2+a_3\partial_3+a_4\partial_4\\ \mathcal{D}_1 = \left\langle \left[\bigotimes^2 \mathbf{a}\right]_5 \otimes \nabla_{\mathbf{a}} \right\rangle_1 &= (a_2a_0-a_1^2)\partial_0+\frac{1}{2}(a_3a_0-a_2a_1)\partial_1+\dots,\end{aligned}$$

and the three dimensional irreducible representation $\left\langle \left[\bigotimes^1 \mathbf{a}\right]_5 \otimes \nabla_{\mathbf{a}} \right\rangle_3$ with basis

$$\begin{aligned}
\mathcal{L}_1 &= 4a_1\partial_0 + 3a_2\partial_1 + 2a_3\partial_2 + a_4\partial_3, \\
\mathcal{L}_0 &= -4a_0\partial_0 - 2a_1\partial_1 + 2a_3\partial_3 + 4a_4\partial_4, \\
\mathcal{L}_{-1} &= -a_0\partial_1 - 2a_1\partial_2 - 3a_2\partial_3 - 4a_3\partial_4,
\end{aligned}$$

which we recognise as parts of the $\mathbf{e}$, $\mathbf{h}$ and $\mathbf{f}$ operators introduced earlier.

Is it straightforward to show that the action of the invariant differential operators on $a(x) = a_0x^4 + 4a_1x^3 + \ldots$ is

$$\mathcal{D}_0(a) = a,$$
$$\mathcal{D}_1(a) = \frac{1}{3}aa'' - \frac{1}{4}a'^2.$$

Then, for example, for $\omega = \frac{dx}{y}$,

$$\begin{aligned}
\mathcal{D}_1\omega &= -\mathcal{D}_1(y)\frac{dx}{y^2} \\
&= -\mathcal{D}_1(a)\frac{dx}{2y^3} \\
&= -(4aa'' - 3a'^2)\frac{dx}{24y^3} \\
&= -\frac{a''dx}{6y} + \frac{a'^2dx}{8y^3} \\
&= -\frac{1}{6}\eta + \frac{a'y'dx}{4y^2} \\
&= -\frac{1}{6}\eta - d(\frac{a'}{4y}) + \frac{a''dx}{4y} \\
&\sim \frac{1}{12}\eta.
\end{aligned}$$

Continuing in this way for all five differential operators we obtain:

$$\mathcal{D}_0 \begin{bmatrix} \omega \\ \eta \end{bmatrix} \sim \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \omega \\ \eta \end{bmatrix},$$

$$\mathcal{D}_1 \begin{bmatrix} \omega \\ \eta \end{bmatrix} \sim \begin{bmatrix} 0 & \frac{1}{12} \\ -I_2 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ \eta \end{bmatrix},$$

and

$$\mathcal{L}_i \begin{bmatrix} \omega \\ \eta \end{bmatrix} \sim 0 \quad \text{for } i = -1, 0, 1.$$

Note that

$$\begin{aligned}
\mathcal{D}_0 I_2 &= 2I_2, \\
\mathcal{D}_0 I_3 &= 3I_3, \\
\mathcal{D}_1 I_2 &= 3I_3, \\
\mathcal{D}_1 I_3 &= 6I_2^2,
\end{aligned}$$

so that

$$\mathcal{D}_0 = 2I_2\frac{\partial}{\partial I_2} + 3I_3\frac{\partial}{\partial I_3}$$

and

$$\mathcal{D}_1 = 3I_3\frac{\partial}{\partial I_2} + \frac{1}{6}I_2^2\frac{\partial}{\partial I_3}.$$

By using invariant operators we have recovered, up to normalizations, the special choice of differential operator and the diagonal/antidiagonal forms for the connection matrices we found earlier (9).

An important property of the invariant operators is that, unlike the ∂_i, they preserve the discriminant variety, $\Delta = 0$:

$$\begin{aligned} \mathcal{D}_0\Delta &= 6\Delta \\ \mathcal{D}_1\Delta &= 0. \end{aligned} \tag{10}$$

10. Variation 4: Genus one by weights

Because the representation theory produces the simple results of the previous section it becomes feasible to analyse the situation purely in terms of the weight structure of $\mathfrak{sl}_2$ as one does when, for instance, decomposing tensor products of representations into irreducibles. This allows us to further simplify the calculation under the assumption that *in the case of the invariant differential operators the coefficients of the connection are polynomial in the invariants.*

To do this we introduce two flavours of *weight*: the coefficient weight,

$$[a_i] = 1, \quad [x] = 0, \quad [y] = \frac{1}{2}, \quad [\omega] = -\frac{1}{2}, \quad [\eta] = \frac{1}{2};$$

and a modified Sato weight,

$$[[a_i]] = i, \quad [[x]] = 1, \quad [[y]] = 2, \quad [[\omega]] = -1, \quad [[\eta]] = 1.$$

These weights are consistent with the defining relation of the curve being of definite weight, coefficient weight 1 and modified Sato weight 4 in this case.

The modified Sato weight is related to the Lie algebraic weight (eigenvalue of $\mathbf{h}$), for general genus, by

$$2(g+1)[X] = 2[[X]] - \text{ weight}_{\mathfrak{sl}_2}(X), \tag{11}$$

where X is any object having definite weights.

Now the argument goes like this.

We seek connection coefficients ${}_0\Gamma_{ij}$ for the $\mathcal{D}_0$ flow,

$$\mathcal{D}_0\begin{bmatrix} \omega \\ \eta \end{bmatrix} \sim \begin{bmatrix} {}_0\Gamma_{11} & {}_0\Gamma_{12} \\ {}_0\Gamma_{21} & -{}_0\Gamma_{11} \end{bmatrix}\begin{bmatrix} \omega \\ \eta \end{bmatrix}.$$

By the representation theory the coefficients must all be invariants because ω and η are invariants in this case. Balancing the weights on each side of the equations we obtain:

$$[\mathcal{D}_0] = 0 \implies [{}_0\Gamma_{11}] = 0,\ [{}_0\Gamma_{12}] = -1,\ [{}_0\Gamma_{21}] = 1;$$

and

$$[[\mathcal{D}_0]] = 0 \implies [[{}_0\Gamma_{11}]] = 0,\ [[{}_0\Gamma_{12}]] = -2,\ [[{}_0\Gamma_{21}]] = 2.$$

Under the assumption the coefficients are polynomial in the a_i they must be polynomial in the invariants I_2 and I_4. Since $[I_2] = 2$, $[[I_2]] = 4$, $[I_3] = 3$ and $[[I_3]] = 6$ we can only have that ${}_0\Gamma_{12} = {}_0\Gamma_{21} = 0$ and ${}_0\Gamma_{11}$ is constant.

In a similar manner for the $\mathcal{D}_1$ connection equations:

$$[\mathcal{D}_1] = 1 \implies [{}_1\Gamma_{11}] = 1,\ [{}_1\Gamma_{12}] = 0,\ [{}_1\Gamma_{21}] = 2;$$
$$[[\mathcal{D}_1]] = 2 \implies [[{}_1\Gamma_{11}]] = 2,\ [[{}_1\Gamma_{12}]] = 0,\ [[{}_1\Gamma_{21}]] = 4.$$

Hence ${}_1\Gamma_{21}$ is a constant multiple of I_2 and the other connection coefficients vanish.

For the the $\mathcal{L}_i$ operators there are no three dimensional irreducible, polynomial representations of the correct weights and hence the corresponding connection coefficients vanish.

In this manner we recover the results of the third variation without any explicit calculation, up to constant factors which can be deduced by, for example, evaluating the expressions at $x = \infty$, $y = \infty$.

11. *Fantasia*: Genus two by weights

As our finale we will now emulate the last approach for the genus two curve, again under the polynomial assumption,

$$y^2 = a(x) = \sum_{i=0}^{6} \binom{6}{i} a_i x^{6-i}.$$

There are seven parameters. The coefficient and modified Sato weights of variables and parameters, consistent with this relation, are

$$[a_i] = 1, \quad [x] = 0, \quad [y] = \frac{1}{2},$$

and

$$[[a_i]] = i, \quad [[x]] = 1, \quad [[y]] = 3.$$

The equivariant basis of cohomology is:

$$\omega_1 = \frac{dx}{y}, \quad \omega_2 = \frac{xdx}{y}, \quad \eta_1 = \frac{a'''dx}{y}, \quad \eta_2 = \mathbf{f}(\eta_1),$$

and the weights,

$$[\omega_1] = [\omega_2] = -\frac{1}{2}, \quad [\eta_1] = [\eta_2] = \frac{1}{2},$$
$$[[\omega_1]] = -2, \quad [[\omega_2]] = -1, \quad [[\eta_1]] = 1, \quad [[\eta_2]] = 2.$$

Invariant differential operators are defined by finding seven dimensional irreducibles inside the tensor products of the a_i's and contracting with the gradient operator. This is equivalent to constructing covariants of degree six in x (of which the ground form is the simplest example) and making the substitution

$$\binom{6}{i} x^{6-i} \to \partial_i.$$

The invariant operators are of the form

$$\mathcal{D}_p = \left\langle \left[\bigotimes^{p+1} \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_1$$

for p taking values $0,\ 2,\ 4,\ 6,\ 10$ and 15.

In this way the covariant ground form itself gives,

$$\mathcal{D}_0 = \left\langle \left[\bigotimes^1 \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_1 = a_0\partial_0 + a_1\partial_1 + \ldots + a_6\partial_6$$

and $\mathcal{D}_2$ has leading term

$$\mathcal{D}_2 = \left\langle \left[\bigotimes^3 \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_1 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \partial_0 + \ldots .$$

Expressions for the covariants are complicated and classical [10, 29, 22], best expressed using transvectants [25, 26], originating in the Ω process of Cayley [9]. Code is also available for generating them [27]

However, the explicit forms of these operators are not needed for our purposes, as we need only know their weights for what follows.

We proceed to calculate the connection matrices. Consider, for example,

$$\mathcal{D}_2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \sim \begin{bmatrix} {}_2\Gamma_{11} & {}_2\Gamma_{12} \\ {}_2\Gamma_{21} & {}_2\Gamma_{22} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} {}_2\Gamma'_{11} & {}_2\Gamma'_{12} \\ {}_2\Gamma'_{21} & {}_2\Gamma'_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

Since $[\mathcal{D}_2] = 2$ and $[[\mathcal{D}_2]] = 6$, we obtain for the coefficient weights,

$$[{}_2\Gamma_{11}] = [{}_2\Gamma_{12}] = [{}_2\Gamma_{21}] = [{}_2\Gamma_{22}] = 2,$$
$$[{}_2\Gamma'_{11}] = [{}_2\Gamma'_{12}] = [{}_2\Gamma'_{21}] = [{}_2\Gamma'_{22}] = 2,$$

and for the modified Sato weights,

$$[[{}_2\Gamma_{11}]] = 6,\ [[{}_2\Gamma_{12}]] = 5,\ [[{}_2\Gamma_{21}]] = 7,\ [[{}_2\Gamma_{22}]] = 6$$
$$[[{}_2\Gamma'_{11}]] = 3,\ [[{}_2\Gamma'_{12}]] = 2,\ [[{}_2\Gamma'_{21}]] = 4,\ [[{}_2\Gamma'_{22}]] = 3.$$

Since the weights of the invariants in this case are

$$[I_2] = 2,\ [I_4] = 4,\ [I_6] = 6,\ [I_{10}] = 10,$$
$$[[I_2]] = 6,\ [[I_4]] = 12,\ [[I_6]] = 18,\ [[I_{10}]] = 30,$$

the only polynomial solution is

$${}_2\Gamma_{11} = c_{11}I_2, \quad {}_2\Gamma_{22} = c_{22}I_2,$$

all other connection coefficients vanishing.

Note, in addition, that whilst in principle the ${}_2\Gamma_{ij}$ or ${}_2\Gamma'_{ij}$ could be part of a three dimensional irreducible representation (since then the product of the connection coefficients and the cohomology basis would belong to $\mathbf{3} \otimes \mathbf{2} = \mathbf{2} \oplus \mathbf{4}$) no such three dimensional representation exists at this degree.

Likewise for the second kind forms,

$$\mathcal{D}_2 \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \sim \begin{bmatrix} {}_2\Upsilon_{11} & {}_2\Upsilon_{12} \\ {}_2\Upsilon_{21} & {}_2\Upsilon_{22} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} {}_2\Upsilon'_{11} & {}_2\Upsilon'_{12} \\ {}_2\Upsilon'_{21} & {}_2\Upsilon'_{22} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$[{}_2\Upsilon_{11}] = [{}_2\Upsilon_{12}] = [{}_2\Upsilon_{21}] = [{}_2\Upsilon_{22}] = 3$$
$$[{}_2\Upsilon'_{11}] = [{}_2\Upsilon'_{12}] = [{}_2\Upsilon'_{21}] = [{}_2\Upsilon'_{22}] = 2$$
$$[[{}_2\Upsilon_{11}]] = 9,\ [[{}_2\Upsilon_{12}]] = 8,\ [[{}_2\Upsilon_{21}]] = 10,\ [[{}_2\Upsilon_{22}]] = 9$$
$$[[{}_2\Upsilon'_{11}]] = 6,\ [[{}_2\Upsilon'_{12}]] = 4,\ [[{}_2\Upsilon'_{21}]] = 7,\ [[{}_2\Upsilon'_{22}]] = 6.$$

The only polynomial solution is that ${}_2\Upsilon'_{11} = c'_{11}I_2$, ${}_2\Upsilon'_{22} = c'_{22}I_2$ all other connection coefficients vanishing.

The symplectic structure, $\omega_1 \wedge \eta_1 + \omega_2 \wedge \eta_2$ implies that $c'_{11} = -c_{11}$ and $c'_{22} = -c_{22}$.

Summarising

$$\mathcal{D}_2 \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \sim \begin{bmatrix} c_{11}I_2 & 0 \\ 0 & -c_{11}I_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$$

$$\mathcal{D}_2 \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \sim \begin{bmatrix} c'_{11}I_2 & 0 \\ 0 & -c'_{11}I_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

At coefficient weight four we have

$$\mathcal{D}_4 = \left\langle \left[\bigotimes^5 \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_1 ,$$

and a since there are four polynomial irreducible representations of dimension 7 at degree 5 we have a choice of invariant differential operator.

However two of the polynomial representations arise from the two seven dimensional irreduicibles at degree 3 by multiplication by I_2 and one by multiplying the seven dimensional degree one representation by I_4.

We therefore choose the remaining irreducible polynomial representation in order to form the invariant operator.

Clearly $[\mathcal{D}_4] = 4$.

To find $[[\mathcal{D}_4]]$ use the relation (11)

$$6[X] = 2[[X]] - \text{ weight}_{\mathfrak{sl}_2}(X).$$

Since the operator is invariant, its Lie algebraic weight is 0 and so $[[\mathcal{D}_4]] = 12$.

For the other polynomial differential operators $[\mathcal{D}_n] = n$ and $[[\mathcal{D}_n]] = 3n$. As before there are also operators $\mathcal{L}_i$ for $i = -1, 0, 1$ corresponding to $\left\langle \left[\bigotimes^1 \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_3$.

Similar arguments to those made for $\mathcal{D}_2$ apply to $\mathcal{D}_4$ and the other invariant differential operators and we summarise the results below:

$\mathcal{D}_0$ $\quad {}_0\Gamma_{11} = -{}_0\Upsilon'_{11} = {}_0\gamma_{11}$, ${}_0\Gamma_{22} = -{}_0\Upsilon'_{22} = {}_0\gamma_{22}$

$\mathcal{D}_2$ $\quad {}_2\Gamma_{11} = -{}_2\Upsilon'_{11} = {}_2\gamma_{11}I_2$, ${}_2\Gamma_{22} = -{}_2\Upsilon'_{22} = {}_2\gamma_{22}I_2$

$\mathcal{D}_4$ $\quad {}_4\Gamma_{11} = -{}_4\Upsilon'_{11}$ and ${}_4\Gamma_{22} = -{}_4\Upsilon'_{22}$ linear in I_4, I_2^2

$\mathcal{D}_6$ $\quad {}_6\Gamma_{11} = -{}_6\Upsilon'_{11}$ and ${}_6\Gamma_{22} = -{}_6\Upsilon'_{22}$ linear in I_6, I_4I_2, I_2^3

$\mathcal{D}_{10}$ $\quad {}_{10}\Gamma_{11} = -{}_{10}\Upsilon'_{11} =$ linear in I_{10}, I_6I_4, $I_6I_2^2$, $I_4^2I_2$, $I_4I_2^3$, I_2^5
${}_{10}\Gamma_{22} = -{}_{10}\Upsilon'_{22} =$ linear in I_{10}, I_6I_4, $I_6I_2^2$, $I_4^2I_2$, $I_4I_2^3$, I_2^5,

$\mathcal{D}_{15}$ $\quad {}_{15}\Gamma_{11} = -{}_{15}\Upsilon'_{11} = {}_{15}\gamma_{11}I_{15}$
${}_{15}\Gamma_{22} = -{}_{15}\Upsilon'_{22} = {}_{15}\gamma_{22}I_{15}$
${}_{15}\Gamma'_{11}$, ${}_{15}\Gamma'_{22}$, ${}_{15}\Upsilon_{11}$, ${}_{15}\Upsilon_{22}$ linear in $I_{10}I_6$, $I_{10}I_4I_2$, $I_6^2I_4, \ldots$

The coefficients can (in principle) be calculated by expansion at ∞ and their numerical values will depend on the explicit forms of the invariant operators.

Recall that I_{15}^2 is polynomial in I_2, I_4, I_6 and I_{10} so the connection coefficients belong to a quadratic extension of $\mathbb{C}[I_2, I_4, I_6, I_{10}]$.

Finally there is the three dimensional operator $\mathcal{L} = (\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1)$,

$$\mathcal{L} = \left\langle \left[\bigotimes^1 \mathbf{a} \right]_7 \otimes \nabla_{\mathbf{a}} \right\rangle_3 .$$

Note that

$$\begin{aligned} \mathcal{L}_{-1} &= \mathbf{e} - \partial_x, \\ \mathcal{L}_1 &= \mathbf{f} - x^2\partial_x - 3xy\partial_y. \end{aligned}$$

We can write the differentials either as, say, $\omega_1 = \frac{dx}{y}$ or $\omega_1 = \frac{dx}{\sqrt{a(x)}}$ and,

$$\begin{aligned} \mathcal{L}_{-1}\omega_1 \sim 0, \qquad & \mathcal{L}_{-1}\omega_2 \sim \omega_1 \\ \mathcal{L}_1\omega_1 \sim -\omega_2, \qquad & \mathcal{L}_1\omega_2 \sim 0. \end{aligned}$$

and so on. We can make this argument using weights too but then the coefficients of the right hand sides will be indeterminate.

So the actions of the $\mathcal{L}_i$ encode the underlying $\mathfrak{sl}_2$ action as we might expect.

12. Conclusions & Prospects

We have seen that we can calculate the shape of components of the Gauß-Manin connection as polynomials in the parameters of the curve in general position in the cases of genus one and two, both hyperelliptic. This depended on the structure of the ring of invariants of the binary forms of degree four and six which in turn depends on partition theory and on the assumption that, for the invariant differential operators, the connection coefficients are polynomial in parameters.

This assumption about the polynomial character of the connection needs a careful treatment and justification but seems reasonable in the light of results in [14, 4].

The approach is effectively an "equivariantization" of Bunkova & Buchstaber's paper [4].

Using the equivariant basis and invariant operators gives relatively simple, for the most part diagonal, expressions for the connection matrices.

Another approach, giving attractive and compact expressions for the connection coefficients, has been described in [17] where, in the case of the genus two curve, the branch points $(y, x) = (0, \alpha_i)$ for $i = 1, \ldots, 6$ are used as parameters. The choice $a_0 = 1$ is also implicit. The invariants are symmetric polynomials in the α_i but, inversely, the α_i are algebraic in the invariants, so it is not easy to see how the two approaches correspond.

Expressions for the invariants and the differential operators (in terms of the parameters of the curve) are cumbersome functions but the general form of the connection coefficients in terms of those invariants and with respect to an equivariant basis is considerably simpler. It remains to calculate the still unspecified constants in the connection coefficients. In general this appears to depend upon some knowledge of the explicit form of the invariants. Although it has not been necessary for the weights argument in the final section, the systematic transvectant notation [25] will be essential for calculating the values of the arbitrary constants.

It is to be hoped that this approach proves useful in other aspects of generalized Picard-Fuchs theory, for instance the results of [8, 11].

It remains to understand how far this approach can be pushed beyond the genus two case. The classical work describing generating functions for invariants and covariants listed in the bibliography affords some hope provided the mechanics of the calculations are feasible.

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He is missed.

Appendix A. Proof of second part of Theorem 7.1

The second differentials η_i are constructed iteratively from

$$\eta_1 = \frac{d^{g+1}a(x)}{dx^{g+1}}\frac{dx}{y}$$

by

$$\eta_{i+1} = \mathbf{f}\eta_i, \quad i = 1 \dots g-1.$$

This definition differs from the classical definition [3, 5] in that η_1 is modified to belong to the kernel of $\mathbf{e}$ and we want it to generate an equivariant basis for cohomology. We have therefore to establish that it is indeed a bona-fide differential of the second kind and generates such a basis.

We have to show that the residues of simple poles of the η_i vanish, that $\mathbf{e}\eta_1 \sim 0$ and that $\mathbf{f}\eta_g \sim 0$. We prove each of these in the following lemmata.

Lemma 1. *η_1 is a differential of the second kind.*

Proof. From

$$a(x) = a_0x^{2g+2} + (2g+2)a_1x^{2g+1} + \dots + \binom{2g+2}{i}a_ix^{2g+2-i} + \dots$$

we have

$$\frac{d^{g+1}a(x)}{dx^{g+1}} = \frac{(2g+2)!}{(g+1)!}\left(a_0x^{g+1} + (g+1)a_1x^g + \dots + \binom{g+1}{i}a_ix^{g+1-i} + \dots\right).$$

Let us call this polynomial $a^{(g+1)}(x)$.

Suppose (x_0, y_0) is a finite, non-branch point of the curve, so $y_0 \neq 0$. Let t be a local parameter and write $x = x_0 + t$ and $y = y_0 + O(t)$. Then

$$\eta_1 = a^{(g+1)}(x_0)\frac{dt}{y_0} + O(t)dt$$

is regular at this point.

Secondly suppose (x_0, y_0) is a finite branch point, that is $y_0 = 0$. Write $x = x_0 + t^2$ and $y = ty_1 + O(t^2)$. Then

$$\eta_1 = 2a^{(g+1)}(x_0)\frac{dt}{y_1} + O(t^2)dt$$

is again regular.

Finally consider the points at infinity which is where the poles live. For the sextic curve $(a_0 \neq 0)$ these are not branch points. Let us make the change of coordinates

$$\begin{aligned} x &= -\frac{1}{t} \\ y &= \frac{s}{t^{g+1}}. \end{aligned} \tag{12}$$

then

$$\eta_1 = a^{(g+1)}(-t^{-1})\frac{t^{g-1}}{s} = \tilde{a}^{(g+1)}(t)\frac{dt}{st^2}$$

where $\tilde{a}^{(g+1)}(t)$ is a polynomial in t of degree $g+1$. In order to confirm that there is no residue at $t = 0$ we need to show that no term linear in t is present in $\tilde{a}^{(g+1)}(t)/s$. This follows by rewriting the ratio:

$$\begin{aligned} \frac{\tilde{a}^{(g+1)}(t)}{s} &= \frac{t^{g+1}a^{(g+1)}(-t^{-1})}{t^{g+1}\sqrt{a(-t^{-1})}} \\ &= \frac{(2g+2)!}{(g+1)!}\frac{a_0 - (g+1)a_1 t + O(t^2)}{\sqrt{a_0 - (2g+2)a_1 t + O(t^2)}} \\ &= \frac{(2g+2)!}{(g+1)!}\sqrt{a_0} + O(t^2). \end{aligned}$$

Hence the result. □

Lemma 2. *If η_k is a differential of the second kind then so is $\mathbf{f}\eta_k$.*

Proof. The form of $\mathbf{f}$ at finite points of the curve is

$$\mathbf{f} = (g+1)xy\partial_y + x^2\partial_x + \sum_{i=0}^{2g+1}(2g+2-i)a_{i+1}\partial_{a_i},$$

and at infinity, under the coordinate change introduced in the lemma above,

$$\mathbf{f}_\infty = \partial_t + \sum_{i=0}^{2g+2}(2g+2-i)a_{i+1}\partial_{a_i}.$$

These are regular operators at the places concerned. In particular $\mathbf{f}_\infty$ cannot generate poles of order one from those of order two. □

For the final part of the proof we need one small extra lemma. First we define a set of polynomials indexed by positive integers n and j,

$$a_j^{(n)}(x) = \sum_{i=0}^{n} \binom{n}{i} a_{j+i} x^{n-i}.$$

In particular $a(x) \equiv a_0^{(2g+2)}(x)$ and $a^{(g+1)}(x) \equiv \frac{(2g+2)!}{(g+1)!} a_0^{(g+1)}(x)$.

Lemma 3.

$$\mathbf{f}\left(x^{-(g+1)} a_j^{(g+1)}(x)\right) = (g+1-j)x^{-(g+1)} a_{j+1}^{(g+1)}(x).$$

Proof. This is straightforward algebra using the definition of $\mathbf{f}$.

Now we prove:

Lemma 4. *η_1 is a highest weight element, $\mathbf{e}\eta_1 \sim 0$, and η_g a lowest weight element, $\mathbf{f}\eta_g \sim 0$ of a g-dimensional irreducible $\mathfrak{sl}_2$ representation.*

Proof. The operator $\mathbf{e}$ commutes with ∂_x and hence $\mathbf{e}a(x) = 0$ implies that $\mathbf{e}a^{(g+1)}(x) = 0$. Also $\mathbf{e}dx = 0$ and $\mathbf{e}y = 0$. Hence $\mathbf{e}\eta_1 = 0$.

Finally we show that $\mathbf{f}\eta_g$ is exact.

We can write η_1 in a form where the denominator is in the kernel of $\mathbf{f}$:

$$\eta_1 = \frac{a^{(g+1)}(x)dx}{\sqrt{a(x)}} = \frac{x^{-(g+1)}a^{(g+1)}(x)dx}{\sqrt{x^{-2(g+1)}a(x)}}$$

i.e. $\mathbf{f}\left(x^{-2(g+1)}a(x)\right) = 0$. We need to show that $\mathbf{f}^g$ is exact.

By the preceeding lemma

$$\mathbf{f}^n(x^{-(g+1)}a_0^{(g+1)}(x)) = \frac{(g+1)!}{(g+1-n)!} x^{-(g+1)} a_n^{(g+1)}(x)$$

and, by an easy induction starting from $\mathbf{f}dx = 2x\,dx$,

$$\mathbf{f}^n\,dx = (n+1)!x^n\,dx.$$

Now then, using $G = \frac{(2g+2)!}{(g+1)!}$ for brevity,

$$\begin{aligned}
\mathbf{f}^g\eta_1 &= \frac{G}{\sqrt{x^{-2(g+1)}a(x)}}\mathbf{f}^g(x^{-(g+1)}a_0^{(g+1)}\,dx) \\
&= \frac{G}{\sqrt{x^{-2(g+1)}a(x)}} \sum_{k=0}^{g} \binom{g}{k} \mathbf{f}^k(x^{-(g+1)}a_0^{(g+1)})\mathbf{f}^{g-k}\,dx \\
&= \frac{(2g+2)!}{\sqrt{a(x)}} \sum_{k=0}^{g} \binom{g}{k} x^{g-k} a_k^{(g+1)}(x)\,dx \\
&= \frac{(2g+2)!a_0^{(2g+1)}(x)\,dx}{\sqrt{a(x)}} \\
&= \frac{g!}{2}\frac{da(x)}{\sqrt{a(x)}} \\
&= g!dy
\end{aligned}$$

Hence $\mathbf{f}\eta_g$ vanishes up to an exact differential and vanishes in cohomology. □

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