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Compactness of nonlinear integral operators with discontinuous and with singular kernels

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ABSTRACT

This paper establishes compactness of nonlinear integral operators in the space of continuous functions. One result deals with operators whose kernel can have jumps across a finite number of curves, which typically arise from the study of ordinary differential equations with boundary conditions of local or nonlocal type. Several other results deal with operators whose kernels have a singularity, which arise from the study of fractional differential equations. We motivate the study of these integral equations by discussing some initial value problems for fractional differential equations of Caputo and Riemann-Liouville type. We prove a compact embedding theorem for fractional integrals in order to give a new treatment for the singular kernel case.

1. Introduction

In studying existence of solution of differential equations it is often convenient to consider the corresponding problem involving a nonlinear integral operator. The advantage is that integral operators are usually compact.

We will study nonlinear integral operators in the space of continuous functions such as a Fredholm type operator

$$N_F u(t) = \int_0^T G(t,s) f(s,u(s))\, ds \tag{1.1}$$

or a Volterra type operator

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$$N_V u(t) = \int_0^t G(t,s) f(s,u(s))\,ds. \tag{1.2}$$

The important point in this paper is that the kernel G will be allowed to have discontinuities along certain curves.

Integral operators such as N_F arise in the study of boundary value problems for ordinary and some fractional differential equations, where G is the Green's function and often can have finite jumps. The operator N_V occurs when initial value problems are studied, often en route to a boundary value problem. In particular, when it arises from the study of fractional differential equations, the kernel is of the form $G(t,s) = (t-s)^{\alpha-1}$ and has a singularity when $0 < \alpha < 1$.

For a problem such as $u''(t)+f(t,u(t)) = 0,\ t \in [0,T]$ with some boundary conditions (BCs), the problem is often equivalent to finding fixed points of an integral operator $Nu(t) = \int_0^T G(t,s)f(s,u(s))\,ds$ in the space $C[0,T]$. The Green's function G is frequently continuous, the integral operator N is compact and continuous (completely continuous) and many theories can be applied to determine existence of unique or of multiple fixed points of N under suitable behaviour of the function f.

There are situations when the kernel G is not continuous at all points but does satisfy Carathéodory conditions, $G(t,s)$ is continuous in t for almost every s and is dominated by an L^1 function of s. Continuity of the function f can also be weakened to Carathéodory conditions. The criterion for compactness in $C[0,T]$ is the Arzelà-Ascoli theorem, N must map bounded sets into bounded equicontinuous sets. A typical result, often quoted, is Proposition V.3.1 of R.H. Martin's book [11]; the result there was refined in [14,15].

When the problem is $u''(t) + f(t,u(t),u'(t)) = 0,\ \ t \in [0,T]$ with some BCs, the corresponding integral operator N must be studied in the space $C^1[0,T]$ so it is required that $(Nu)'(t) = \int_0^T \partial_t G(t,s)f(s,u(s),u'(s))\,ds$ also defines a compact map. Even in simple cases the kernel $\partial_t G(t,s)$ may not be continuous, a typical situation is when there is a jump across the line $s = t$, thus it is not continuous in t for almost all s and the previous result is not applicable. A simple example is the problem $u'' + f(t,u,u') = 0,\ u(0) = 0, u(1) = 0$ where

$$G(t,s) := \begin{cases} s(1-t), & \text{if } s \le t, \\ t(1-s), & \text{if } s > t, \end{cases} \text{ and } \partial_t G(t,s) := \begin{cases} -s, & \text{if } s < t, \\ 1-s, & \text{if } s > t, \end{cases}$$

and $\partial_t G$ is not defined on the diagonal $s = t$; there is a finite jump discontinuity with both left and right derivatives existing on the diagonal.

In a simple case, such as the example just given, it is not hard to carefully prove that the Arzelà-Ascoli theorem works even with the jump. The proof is often omitted, for example [10] study some fourth order problems with third derivative dependence and simply claim the previous result applies, citing [15]. However, this is not applicable to their two problems, the third derivative of the Green's function has a jump on the diagonal $s = t$; direct use of the Arzelà-Ascoli theorem would work.

However, there are other problems where this is less simple. One goal of this paper is to give a general result which allows for the kernel to have a finite number of discontinuities across certain curves, which will include the simple case of a jump across the 'curve' $s = t$. One relatively weak hypothesis we use is that $|G(t,s)| \le \Phi(s)$ for some $\Phi \in L^1$.

For problems involving higher order derivatives, the highest order derivative operator may often have discontinuities in the corresponding partial derivative of the Green's function. López-Somoza and Minhós [9] studied some integral equations of the form

$$Tu(t) = \int_0^1 K(t,s) f(s,u(s),\ldots,u^{(m)}(s))\,ds$$

with nonlinearities depending on derivatives up to order m. To get compactness in the space $C^m[0,1]$ of m times continuously differentiable functions they used the following hypotheses on K_m, the m-th partial derivative with respect to t of the kernel:

For every $\varepsilon > 0$ and every fixed $\tau \in [0,1]$, there exist a set $Z_\tau \subset [0,1]$ with measure zero and some $\delta > 0$ such that $|t-\tau| < \delta$ implies that $|K_m(t,s) - K_m(\tau,s)| < \varepsilon$, for all $s \in [0,1] \setminus Z_\tau$ such that $s < \min\{t,\tau\}$ or $s > \max\{t,\tau\}$.

Our result will apply to this example with a much simpler looking assumption.

The second type of integral operator is with a singular kernel and arises in the study of fractional equations. These often involve the Riemann-Liouville fractional integral

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s))\, ds.$$

For $\alpha \geq 1$ the kernel $(t-s)^{\alpha-1}$ is continuous for $s \leq t$, and then we may write

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^T g(t,s) f(s,u(s))\, ds$$

$$\text{where } \; g(t,s) = \begin{cases} (t-s)^{\alpha-1}, & \text{if } 0 \leq s \leq t \leq T \\ 0, & \text{if } 0 \leq t < s \leq T, \end{cases}$$

and the result of first type applies. The trickier and more interesting case is when $0 < \alpha < 1$. A different result is necessary since the kernel for $0 < \alpha < 1$ does not satisfy the bound employed in the first type of problem. When $\alpha > 1$ and f depends on (possibly fractional) derivatives the (fractional) derivative operator has also to be proved compact. Such a derivative operator is then often one having a singular kernel, of the type similar to the case with $\alpha < 1$.

We motivate the study of these integral equations by discussing some initial value problems for fractional differential equations of Caputo and Riemann-Liouville type. We will give some new concise proofs of compactness of these operators, and more general ones, by proving and utilizing some new properties of fractional integrals, including a compact embedding theorem for fractional integrals, in order to give a new treatment for the singular kernel case.

This approach avoids the longer calculations using the Arzelà-Ascoli theorem that have previously been used on an *ad hoc* basis for such problems.

2. Some function spaces

For simplicity we consider functions defined on an arbitrary finite interval $[0,T]$, which is, by a simple change of variable, equivalent to any finite interval. In this paper all functions are assumed to be measurable, and all integrals are Lebesgue integrals, even if not explicitly stated.

$L^p = L^p[0,T]$ $(1 \leq p < \infty)$ denotes the usual space of functions whose p-th power is Lebesgue integrable; endowed with the norm $\|u\|_p = \left(\int_0^T |u(s)|^p\, ds\right)^{1/p}$ it is a Banach space. L^∞ will denote the essentially bounded functions with norm $\|u\|_\infty = \operatorname{esssup}_{t\in[0,T]} |u(t)|$.

The space of functions that are continuous on $[0,T]$ is denoted by $C[0,T]$ or sometimes simply C and is a Banach space when endowed with the supremum norm $\|u\|_\infty := \max_{t\in[0,T]} |u(t)|$. For $n \in \mathbb{N}$ we will write $C^n = C^n[0,T]$ to denote those functions u whose n-th derivative $u^{(n)}$ is continuous on $[0,T]$.

The space of absolutely continuous functions is denoted $AC = AC[0,T]$. For $n \in \mathbb{N}$, $AC^n = AC^n[0,T]$ will denote those functions u whose n-th derivative $u^{(n)}$ is in $AC[0,T]$, hence $u^{(n+1)}(t)$ exists for a.e. t and is an L^1 function.

A note of caution: some authors denote this space as AC^{n+1}.

The space AC is the appropriate space for the fundamental theorem of the calculus for Lebesgue integrals. In fact, we have the following equivalence.

$$u \in AC[0,T] \text{ if and only if } u'(t) \text{ exists for almost every (a.e.) } t \in [0,T]$$
$$\text{with } u' \in L^1[0,T] \text{ and } u(t) - u(0) = \int_0^t u'(s)\,ds \text{ for all } t \in [0,T]. \quad (2.1)$$

For $\eta \geq -1$, we define a space which allows pointwise singularities at 0, which, as far as we are aware is newly introduced here, and will be utilised in this paper.

$$L_\eta[0,T] := \{f : [0,T] \to \mathbb{R} : f(t) = t^\eta g(t) \text{ a.e., for some } g \in L^\infty[0,T]\}. \quad (2.2)$$

For $f \in L_\eta$ with $f(t) = t^\eta g(t)$ we define the norm $\|f\|_\eta = \|g\|_\infty$. The spaces with a singularity at zero are $L_{-\eta}$ where $\eta > 0$. For situations that arise from studying fractional derivatives, it is the case $L_{-\eta}$ with $0 \leq \eta < 1$ that is important, where $\eta < 1$ is required to ensure that certain integrals exist.

We recall similar spaces of functions that are continuous except at 0. The space C_η is defined by

$$C_\eta[0,T] := \{f : [0,T] \to \mathbb{R} : f(t) = t^\eta g(t) \text{ a.e., for some } g \in C[0,T]\}. \quad (2.3)$$

When endowed with the norm $\|f\|_\eta$, C_η is a Banach space. The spaces with a singularity at zero are $C_{-\eta}$ with $\eta > 0$. Functions f in $C_{-\eta}$ with $0 \leq \eta < 1$ are integrable and are continuous on $(0,T]$ with $\lim_{t\to 0+} t^\eta f(t)$ exists. It is an appropriate class for the study of Riemann-Liouville fractional integrals and derivatives, see for example [1].

Note that, for $0 \leq \gamma \leq \eta < 1$, $L^\infty = L_0 \subset L_{-\gamma} \subset L_{-\eta}$, similarly $C[0,T] = C_0 \subset C_{-\gamma} \subset C_{-\eta}$. Moreover, for $0 < \eta < 1$, the inclusion $L_{-\eta} \subset L^p$ holds when $p > 1/\eta$.

When $0 < \eta < 1$, the main difference between the space L^p with $p > 1/\eta$ and the space $L_{-\eta}$ is (roughly) that functions in $L_{-\eta}$ can be considered as having only one pointwise singularity at 0, functions in L^p can have many pointwise singularities. For the functions that arise in applications, these differences rarely occur and the space $L_{-\eta}$ is often adequate, indeed usually the space $C_{-\eta}$ suffices. However, we will find the space $L_{-\eta}$ useful, particularly in Theorem 7.2 below.

The Hölder space denoted $C^{0,\alpha}[0,T]$, $0 < \alpha \leq 1$, consists of all functions f such that there is a positive real constant $M = M(f)$, such that

$$|f(t) - f(\tau)| \leq M|t-\tau|^\alpha. \quad (2.4)$$

$C^{0,\alpha}[0,T]$ is a Banach space when endowed with the following norm

$$\|f\|_{0,\alpha} := \sup_{t\in[0,T]} |f(t)| + \sup_{t,\tau\in[0,1],t\neq\tau} \frac{|f(t)-f(\tau)|}{|t-\tau|^\alpha}. \quad (2.5)$$

Note that there are functions that are Hölder continuous but are not AC, for example a Weierstrass function, and AC functions that are not Hölder continuous, for example

$$f(x) = \begin{cases} 1/\log x, & \text{if } x \in (0,1/2], \\ 0 & \text{if } x = 0. \end{cases}$$

We will use the known fact that for $0 < \alpha < 1$ the Hölder space $C^{0,\alpha}$ embeds continuously and compactly into $C[0,T]$; since it makes this paper more self contained we include the short proof.

Lemma 2.1. *If $\{u_n\}$ is a bounded sequence in $C^{0,\alpha}[0,T]$, then $\{u_n\}$ is relatively compact in $C[0,T]$.*

Proof. Let $\|u_n\|_{0,\alpha} \le M$, then $\|u_n\|_\infty \le M$ so the embedding is continuous. For $t \neq \tau$ we have

$$|u_n(t) - u_n(\tau)| = \frac{|u_n(t) - u_n(\tau)|}{|t-\tau|^\alpha}|t-\tau|^\alpha \le M|t-\tau|^\alpha$$

so $\{u_n\}$ is bounded and equicontinuous hence relatively compact in $C[0,T]$ by the Arzelà-Ascoli theorem. □

3. Compactness of Fredholm integral operators with discontinuous kernels

We say that a map $N : C[0,T] \to C[0,T]$ is *compact* when N maps each bounded subset of $C[0,T]$ into a compact subset of $C[0,T]$, that is N(bounded set) is relatively compact. We say $N : C[0,T] \to C[0,T]$ is *completely continuous* if N is compact and continuous.

We will extend a result given for linear operators in the book by Kolmogorov and Fomin [6], our proof uses the method from [6], but our new result deals with nonlinear operators and has weaker conditions.

Definition 3.1. A function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is said to be L^q-Carathéodory, $(1 \le q \le \infty)$, if f is measurable, $u \mapsto f(t,u)$ is continuous for a.e. t, and, for each $R > 0$, there exists a function $f_R \in L^q[0,T]$ such that $|f(t,u)| \le f_R(t)$ for all u with $|u| \le R$.

Theorem 3.2. *Let $g : [0,T] \times [0,T] \to \mathbb{R}$ be a measurable function such that there exists $\Phi \in L^1[0,T]$ satisfying $|g(t,s)| \le \Phi(s)$ for all t,s. Further suppose that g is continuous for $(t,s) \in [0,T] \times [0,T]$ except on a finite number of curves $s = \varphi_k(t)$, $k = 1,2,\dots,n$, where each $\varphi_k : [0,T] \to [0,T]$ is continuous. Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be L^1-Carathéodory, and such that $\Phi f_R \in L^1[0,T]$ for each $R > 0$. Then the integral operator*

$$Nu(t) := \int_0^T g(t,s) f(s,u(s))\, ds \tag{3.1}$$

is completely continuous in the space $C[0,T]$.

Proof. We will prove that N is bounded and equicontinuous on each bounded set $D \subset C[0,T]$. Suppose that $\|u\| \le R$ for $u \in D$. Then we have

$$|Nu(t)| \le \int_0^T |g(t,s)||f(s,u(s))|\, ds \le \int_0^T \Phi(s) f_R(s)\, ds = \|\Phi f_R\|_1$$

so that $\|Nu\|_\infty \le \|\Phi f_R\|_1$ and $N(D)$ is bounded.

Now we turn to the equicontinuity. Let $\varepsilon > 0$, we have to prove that there exists $\delta > 0$, depending only on ε, such that $|Nu(t_1) - Nu(t_2)| < \varepsilon$ for $|t_1 - t_2| < \delta$ and all $u \in D$. Let $\|u\| \le R$ for $u \in D$ then $|f(t,u(t))| \le f_R(t)$ for all $u \in D$. Let $\delta_1 > 0$ be such that for $E \subset [0,T]$, $\text{meas}(E) < \delta_1$ implies $\int_E \Phi(s) f_R(s)\, ds < \varepsilon/4$. Let $\varepsilon_1 = \dfrac{\delta_1}{4n}$ and let

$$G_k := \{(t,s) \in [0,T] \times [0,T] : |s - \varphi_k(t)| < \varepsilon_1\}, \text{ and } G := \cup_{k=1}^n G_k,$$

and let F be the complement of G with respect to $[0,T]\times[0,T]$. As G is open, F is closed, and g is uniformly continuous on F. Let $\varepsilon_2 = \dfrac{\varepsilon}{2\|f_R\|_1}$ and let $\delta_2 > 0$ be such that $|g(t_1,s) - g(t_2,s)| < \varepsilon_2$ for $(t_1,s),(t_2,s) \in F$ and $|t_1 - t_2| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. For $|t_1 - t_2| < \delta$ we have

$$\begin{aligned}
|Nu(t_1) - Nu(t_2)| &\le \int_0^T |g(t_1,s) - g(t_2,s)||f(s,u(s))|\,ds \\
&= \int_{(t_1,s)\in G\cup(t_2,s)\in G} |g(t_1,s) - g(t_2,s)||f(s,u(s))|\,ds \\
&\quad + \int_{(t_1,s)\in F\cap(t_2,s)\in F} |g(t_1,s) - g(t_2,s)||f(s,u(s))|\,ds \\
&\le \int_{(t_1,s)\in G\cup(t_2,s)\in G} 2\Phi(s) f_R(s)\,ds + \int_{(t_1,s)\in F\cap(t_2,s)\in F} \varepsilon_2 f_R(s)\,ds \\
&\le \int_{(t_1,s)\in G\cup(t_2,s)\in G} 2\Phi(s) f_R(s)\,ds + \int_0^T \varepsilon_2 f_R(s)\,ds = I_G + I_F, \text{ say.}
\end{aligned}$$

The set $E := \{s : (t_1,s) \in G \cup (t_2,s) \in G\}$ has length less than $4n\varepsilon_1 = \delta_1$ so the first integral $|I_G| < \varepsilon/2$. The second integral $|I_F| < \varepsilon/2$ by the choice of ε_2. These two steps prove that N is a compact map by the Arzelà-Ascoli theorem.

Finally we prove that N is continuous. For this, let $u_n \to u$ in $C[0,T]$ and let $R > 0$ be such that $\|u_n\| \le R$. Then $f(s,u_n(s)) \to f(s,u(s))$ for a.e. s. For each (fixed) t, $|g(t,s)f(s,u_n(s))| \le \Phi(s) f_R(s) \in L^1$. By the dominated convergence theorem, $Nu_n(t) = \int_0^T g(t,s) f(s,u_n(s))\,ds \to Nu(t)$ as $n \to \infty$, that is we have pointwise convergence. Since $\{Nu_n\}$ is relatively compact, by a standard argument, the convergence is uniform in t. This completes the proof that $N : C[0,T] \to C[0,T]$ is completely continuous. □

Remark 3.3. The result in the book [6, Theorem 1, page 112] is for the linear case when $f(s,u(s)) = u(s)$ and it is assumed that $|g(t,s)|$ is uniformly bounded for all t, s. If $f(s,u(s))$ is continuous, the substitution operator $Fu(t) := f(t,u(t))$ maps $C[0,T]$ into itself and the result for linear operators can be used for the nonlinear case, via composition of functions, under stronger restrictions than we have. When f satisfies Carathéodory conditions but is not continuous this is no longer valid.

Theorem 3.2 is not true for general curves instead of graphs of functions as the following simple example given in [6] shows.

Example 3.4. Let

$$g(t,s) := \begin{cases} 1, & \text{if } t < 1/2, \\ 0, & \text{if } t \ge 1/2. \end{cases}$$

Then $Nu(t) = \int_0^1 g(t,s)u(s)\,ds$ maps the continuous function $u \equiv 1$ into a discontinuous function.

Here the curve is $t = 1/2$ which is not a graph; it is not to be confused with $s = 1/2$ which is an allowed curve. A nonlocal boundary value problem where the corresponding integral operator has a discontinuity on a line $s = \eta$ is studied in [4].

4. Some properties of fractional integrals

In the study of fractional integrals and fractional derivatives the Gamma and Beta functions occur naturally and frequently. The Gamma function is, for $p > 0$, given by

$$\Gamma(p) := \int_0^\infty s^{p-1} \exp(-s)\, ds \tag{4.1}$$

which is an improper Riemann integral but is well defined as a Lebesgue integral, and is an extension of the factorial function: $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. The Beta function is defined by

$$B(p,q) := \int_0^1 (1-s)^{p-1} s^{q-1}\, ds \tag{4.2}$$

which is a well defined Lebesgue integral for $p > 0, q > 0$. It is well known, and proved in calculus texts, that $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Definition 4.1. The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of a function $u \in L^1[0,T]$ is defined for a.e. t by

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

The integral $I^\alpha u$ is the convolution of the L^1 functions h, u where $h(t) = t^{\alpha-1}/\Gamma(\alpha)$, so, by the well known results on convolutions, $I^\alpha u$ is defined as an L^1 function, in particular $I^\alpha u(t)$ is finite for a.e. t. $I^\alpha u(0)$ is not necessarily defined for $u \in L^1$, for example if $0 < \alpha < \alpha + \varepsilon < 1$ and $u(t) = t^{-\alpha-\varepsilon}$ then $I^\alpha u(t) = \frac{\Gamma(1-\alpha-\varepsilon)}{\Gamma(1-\varepsilon)}\, t^{-\varepsilon}$. If $\alpha = 1$ this is the usual integration operator which we denote I. We set $I^0 u = u$.

Interchanging the order of integration, using Fubini's theorem, shows that these fractional integral operators satisfy a semigroup property as follows:

Lemma 4.2. *Let $\alpha, \beta > 0$ and $u \in L^1[0,T]$. Then $I^\alpha(I^\beta u)(t) = I^{\alpha+\beta}u(t)$ for a.e. $t \in [0,T]$, and for all t if $\alpha + \beta \geq 1$.*

The proof is given in [12, (2.21)]. A detailed proof is given in [17, Lemma 3.4] where it is also shown that this holds for all t when $u \in C_{-\gamma}$ ($\gamma \in [0,1)$) and $\alpha + \beta \geq \gamma$.

Hardy and Littlewood [3] showed that, for $0 < \alpha < 1$ and $\alpha > 1/p$, the R-L fractional integral operator I^α maps $L^p[0,T]$ into the Hölder space $C^{0,\alpha-1/p}$. We will prove in Theorem 4.5 below a slightly less general result, but with a simpler argument, that for $0 < \gamma < \alpha < 1$, I^α maps $L_{-\gamma}[0,T]$ into the Hölder space $C^{0,\alpha-\gamma}[0,T]$.

We first give the following result which is novel and a small improvement on the result in [17], the extension proves useful in Theorems 4.5 and 7.2 below.

Theorem 4.3. *Let $0 < \alpha < 1$ and $0 \leq \gamma < 1$. Then I^α maps $L_{-\gamma}[0,T]$ into $L_{\alpha-\gamma}[0,T]$ and maps $C_{-\gamma}[0,T]$ into $C_{\alpha-\gamma}[0,T]$. Moreover, for $f \in L_{-\gamma}$, $\|I^\alpha f\|_{\alpha-\gamma} \leq \dfrac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\|f\|_{-\gamma}$.*

Proof. Let $f \in L_{-\gamma}[0,T]$ and $f(t) = t^{-\gamma} g(t)$ for $g \in L^\infty[0,T]$. We have

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} g(s)\, ds$$
$$= \frac{1}{\Gamma(\alpha)} t^{\alpha-\gamma} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{-\gamma} g(t\sigma)\, d\sigma.$$

Let $h(t) := \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\gamma} g(t\sigma)\, d\sigma$. Then

$$|h(t)| \le \|g\|_\infty \frac{B(\alpha, 1-\gamma)}{\Gamma(\alpha)} = \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \|g\|_\infty,$$

so h is an L^∞ function, and $\|I^\alpha f\|_{\alpha-\gamma} \le \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \|f\|_{-\gamma}$.

When g is continuous, h is continuous by the dominated convergence theorem, which completes the proof. □

Remark 4.4. The assertion concerning the space $C_{-\gamma}[0,T]$ was proved in [17]. In particular, if $0 \le \gamma < \alpha < 1$ and $u \in C_{-\gamma}[0,T]$, then $I^\alpha u$ is a continuous function.

The promised embedding theorem is as follows.

Theorem 4.5. *Let $0 \le \gamma < \alpha < 1$. Then*

1. *I^α maps $L^\infty[0,T]$ into $C^{0,\alpha}[0,T]$ and $\|I^\alpha f\|_{C^{0,\alpha}} \le \big(1 + \frac{2}{\Gamma(\alpha+1)}\big)\|f\|_\infty$. Moreover, $\lim_{t\to 0} I^\alpha f(t) = I^\alpha f(0) = 0$.*
2. *I^α maps $L_{-\gamma}[0,T]$ into $C^{0,\alpha-\gamma}[0,T]$. Moreover, $\lim_{t\to 0} I^\alpha f(t) = 0$.*

Proof. (1) Let $0 \le t < t+h \le T$. We must prove that, for $f \in L^\infty[0,T]$, there exists a constant M, which can depend on f, such that

$$|I^\alpha f(t+h) - I^\alpha f(t)| \le M|h|^\alpha.$$

We have

$$|I^\alpha f(t+h) - I^\alpha f(t)| = \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t+h} (t+h-s)^{\alpha-1} f(s)\, ds - \int_0^t (t-s)^{\alpha-1} f(s)\, ds \Big|$$
$$\le \frac{1}{\Gamma(\alpha)} \Big\{ \Big| \int_0^t \big[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\big] f(s)\, ds \Big| + \int_t^{t+h} (t+h-s)^{\alpha-1} |f(s)|\, ds \Big\}$$
$$\le \frac{\|f\|_\infty}{\Gamma(\alpha)} \Big| \int_0^t \big[(t+h-s)^{\alpha-1} - (t-s)^{\alpha-1}\big]\, ds \Big| + \frac{\|f\|_\infty}{\Gamma(\alpha)} \int_t^{t+h} (t+h-s)^{\alpha-1}\, ds$$
$$= I_1 + I_2.$$

The second integral I_2 evaluates to $\dfrac{\|f\|_\infty}{\Gamma(\alpha)}\dfrac{h^\alpha}{\alpha} = \dfrac{\|f\|_\infty}{\Gamma(\alpha+1)}h^\alpha$. The first integral I_1 evaluates to

$$\frac{\|f\|_\infty}{\Gamma(\alpha)}\Big|\frac{(t+h)^\alpha}{\alpha} - \frac{h^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}\Big| = \frac{\|f\|_\infty}{\Gamma(\alpha+1)}\big[t^\alpha + h^\alpha - (t+h)^\alpha\big].$$

Therefore

$$|I^\alpha f(t+h) - I^\alpha f(t)| \le \frac{\|f\|_\infty}{\Gamma(\alpha+1)}\big[t^\alpha + h^\alpha - (t+h)^\alpha + h^\alpha\big] \le 2\frac{\|f\|_\infty}{\Gamma(\alpha+1)}h^\alpha.$$

This proves that $\|I^\alpha f\|_{C^{0,\alpha}} \le \|f\|_\infty + 2\dfrac{\|f\|_\infty}{\Gamma(\alpha+1)}$.

As $I^\alpha f \subset C^{0,\alpha}[0,T] \subset C[0,T]$, $I^\alpha f(t)$ is a continuous function of t, therefore $\lim_{t\to 0} I^\alpha f(t) = I^\alpha f(0) = 0$.

(2) Let $f \in L_{-\gamma}$. Then, by the semigroup property of fractional integrals, we have $I^\alpha f(t) = I^{\alpha-\gamma}I^\gamma f(t)$ a.e. We obtain $I^\gamma f \in L_0 = L^\infty$ by Theorem 4.3 and, from part (1), $I^{\alpha-\gamma}$ maps L^∞ into the Hölder space $C^{0,\alpha-\gamma}[0,T]$. Hence $I^\alpha f$ is a continuous function and the last assertion is shown. □

Remark 4.6. As mentioned above this result was proved for the somewhat more general case when $f \in L^p$ with $p > 1/\alpha$ by Hardy and Littlewood [3]. When $\alpha > 1$ the cases $\alpha - 1/p \in \mathbb{N}$ are a little different, see [3] or [12, Theorem 3.6]. The (two page long) proof for $f \in L^p$ can also be found in Theorem 2.6 of [2]. A different proof is given in [7, Theorem 3.5]. We believe our new idea of proving the easier part (1) and then deducing part (2) gives a simpler, shorter, proof for this case, which has interest because of equivalences with initial value problems of fractional differential equations where the space $C_{-\gamma}$ occurs quite naturally. We discovered that the first part of (1), I^α is bounded from L^∞ into $C^{0,\alpha}$, is given after Corollary 2 in Section 3.1 of [12] in less detail and with some misprints.

Remark 4.7. This result gives that I^α maps $C[0,T]$ into the Hölder space $C^{0,\alpha}$; but I^α does not map $C[0,T]$ into $AC[0,T]$ as is shown in §5.5 of [3] using a Weierstrass type function.

5. Some fractional differential equations and their integral versions

The R-L fractional integral of order $\alpha > 0$ for $u \in L^1[0,T]$ is

$$I^\alpha u(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \text{ for a.e. } t.$$

In the important case $0 < \alpha < 1$, the kernel $(t-s)^{\alpha-1}$ is singular on the line $s = t$. For a real number $\beta \ge 0$ let $\lceil\beta\rceil$ (the ceiling function acting on β) denote the smallest integer greater than or equal to β.

Definition 5.1. The Riemann-Liouville (R-L) fractional differential operator of order β is defined when, for $n = \lceil\beta\rceil$, $D^{n-1}(I^{n-\beta}u) \in AC$, that is $I^{n-\beta}u \in AC^{n-1}$, by

$$D^\beta u := D^n I^{n-\beta}u.$$

The Caputo differential operator of order β, (or Caputo derivative for short) is defined ([2, Definition 3.2]) when $I^{n-\beta}u \in AC^{n-1}$ and $T_{n-1}u$ exists by

$$D_*^\beta u = D^\beta(u - T_{n-1}u) = D^n I^{n-\beta}(u - T_{n-1}u),$$

where $T_{n-1}u$ is the Taylor polynomial of degree $n-1$, $T_{n-1}u(t) := \sum_{k=0}^{n-1} \frac{t^k D^k u(0)}{k!}$ where D^k are k-th order ordinary derivatives.

Under the given conditions each fractional derivative exists a.e. and involves a fractional integral of order in $(0,1)$, so a singular kernel is always involved.

Remark 5.2. The Caputo derivative is often defined for $u \in AC^{n-1}$, by the equation

$$D_C^{\beta} u(t) := (I^{n-\beta} D^n u)(t), \text{ for a.e. } t.$$

The disadvantage of the simpler definition $D_C^{\beta} u$ is that the equivalence between a Caputo fractional derivative equation and an integral equation is only valid for the definition $D_*^{\beta} u$ because of the fact that I^{α} does not map all of $C[0,T]$ into $AC[0,T]$; some extra hypotheses are necessary.

We state the equivalence result for the Caputo fractional equation of order $\alpha \in (0,1)$ when the nonlinear term is in the space $C_{-\gamma}$; the higher order case is proved in [17, Theorem 5.1]. The case when there is no singular term, $s^{-\gamma}$, is essentially well known, for example, Diethelm [2, Lemma 6.2].

Proposition 5.3. *[16, Theorem 4.6] Let f be continuous on $[0,T] \times \mathbb{R}$, let $0 < \alpha < 1$ and let $0 \leq \gamma < \alpha$. If a function $u \in AC$ satisfies the Caputo fractional initial value problem*

$$D_C^{\alpha} u(t) = t^{-\gamma} f(t, u(t)), \textit{ for a.e. } t \in (0,T], \; u(0) = u_0, \tag{5.1}$$

then u satisfies the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} f(s, u(s))\, ds, \; t \in [0,T]. \tag{5.2}$$

Secondly, if $u \in C[0,T]$ satisfies (5.2) then $I^{\alpha-1}(u - u_0) \in AC$ and u satisfies

$$D_*^{\alpha}(u)(t) = t^{-\gamma} f(t, u(t)), \textit{ for a.e. } t \in (0,T], \; u(0) = u_0. \tag{5.3}$$

Thirdly, if $u \in C[0,T]$ and $I^{\alpha-1}(u - u_0) \in AC$, and u satisfies (5.3), then u satisfies (5.2).

For the R-L fractional differential equation an equivalence result is as follows.

Proposition 5.4. *[17, Theorem 6.2] Let $u \in L^1$ be such that $I^{1-\alpha} u \in AC$ and suppose that $t \mapsto f(t, u(t)) \in L^1$. Then $D^{\alpha} u(t) = f(t, u(t))$ a.e. and $I^{1-\alpha} u(0) = c\Gamma(\alpha)$ if and only if $u \in L^1$ with $t \mapsto f(t, u(t)) \in L^1$ satisfies $u(t) = ct^{\alpha-1} + I^{\alpha} f(t, u(t))$ a.e., where $c = I^{1-\alpha} u(0)/\Gamma(\alpha)$.*

This is stated in an equivalent form, also for the higher order case, as [5, Lemma 2.5(b)] and is proved in [12, Theorem 2.4]. It is also proved, with an 'initial condition' given in terms of $\lim_{t \to 0+} u(t) t^{1-\alpha}$ under some slightly different hypotheses, in [1, Theorems 4.10 and 5.1]. The higher order case is also discussed in [8].

A result used in [1] is as follows.

Lemma 5.5. *Let $0 < \alpha < 1$ and suppose that $u \in L^1$. Then*

$$\lim_{t \to 0+} u(t) t^{1-\alpha} = c \textit{ implies that } \lim_{t \to 0+} I^{1-\alpha} u(t) = c\Gamma(\alpha).$$

A proof is given in [17, Lemma 6.3] and it is proved for $\alpha \in \mathbb{C}$ with $0 < \mathrm{Re}(\alpha) < 1$ in [5, Lemma 3.2]. It is also proved in [1, Theorem 6.1] where a converse is also claimed, but the converse part of the proof has a gap. Combining the previous two results we have

Proposition 5.6. *If $u \in C_{\alpha-1}$ and $t \mapsto f(t, u(t)) \in C_{-\gamma}$ for some $\gamma < 1$ and satisfies the Volterra integral equation*

$$u(t) = u^0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))\, ds,\ t \in [0, T], \tag{5.4}$$

then $D^\alpha u(t) = f(t, u(t))$ and $\lim_{t\to 0+} u(t) t^{1-\alpha} = u^0$.

The hypothesis $f(t, u(t)) \in C_{-\gamma}$ implies that $I^\alpha f \in C_{\alpha-\gamma} \subset C_{\alpha-1}$ by Theorem 4.3 so all terms make sense.

6. Compactness of integral operator arising from Caputo IVP

We now prove the complete continuity of the integral operator that arises from the initial value problem for the Caputo differential equation as in Proposition 5.3.

Theorem 6.1. *Let $0 \le \gamma < \alpha \le 1$ and $f : [0, T] \times \mathbb{R}$ be continuous and let u_0 be a constant. Then the integral operator*

$$Nu(t) := u_0 + I^\alpha(s^{-\gamma} f(s, u(s)))(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\gamma} f(s, u(s))\, ds$$

is a completely continuous map from $C[0, T]$ into $C[0, T]$.

Proof. Clearly the term u_0 has no bearing on the proof. For $u \in C[0, T]$ write $(Fu)(t) = t^{-\gamma} f(t, u(t))$. For u in a bounded subset of $C[0, T]$, $\{Fu\}$ is bounded in $C_{-\gamma}$. By Theorem 4.5 I^α maps $C_{-\gamma}$ into the Hölder space $C^{0,\alpha-\gamma}$. Since the Hölder space is compactly embedded in $C[0, T]$, see Lemma 2.1, this proves that $\{I^\alpha Fu\}$ belongs to a compact subset of $C[0, T]$, thus $N : C[0, T] \to C[0, T]$ is compact. To see that N is continuous, let $u_n \to u$ in $C[0, T]$, then there exists $M > 0$ such that $\|u_n\|_\infty \le M$. Let $(Fu_n)(t) = t^{-\gamma} f(t, u_n(t))$. As f is (uniformly) continuous on $[0, T] \times [-M, M]$, there exists M_1 such that $|f(s, u_n(s))| \le M_1$ for all $s \in [0, T]$. Then we have

$$\begin{aligned}
\int_0^t (t-s)^{\alpha-1} |(Fu_n)(s)|\, ds &= t^{\alpha-\gamma} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{-\gamma} |f(t\sigma, u_n(t\sigma))|\, d\sigma \\
&\le M_1 t^{\alpha-\gamma} \int_0^1 (1-\sigma)^{\alpha-1} \sigma^{-\gamma}\, d\sigma \\
&\le M_1 T^{\alpha-\gamma} B(\alpha, 1-\gamma).
\end{aligned}$$

For $t = 0$, we have $I^\alpha(Fu_n)(0) = 0 = I^\alpha(Fu)(0)$. For each (fixed) $t > 0$, we have $(t-s)^{\alpha-1}(Fu_n)(s) \to (t-s)^{\alpha-1}(Fu)(s)$ for $s \ne 0$ and $s \ne t$, thus, by the dominated convergence theorem, we conclude that $I^\alpha(Fu_n)(t) \to I^\alpha(Fu)(t)$ pointwise. Since $\{I^\alpha(Fu_n)\}$ belongs to a compact subset of $C[0, T]$ the convergence is therefore uniform in t, thus, $Nu_n \to Nu$ in $C[0, T]$. □

Remark 6.2. Theorem 6.1 can be used when f depends on fractional derivatives of lower order than α, say $f(t, u, u^{\{1\}}, \dots, u^{\{n-1\}})$ when $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and $t \mapsto (t, u(t), u^{\{1\}}(t), \dots, u^{\{n-1\}}(t))$ is continuous. For example, let $0 < \beta < \alpha < 1$ and $0 \le \gamma < \alpha - \beta$. Let

$$X := \{u \in C[0, T], D_*^\beta u \in C[0, T]\}$$

be endowed with the norm $\|u\|_X := \|u\|_\infty + \|D_*^\beta u\|_\infty$. Thus $u \in X$ means $u \in C[0, T]$ and $I^{1-\beta}(u - u_0) \in C^1[0, T]$. X is a Banach space, for example Su [13] proves the R-L case, [18, Proposition 6.7] proves the Caputo case. If $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and we write $(Fu)(t) := t^{-\gamma} f(t, u(t), D_*^\beta u(t)$, then, for u in a bounded subset of X, $\{I^\alpha Fu\}$ and $\{I^{\alpha-\beta} Fu\}$ belong to compact subsets of $C[0, T]$, which proves the corresponding operators N and $D_*^\beta N$ are compact from X into X. Further details of this case are given in Theorem 8.3.

Remark 6.3. For N as in Theorem 6.1, it is proved in [16, Theorem 4.8] that N is completely continuous using a much longer argument directly with the Arzelà-Ascoli theorem, which is only used here, Lemma 2.1, to prove the Hölder space embeds compactly into the space of continuous functions. Our new proof combines some useful properties of the Riemann-Liouville fractional integral.

Recently Lan [7] proved, with longer, different methods, that the fractional integral operator I^α is compact from L^p to C when $p > 1/\alpha$, which can also be used to deduce the compactness result of Theorem 6.1.

7. Integral operator arising from a R-L fractional derivative IVP

Recall that, for $0 < \alpha < 1$, from Proposition 5.6, if $u \in C_{\alpha-1}$ then

$$u(t) = u^0 t^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, u(s))\, ds,\ t \in [0, T], \tag{7.1}$$

implies that $D^\alpha u(t) = f(t, u(t))$ and $\lim_{t\to 0+} u(t)t^{1-\alpha} = u^0$.

Note that u is a solution in $C_{\alpha-1}$ of (7.1) if and only if $v(t) := t^{1-\alpha} u(t)$ is a solution in $C[0, T]$ of

$$v(t) = u^0 + t^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, s^{\alpha-1} v(s))\, ds,\ t \in (0, T]. \tag{7.2}$$

Therefore we will give conditions so that

$$Nv(t) := u^0 + t^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s, s^{\alpha-1} v(s)))\, ds \tag{7.3}$$

defines a completely continuous operator in $C[0, T]$. For this, writing $\hat{f}(t, v) = f(t, t^{\alpha-1} v)$, it is enough to prove that the following fractional integral operator

$$\hat{N} v(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \hat{f}(s, v(s))\, ds \tag{7.4}$$

is continuous and compact in $C[0, T]$.

Definition 7.1. A function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is said to be $L^q_{\rm loc}$, $(1 \le q \le \infty)$ if f is measurable and, for each $R > 0$, there exists a function $f_R \in L^q[0,T]$ such that $|f(t,u)| \le f_R(t)$ for all u with $|u| \le R$.

This is similar to the L^q-Carathéodory condition but no continuity condition in the u variable is imposed. We believe the next result is new. It is precisely what is needed for proving an existence result.

Theorem 7.2. *Let $f : [0,T] \times \mathbb{R} \setminus \{0\}$ be continuous and suppose there exist $0 \le \gamma < \alpha < 1$ and an L^∞_{loc} function $g : [0,T] \times \mathbb{R} \to [0,\infty)$ such that $|f(t,t^{\alpha-1}v)| \le t^{-\gamma} g(t,v)$ for $t > 0$. Then $\hat{N}$ defined in (7.4) is completely continuous in $C[0,T]$.*

Proof. Let $\{v_n\}$ be bounded in $C[0,T]$, say $\|v_n\|_\infty \le R$. Write $f_n(t) = \hat{f}(t, v_n(t))$ and $g_n(t) := g(t, v_n(t))$, then there exists $g_R \in L^\infty$ such that $g_n(t) \le g_R(t)$ for all $t \in [0,T]$ and all n. The operator $\hat{N}$ can be written as $\hat{N} v_n = I^\alpha f_n$. Firstly we suppose that $f_n \ge 0$. Then we have $f_n(t) \le t^{-\gamma} g_n(t)$, so there exists $\lambda_{n,t} \in [0,1]$ such that $f_n(t) = \lambda_{n,t} t^{-\gamma} g_n(t)$ and

$$I^\alpha(\lambda_{n,s} s^{-\gamma} g_n(s)) = I^\alpha(\text{bounded set in } L_{-\gamma})$$

which lies in a compact subset of $C[0,T]$ by Theorem 4.5 and Lemma 2.1. This shows that $\{I^\alpha f_n\}$ belongs to a compact subset of $C[0,T]$ when $f_n \ge 0$. In the general case, write f_n^+, f_n^- for the positive and negative parts of f_n so that $f_n = f_n^+ - f_n^-$ and $|f_n| = f_n^+ + f_n^-$, and $f_n^\pm(t) \le |f_n(t)| \le t^{-\gamma} g_n(t)$. By the first part, $I^\alpha f_n^+$ and $I^\alpha f_n^-$ belong to a compact subset of $C[0,T]$. Thus, there is a subsequence of $\{v_n\}$ say $\{v_n^+\}$ such that $\{I^\alpha(\hat{f}(t, v_n^+(t)))^+\}$ converges in $C[0,T]$, and there is a subsequence of $\{v_n^+\}$, say $\{v_n^\pm\}$ such that $\{I^\alpha(\hat{f}(t, v_n^\pm(t)))^-\}$ converges in $C[0,T]$. Then

$$I^\alpha(\hat{f}(t, v_n^\pm(t))) = I^\alpha(\hat{f}(t, v_n^\pm(t)))^+ - I^\alpha(\hat{f}(t, v_n^\pm(t)))^-$$

converges in $C[0,T]$, that is, $\hat{N}$ is compact.

Now we show that $\hat{N}$ is continuous in $C[0,T]$. Let $v_n \to v$ in $C[0,T]$, then $\|v_n\|_\infty \le R$ for some R and $g(s, v_n(s)) \le g_R(s)$ for all $s \in [0,T]$ and all n. Also $f(s, s^{\alpha-1} v_n(s)) \to f(s, s^{\alpha-1} v(s))$ for each $s > 0$. For each fixed $t > 0$, as

$$(t-s)^{\alpha-1} f(s, s^{\alpha-1} v_n(s)) \le (t-s)^{\alpha-1} s^{-\gamma} g(s, v_n(s)) \le (t-s)^{\alpha-1} s^{-\gamma} g_R(s)$$

which is an L^1 function of $s \in [0,t]$, by the dominated convergence theorem we see that $\hat{N} v_n(t) \to \hat{N} v(t)$, that is we have pointwise convergence. Since $\{\hat{N} v_n\}$ belongs to a compact subset of $C[0,T]$ the convergence is uniform in t, that is $\hat{N} v_n \to \hat{N} v$ in $C[0,T]$. □

Corollary 7.3. *Suppose that $f : [0,T] \times \mathbb{R} \setminus \{0\}$ is continuous and that there exist $0 \le \gamma < \alpha < 1$ and $\beta > 0$ and two functions $\phi_i \in L^\infty[0,T]$ $(i = 1,2)$ such that*

$$|f(t,u)| \le t^{-\gamma} \phi_1(t) + t^{(1-\alpha)\beta - \gamma} \phi_2(t) |u|^\beta, \ t \in (0,T]. \tag{7.5}$$

Then the integral operator

$$Nv(t) := v^0 + t^{1-\alpha} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, s^{\alpha-1} v(s))\, ds \tag{7.6}$$

is completely continuous in $C[0,T]$.

Proof. We have

$$|f(t,t^{\alpha-1}v)| \le t^{-\gamma}\phi_1(t) + t^{(1-\alpha)\beta-\gamma}\phi_2(t)t^{(\alpha-1)\beta}|v|^\beta$$
$$= t^{-\gamma}\big(\phi_1(t)+\phi_2(t)|v|^\beta\big) := t^{-\gamma}g(t,v).$$

Clearly $g \in L^\infty_{\rm loc}$, thus the result follows from Theorem 7.2. □

Remark 7.4. I believe the cases $\beta \ne 1$ are new. The case $\beta < 1$ could be deduced from the case $\beta = 1$ since $|u|^\beta \le 1 + |u|$. For the special case $\beta = 1$, a slightly more general result is essentially proved in [19, Lemma 4.1] where it is assumed that $|f(t,u)| \le k(t) + l(t)|u|$ where $k \in L^p$ and $t^{\alpha-1}l(t) \in L^p$ for some $p > 1/\alpha$. The longer proof in [19] does not use any of the properties of fractional integrals that we have used, it is essentially using the Arzelà-Ascoli theorem. For the case when $\beta = 1$ there are Gronwall type inequalities [16,19] and, using these and the complete continuity of the integral operator, global existence of solutions of initial value problems of fractional equations can be proved, as is done in [16,19].

8. Order $\alpha \in (0,1)$ when f depends on fractional derivatives

We consider Caputo fractional equations when the nonlinearity depends on fractional derivatives. Let $0 < \beta < \alpha < 1$ and f be continuous. Consider the problem:

$$D^\alpha_* u(t) = f(t,u(t),D^\beta_* u(t)), \text{ for a.e. } t > 0, \;\; u(0) = u_0. \tag{8.1}$$

If u is a sufficiently regular solution of (8.1) then u is a solution of the integral equation

$$\begin{aligned} u(t) &= u_0 + I^\alpha f(t,u(t),D^\beta_* u(t)) \\ &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),D^\beta_* u(s))\,ds. \end{aligned} \tag{8.2}$$

However, the two problems are not necessarily equivalent. Equivalence depends on how 'solution' of each problem is defined. An appropriate Banach space is

$$X := \{u \in C[0,T], D^\beta_* u \in C[0,T]\}.$$

Definition 8.1. For a continuous function f we say that $u \in X$ is a solution of (8.1), on an interval $[0,T]$, if $D^\alpha_* u$ exists and equals f a.e. We say that $u \in X$ is a solution of (8.2) if u satisfies (8.2) for all $t \in [0,T]$.

Define the operator $N : X \to X$ by

$$Nu(t) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s),D^\beta_* u(s))\,ds \tag{8.3}$$

The following is readily proved.

Proposition 8.2. *Let f be continuous. Then $u \in X$ is a solution of (8.1) if and only if $u \in X$ is a solution of (8.2), that is u is a fixed point of N in X.*

To obtain fixed points it is useful to have the following result.

Theorem 8.3. *Let f be continuous. Then $N : X \to X$ is completely continuous.*

Proof. For $u \in X$ let $(Fu)(t) := f(t, u(t), D_*^\beta u(t))$, then $t \mapsto (Fu)(t)$ is continuous and, since u_0 plays no part, we want to prove that

$$I^\alpha(Fu)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(Fu)(s)\, ds$$

maps into a compact subset of $C[0,T]$. We also want $D_*^\beta N$ to do the same. Since the Caputo derivative of a constant is zero we have

$$\begin{aligned} D_*^\beta Nu(t) &= D_*^\beta I^\alpha(Fu) = D^\beta(I^\alpha(Fu)(s) - I^\alpha(Fu)(0))(t) \\ &= DI^{1-\beta} I^\alpha(Fu)(s)(t), \text{ since } I^\alpha(Fu)(0) = 0 \text{ by Theorem 4.5 (1)}, \\ &= I^{\alpha-\beta}(Fu)(t), \text{ where we used the semigroup property.} \end{aligned}$$

Thus we need to prove that both $I^\alpha F$ and $I^{\alpha-\beta}F$ map into compact subsets of $C[0,T]$. In fact, these results follow immediately from Remark 6.2. Continuity of these maps is proved as in the proof of Theorem 6.1. □

Remark 8.4. By Remark 6.2 the result is also valid if $f(t, u(t), D_*^\beta u(t))$ in (8.2) is replaced by $t^{-\gamma} f(t, u(t), D_*^\beta u(t))$ provided that $0 \le \gamma < \alpha - \beta$.

9. Order $\alpha \in (1, 2)$ when f depends on fractional derivatives

We now consider the problem

$$u(t) = u_0 + b_1 t^\beta / \Gamma(\beta+1) + I^{\alpha+\beta} f(t, u(t), D_*^\gamma u(t)), \tag{9.1}$$

for $0 < \gamma \le \beta \le 1$, $0 < \alpha \le \alpha + \beta - \gamma < 1$ and $\alpha + \beta > 1$ and f is continuous. We do not impose an ordering between α and β. This is the integral equation version of the initial value problem of the sequential fractional differential problem

$$\begin{aligned} D_*^\alpha \big(D_*^\beta u(t)\big) &= f(t, u(t), D_*^\gamma u(t)), a.e. \\ u(0) = u_0,&\ D_*^\beta u(0) = b_1. \end{aligned} \tag{9.2}$$

Note that this is not the same as having $D_*^{\alpha+\beta} u$ on the left since fractional derivatives do not commute or satisfy a semigroup property in general,

$$D_*^\alpha(D_*^\beta)u \ne D_*^\beta(D_*^\alpha)u \ne D_*^{\alpha+\beta}u,$$

see Examples 2.6 and 2.7 in Diethelm's book [2]. Informally we can 'solve' (9.2) as follows:

$$\begin{aligned} &D_*^\alpha \big(D_*^\beta u(t)\big) = f(t, u(t), D_*^\gamma u(t)), \\ &\implies D_*^\beta u(t) = (D_*^\beta u)(0) + I^\alpha f = b_1 + I^\alpha f \\ &\implies u(t) = u_0 + I^\beta b_1 + I^{\alpha+\beta} f = u_0 + \frac{b_1}{\Gamma(\beta+1)} t^\beta + I^{\alpha+\beta} f. \end{aligned}$$

We also have, writing $(Fu)(t) := f(t, u(t), D_*^\gamma u(t))$,

$$D_*^{\beta}u(t) = b_1 + I^{\alpha}(Fu)(t). \tag{9.3}$$

Equations (9.1) and (9.3) should be studied in a space where at least u and $D_*^{\beta}u$ are continuous such as the space $X = \{u \in C[0,T], D_*^{\beta}u \in C[0,T]\}$ as in the previous section, but we will not study existence of solutions here. We only discuss the compactness requirements in this space. For $u \in X$, $t \mapsto (Fu)(t)$ is continuous. We want both $I^{\alpha+\beta}(Fu)$ and $I^{\alpha}(Fu)$ to be completely continuous in $C[0,T]$. We have

$$I^{\alpha+\beta}(Fu)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1}(Fu)(s)\,ds.$$

We can write this as $I^{\alpha+\beta}(Fu)(t) = \frac{1}{\Gamma(\alpha+\beta)} \int_0^T g(t,s)(Fu)(s)\,ds$ where

$$g(t,s) = \begin{cases} (t-s)^{\alpha+\beta-1}, & \text{if } 0 \le s \le t \le T, \\ 0, & \text{if } 0 \le t < s \le T. \end{cases}$$

Since $\alpha + \beta > 1$ this is a special case of Theorem 3.2. Furthermore, $I^{\alpha}(Fu)$ is completely continuous in $C[0,T]$ by Remark 6.2.

10. Comments on boundary value problems

For Caputo fractional differential equations one can impose many types of boundary condition and, under appropriate conditions, find a Green's function. For examples, for a problem of order $\alpha \in (1,2)$ such as

$$D_*^{\alpha}u(t) = f(t,u(t)),\ t \in [0,T],\ \ a_0u(0) - b_0u'(0) = c_0, a_1u(1) + b_1u'(1) = c_1,$$

if we write $(Fu)(t) = f(t,u(t))$, the first step is to say solutions are of the form $u(t) = u_0 + tu_1 + I^{\alpha}(Fu)(t)$ and solve the algebraic equations obtained from the boundary conditions to determine u_0, u_1. Since it is necessary that $I^{\alpha}(Fu)$ is a compact operator it is useful to know when this is the case; our results give this. In particular, if it is shown that u_0, u_1 belong to uniformly bounded sets then the operator is compact.

For example, for the case

$$D_*^{\alpha}u(t) = f(t,u(t)),\ t \in [0,T],\ \ u(0) = a, u(1) = b,$$

we find $u(t) = (1-t)a + tb - tI^{\alpha}(Fu)(1) + I^{\alpha}(Fu)(t)$ and would consider the operator $Nu(t) = (1-t)a + tb - tI^{\alpha}(Fu)(1) + I^{\alpha}(Fu)(t)$. The terms $(1-t)a$ and tb are clearly completely continuous in $C[0,T]$. For u in a bounded subset of $C[0,T]$, if $f(t,u) = t^{-\gamma}g(t,u)$ with g continuous, $I^{\alpha}(Fu)(t)$ will belong to a compact set so the endpoint $I^{\alpha}(Fu)(1)$ will be uniformly bounded and thus all terms are completely continuous.

When f depends on a fractional derivative, the 'derivative' operator must also be considered as discussed in Section 9 above. With more complicated problems and more general boundary conditions this procedure might be difficult to implement.

For boundary value problems for R-L fractional differential equations with $\alpha \in (1,2)$ such as $D^{\alpha}u(t) = f(t,u(t))$, the first step is to claim that solutions are of the form $u(t) = c_1t^{\alpha-2} + c_2t^{\alpha-1} + I^{\alpha}f$. This is only correct if solutions are sought in an appropriate space such as either L^1 or $C_{\alpha-2}$. Using the space $C_{\alpha-2}$ the integral operator is similar to the ones studied in the Caputo case and Theorems we have proved will be applicable. The space L^1 has a different compactness criterion. If solutions are sought in the space $C[0,T]$ then the term $c_1t^{\alpha-2}$ cannot occur and the only consistent boundary condition at 0 is $u(0) = 0$. With the boundary condition $u(0) = 0$ many types of boundary condition can be considered at $t = 1$; the integral operator is again similar to the ones studied in the Caputo case.

11. Conclusion

We have established several compactness results which can be applied in many situations, more than we have described. We hope this can replace many longer calculations using the Arzelà-Ascoli theorem.

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References

[1] L.C. Becker, T.A. Burton, I.K. Purnaras, Complementary equations: a fractional differential equation and a Volterra integral equation, Electron. J. Qual. Theory Differ. Equ. 12 (2015) 1–24.
[2] K. Diethelm, The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type, Lecture Notes in Mathematics, vol. 2004, Springer-Verlag, Berlin, 2010.
[3] G.H. Hardy, J.E. Littlewood, Some properties of fractional integrals. I., Math. Z. 27 (1928) 565–606.
[4] G. Infante, J.R.L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, J. Math. Anal. Appl. 272 (2002) 30–42.
[5] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204, Elsevier Science: B.V., Amsterdam, 2006.
[6] A.N. Kolmogorov, S.V. Fomin, Elements of the Theory of Functions and Functional Analysis. Vol. 1. Metric and Normed Spaces, Translated from the first Russian edition by Leo F. Boron, Graylock Press, Rochester, N.Y., 1957, ix+129 pp.
[7] K.Q. Lan, Compactness of Riemann-Liouville fractional integral operators, Electron. J. Qual. Theory Differ. Equ. (2020) 84.
[8] K.Q. Lan, Equivalence of higher order linear Riemann-Liouville fractional differential and integral equations, Proc. Am. Math. Soc. 148 (12) (2020) 5225–5234.
[9] L. López-Somoza, F. Minhós, Existence and multiplicity results for some generalized Hammerstein equations with a parameter, Adv. Differ. Equ. (2019) 423.
[10] Y. Ma, C. Yin, G. Zhang, Positive solutions of fourth-order problems with dependence on all derivatives in nonlinearity under Stieltjes integral boundary conditions, Bound. Value Probl. (2019) 41.
[11] R.H. Martin, Nonlinear Operators and Differential Equations in Banach Spaces, Wiley, New York, 1976.
[12] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach Science Publishers, Yverdon, 1993.
[13] X. Su, Boundary value problem for a coupled system of nonlinear fractional differential equations, Appl. Math. Lett. 22 (2009) 64–69.
[14] J.R.L. Webb, G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, J. Lond. Math. Soc. (2) 74 (2006) 673–693.
[15] J.R.L. Webb, G. Infante, Nonlocal boundary value problems of arbitrary order, J. Lond. Math. Soc. (2) 79 (2009) 238–258.
[16] J.R.L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, J. Math. Anal. Appl. 471 (2019) 692–711.
[17] J.R.L. Webb, Initial value problems for Caputo fractional equations with singular nonlinearities, Electron. J. Differ. Equ. (2019) 117.
[18] J.R.L. Webb, A fractional Gronwall inequality and the asymptotic behaviour of global solutions of Caputo fractional problems, Electron. J. Differ. Equ. (2021) 80.
[19] T. Zhu, Fractional integral inequalities and global solutions of fractional differential equations, Electron. J. Qual. Theory Differ. Equ. (2020) 5.