Guarantees in Fair Division: general or monotone preferences

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Abstract When dividing a "manna" Ω of private items (commodities, workloads, land, time slots) between n agents, the individual guarantee is the welfare each agent can secure in the worst case of other agents' preferences and actions. If the manna is non atomic and utilities are continuous (not necessarily monotone or convex) the minMax utility, that of our agent's best share in her worst partition of the manna, is guaranteed by Kuhn's generalisation of Divide and Choose. The larger Maxmin utility – of her worst share in her best partition – cannot be guaranteed, even for two agents.

If for all agents more manna is better than less (or less is better than more), the new Bid & Choose rules offer guarantees between minMax and Maxmin by letting agents bid for the smallest (or largest) size of a share they find acceptable.

1 Introduction and the punchlines

The fair division of a common property manna – resources privately consumed – is a complicated problem if its joint owners have heterogenous preferences over the manna. A coarse yet important benchmark is the welfare *guarantee* a division rule offers to each participant: this is the highest welfare that a given agent can secure in this rule, irrespective of the preferences and actions of other agents, even if our agent is clueless about the latter and assumes the worst. The more an agent is risk averse and the less she knows about others' preferences, the more this worst case benchmark matters to her.

Our goal is to throw some light on the feasible guarantees in the very general class of *non atomic* fair division problems, where small changes in the size of a share result in small utility changes (a continuity property explained below). Our model places no other restrictions on the structure of preferences and corresponding utilities, or their direction: the manna may contain some desirable parts (money, tasty cake, valuable commodities), some not (unpleasant tasks, financial liabilities, burnt parts of the cake that must still be eaten: Segal-Halevi [33]); agents may disagree over which parts are good or bad; utilities can be single-peaked over some parts (teaching loads, volunteering time, shares of a risky project), single-dipped on others, etc..

Assume that the manna Ω and the domain $\mathcal{D}$ of potential preferences, described for clarity as utility functions u_i, are common knowledge. A *fair and feasible guarantee* is a mapping $(u_i, n) \rightarrow \Gamma(u_i; n)$ selecting a utility level for each u_i in $\mathcal{D}$ and each number n of joint owners. The mapping is fair because it ignores agent i's identity. The guarantee is *feasible* if for any profile $(u_i)_{i=1}^n$ in $\mathcal{D}^n$ there exists a partition $(S_i)_{i=1}^n$ of Ω such that $u_i(S_i) \geq \Gamma(u_i; n)$ for all i.

Given the division problem $(\Omega, \mathcal{D})$ we ask what are the best (highest) feasible and fair guarantees? and what mechanism implements them in the simple sense of implementation described in the last paragraph of this section?

The first observation is that any such guarantee $\Gamma(u; n)$ is bounded above by the utility, denoted $Maxmin(u; n)$, of the worst share for u in the best n-partition of the manna. Formally we have, for all $u \in \mathcal{D}$ and n:

$$\Gamma(u; n) \leq Maxmin(u; n) = \max_{\Pi=(S_i)_{i=1}^n} \min_{1 \leq i \leq n} u(S_i) \tag{1}$$

where the maximum (that may not be achieved exactly) bears on all n-partitions $\Pi = (S_i)_{i=1}^n$ of Ω. This follows by fairness and feasibility: at the unanimous profile where $u_i = u$ for all i there is a partition Π such that $u(S_i) \geq \Gamma(u; n)$ for all i, hence $\Gamma(u; n) \leq \min_{1 \leq i \leq n} u(S_i) \leq Maxmin(u; n)$.

Therefore if $(u, n) \to Maxmin(u; n)$ is feasible it is the best fair and feasible guarantee, which answers the first of the two general questions above. This happens in two well known and much discussed fair division models.

In the cake-cutting model due to Steinhaus [35] the manna Ω is a measurable space endowed with a non atomic measure, and utilities are additive measures, absolutely continuous with respect to the base measure. Additivity of u implies $Maxmin(u; n) \leq \frac{1}{n} u(\Omega)$; this is in fact an equality because the cake can be partitioned in n shares of equal utility. Agent i's share S_i is *proportionally fair* if $u_i(S_i) \geq \frac{1}{n} u_i(\Omega)$: this is feasible for all agents at any preference profile $(u_i)_{i=1}^n$, therefore proportional fairness offers the best possible guarantee in this model. It is the weakest and least controversial test of fairness throughout the cake-cutting literature (Brams and Taylor [14] and Robertson and Webb [32]).

In the microeconomic model of fair division the manna is a bundle $\omega \in \mathbb{R}_+^K$ of K divisible and non disposable items, and $\mathcal{D}$ is the set of convex and continuous preferences over $[0, \omega]$ (not necessarily monotonic). In $\mathcal{D}$ the inequality $Maxmin(u; n) \leq u(\frac{1}{n}\omega)$ is also true[1], and it is feasible to give an

[1] Pick a hyperplane H supporting the upper contour of u at $\frac{1}{n}\omega$; the *lower* contour of u at $\frac{1}{n}\omega$ contains one closed half-space cut by H, and every division of ω as $\omega = \sum_1^n z_i$ includes at least one z_j in that half-space.

equal share $\frac{1}{n}\omega$ to every agent. Therefore the *equal split lower bound* $u_i(z_i) \geq u(\frac{1}{n}\omega)$ (where z_i is i's share of ω) is the best fair and feasible guarantee. Here too it is the starting point of the discussion of fairness (see e. g., Thomson [38] and Moulin [29]).

As soon as we drop either additivity in the former model or convexity in the latter one, the Maxmin guarantee is not feasible any more, even in two person problems. In a simple example Ann and Bob share 10 units of a single non disposable divisible item (e.g., time spent in a given activity). Ann's preferences are single-peaked (hence convex), while Bob's are single-dipped (see Figure 1: the Figures are collected in Section 8):

$$u_A(x) = x(12 - x) \;\; ; \;\; u_B(x) = x(x - 6) \text{ for } 0 \leq x \leq 10$$

Compute

$$Maxmin(u_A) = 35 \text{ at } \Pi_1 = \{5, 5\} \; ; \; Maxmin(u_B) = 0 \text{ at } \Pi_2 = \{0, 10\}$$

If Bob's share is worth at least $Maxmin(u_B)$ then Ann gets either the whole manna or at most 4 units: so her utility is at most 32 therefore $(Maxmin(u_A), Maxmin(u_B))$ is not feasible.

A second critical benchmark utility is $minMax(u; n)$, the utility of the best share for u in the worst possible n-partition of Ω:

$$minMax(u; n) = \min_{\Pi=(S_i)_{i=1}^n} \max_{1 \leq i \leq n} u(S_i)$$

where as before the minimum bears on all n-partitions of Ω.

Our first main result, Theorem 1 in Section 4, says that in any non atomic problem, the mapping $u \rightarrow minMax(u; n)$ is a feasible and fair guarantee; in particular $minMax(u; n) \leq Maxmin(u; n)$ for all $u \in \mathcal{D}$ and n (by (1)). Moreover the $minMax$ guarantee is implemented by Kuhn's little known n-person generalisation of Divide and Choose (Kuhn [22]), denoted here D&C$_n$.

The result is clear in two person problems, where the simple Divide and Choose rule guarantees her $Maxmin$ to the Divider and his $minMax$ to the Chooser. For instance in the example above Ann would divide as $\Pi_1 = \{5, 5\}$ and Bob would get utility -5, exactly his $minMax(u_B; 2)$; while Bob would divide as $\Pi_2 = \{0, 10\}$, and Ann would choose 10, thus achieving $minMax(u_A; 2) = 20$.[2]

[2] We give in subsection 3.4 a n-person example where for everyone $Maxmin$ is the best utility and $minMax$ is the worst; and when any agent gets her $Maxmin$ utility, everyone else gets his $minMax$ utility.

In three persons problems D&C$_3$ works as follows. The Divider Ann offers a 3-partition $\Pi = \{S_1, S_2, S_3\}$ where all shares are of equal value to her; Bob accepts all shares worth at least $minMax(u_B; 3)$, and Charles all those worth at least $minMax(u_C; 3)$. If Bob and Charles can each be assigned a share they accept, we do so and Ann gets the last piece; if more than one such assignment is feasible any choice implements the target guarantee, which is all we need. If both accept a single share in Π, the same one, we give one of the remaining shares S_k to Ann (it does not matter which one) and then run D&C$_2$ between Bob and Charles for $\Omega \diagdown S_k$ (it does not matter who divides or chooses).

The n-person division rule D&C$_n$ proceeds similarly in at most $n-1$ steps of Division and Acceptance between a shrinking set of agents sharing a shrinking manna. Its only subtlety is a simple combinatorial matching step (Lemma 2 in Section 4) after each partitioning of the remaining manna.

The hard step in proving Theorem 1 is Lemma 1 in Subsection 3.2, stating that in each round of D&C$_n$ the current Divider can find an *equipartition*: a partition of the remaining manna where all shares are equally valuable to this Divider. Because we only assume that the manna is measurable and endowed with a non atomic measure and that utilities are continuous in that measure, the proof of Lemma 1 requires advanced tools in algebraic geometry: this is the object of the companion paper Avvakumov and Karasev [4] also discussed in the next Section.

Our second main result, Theorem 2 in subsection 5.2, focuses on non atomic problems where preferences are also *co-monotone*: that is, *increasing* if enlarging a share cannot make it worse and we speak of a *good manna*; or *decreasing* if the opposite holds and we have a *bad manna*. Either restriction on preferences opens the door to a new family of division rules significantly simpler than D&C$_n$ and implementing a better guarantee than the $minMax$ (weakly better and for some problems strictly better). These rules are inspired by the well known *Moving Knife* (MK$_n$) rules (Dubins and Spanier [20]) that we recall first.

Assume the manna is good: a knife cuts continuously an increasing share of the cake; agents can stop the knife at any time; the *first* agent who does gets the share cut so far. Repeat between the remaining agents and manna. For a bad manna, agents can drop at any time and the *last* one to drop gets the share cut so far.

A Moving Knife (MK) rule chooses a single arbitrary path for the knife, which tightly restricts the range of individual shares and partitions, hence can result in a very inefficient allocation. We introduce the large family of *Bid & Choose* (B&C$_n$) rules: they resemble the MK rules but allow all partitions in their range. Each rule is defined by fixing a benchmark *additive measure* of the shares, diversely interpreted as their size, their market price, etc.. If the manna is good a bid b_i by agent i is the smallest measure of a share that i finds acceptable: the smallest bidder i^* chooses freely a share of measure at most b_{i^*}, then we repeat between the remaining agents and manna. For a bad manna the bid b_i is the largest size of a share that i finds acceptable, and the largest bidder i^* picks any share of size at least b_{i^*}.

Theorem 2 in section 5 shows that all B&C$_n$ rules, as well as all MK$_n$ rules implement a guarantee between the $minMax$ and $Maxmin$ level.

A handful of examples in subsection 5.3 show that the B&C$_n$ guarantee improves substantially the $minMax$ guarantee in the microeconomic model of fair division. There the equal split lower bound is the $Maxmin$ benchmark (the best possible) for agents with convex preferences, while for agents with "concave" preferences (convex lower contours) equal split is the $minMax$ guarantee, significantly below the B&C$_n$ guarantee.

Throughout the paper we speak of implementation in the very simple sense adopted by most of the cake cutting literature (e. g., Brams and Taylor [14]; see also the general concept of implementation in "protective equilibrium" by Barbera and Dutta [7]). A rule implements (guarantees) a certain utility level γ means this: no matter what her preferences, each agent has a strategy that depends also upon Ω, n and $\mathcal{D}$, such that whatever other agents do the utility of her share is no less than γ. Moreover the "guaranteeing strategy" is essentially unique.

2 Relevant literature

The two welfare levels $Maxmin$ and $minMax$ are key to our results. In the *atomic* model where the manna is a set of indivisible items, they are introduced by Budish [16] and Bouveret and Lemaitre [12] respectively. If utilities are additive in that model, the basic inequality of our *non atomic* model is reversed:

$$Maxmin(u;n) \leq \frac{1}{n}u(\Omega) \leq minMax(u;n)$$

and $minMax(u;n)$ is obviously not a feasible guarantee. It took a couple of years and many brain cells to check that the $Maxmin$ lower bound may not be feasible either for three or more agents (Procaccia and Wang [31]), though this happens in rare instances of the model (Kurokawa et al. [24]).[3] Our paper is the first general discussion of these two bounds in the non atomic model of cake division.

Kuhn's 1967 [22] n person generalisation of Divide and Choose promptly implements the $minMax$ guarantee in our model (Theorem 1). Except for a recent discussion in Aigner-Horev and Segal-Halevi [1] for additive utilities D&C$_n$ has not received much attention, a situation which our paper may help to correct. In particular, unlike the Diminishing Share (Steinhaus [35]) Moving Knife (Dubins and Spanier [20]), and Bid and Choose rules, it is very well suited to divide *mixed manna,* i. e., containing subjectively good and bad parts, as when we divide the assets and liabilities of a dissolving partnership. Introduced in Bogomolnaia et al. [11], [10] for the competitive fair division of microeconomic commodities, the mixed manna model is discussed in Segal-Halevi [33] for a general cake and in Aziz et al. [5] for indivisible items.

Privacy preservation is a growing concern in a world of ever expanding information flows. The D&C$_n$ rule stands out for its informational parsimony: each Divider only reports a partition with the understanding that she is indifferent between its shares, and Choosers only only accept a subset of these shares. If the manna is mixed, no one is asked to explain which parts they view as good or bad: for instance if we divide tasks, I may not want others to know which tasks I am actually happy to perform and which ones I am not.

The "cuts" selected by Dividers and "queries" answered by Choosers require only a modest cognitive effort: no one needs to form complete preference relations over all shares of the cake. Taking this feature to heart, a large literature in the cake cutting model evaluates the informational complexity of various mechanisms by the number of cuts and queries they involve: see Brams and Taylor [14] or Robertson and Webb [32], and more recently Cseh and Fleiner [17] and Crew et al. [18]. This line of research goes beyond the test of proportional guarantee, using cuts and queries more complex than in D&C$_n$ to reach an Envy-free division of the cake. The algorithms in Brams and Taylor [13], and more recently Aziz and McKenzie [6], do exactly this

[3] If the manna is atomic and utilities are not necessarily additive, it is easy to construct examples showing that all six orderings of $Maxmin$, $minMax$, and $\frac{1}{n}u(\Omega)$ are possible.

when utilities are additive and non atomic; but because they involve an astronomical number of cuts and queries they are of no practical interest and squarely contradict informational parsimony. See Branzei [15] and Kurokawa et al. [23] for some fine tuning of these general facts.

For microeconomic fair division under additive utilities, D'All'Aglio [19] suggests to use an objective "market value" of the manna to limit the discrepancies generated by differences in subjective preferences: this is the most natural interpretation of the benchmark measure defining our Bid and Choose rules.

The "equipartition" Lemma (section 3.2) is critical to the proof of Theorem 1 and proved in Avvakumov and Karasev [4] by algebraic geometry techniques. The latter, or subtle variants of Sperner's Lemma, demonstrate the existence of an Envy-free division under very general preferences, where which share I like best in a given partition can depend upon the partition itself, not just upon my own share: the seminal insights in Stromquist [36] and Woodall [39] are considerably strenghtened by the recent results in Su [37], Segal-Halevi [33], Meunier and Zerbib [26] and Avvakumov and Karasev [3]. However these results do not apply to a mixed manna because they assume, either that all agents (weakly) prefer any non empty share to the empty share, or that all weakly prefer the empty share to any non empty one.

When we divide private goods and preferences are convex, the equal split lower bound corresponds to the *unanimity utility*: the common efficient utility level in the economy where everyone has the same preferences (Footnote 1). When applied to fair division problems involving production, the unanimity utility delivers some compelling fair and feasible guarantees as well as some meaningful *upper bounds* on individual welfare: Moulin [28], [27].

3 Non atomic fair division

3.1 Basic definitions

The manna Ω is a bounded measurable set in an euclidian space, endowed with the Lebesgue measure $|\cdot|$, and such that $|\Omega| > 0$. A share S is a possibly empty measurable subset of Ω, and $\mathcal{B}$ is the set of all shares. A n-partition of Ω is a n-tuple of shares $\Pi = (S_i)_{i=1}^n$ such that $\cup_{i=1}^n S_i = \Omega$ and $|S_i \cap S_j| = 0$ for all $i \neq j$; and $\mathcal{P}_n(\Omega)$ is the set of all n-partitions of Ω. We define similarly an n-*partition of* S for any share $S \in \mathcal{B}$, and write their set as $\mathcal{P}_n(S)$.

If $S \otimes T = (S \cup T) \diagdown (S \cap T)$ is the symmetric difference of shares, recall that $\delta(S,T) = |S \otimes T|$ is a pseudo-metric on $\mathcal{B}$ (a metric except that $\delta(S,T) = 0$ iff S and T differ by a set of measure zero).

A utility function u is a mapping from $\mathcal{B}$ into $\mathbb{R}$ such that $u(\varnothing) = 0$ and u is continuous for the pseudo-metric δ and bounded. So u does not distinguish between two shares at pseudo-distance zero (equal up to a set of measure zero): for instance $u(S) = 0$ if $|S| = 0$. Also if the sequence $|S^t|$ converges to zero in t, so does $u(S^t)$. We write $\mathcal{D}(\Omega)$ for this domain of utility functions.

A non atomic division problem consists of $(\Omega, \mathcal{B}, (u_i)_{i=1}^n \in \mathcal{D}(\Omega)^n)$. Several subdomains of $\mathcal{D}(\Omega)$ play a role below:

- additive utilities: $u \in \mathcal{A}dd(\Omega)$ *iff* $u(S) = \int_S f(x)dx$ for all S, where f is bounded and measurable in Ω;

- monotone increasing: $u \in \mathcal{M}^+(\Omega)$ *iff* $S \subset T \Longrightarrow u(S) \leq u(T)$ for all S, T;

- monotone decreasing: $u \in \mathcal{M}^-(\Omega)$ *iff* $S \subset T \Longrightarrow u(S) \geq u(T)$ for all S, T;

- separable: $u \in \mathcal{S}(\Omega)$ *iff* there is a finite set A, a partition $(C_a)_{a\in A} \in \mathcal{P}_{|A|}(\Omega)$ of Ω, and a continuous function v from $\mathbb{R}_+^A$ into $\mathbb{R}$, such that $u(S) = v((|S \cap C_a|)_{a \in A})$ for all $S \in \mathcal{B}$.

The separable domain $\mathcal{S}(\Omega)$ captures the standard microeconomic fair division model: A is a set of divisible commodities, the manna is the bundle $\omega \in \mathbb{R}_+^A$ such that $\omega_a = |C_a|$ for all a, a share S_i gives to agent i the amount $z_{ia} = |S_i \cap C_a|$ of commodity a, and the partition $\Pi = (S_i)_{i=1}^n$ corresponds to the division of the manna as $\omega = \sum_1^n z_i$.

In the general non atomic division problem, the set of shares $\mathcal{B}$ is not compact for the pseudo-metric δ. It follows that when we maximize or minimize utilities over shares, or look for a partition achieving a benchmark utility $minMax$ or $Maxmin$, we cannot claim the existence of an exact solution to the program: the $minMax$ is not a true minimum, only an infimum, and $Maxmin$ is only a supremum, not a true maximum. As this will cause no confusion, we stick to the min and Max notation throughout.

However in the microeconomic model, the set of shares and of partitions are both compact so for this important set of problems (where all our examples live) the min and Max notation are strictly justified.

One can also specialise the general model by imposing constraints on the set of feasible shares. The most important instance is the familiar *interval model*, where the manna is $\Omega = [0, 1]$ and a share must be an interval, so an n-partition is made of n adjacent intervals. Other instances assume Ω is a polytope, and shares are polytopes of a certain type: e.g. triangles or tetrahedrons (Segal-Halevi et al. [34]). Sometimes shares must be connected subsets of Ω (Berliant et al. [8], Aumann and Dombb [2]) or even separated by a minimal gap (Elkind et al. [21]).

The Divide and Choose$_n$ rules, as well as our Bid and Choose$_n$ rules, do not work in these models. Consider for instance the D&C$_n$ rule (section 4) in the interval model. The first divider can find an equipartition made of adjacent intervals (Lemma 1), but the agent called next to divide is handed a set of typically disconnected intervals, so our Theorems 1 and 2 do not apply. But the interval model is still useful here in a technical sense: the proof of the critical Lemma 1 in the next subsection starts by projecting the general problem onto an interval model and proving existence of an equipartition there.

3.2 Equipartitions

Definition 1 *An n-equipartition of the share $T \in \mathcal{B}$ for utility $u \in \mathcal{D}(T)$ is a partition $\Pi^e = (S_i)_{i=1}^n \in \mathcal{P}_n(T)$ such that $u(S_i) = u(S_j)$ for all $i, j \in \{1, \cdots, n\}$; we write $u(\Pi^e)$ for this common value, and $\mathcal{EP}_n(T; u)$ for the set of these n-equipartitions.*

It is clear that $\mathcal{EP}_n(S; u)$ is non empty if u is additive. Let $\mathcal{B}[S]$ be the set of shares included in S: Lyapunov Theorem implies that the range $u(\mathcal{B}[S])$ is convex, so it contains $\frac{1}{n}u(S)$; then we replace n by $n - 1$ and repeat the argument on the remaining share.

The same is true if u is monotone ($u \in \mathcal{M}^{\pm}(\Omega)$): the proof, outlined in Remark 1 below, is fairly simple. But proving the next statement is much harder.

Lemma 1 Avvakumov and Karasev [4]
Fix a share $S \in \mathcal{B}$ and a utility $u \in \mathcal{D}(\Omega)$. The set $\mathcal{EP}_n(S; u)$ of n-equipartitions of S at u is non empty.

Proof. Avvakumov and Karasev's Theorem proves Lemma 1 for the interval model (which, as mentioned above, is not a special case of our model). Fix a real valued function f on the set of intervals $[a,b] \subset [0,1]$, continuous in the standard topology and such that $f(a,a) = 0$ for all $a \in [0,1]$. Then there exist n subintervals $[0 = x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n = 1]$ of $[0,1]$ forming an equipartition of f: $f(x_{i-1}, x_i)$ is constant for $i = 1, \cdots, n$.

Start now from a share S in the statement of Lemma 1 and pick a point ω in S; let ρ be the radius of the smallest ball $B(\rho;\omega)$ centered at ω and containing S up to a set of measure zero (recall Ω is bounded). Then we define

$$f(a,b) = u(\{B(b\rho;\omega) \diagdown B(a\rho;\omega)\} \cap S) \text{ for all } 0 \leq a \leq b \leq 1$$

(this is an instance of a *moving knife* through S, formally define in section 5.1)

The function f is continuous because u is continuous in $\mathcal{B}$, and $f(a,a) = 0$. Then an f-equipartition $([x_{i-1}, x_i])_{i=1}^n$ of $[0,1]$ yields the desired u-equipartition $(\{B(x_t\rho;\omega) \diagdown B(x_{t-1}\rho;\omega)\} \cap S)_{i=1}^n$ of S. ∎

Remark 1 It is easy to prove Lemma 1 if we assume that the sign of u is constant: all shares are weakly preferred to the empty share, or all are weakly worse. Assume the former and use as above a moving knife to project S onto $[0,1]$, where a n-partition is identified with a point in the simplex of dimension $n-1$. Then apply the Knaster–Kuratowski–Mazurkiewicz Lemma to the sets Q_i of partitions of the interval where the i-th interval gives the lowest utility: each Q_i is closed, contains the i-th face of the simplex, and their union covers it entirely. Thus these sets intersect.

*One can also invoke the stronger results in Stromquist [36] and Su [37] showing the existence of an Envy-free partition under this assumption. But recall that a key feature in the division of a mixed manna is that the sign of u is **not** constant across shares.*

3.3 Two utility benchmarks

Definition 2 *Fix n, the manna $(\Omega, \mathcal{B})$ and $u \in \mathcal{D}(\Omega)$:*

$$minMax(u;n) = \min_{\Pi \in \mathcal{P}_n(\Omega)} \max_{1 \leq i \leq n} u(S_i) \ ; \ Maxmin(u;n) = \max_{\Pi \in \mathcal{P}_n(\Omega)} \min_{1 \leq i \leq n} u(S_i) \tag{2}$$

Recall that $minMax$ is the utility agent u can achieve by having first pick in the worst possible n-partition of Ω, and $Maxmin$ by having last pick in the best possible n-partition of Ω.

Proposition 1

i) If $u \in \mathcal{A}dd(\Omega)$ *then* $minMax(u;n) = Maxmin(u;n) = \frac{1}{n}u(\Omega)$

ii) If $u \in \mathcal{M}^{\pm}(\Omega)$

$$minMax(u;n) = \min_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \ ; \ Maxmin(u;n) = \max_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \quad (3)$$

iii) If $u \in \mathcal{D}(\Omega)$

$$minMax(u;n) \leq \min_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \leq \max_{\Pi^e \in \mathcal{EP}_n(\Omega;u)} u(\Pi^e) \leq Maxmin(u;n) \quad (4)$$

Proof

Statement iii) If Π^e is an n-equipartition, $u(\Pi^e)$ is the utility of its best share, hence $minMax(u;n) \leq u(\Pi^e)$; proving the other inequality in (4) is just as easy.

Statement i) By additivity of u, for any n-partition Π we have $\max_i u(P_i) \geq \frac{1}{n}u(\Omega)$ implying $minMax(u;n) \geq \frac{1}{n}u(\Omega)$; we check symmetrically $\frac{1}{n}u(\Omega) \geq Maxmin(u;n)$, and the conclusion follows by comparing these inequalities to those in (4).

Statement ii) Assume $u \in \mathcal{M}^{+}(\Omega)$; the proof for $\mathcal{M}^{-}(\Omega)$ is identical. The continuity and monotonicity of u imply: if S, T are two disjoints shares such that $u(S) > u(T)$, we can trim part of S and add it to T to get two disjoint shares with equal utility in between $u(S)$ and $u(T)$. To check this let $B(r)$ be a ball with an arbitrary fixed center and radius r; we trim S to $S(r) = S \diagdown B(r)$ and padd T to $T(r) = T \cup \{S \cap B(r)\}$: for some choice of r we get $u(S(r)) = u(T(r))$.

Expanding this argument, if $S_1, \cdots, S_k$ and T are disjoint shares such that

$$u(S_1) = u(S_2) = \cdots = u(S_k) > u(T)$$

we can simultaneously trim $S_1, \cdots, S_k$, keeping them of equal utility, and add the trimming to T, so that the resulting $k+1$ shares are all equally good and their common utility is between the two utilities above. Iterating this process, we see that if $\Pi = (S_i)_{i=1}^{n} \in \mathcal{P}_n(\Omega)$ is such that $\max_{1 \leq i \leq n} u(S_i) >$

$\min_{1\le j\le n} u(S_j)$, we can construct an equipartition $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that

$$\max_{1\le i\le n} u(S_i) > u(\Pi^e) > \min_{1\le j\le n} u(S_j)$$

Now fix $\varepsilon > 0$, arbitrarily small, pick $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ such that $\min_{1\le j\le n} u(S_j) \ge Maxmin(u; n) - \varepsilon$, and assume that Π is not an equipartition. By the argument above we can find $\Pi^e \in \mathcal{EP}_n(\Omega; u)$ such that $u(\Pi^e) > \min_{1\le j\le n} u(S_j)$, therefore Π^e too is an ε-approximation of $Maxmin(u; n)$, and the right-hand inequality in (3) follows. The proof of the left-hand inequality is similar. ■

We illustrate statement *ii)* in a microeconomic example with two goods and the manna $\omega = (1, 1)$. For the Leontief utility $u(x, y\} = \min\{x, y\}$ we have $Maxmin(u; 2) = \frac{1}{2}$ at the equal split partition, while $minMax(u; 2) = 0$ at the equipartition $\{(1, 0); (0, 1)\}$; for the anti-Leontief utility $v(x, y) = \max\{x, y\}$ the same two equipartitions give dually $minMax(v; 2) = \frac{1}{2}$ and $Maxmin(v; 2) = 1$. Clearly the utility profile $(Maxmin(u; 2), Maxmin(u; 2))$ is not feasible. See more striking examples of the incompatibility in the next subsection.

For statement *iii)* we use the example of section 1 to show that both inequalities in (4) can be strict. Recall that Ann's $Maxmin(u_A; 2) = 35$ is achieved by the equal split partition, while $minMax(u_A; 2) = 20$ obtains by the partition$\{0, 10\}$, not an equipartition. Next Bob has single-dipped preferences $u_B(x) = x(x - 6)$: among three agents he can propose two 3-equipartitions: equal split achieving $minMax(u_B; 3) = -8.9$; and the more appealing $\{2, 4, 4\}$ with the common value -8. But $Maxmin(u_B; 3) = 0$ is only achieved at the partition $\{0, 0, 10\}$.

Remark 2: In the interval model with a monotone utility u, it is easy to check that any two n-equipartitions have the same utility and in turn this implies $minMax(u; n) = Maxmin(u; n)$: hence this is the best guarantee. The example of section 1 can be viewed as an instance of the interval model where the two agents are indifferent between $[0, x]$ and $[1 - x, 1]$ for all x. It shows that only the inequality (4) holds true in the general (non monotone) interval model.

3.4 Guarantees (fair and feasible)

Definition 3 *Fix the manna $(\Omega, \mathcal{B})$ and a subdomain $\mathcal{D}^*$, $\mathcal{D}^* \subseteq \mathcal{D}(\Omega)$. A (fair and feasible) guarantee in $\mathcal{D}^*$ is a mapping $\Gamma : u \to \Gamma(u; n)$ such that* *for any profile* $(u_i)_{i=1}^n \in (\mathcal{D}^*)^n$ *there exists* $\Pi = (S_i)_{i=1}^n \in \mathcal{P}_n(\Omega)$ *such that* $u_i(S_i) \geq \Gamma(u_i; n)$ *for all* i*.*

As this will cause no confusion we simply speak of a guarantee without repeating the hard-wired properties of fairness and feasibility.

In section 1 we observed, by looking at unanimity profiles, that $Maxmin(\cdot; n)$ is an upper bound for *any* guarantee: inequality (1). We also mentioned two subdomains where $Maxmin(\cdot; n)$ itself is a (hence the optimal) guarantee: the additive domain $\mathcal{A}dd(\Omega)$ and the subdomain of the separable one $\mathcal{S}(\Omega)$ where preferences are also convex. Finally we used the Ann and Bob microeconomic example with a single commodity to show that $Maxmin(\cdot; n)$ is not a guarantee in $\mathcal{D}(\Omega)$, even for $n = 2$ and a one dimensional manna.

As announced in the introduction (footnote 2) we give now some simple microeconomic n-person examples where for everyone $Maxmin$ is the best utility and $minMax$ is the worst, and when any agent gets her $Maxmin$ utility, everyone else gets his $minMax$ utility. In particular the utility profile where i gets her $Maxmin$ while others get their $minMax$ is Pareto optimal in this economy.

We start with a simple three person example with one unit of nine goods $a_k, k = 1, \cdots, 9$. With the compact notation $\wedge, \vee$ for min and max respectively we define the three utilities as follows:

$$u_A(z) = (z_1 \wedge z_2 \wedge z_3) \vee (z_4 \wedge z_5 \wedge z_6) \vee (z_7 \wedge z_8 \wedge z_9)$$

$$u_B(z) = (z_1 \wedge z_5 \wedge z_9) \vee (z_2 \wedge z_6 \wedge z_7) \vee (z_3 \wedge z_4 \wedge z_8)$$

$$u_C(z) = (z_1 \wedge z_6 \wedge z_5) \vee (z_2 \wedge z_4 \wedge z_9) \vee (z_3 \wedge z_5 \wedge z_7)$$

An agent gets his $Maxmin$ utility 1 only by eating all three goods in one of his complementary triples, and in each case this implies that the other two agents get their $minMax$ utility 0.

For a general n-person example with the announced features, we select for each pair of (distinct) agents i, j a set of n^2 goods $a_{ij}(k, \ell), 1 \leq k, \ell \leq n$. To get his top utility 1 agent i must, for each $j \neq i$, eat all ij-goods of a certain type k (possibly depending on j), while agent j must, for each $i \neq j$, eat all ij-goods of a certain type ℓ. Formally

$$u_i(z) = \wedge_{j \in N \setminus i} \vee_{k=1}^n \wedge_{\ell=1}^n z_{ij}(k, \ell)$$

4 The Divide & Choose$_n$ rule

Start by a combinatorial observation. Let G be a bilateral graph between the sets M of agents and R of shares: interpret $(m,r) \in G$ as agent m *likes* share r. We say that the subset $\widetilde{M}$ of agents are *properly matched* to the subset $\widetilde{R}$ of shares if $|\widetilde{M}| = |\widetilde{R}|$, agents in $\widetilde{M}$ are each matched (one-to-one) to a share they like in $\widetilde{R}$, and no one outside $\widetilde{M}$ likes any share in $\widetilde{R}$.

Lemma 2. *Assume $|M| = |R|$ and some agent i^1 likes all objects in R. Then there is a (non empty) largest set M^1 of properly matchable agents containing i^1: if $\widetilde{M}$ is properly matched to $\widetilde{R}$, then $\widetilde{M} \subseteq M^1$.*

Proof. We apply the Gallai-Edmonds decomposition of a bipartite graph: see e.g. Lovasz and Plummer [25] Chapter 3 or Lemma 1 in Bogomolnaia and Moulin [9]. If M can be matched with R this is a proper match and the statement holds true. If M and R cannot be matched, then we can uniquely partition M as (M^2, M^1) and R as (R^2, R^1) such that:
1. $|M^2| > |R^2|$, the agents in M^2 do not like any object in R^1, and they compete for the over-demanded objects in R^2: every subset of R^2 is liked by a strictly larger subset of agents in M^2;
2. $|M^1| < |R^1|$ and the agents in M^1 can be matched with some subset of R^1.

By the general Gallai-Edmonds result, M^2 and R^1 are non empty. Here M^1 is non empty as well because it contains the special agent i^1. Every match of M^1 to a subset of R^1 is proper. Finally suppose $\widetilde{M}$ is properly matched to $\widetilde{R}$ and $\widehat{M} = \widetilde{M} \cap M^2$ is non empty. Then $\widehat{M}$ is matched to some subset $\widehat{R}$ of R^2 but $\widehat{R}$ is liked by more agents in M^2 than there are in $\widehat{M}$, therefore the match is not proper: contradiction. So $\widetilde{M}$ does not intersect M^2 as was to be proved.■

Definition 4: *the D&C$_n$ rule.*
Fix the manna $(\Omega, \mathcal{B})$ and the ordered set of agents $N = \{1, \cdots, n\}$, each with a utility in $\mathcal{D}(\Omega)$.
Step 1. Agent 1 proposes a partition $\Pi^1 \in \mathcal{P}_n(\Omega)$; all other agents report which shares in Π^1 they like. In the resulting bipartite graph between N and the shares in Π^1, where agent 1 likes all the shares, we use Lemma 2 to match properly the largest possible set of agents N^1 (it contains agent 1) with some set of shares R; if $N^1 = N$ we are done, otherwise we *go to*
Step 2. Repeat with the remaining manna Ω^2 and agents in $N \diagdown N^1$. Ask the first agent in the exogenous ordering to propose a partition $\Pi^2 \in \mathcal{P}_{n-|N_1|}(\Omega^2)$,

while others report which of these new shares they like. And so on.

At least one agent, the Divider, is served in each step, thus the algorithm just described takes at most $n-1$ steps. But the algorithm matches as many agents as possible so as to minimize the number of cuts (as well as information disclosure), and typically takes fewer steps.

There is some flexibility in the Definition of the rule: although the set of agents matched in each step is unambiguous, we may have several choices for the set R of shares to assign in each step, and multiple ways to assign these to the agents.

Our first main result is that $minMax$ is a guarantee (Definition 3) implemented by the D&C$_n$ rule in the full domain $\mathcal{D}(\Omega)$.

Theorem 1

Fix the manna $(\Omega, \mathcal{B})$ *and* n.

*i) In the D&C$_n$ rule, an agent with utility $u \in \mathcal{D}(\Omega)$ is guaranteed the $minMax(u;n)$ utility level by 1) when called to divide, proposing an equipartition $\Pi^e \in \mathcal{EP}_m(S;u)$ of the remaining share S of manna among the m remaining agents, and 2) when reporting shares he likes, accepting **all** shares, and only those, not worse than $minMax(u;n)$ (the $minMax$ level in the **initial** problem).*

ii) The first Divider (and no one else) is guarantees her $Maxmin$ utility by proposing her $Maxmin$ partition in Step 1. Other agents are guaranteed more than their $minMax$ utility.

Proof. *Statement i)*. Consider agent u using the strategy in the statement. At a step where he must report which shares he likes among those offered at that step, he can for sure find one worth at least $minMax(u;n)$: indeed all shares previously assigned are worth to him strictly less than $minMax(u;n)$, and together with the freshly cut shares they form a partition in $\mathcal{P}_n(\Omega)$; in any partition at least one share is worth $minMax(u;n)$ or more.

At a step where our agent is called to cut, he proposes to the remaining agents an m-equipartition $\Pi^e \in \mathcal{EP}_m(S;u)$ of the remaining manna S. To check the inequality $u(\Pi^e) \geq minMax(u;n)$ note that Π^e together with the previously assigned shares is a partition of Ω in which the old shares are worth strictly less than $minMax(u;n)$. Note that after Step 1 an agent can secure his $Maxmin$ utility for the smaller manna S among m agents, but this may be below the $Maxmin$ utility in the initial problem.

Statement ii). This is clear for the first Divider. Fix now an agent i with utility u and check that if he is not the first Divider, for certain moves of the other agents he gets exactly his $minMax$ utility. Pick a partition $\Pi \in \mathcal{P}_n(\Omega)$ achieving $minMax(u;n)$ (as usual, the existence assumption is without loss). Suppose that the first Divider, who is not agent i, offers Π, and all agents other than i (including the Divider) find all shares acceptable. If agent i accepts at least one share on offer, then a full match is feasible; if agent i rejects all shares on offer, the other $n-1$ agents get a share in the first round and agent i still eats one component of Π. $\blacksquare$

5 Bid and Choose and Moving Knives for good or bad manna

We now assume that the manna is unanimously good, $u \in \mathcal{M}^+(\Omega)$, or unanimously bad, $u \in \mathcal{M}^-(\Omega)$. Because $u(\varnothing) = 0$, we have $u(S) \geq 0$ for all S in the former case and $u(S) \leq 0$ in the latter. Recall that in these two domains, the $minMax$ (resp. $Maxmin$) utility is the smallest (resp. largest) equipartition utility: property (3) in Proposition 1.

The profile of $Maxmin$ utility levels may still not be feasible with monotone preferences, as illustrated in the microeconomic example following the proof of Proposition 1 (subsection 3.3). There agent u with Leontief utilities has $minMax(u;2) = 0$ and $Maxmin(u;2) = \frac{1}{2}$ whereas the Bid and Choose rules described in section 5.3 give her a guarantee of $\frac{1}{3}$. Similarly agent v with anti-Leontief utilities is guaranteed utility $\frac{2}{3}$ by B&C$_n$, compared to $minMax(v;2) = \frac{1}{2}$ and $Maxmin(v;2) = 1$.

5.1 MK$_n^\kappa$ and B&C$_n^\theta$ rules

A *moving knife* through the manna $(\Omega, \mathcal{B}, |\cdot|)$ is a path $\lambda : [0,1] \ni t \to \Lambda(t) \in \mathcal{B}$ from $\Lambda(0) = \varnothing$ to $\Lambda(1) = \Omega$, continuous for the pseudo-metric δ on $\mathcal{B}$ and inclusion increasing:

$$0 \leq t < t' \leq 1 \Longrightarrow \Lambda(t) \subset \Lambda(t')$$

The moving knife λ arranges shares (of a good manna) increasingly valuable to all participants along the specific path of the knife. An example is $\Lambda(t) = B(t\rho) \cap \Omega$, where $t \to B(t\rho)$ is a path of balls with a fixed center and radius growing from 0 to ρ, where $B(\rho)$ contains Ω. Moving knifes can take many other shapes, for instance half-spaces of parallel hyperplanes.

Our Bid and Choose rules offer more choices than Moving Knives to the agents. The designer picks a benchmark *measure* θ of the shares: θ is a positive σ-additive measure on $(\Omega, \mathcal{B})$, normalised to $\theta(\Omega) = 1$. It is absolutely continuous w.r.t. the Lebesgue measure $|\cdot|$ with a strictly positive density. In particular θ is strictly inclusion increasing:

$$\forall S, T \in \mathcal{B} : S \subset T \text{ and } |T \diagdown S| > 0 \Rightarrow \theta(S) < \theta(T)$$

In applications θ can evaluate for instance the market value, physical size, or weight of a share.

Fixing a moving knife λ and a measure θ, we give parallel definitions of the Moving Knife (MK_n^λ) and the Bid and Choose ($\mathrm{B\&C}_n^\theta$) rules. In both cases a clock t runs from $t = 0$ to $t = 1$.

Definition 5 *the MK_n^λ and $B\&C_n^\theta$ rules with increasing utilities*
Step 1. The first agent i_1 to stop the clock, at t^1, gets the share $\Lambda(t^1)$ in MK_n^λ, or in $B\&C_n^\theta$ chooses any share in Ω s.t. $\theta(S) = t^1$, say S_{i_1}, and leaves;
Step k: Whoever stops the clock first at t^k gets the share $\Lambda(t^k)\diagdown\Lambda(t^{k-1})$ in MK_n^λ, or in $B\&C_n^\theta$ chooses any share in $\Omega\diagdown\cup_{\ell=1}^{k-1} S_{i_\ell}$ s.t. $\theta(S) = t^k - t^{k-1}$, say S_{i_k}, and leaves;
In Step $n-1$ the single remaining agent who did not stop the clock takes the remaining share $\Omega\diagdown\Lambda(t^{n-1})$ or $\Omega\diagdown\cup_{\ell=1}^{n-1} S_{i_\ell}$.

Definition 5* *with decreasing utilities*
In each step all agents must choose a time to "drop", and the last agent i_1 who drops, at t^1, gets $\Lambda(t^1)$ in MK_n^κ, or in $B\&C_n^\theta$ chooses S_{i_1} s.t. $\theta(S_{i_1}) = t^1$. The other steps are similarly adjusted.

Breaking ties between agents stopping the clock (or dropping) at the same time is the only indeterminacy in these rules, much less severe than in $\mathrm{D\&C}_n$, where we serve at each step an unambiguous set of agents but there are typically several ways to match them properly.

Up to tie-breaking, the rules $\mathrm{B\&C}_n^\theta$ and MK_n^λ are anonymous but not neutral: they do not give a special role to any agent but restrict the choices of shares along the knife or according to their θ-measure. Compare with $\mathrm{D\&C}_n$ that is neutral (placing no restictions on the partitions selected by successive Dividers) but not anonymous.

In MK_n^λ the share of an agent takes the form $\Lambda(t)\diagdown\Lambda(t')$ so it varies in a set of dimension 2 (and feasible partitions move in a set of dimension $n-1$). By contrast every partition in $\mathcal{P}_n(\Omega)$ is feasible under the $\mathrm{B\&C}_n^\theta$ rule.

To check this fix $\Pi = (S_i)_{i=1}^n$ and assume first $|S_i| > 0$ for all i. Consider n agents deciding (cooperatively) to achieve Π. By construction of θ the sequence $t^i = \theta(\cup_{j=1}^i S_j)$ increases strictly therefore they can stop the clock (or drop) at these successive times and choose the corresponding shares in Π. If there are shares of measure zero they can all be distributed at time 0.

Remark 3. We can also define static versions of MK_n^λ and $B\&C_n^\theta$ where agents bid all at once for potential stopping times, and implementing the same guarantees as in the next subsection; for brevity we do not discuss these rules.

5.2 B&C$^\theta$ and MK$^\lambda$ guarantees

In the rest of this section utilities in $\mathcal{M}^+(\Omega)$ or $\mathcal{M}^-(\Omega)$ are *uniformly* continuous w.r.t. the pseudo-metric induced by the measure θ (itself absolutely continuous w.r.t. the Lebesgue neasure)

$$\forall \varepsilon > 0 \exists \eta > 0 : \theta(S \otimes T) \leq \eta \Longrightarrow |u(S) - u(T)| \leq \varepsilon \tag{5}$$

All separable utilities in $\mathcal{S}(\otimes)$, in particular all in the examples of the next section 5.3, are uniformly continuous. Because the Lebesgue measure is absolutely continuous w.r.t. θ, the uniform continuity property holds.

We fix a uniformly continuous increasing utility $u \in \mathcal{M}^+(\Omega)$. The results are identical, and identically phrased, for a bad manna $u \in \mathcal{M}^-(\Omega)$. See also Remark 4 at the end of this section.

Define the triangle $\mathcal{T} = \{(t^1, t^2) | 0 \leq t^1 \leq t^2 \leq 1\}$ in $\mathbb{R}_+^2$ and the set $\Upsilon(n)$ of increasing sequences $\tau = (t^k)_{0 \leq k \leq n}$ in $[0, 1]$ s.t.

$$t^0 = 0 \leq t^1 \leq \cdots \leq t^{n-1} \leq 1 = t^n$$

For a moving knife λ, utilities of the shares in MK$_n^\lambda$ are described by the function u^λ on $\mathcal{T}$:

$$u^\lambda(t^1, t^2) = u(\Lambda(t^2) \diagdown \Lambda(t^1)) \text{ for all } (t^1, t^2) \in \mathcal{T}$$

For a measure θ, the corresponding definition in B&C$^\theta$ is the *indirect utility* u^θ:

$$u^\theta(t^1, t^2) = \min_{T: \theta(T) = t^1} \max_{S: S \cap T = \varnothing; \theta(S) = t^2 - t^1} u(S) \text{ for all } (t^1, t^2) \in \mathcal{T} \tag{6}$$

(where because $\mathcal{B}$ is not compact for the pseudo-metric, the min and the max may not be reached)

Both u^λ and u^θ decrease (weakly) in t^1 and increase (weakly) in t^2. We show below that u^λ and u^θ are continuous in t^1, t^2: and that the guarantees Γ^λ and Γ^θ implemented by MK_n^λ and $\mathrm{B\&C}_n^\theta$ respectively are computed as

$$\Gamma^\alpha(u;n) = \max_{\tau\in\Upsilon(n)} \min_{0\le k\le n-1} u^\alpha(t^k;t^{k+1}) \text{ where } \alpha \text{ is } \lambda \text{ or } \theta \tag{7}$$

For instance in MK_2^λ with two agents, define τ_λ to be a (not necessarily unique) position of the knife making our agent indifferent between the share $\Lambda(\tau_\lambda)$ and its complement. Then

$$\Gamma^\lambda(u;2) = \max_{0\le t^1\le 1} \min\{u(\Lambda(t^1)), u(\Omega\diagdown\Lambda(t^1))\} = u(\Lambda(\tau_\lambda)) = u(\Omega\diagdown\Lambda(\tau_\lambda))$$

In $\mathrm{B\&C}_2^\theta$ the critical bid τ_θ makes the best share of size τ_θ as good as the worst share of size $1-\tau_\theta$:

$$\Gamma^\theta(u;2) = \max_{0\le t^1\le 1} \min\{\max_{\theta(S)=t^1} u(S), \min_{\theta(S)=t^1} u(\Omega\diagdown S)\} = \max_{\theta(S)=\tau_\theta} u(S) = \min_{\theta(S)=\tau_\theta} u(\Omega\diagdown S) \tag{8}$$

Lemma 4
i) The utility u^λ and the indirect utility u^θ are continuous.
ii) The maximum of problem (7) (for both rules) is achieved at some $\tau\in\Upsilon(n)$ where the sequence t^k increases in k, all the $u^\alpha(t^k;t^{k+1})$ are equal, and this common utility $\Gamma^\lambda(u;n)$ or $\Gamma^\theta(u;n)$ is the optimal value of (7).

Proof
Statement i). For u^λ we have for any $t^1\le t^2$ and $s^1\le s^2$

$$\delta(\Lambda(t^2)\diagdown\Lambda(t^1), \Lambda(s^2)\diagdown\Lambda(s^1)) \le \delta(\Lambda(t^2),\Lambda(s^2)) + \delta(\Lambda(t^1),\Lambda(s^1))$$

and both $S\to u(S)$ and $t\to\Lambda(t)$ are δ-continuous.

For u^θ we replace first in definition (*6*) the equalities like $\theta(T)=t^1$ with inequalities $\theta(T)\le t^1$. Then we fix t^1, t^2, $\varepsilon>0$, and a corresponding η in the uniform continuity property (5). Next we assume s^1, s^2 is s.t. $|s^1-t^1| + |s^2-t^2|\le\eta$ and we distinguish two cases.

Case 1. $u^\theta(t^1,t^2)\ge u^\theta(s^1,s^2)$. Start with an arbitrary share T s.t. $\theta(T)\le s^1$. We can choose $T^*\subseteq T$ s.t. $\theta(T^*)\le t^1$ and $\theta(T^*\otimes T)\le\eta$ (if $t^1\ge s^1$ then $T^*=T$ will do). By definition of $u^\theta(t^1,t^2)$ we can then choose a share S^* disjoint from T^* and such that

$$\theta(S^*)\le t^2-t^1 \text{ and } u(S^*)\ge u^\theta(t^1,t^2)-\varepsilon$$

Trimming S^* of its intersection with $T\diagdown T^*$, if any, leaves us with S^{**} disjoint from T and s.t. $u(S^{**}) \geq u^\theta(t^1, t^2) - 2\varepsilon$ (because $\theta(T\diagdown T^*) \leq \eta$). If $s^2 - s^1 < t^2 - t^1$ we again trim S^{**} to S such that $\theta(S) = s^2 - s^1$ and $\theta(S^{**}\diagdown S) \leq \eta$; or we simply set $S = S^{**}$ if $t^2 - t^1 \leq s^2 - s^1$. Thus we have $u(S) \geq u^\theta(t^1, t^2) - 3\varepsilon$.

If the initial choice of T is optimal up to ε for $u^\theta(s^1, s^2)$ any S disjoint from T of θ-size $s^2 - s^1$ is s.t. $u(S) \leq u^\theta(s^1, s^2) + \varepsilon$. We conclude $u^\theta(t^1, t^2) \leq u^\theta(s^1, s^2) + 4\varepsilon$.

Case 2. $u^\theta(s^1, s^2) \geq u^\theta(t^1, t^2)$. The symmetric argument simply exchanges the roles of t^i and s^i.

Statement ii). We check first the existence of $\tau \in \Upsilon(n)$ where all the $u^\alpha(t^k; t^{k+1})$ are equal. For simplicity we assume $n = 3$, the general proof is entirely similar. Fixing u and t^1 there is some t^2 such that $u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. To see this note that the function $f(x) = u^\theta(x, 1) - u^\theta(t^1, x)$ is well defined, continuous and weakly decreasing on $[t^1, 1]$, while $f(t^1) \geq 0 \geq f(1)$.

This common value is unique (though t^2 may not be) and defines a function $g(t^1) = u^\theta(t^1; t^2) = u^\theta(t^2; 1)$. The continuity and monotonicity properties of u^θ imply easily that g is weakly decreasing and continuous; moreover $g(0) \geq 0 = g(1)$. Then the function $x \to g(x) - u^\theta(0; x)$ decreases weakly and changes sign in $[0, 1]$. So for t^1 s.t. $g(t^1) = u^\theta(0; t^1)$, with associated t^2 we have

$$u^\theta(0; t^1) = u^\theta(t^1; t^2) = u^\theta(t^2; 1)$$

as desired.

Check finally that if at $\tau_* \in \Upsilon(n)$, for $\alpha = \theta$ or λ, all the terms $u^\alpha(t^k_*; t^{k+1}_*)$, $0 \leq k \leq n-1$, are equal to a common value δ, then τ_* solves program (7). If it does not there is a τ such that $u^\alpha(t^k; t^{l+1}) > \delta$ for $0 \leq k \leq n-1$. Applying this inequality at $k = 0$ gives $t^1 > t^1_*$; next at $k = 1$ we get $u^\alpha(t^1, t^2) > u^\alpha(t^1_*, t^2_*)$ implying $t^2 > t^2_*$; and so on until we reach a contradiction with the fact that both τ and τ_* are in $\Upsilon(n)$.

Finally, the optimal sequence t^k increases in k, strictly if u is not everywhere zero because $u(t, t) = 0$ for all t. ■

Theorem 2

Fix the manna $(\Omega, \mathcal{B})$, the number of agents n, and a utility $u \in \mathcal{M}^+(\Omega)$. Write $\tau_\lambda, \tau_\theta$ the solutions of program (7) for the rules MK^λ_n and $B\&C^\theta_n$.
i) With the MK^λ_n rule, an agent is guaranteed the utility $\Gamma^\lambda(u; n)$ by committing, for all k, to stop the knife at t^k_λ if exactly $k - 1$ other agents have been served before;

ii) With the B&C$_n^\theta$ rule, she is guaranteed $\Gamma^\theta(u;n)$ *by stopping the clock at* t_θ^k *if exactly* $k-1$ *other agents have been served before; and choosing then the best available share of size* $t^k - t^{k-1}$.
iii) $minMax(u;n) \leq \Gamma^\alpha(u;n) \leq Maxmin(u;n)$ where α is λ or θ.

Proof.
Statement i) and iii) for MK_n^λ. By construction of τ_λ the equipartition $\Pi = (\Lambda(t_\lambda^k)\diagdown\Lambda(t_\lambda^{k-1}))_1^n$ has $u(\Pi) = \Gamma^\lambda(u;n)$. Thus (3) in Proposition 1 implies the inequalities *iii*). Next if the knife has been stopped $k-1$ times before our agent is served, the last stop occured at or before t_λ^{k-1} therefore if she does stop the knife at t_λ^k (and wins the possible tie break) her share is at least $\Lambda(t_\lambda^k)\diagdown\Lambda(t_\lambda^{k-1})$. If she never gets to stop the knife, the last stop is at or before t_λ^{n-1} so she gets at least $\Omega\diagdown\Lambda(t_\lambda^{n-1}) = \Lambda(1)\diagdown\Lambda(t_\lambda^{n-1})$ and the utility $u^\lambda(t_\lambda^{n-1}, 1)$.
Statement ii). If she is the first to stop the clock (perhaps also winning the tie break) at step k, in step $k-1$ the clock stopped at $t^{k-1} \leq t_\theta^{k-1}$ and the share T already distributed at that time had $\theta(T) = t^{k-1}$: therefore she can choose a share with utility $u^\theta(t^{k-1}; t_\theta^k) \geq u^\theta(t_\theta^{k-1}; t_\theta^k) = \Gamma^\theta(u;n)$. If she is the last to be served, having never stopped the clock (or lost some tie breaks) the share assigned to all other agents has $\theta(T) = t^{n-1} \leq t_\theta^{n-1}$ therefore her share is worth $u^\theta(t^{n-1}; 1) \geq u^\theta(t_\theta^{n-1}; 1) = \Gamma^\theta(u;n)$.
Statement iii) for B&C$_n^\theta$.

Right hand inequality. We know since Definition 3 (and section 1) that any guarantee is bounded above by $Maxmin(u;n)$, and we just proved that $\Gamma^\theta(u;n)$ is such a guarantee.

Left hand inequality. Recall from Lemma 4 that $\Gamma^\theta(u;n) = u^\theta(t_\theta^{k-1}; t_\theta^k)$ for each $k = 1, \cdots, n$. We construct now a partition $\Pi = (R_k)_1^n$ s. t. $u(R_k) \leq u^\theta(t_\theta^{k-1}; t_\theta^k)$ for $1 \leq k \leq n$: this implies $minMax(u;n) \leq \max_{k=1,\cdots,n} u(R_k) \leq \Gamma^\theta(u;n)$ and the claim. The construction is by a decreasing induction in n.

In (the first) step n of the induction we define the 2-partition $\Pi^n = (T_{n-1}, R_n)$ of Ω where T_{n-1} is any solution of the program $\min_{T:\theta(T)=t_\theta^{n-1}} u(\Omega\diagdown T)$, and $R_n = \Omega\diagdown T_{n-1}$. If exact solutions are not available, it is enough to pick approximate solutions; we omit the straightforward details.

Thus $u(R_n) = u^\theta(t_\theta^{n-1}; 1)$ and $\theta(T_{n-1}) = t_\theta^{n-1}$.

Assume that in step k we constructed the $(n-k+2)$-partition $\Pi^k = (T_{k-1}, R_k, R_{k+1}, \cdots, R_n)$ s.t. $\theta(T_{k-1}) = t_\theta^{k-1}$ and $u(R_\ell) \leq u^\theta(t_\theta^{\ell-1}; t_\theta^\ell)$ for

$k \leq \ell \leq n$. Pick $\widetilde{T}$ a solution of

$$\min_{T:\theta(T)=t_\theta^{k-2}} \max_{S:S\cap T=\varnothing;\theta(S)=t_\theta^{k-1}-t_\theta^{k-2}} u(S) = u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$$

As $\theta(\widetilde{T} \cap T_{k-1}) \leq t_\theta^{k-2}$ and $\theta(T_{k-1}) = t_\theta^{k-1}$ we can choose T_{k-2} s.t. $\widetilde{T} \cap T_{k-1} \subseteq T_{k-2} \subseteq T_{k-1}$ and $\theta(T_{k-2}) = t_\theta^{k-2}$. Then we set $R_{k-1} = T_{k-1} \diagdown T_{k-2}$ so that $u(R_{k-1}) \leq u^\theta(t_\theta^{k-2}; t_\theta^{k-1})$ follows from $R_{k-1} \cap \widetilde{T} = \varnothing$ and the definition of $\widetilde{T}$. This completes the induction step. We note finally that each set R_k thus constructed is of θ-size $t_\theta^k - t_\theta^{k-1}$, and that $\max_k u(R_k) = \Gamma^\theta(u; n)$. ■

It is easy to check that no agent can secure more utility than Γ_n^λ in MK_n^λ or Γ_n^θ in $\mathrm{B\&C}_n^\theta$.

Remark 4. The minMax guarantee and Maxmin upper bound for $u \in \mathcal{M}^\varepsilon(\Omega)$ *and* $-u \in \mathcal{M}^{-\varepsilon}(\Omega)$*, where* $\varepsilon = \pm$*, are related:* $minMax(-u; n) = -Maxmin(u; n)$*. With two agents the guarantees* $\Gamma^\lambda(u; 2)$ *and* $\Gamma^\theta(u; 2)$ *are similarly antisymmetric:*

$$\Gamma^\alpha(-u; 2) = -\Gamma^\alpha(u; 2) \text{ where } \alpha \text{ is } \lambda \text{ or } \theta \tag{9}$$

and they obtain from the same partition. This is clear for Γ^λ *and it follows for* Γ^θ *by the change of variable* $S \to S' = \Omega \diagdown S$*:*

$$\Gamma^\theta(-u; 2) = -\min_{0\leq t^1\leq 1} \max\{\min_{\theta(S)=t^1} u(S), \max_{\theta(S)=t^1} u(\Omega \diagdown S)\} =$$

$$-\min_{0\leq t^1\leq 1} \max\{\min_{\theta(S')=1-t^1} u(\Omega \diagdown S'), \max_{\theta(S')=1-t^1} u(S')\} =$$

$$-\min_{0\leq t'\leq 1} \max\{\max_{\theta(S')=t'} u(S'), \min_{\theta(S')=t'} u(\Omega \diagdown S')\}$$

Where the last equality is because if two continuous functions $t \to f(t)$ *and* $t \to g(t)$ *intersect in* $[0, 1]$ *and one increases while the other decreases, then* $\min_{0\leq t\leq 1} \max\{f(t), g(t)\} = \max_{0\leq t\leq 1} \min\{f(t), g(t)\}$ *are both attained at the intersection point.*

The identity (9) generalises to $n \geq 3$ *for the* MK^λ guarantee, *but not for the B&C*$^\theta$ *one.*

5.3 Microeconomic fair division

We must divide a good manna $\omega \in \mathbb{R}_+^K$ in n shares $z_i \in \mathbb{R}_+^K$. Utilities $u \in \mathcal{M}^+(\omega)$ are continuous and weakly increasing on $[0, \omega]$.

A Moving Knife is a continuous increasing path $t \rightarrow \Lambda(t)$ from 0 to ω: a natural choice is $\Lambda(t) = t\omega, 0 \leq t \leq 1$: the corresponding guarantee $\Gamma^{\lambda}(u;n) = u(\frac{1}{n}\omega)$ is the *equal split* utility $u(\frac{1}{n}\omega)$. A positive and additive measure θ defining B&C$^{\theta}$ is a "price" $\theta(z) = p \cdot z$, $p \in \mathbb{R}_{+}^{K} \diagdown \{0\}$, and we write the corresponding guarantee as Γ^{p}.

Recall from section 1 that if an agent's preferences are convex her *equal split* utility is her $Maxmin$ utility, the upper bound on all guarantees ((1)), therefore it is weakly larger than the B&C^{p} guarantee for any p. The converse inequality holds for concave preferences.

Lemma 5
i) If the upper contours of the utility $u \in \mathcal{M}^{+}(\omega)$ *are convex, then* $\Gamma^{p}(u;n) \leq u(\frac{1}{n}\omega) = Maxmin(u;n)$.
ii) If the lower contours of the utility $u \in \mathcal{M}^{+}(\omega)$ *are convex, then* $minMax(u;n) = u(\frac{1}{n}\omega) \leq \Gamma^{p}(u;n)$.

The proof of statement *ii)* mimicks that of statement *i)* in footnote 2 (section 1).

We turn to a handful of numerical examples where $K = 2$, $\omega = (1,1)$, and $p \cdot z = \frac{1}{2}(x+y)$. Shares are $z = (x,y)$, utilities are 1-homogenous and normalised so that $u(\omega) = 10$. We compute our three guarantees: Bid and Choose Γ^{p}, equal split, and $minMax$, and compare them to the $Maxmin$ upper bound.

The first three utilities (Leontief, Cobb Douglas and CES) define convex preferences, the last two define concave preferences (represented by quadratic and "anti-Leontief" utilities).

Our first table assumes two agents, $n = 2$, and illustrates Lemma 5. An agent with convex preferences prefers the equal split guarantee to Γ^{p}; the opposite is true for agents with concave preferences.

$u(x,y)$	$minMax(u;2)$	$\Gamma^{p}(u;2)$	$u(\frac{1}{2}\omega)$	$Maxmin(u;2)$
$10\min\{x,y\}$	0	3.3	5	5
$10\sqrt{x \cdot y}$	0	4.1	5	5
$\frac{5}{2}(\sqrt{x}+\sqrt{y})^{2}$	2.5	4.4	5	5
$5(x+y)$	5	5	5	5
$5\sqrt{2(x^{2}+y^{2})}$	5	5.9	5	7.1
$10\max\{x,y\}$	5	6.7	5	10

The equal split partition delivers the $Maxmin$ utility for the first four preferences, and the $minMax$ utilities for the last three. The equipartition

$\Pi = \{(1,0),(0,1)\}$ gives similarly the $minMax$ utilities of the first four, and the $Maxmin$ ones for the last three.

To compute $\Gamma^p(u;2)$ we know from (8) that the optimal bid t^1 (denoted t for simplicity) solves

$$\max_{\frac{1}{2}(x+y)\leq t} u(x,y) = \min_{\frac{1}{2}(x+y)\leq t} u(1-x,1-y) = \min_{\frac{1}{2}(x+y)\geq 1-t} u(x,y)$$

This equality implies $0 \leq t \leq \frac{1}{2}$. If u represents convex preferences symmetric in the two goods, $u(x,y)$ is maximal under $\frac{1}{2}(x+y) \leq t$ at $x = y = t$, and minimal under $x + y \geq 2(1-t)$ at $x = 1, y = 1-2t$. So we must solve $u(t,t) = u(1,1-2t)$: see Figure 2.

If u represents concave symmetric preferences its maximum under $\frac{1}{2}(x + y) \leq t$ is at $x = 0, y = 2t$, and its minimum under $x + y \geq 2(1-t)$ at $x = y = 1-t$, so we solve $u(0,2t) = u(1-t,1-t)$: see Figure 3.

We compute finally the same guarantees with three agents:

$u(x,y)$	$minMax(u;3)$	$\Gamma^p(u;3)$	$u(\frac{1}{3}\omega)$	$Maxmin(u;3)$
$10\min\{x,y\}$	0	2	3.3	3.3
$10\sqrt{x\cdot y}$	0	2.4	3.3	3.3
$\frac{5}{2}(\sqrt{x}+\sqrt{y})^2$	2	2.5	3.3	3.3
$5(x+y)$	3.3	3.3	3.3	3.3
$5\sqrt{2(x^2+y^2)}$	3.3	4.1	3.3	4.1
$10\max\{x,y\}$	3.3	5	3.3	5

The $minMax$ equipartition for $u = \frac{5}{2}(\sqrt{x}+\sqrt{y})^2$ and the $Maxmin$ equipartition for $u' = 5\sqrt{2(x^2+y^2)}$ have the same form $\Pi = \{(x,0),(0,x),(1-x,1-x)\}$: in the former case we find $x = \frac{4}{5}$ and $minMax(u;3) = 2$, in the latter we get $x = 2-\sqrt{2}$ and $Maxmin(u';3) = 10(\sqrt{2}-1)$. Lemma 5 and the partition $\Pi' = \{(1,0),(0,\frac{1}{2}),(0,\frac{1}{2})\}$ fill the remaining values of $minMax$ and $Maxmin$.

To compute $\Gamma^p(u;3)$ we know by Lemma 4 that the three terms in (7) are equal. They are

$u^p(0,t^1) = \max_{\frac{1}{2}(x+y)\leq t^1} u(x,y)$

$u^p(t^1,t^2) = \min_{\frac{1}{2}(x^1+y^1)\leq t^1} \max_{\frac{1}{2}(x+y)\leq t^2-t^1 \text{ and } (x^1+x,y^1+y)\leq(1,1)} u(x,y)$

$u^p(t^2,1) = \min_{\frac{1}{2}(x^2+y^2)\leq t^2} u(1-x^2,1-y^2)$

Clearly $t^1 \leq \frac{1}{3}$ (as $t^2 - t^1 < \frac{1}{3} < t^1$ and $1-t^2 < \frac{1}{3} < t^1$ are both impossible). Therefore $u^p(0,t^1) = u^p(t^1,t^2)$ is achieved by $t^2 = 2t^1$ (the constraint $(x^1+x, y^1+y) \leq (1,1)$ does not bind). Writing $t = t^1 = t^2 - t^1$ it remains to solve

$$\max_{\frac{1}{2}(x+y)\leq t} u(x,y) = \min_{\frac{1}{2}(x^2+y^2)\leq 2t} u(1-x^2, 1-y^2) = \min_{\frac{1}{2}(x+y)\geq 1-2t} u(x,y)$$

When u represents convex preferences symmetric in the two goods, the minimum on the right-hand side is achieved by $(x,y) = (1-4t, 1)$ so we solve $u(t,t) = u(1-4t, 1)$. See Figure 4.

If u represents concave symmetric preferences, the minimum on the right-hand side is achieved by $(x,y) = (1-2t, 1-2t)$ so we solve $u(2t, 0) = u(1-2t, 1-2t)$. See Figure 5.

6 Concluding comments

Comparing B&C$_n$ and D&C$_n$ rules The exogenous ordering of the agents greatly affects the outcome of D&C$_n$, whereas B&C$_n$ treats the agents symmetrically. On the other hand the choice of the benchmark measure in B&C$_n$ is exogenous, which allows much, perhaps too much flexibility to the designer.

In D&C$_n$ the dividing agent may have many different strategies guaranteeing her $minMax$ utility. By contrast in B&C$_n$ the solution to programs (8) and (7) is often unique. Multiple choices and the resulting indeterminacy of the outcome may be appealing for the sake of privacy preservation, less so from the implementation viewpoint.

Some challenging open questions 1). Fix the manna $(\Omega, \mathcal{B})$ as in Theorem 1, and each of the n agents with his own utility in $\mathcal{D}(\Omega)$. As mentioned in sections 2 and 3.2 (Remark 1), Stromquist ([36]) showed that an Envy-free partition of Ω exists *if all utilities are non negative for all shares.* Without the sign assumption on utilities, Avvakumov and Karasev ([3]) prove existence of an Envy-free partition if n is a power of a prime number. Whether the latter remains true for all n is still an open question.

2) If the utilities vary in a domain $\mathcal{U}(\Omega)$ where the $Maxmin$ utility is not feasible, we would like to describe the family of *undominated* guarantees $u \to \Gamma(u;n)$. For instance in the microeconomic domain $\mathcal{M}^+(\omega)$ of Subsection 5.3, the equal split guarantee is clearly undominated. We conjecture that in the domains $\mathcal{M}^\pm(\Omega)$ the B&C$_n$ guarantees Γ^θ are undominated as well.

3) Divide and Choose, even with two agents, has no easy generalisation when agents have unequal rights to the good manna (or unequal liability for the bad manna). Consider two agents with shares $\frac{3}{7}, \frac{4}{7}$: if utilities are additive, the Divider with a $\frac{3}{7}$th share can partition the manna in seven pieces and let the Chooser pick four of them. This becomes quickly unmanageable if the shares have high denominators, or utilities are not additive. It is also unclear how to adapt the Bid and Choose rules to account for unequal rights.

7 Figures

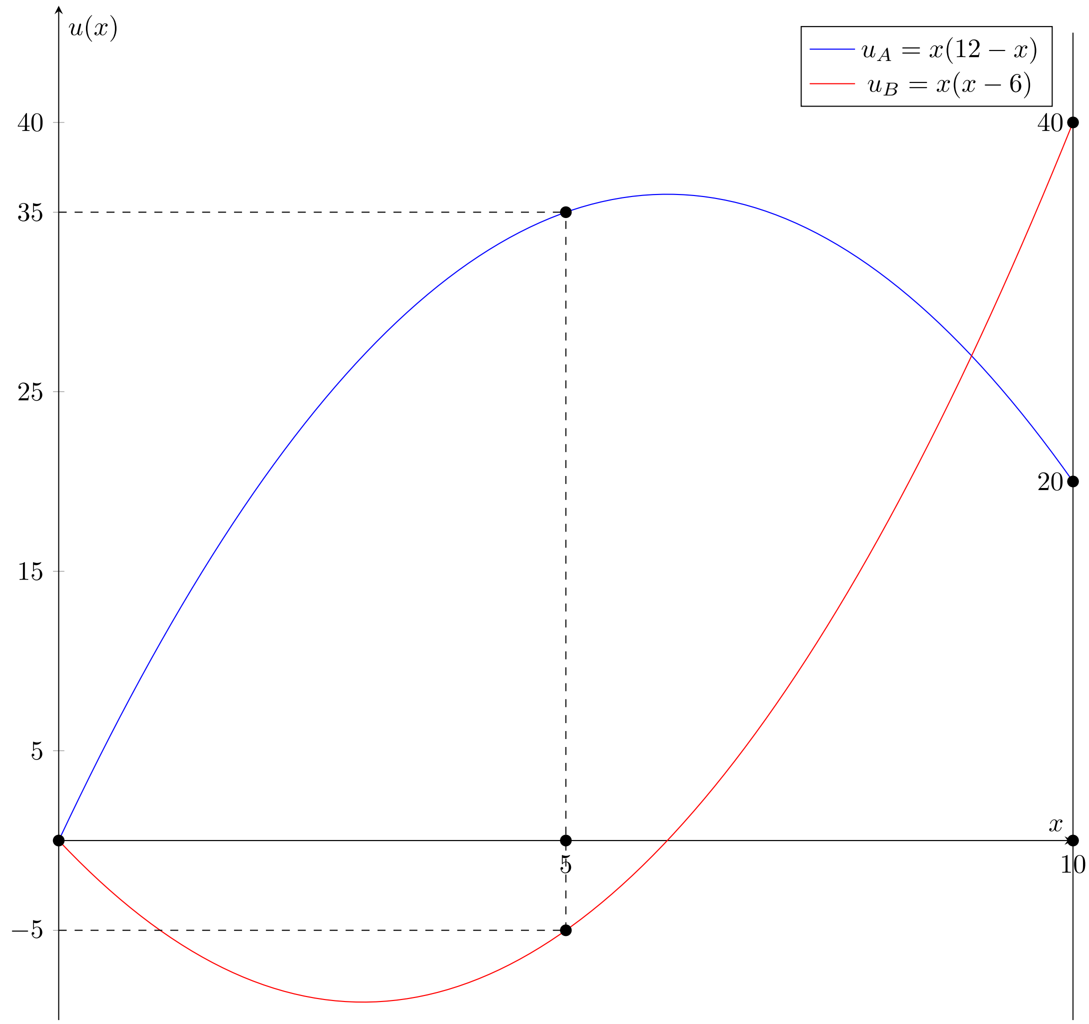

Figure 1

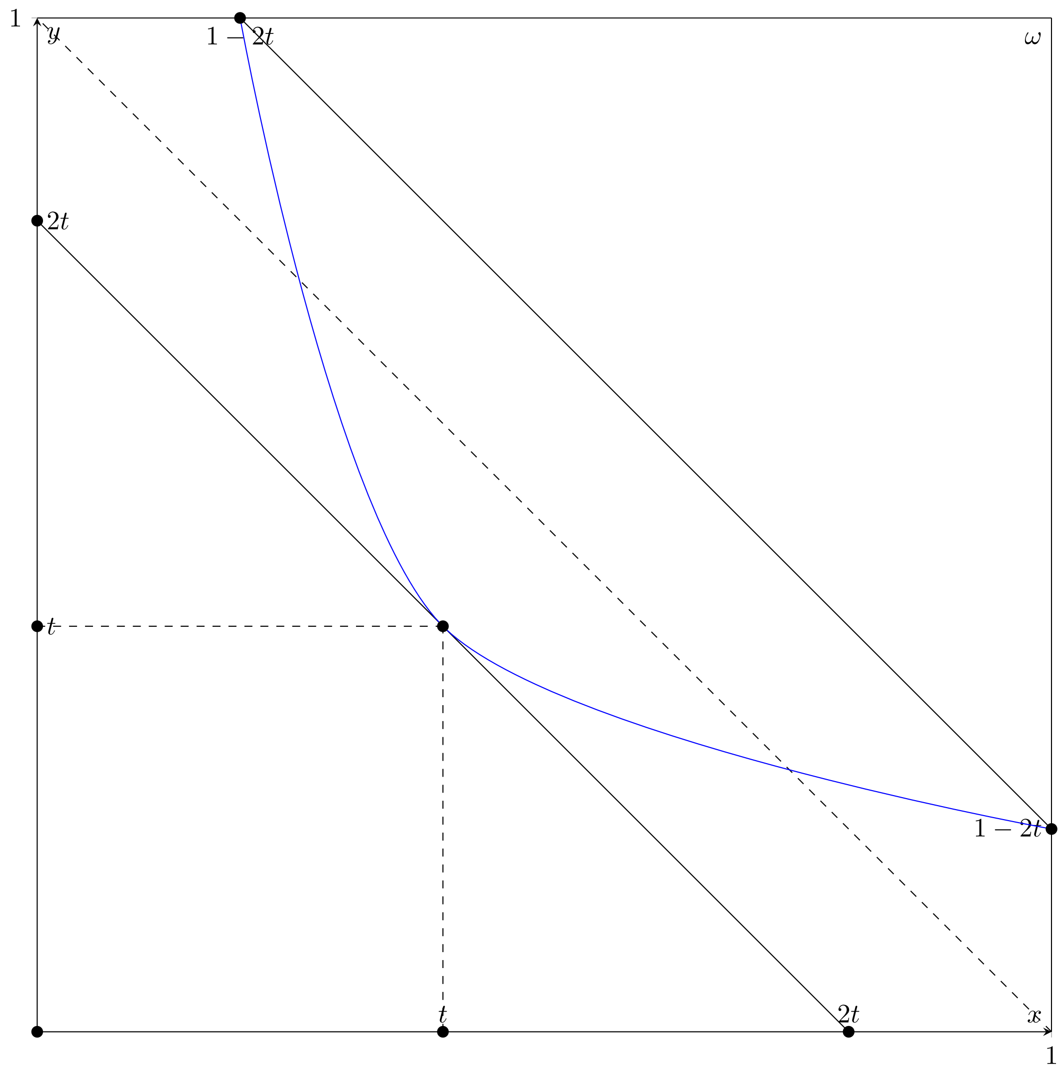

Figure 2: $u(t,t) = u(1-2t, 1)$

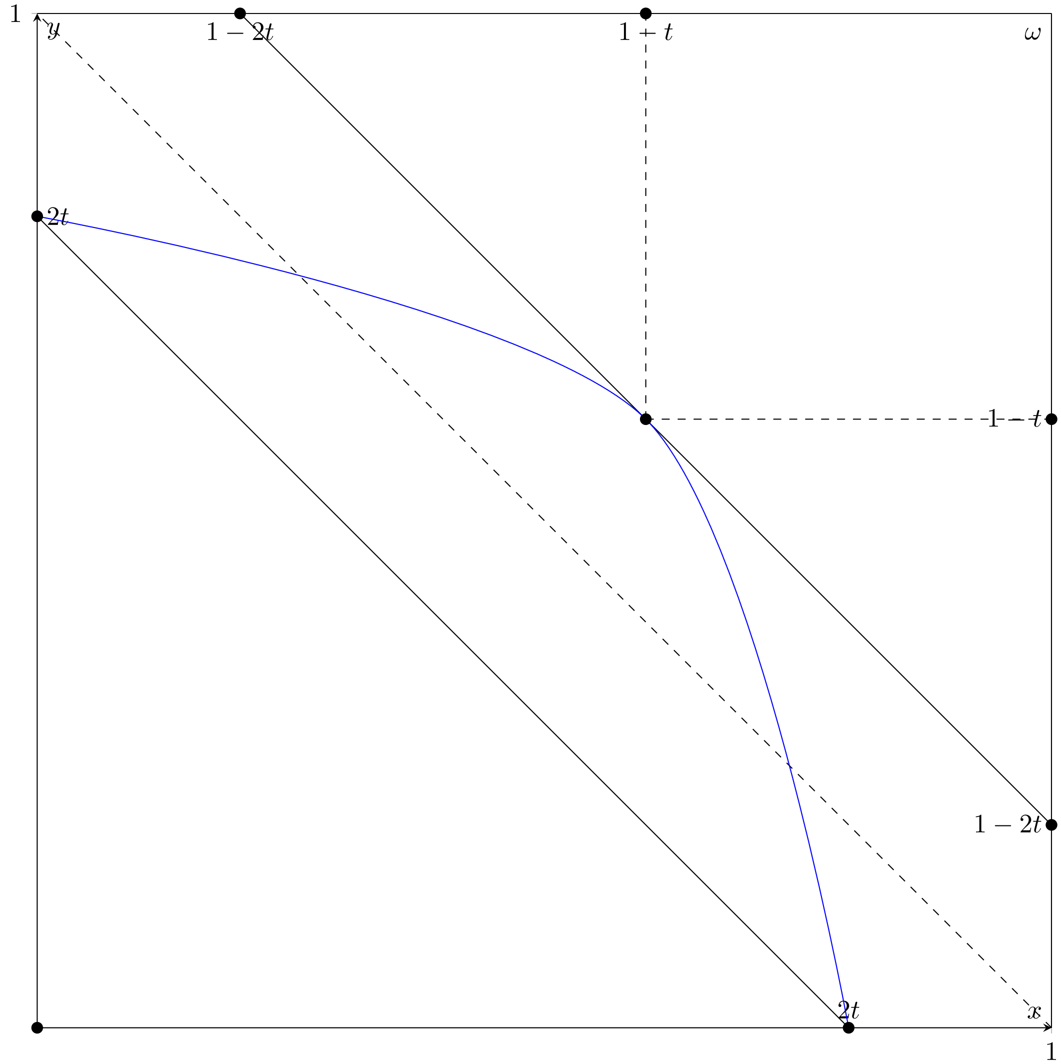

Figure 3: $u(2t, 0) = u(1 - t, 1 - t)$

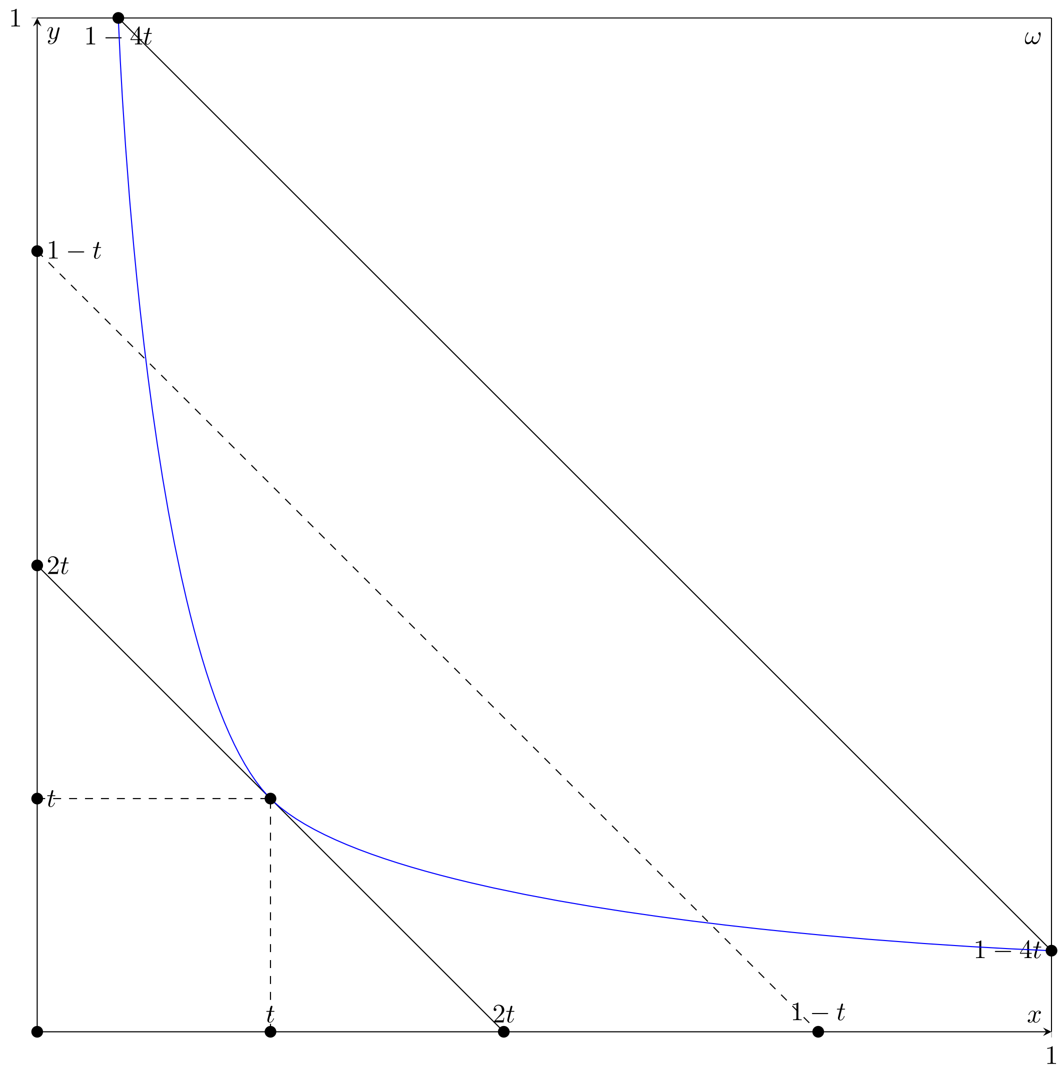

Figure 4: $u(t,t) = u(1-4t,1)$

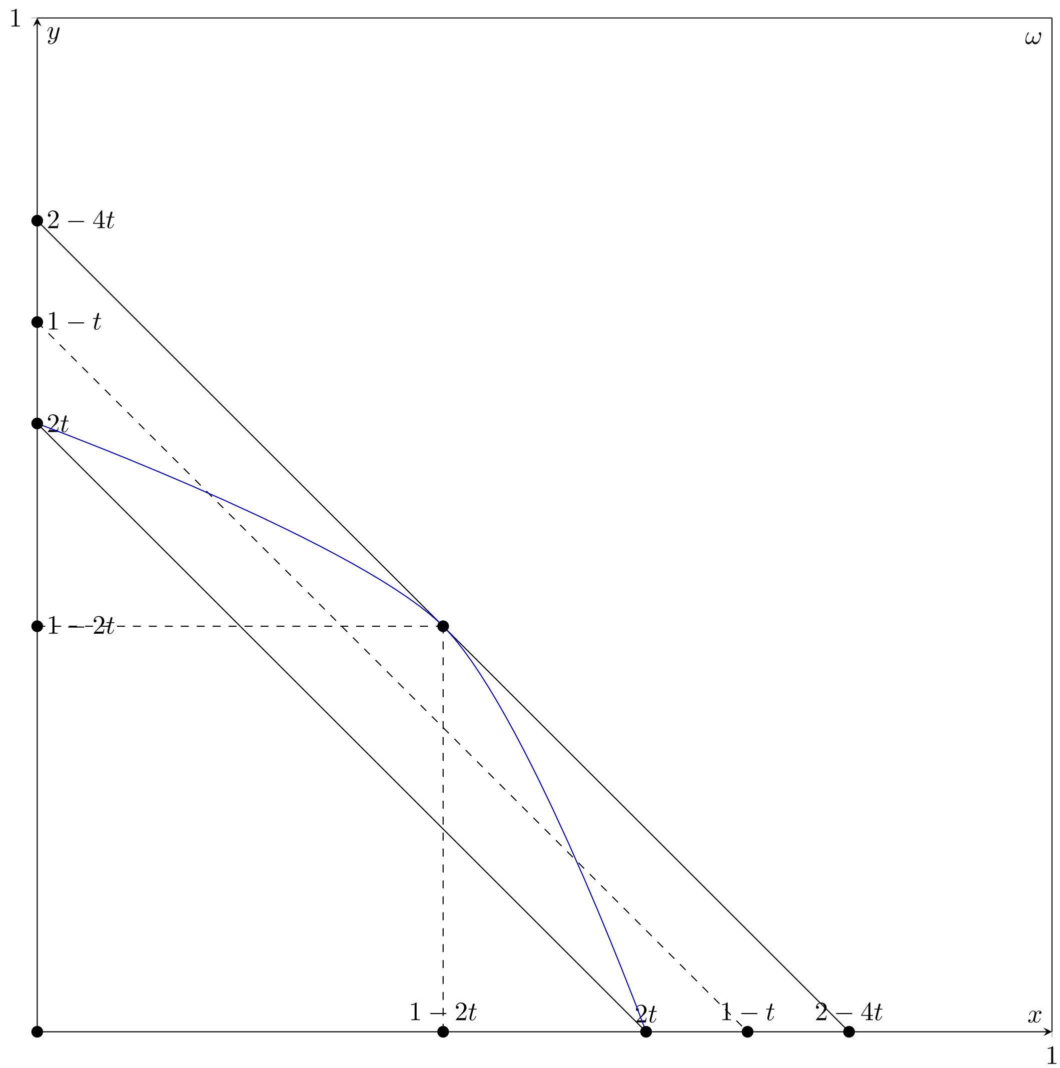

Figure 5: $u(2t,0) = u(1-2t,1-2t)$

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