Contents lists available at ScienceDirect

Journal of Economic Dynamics & Control

journal homepage: www.elsevier.com/locate/jedc

Computing time-consistent equilibria: A perturbation approach^{*}

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ARTICLE INFO

Article history: Received 29 June 2021 Revised 25 February 2022 Accepted 27 February 2022 Available online 01 March 2022

JEL classification: C63 E52 E70

Keywords: Time-consistent equilibrium Monetary policy Fiscal policy Quasi-geometric discounting

ABSTRACT

Time-consistency is a key feature of many important policy problems, such as those relating to optimal fiscal policy, optimal monetary policy, macro-prudential policy, and sovereign lending. It is also important for private-sector decision-making through mechanisms such as quasi-geometric discounting. These problems are generally solved using some form of projection method. The difficulty with projection methods is that their computational complexity increases rapidly with the number of state variables, limiting the so-phistication of the models that can be solved. This paper develops a perturbation method for solving models with time-inconsistency. The method operates on a model's (generalized) Euler equations; it does not require forming a quadratic approximation to household welfare and it does not require that the model's steady state be efficient. We illustrate the method and its applicability to different environments by applying it to a range of models featuring time-inconsistency.

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1. Introduction

Time-consistency features importantly in many areas of macroeconomics: fiscal policy, monetary policy, banking regulation, and sovereign lending, to name just a few. In the context of optimal monetary policy time-inconsistency emerges because some constraints on the central bank's decision problem, such as the Phillips curve, involve expectations and only bind ex ante. The central bank's well-intentioned efforts to leverage the expectations channel into inflation guide it towards announcing policies for the future that it ultimately has no incentive to implement. In the absence of a commitment mechanism, the equilibrium of interest in such problems is usually the Markov-perfect equilibrium, which is time-consistent by construction. Time-consistency problems also feature in behavioral macroeconomics, where decision-makers may have hyperbolic or quasi-geometric preferences. The difficulty with solving models involving time-consistency problems is that although private sector expectations affect current behavior, policy decisions must be made taking as given the process by which those expectations are formed. To solve for equilibrium in such models one often must make strong assumptions in order to gain analytic tractability and/or employ numerical methods whose computational intensity increases rapidly, usually exponentially, with the number of state variables. Although recent advances have ameliorated the curse of dimensionality,¹ the application of projection methods remains time-consuming, even for models with relatively few state variables.

¹ Examples include sparse-grid methods, such as Smolyak (1963), which was introduced into Economics by Krueger and Kubler (2004) (see Judd et al., 2014 for a nice presentation) and the hyperbolic cross approximation method (Dennis, 2021). Other advances include solutions based on endogenous

https://doi.org/10.1016/j.jedc.2022.104349

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^{*} I would like to thank the editor, two anonymous referees, Tatiana Damjanovic, and participants at the Computational Economics and Finance, 2021, conference for comments. All simulation results and figures were produced using Julia 1.6.5.

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This paper presents a perturbation method to solve optimization-based dynamic macroeconomic models for first-order accurate time-consistent equilibria. The method is based on the tools used to obtain first- and second-order accurate solutions to rational expectations models, which are now widely available. When expressed in terms of first-order conditions, models that feature time-inconsistency contain what are known as generalized-Euler equations. Generalized-Euler equations differ from standard Euler equations in that they contain the levels of variables as well as their derivatives with respect to endogenous state variables, derivatives that arise when today's choices affect tomorrow's behavior. These derivatives are problematic for perturbation methods because they imply that the model's steady state cannot be solved independently of its dynamics. We overcome the challenges that these derivatives pose by applying an iterative scheme whereby a first-order accurate solution emerges from the repeated application of a second-order perturbation method. In doing so we dispel the common belief that perturbation methods cannot be applied to models with time-inconsistency unless the steady state is efficient. Although we illustrate the approximation method in terms of a first-order perturbation scheme, we also demonstrate that the method can be used to produce higher-orders of approximation.

Because our solution procedure is an iterative one that requires second-order methods to obtain a first-order accurate solution, it is slightly more demanding than a standard first-order perturbation method, but it remains significantly less demanding than projection methods.² Moreover, our procedure inherits the scalability of perturbation methods, allowing it to be applied to models with larger numbers of state variables, ones that cannot quickly be solved using projection methods, even using sparse grid technology. Further, because it is based on (generalized) Euler equations, our method does not require second-order welfare approximations and nor does it ask that the perturbation point be the model's efficient steady state. This latter point is worth emphasizing because many important applications involving time-inconsistency–such as applications involving fiscal policy (Balke and Ravn, 2016), macro-prudential policy (Bianchi and Mendoza, 2018), and behavioral macroeconomics (Brunnermeier et al., 2016) are ones where the steady state is inefficient.

We illustrate our approach by applying it to a range of models drawn from different areas in macroeconomics. For purposes of exposition and to aid comparisons, these models are intentionally kept comparatively simple. They each contain just two state variables and they can all be solved relatively quickly using projection methods. The fact that they can be solved using projection methods without too much difficulty enables us to obtain highly accurate global solutions and allows us to illustrate the accuracy of our first-order accurate solution. Furthermore, the models' simplicity allows us to provide a clearer description of how the solution method can be applied.³ Having illustrated some of the many environments to which the method can be applied, we show how welfare can be approximated and use our perturbation method to solve a larger model to second-order accuracy.

We are not the first to apply perturbation methods to models featuring time-inconsistency. In the monetary policy literature, it is common to fit a linear-quadratic (LQ) approximation to the non-linear problem (Benigno and Woodford, 2005; 2012) and compute a linear solution from the LQ approximation. However, applications of the LQ approach invariably assume that monetary policy is conducted according to the timeless perspective (Woodford, 1999) whereas our focus is on discretion. When the LQ approach is applied to problems where monetary policy is conducted with discretion, it is generally necessary to approximate about an efficient steady state with zero inflation.⁴ Although that approach allows the steady state to be obtained independently of the model's dynamics, the requirement that the steady state be efficient limits the set of models that can be analyzed. In addition, deriving a second-order accurate approximation to welfare is often analytically demanding making the method challenging to apply to even quite simple models.

Earlier work by Oudiz and Sachs (1985), Backus and Driffill (1986), Söderlind (1999), and Dennis (2007) developed methods for solving LQ policy problems for time-consistent equilibria. However, these methods were based on a quadratic loss function, but were essentially silent on how that loss function was obtained. Dotsey and Hornstein (2003) present a numerical method to form a LQ approximation for a discretionary policy problem where the steady state is inefficient. Their method is a form of successive approximations that begins with a guess at the steady state around which to perturb the model and then iterates over the steady state until convergence is reached. Unfortunately, their approach is generally inapplicable to models whose steady state is inefficient and it is subject to the pitfalls documented in Kim and Kim (2003), Kim and Kim (2006).

A method related to the one developed here is described in Krusell et al. (2002). Like ourselves, they express the problem to be solved in terms of a system containing a generalized Euler equation and solve simultaneously for the model's steady state and equilibrium dynamics. Our solution strategy extends their approach to stochastic economies and to models with multiple state variables, and it differs from their method by explicitly imposing saddle-point stability on the first-order dynamics.

The remainder of this paper is organized as follows. In the following section we outline the five two-state-variable models that we use to demonstrate our solution method. In section three we present our perturbation-based solution strat-

grid methods Carroll (2006) and the Taylor projection method developed by Levintal (2018). There have also been advances extending the application of perturbation methods to higher orders (fifth in the case of Levintal, 2017) and to nonlinear estimation (Andreasen, 2011).

 $^{^{2}\,}$ In our applications we found the linear solution to converge quickly, usually in just 4 or 5 iterations.

³ Of course, there are some models for which a first-order perturbation solution would be inappropriate or inaccurate. Examples include models containing uncertainty shocks, occasionally binding constraints, or models with rare disasters. To solve such models accurately, one may need to employ a higher-order perturbation or a projection method. Similar considerations apply to applications where the focus is on tail-regions.

⁴ See Appendix A.1 for a discussion of why an efficient steady state is important for standard LQ methods.

egy. Section four applies the method to each of the five models and compares the results to those from a highly accurate projection-based solution. Section five discusses how household welfare can be recovered once the solution is obtained. Section six considers a more sophisticated labor-search model and solves it to second-order accuracy. Section seven concludes. Appendix A identifies and discusses special cases where LQ approximations can be employed to solve for time-consistent equilibria and shows why LQ methods cannot be applied generally. Appendix B presents the algorithm we used when solving the models using projection methods.

2. The models

In this section we outline five models to which we apply our solution method. The models are of varying complexity, but they are all simple enough that they can be easily described and their solution using projection methods is not too time-consuming, and can be measured in minutes or hours rather than days. The latter is important because we use a highly accurate global solution as the benchmark to assess the accuracy of the first-order accurate solution. The first of the five models-the stochastic growth model-does not involve time-inconsistency. Its inclusion in the analysis establishes a benchmark for the accuracy we might hope to obtain for the models that do involve time-inconsistency. The remaining four models all involve time-inconsistent decision-making and are drawn from various literatures: optimal fiscal policy, quasi-geometric preferences, and optimal monetary policy. We provide a brief description of each model and document its key equations; readers are referred to the original sources for complete derivations.

2.1. Model one - stochastic growth model

The stochastic growth model (Brock and Mirman, 1972) needs little introduction. A representative consumer/producer has capital stock, k_t , and makes decisions regarding consumption, c_t , and future capital in order to maximize expected discounted lifetime utility, which depends on the sequence of goods consumed. We assume that period-utility is of the isoelastic form. With goods produced according to a Cobb-Douglas technology and with aggregate technology, a_t , obeying a standard stationary AR(1) process, the key equations characterizing equilibrium are:

$$u_{t+1} = \rho u_t + \varepsilon_{t+1},\tag{1}$$

$$k_{t+1} = (1-\delta)k_t + e^{a_t}k_t^{\alpha} - c_t,$$
⁽²⁾

$$c_t^{-\sigma} = \beta E_t \Big[c_{t+1}^{-\sigma} \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) \Big]. \tag{3}$$

Equation (2) is the law-of-motion for capital, which allows the capital stock to be augmented by unconsumed production and to depreciate at rate $\delta \in (0, 1]$. Equation (3) is the standard consumption-Euler equation in which $\beta \in (0, 1)$ is the discount factor, $\sigma \in (0, \infty)$ is the inverse of the elasticity of intertemporal substitution, $\alpha \in (0, 1)$ is capital's share of income, and E_t is the mathematical expectations operator. When solving the model we set $\beta = 0.99$, $\sigma = 1$, $\alpha = 0.3$, $\delta = 0.015$, and $\rho = 0.95$. The standard deviation of the technology innovation, ε_t , is set to 0.01.

2.2. Model two - time-consistent fiscal policy

This model is taken from Ambler and Pelgrin (2010) (which draws on Klein et al., 2008), who used it as a vehicle to illustrate how to apply control methods to compute Markov-perfect policies for stochastic non-linear models. The model was further analyzed by Dennis and Kirsanova (2016) who showed that it could be solved efficiently using a projection method based on Chebyshev polynomials applied to a system of equilibrium conditions containing a generalized Euler equation.

The environment is one in which a representative consumer/producer owns the capital stock, produces using a Cobb-Douglas technology, and receives utility from consuming goods and government services. The government purchases goods, transforms them costlessly into government services, and provides them free to consumers. Government expenditure is financed through a tax levied on household-income with an allowance made for capital depreciation. The household's problem is to choose consumption and future capital to maximize expected discounted life-time utility while taking taxes and the provision of government services as given. The government's problem is to choose the level of services to provide in order to maximize household welfare, taking into account its balanced budget condition and the impact that income taxation has on households' incentives to accumulate capital. Complete descriptions of the model can be found in Ambler and Pelgrin (2010) and Dennis and Kirsanova (2016).

With the household's expected discounted lifetime utility given by:

$$U_t = E_t \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}}{1-\sigma} + \mu \frac{g_t^{1-\eta}}{1-\eta} \right) \right],\tag{4}$$

 $\beta \in (0, 1), \sigma \in (0, \infty), \mu \in (0, \infty), \eta \in (0, \infty)$, welfare maximization by the consumer and the government leads to the following system of constraints and first-order conditions:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{5}$$

$$k_{t+1} = (1 - \delta)k_t + e^{a_t}k_t^{\alpha} - c_t - g_t,$$
(6)

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 + \left(1 - \frac{g_{t+1}}{e^{a_{t+1}} k_{t+1}^{\alpha} - \delta k_{t+1}} \right) \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} - \delta \right) \right) \right],\tag{7}$$

$$\mu g_t^{-\eta} = \beta E_t \Big[\Big(c_{t+1}^{-\sigma} - \mu g_{t+1}^{-\eta} \Big) c_k(a_{t+1}, k_{t+1}) + \mu g_{t+1}^{-\eta} \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) \Big].$$
(8)

Equation (5) describes the process for aggregate technology while Eq. (6) summarizes the law-of-motion for capital. Relative to the stochastic growth model, Eq. (6) only differs in that government purchases of goods, g_t , in addition to consumption subtracts from production in determining the level of investment. The consumption-Euler equation is summarized by Eq. (7). In this equation it is the after-tax return on capital that matters for consumption, where the tax rate is applied to production minus depreciated capital and is determined importantly by the level of government services provided.

The final equation in the system, Eq. (8), is the first-order condition associated with government services. We denote the household's decision rule for consumption by $c(a_t, k_t)$. Equation (8) takes the form of a generalized Euler equation because it depends on the derivative of the household's consumption decision rule with respect to capital. This derivative enters the Euler equation because the government must account for the effect an increase in its provision of services-funded through higher income-taxation-has on household consumption via lower capital accumulation. When solving this model we take the parameterization from Ambler and Pelgrin (2010). Specifically, we set $\beta = 0.987$, $\alpha = 0.3$, $\delta = 0.05$, $\sigma = 1$, $\mu = 0.3$, $\eta = 1$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.03.

2.3. Model three – quasi-geometric discounting

This model comes from Krusell et al. (2002) and Maliar and Maliar (2005). Like the previous two models, we can think of this one in terms of a representative consumer/producer that owns the capital stock and that chooses consumption and future capital in order to maximize expected discounted lifetime utility. However, in this model the household/producer has quasi-geometric discounting, which is to say that its expected discounted lifetime utility is given by:

$$U_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \theta \beta E_t[V_{t+1}],\tag{9}$$

 $\theta \in (0, 1], \sigma \in (0, \infty)$, with:

$$V_t = \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E_t[V_{t+1}],$$
(10)

 $\beta \in (0, 1)$, where we have chosen period-utility to be of the isoelastic form for simplicity. Together, Eqs. (9) and (10) imply that the household discounts between today and tomorrow at rate $\beta\theta$, and between tomorrow and the next day at rate β . When θ is less than one the short-run discount rate is greater than the long-run discount rate, leading to a Strotz (1956) form of time-inconsistency.

Solving the household/producer's decision problem leads to the following key equations:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{11}$$

$$k_{t+1} = (1-\delta)k_t + e^{a_t}k_t^{\alpha} - c_t, \tag{12}$$

$$c_t^{-\sigma} = \beta E_t \Big[c_{t+1}^{-\sigma} \Big(\theta \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) + (1 - \theta) k_k(a_{t+1}, k_{t+1}) \Big) \Big].$$
(13)

Equations (11) and (12) are familiar and standard. The crucial difference between this model and the stochastic growth model lies in Eq. (13), which takes the form of a generalized Euler equation because it depends on the derivative of the decision rule for future capital with respect to capital, $k_k(a_t, k_t)$, in addition to the level of capital itself. This derivative enters because the current-period household can use its decision regarding future capital to alter the future state and thereby alter the decisions made by its future self. As a consequence capital accumulation has a pecuniary return in the form of the marginal product of capital and a non-pecuniary return related to the effect current-period saving has on how the future household determines its consumption. If $\theta = 1$, then this non-pecuniary return disappears, Eq. (13) simplifies to Eq. (3), and there is no time-inconsistency. When solving this model we are guided by Maliar and Maliar (2005) and set $\beta = 0.95$, $\theta = 0.95$, $\alpha = 0.36$, $\delta = 0.1$, $\sigma = 2$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.01.

2.4. Model four - time-consistent monetary policy

Our fourth model is an application of optimal discretionary monetary policy in a new Keynesian model. Our analysis of this model builds on Comincini, 2022 who who solved it using a value-functioniteration, but we recast it as a system of constraints and first-order conditions, one of which is a generalized Euler equation. Related models, but log-linearized and studied for an ad-hoc loss function and/or non-discretionary policy can be found in a variety of places, including Amato and Laubach (2004) and Dennis and Söderström (2006). Comincini, 2022 provides a full description of the model.

The model is one in which households receive utility from consumption and dis-utility from labor, h_t , and there are external habits in consumption. We assume that expected discounted lifetime utility is additively separable in consumption and labor and takes the form:

$$U_{t} = E_{0} \left[\sum_{t=0}^{\infty} \beta^{t} \left(\frac{(c_{t} - \gamma C_{t-1})^{1-\sigma}}{1-\sigma} - \nu \frac{h_{t}^{1+\chi}}{1+\chi} \right) \right],$$
(14)

 $\beta \in (0, 1), \sigma \in (0, \infty), \nu \in (0, \infty), \chi \in (0, \infty), \text{ and } \gamma \in (0, 1), \text{ where } C_t \text{ denotes period-} t \text{ aggregate consumption. Monopolis-tically competitive firms employ labor and produce according to the technology:$

$$y_t = e^{a_t} h_t, \tag{15}$$

and set prices subject Rotemberg (1982) quadratic adjustment cost. The household's utility maximization leads to the labor supply equation:

$$wh_t^{\chi} = w_t (c_t - \gamma C_{t-1})^{-\sigma},$$
 (16)

where w_t denotes the real wage, while firm's cost minimization causes real marginal costs, ω_t , to be given by:

$$\omega_t = \frac{w_t}{e^{a_t}}.\tag{17}$$

Firms are assumed to set the price for their good to maximize their expected discounted net cash flow, where the cashflows are paid to households in the form of a dividend and valued in terms of the utility that dividend provides. From the first-order condition for price-setting we get (in a symmetric equilibrium and after aggregating across firms) the following Phillips curve for inflation, π_t :

$$\pi_t(1+\pi_t) = \frac{1-\epsilon}{\phi} + \frac{\epsilon}{\phi}\omega_t + \beta E_t \left[\frac{(C_{t+1}-\gamma C_t)^{-\sigma} e^{a_{t+1}} H_{t+1} \pi_{t+1} (1+\pi_{t+1})}{(C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t} \right],\tag{18}$$

where $\phi \in (0, \infty)$ governs the magnitude of the price-adjustment cost and $\epsilon \in (1, \infty)$ represents the price elasticity of demand. Finally, we have the resource constraint:

$$C_t = \left(1 - \frac{\phi}{2}\pi_t^2\right)e^{a_t}H_t.$$
(19)

We substitute Eqs. (16) and (17) into the Phillips curve, then the central bank's decision problem is to choose π_t to maximize household welfare (Eq. (14)), subject to the Phillips curve and the resource constraint. The resulting system of first-order conditions is:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{20}$$

$$C_t = \left(1 - \frac{\phi}{2}\pi_t^2\right)e^{a_t}H_t,\tag{21}$$

$$\pi_{t}(1+\pi_{t}) = \frac{1-\epsilon}{\phi} + \frac{\epsilon}{\phi} \frac{1}{e^{a_{t}}} \frac{\nu H_{t}^{\chi}}{(C_{t}-\gamma C_{t-1})^{-\sigma}} + \beta E_{t} \left[\frac{M_{t+1}}{(C_{t}-\gamma C_{t-1})^{-\sigma} e^{a_{t}} H_{t}} \right],$$
(22)

$$M_t = (C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t \pi_t (1 + \pi_t),$$
(23)

$$0 = -\phi \pi_t \lambda_{1t} - (1 + 2\pi_t) (C_t - \gamma C_{t-1})^{-\sigma} \lambda_{2t},$$
(24)

$$\nu H_t^{\chi} = \left(1 - \frac{\phi}{2}\pi_t^2\right)e^{a_t}\lambda_{1t} + \left[\frac{\epsilon(1+\chi)}{\phi}H_t^{\chi} + \left(\frac{1-\epsilon}{\phi} - \pi_t(1+\pi_t)\right)(C_t - \gamma C_{t-1})^{-\sigma}e^{a_t}\right]\lambda_{2t},\tag{25}$$

$$\lambda_{1t} = (C_t - \gamma C_{t-1})^{-\sigma} + \beta E_t [M_c(a_{t+1}, C_t)] \lambda_{2t}$$
$$-\sigma \left(\frac{1-\epsilon}{\phi} - \pi_t (1+\pi_t)\right) (C_t - \gamma C_{t-1})^{-\sigma} e^{a_t} H_t \lambda_{2t}$$

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$$+\beta E_{t} \Big[\Big(\sigma \gamma \Big(\frac{1-\epsilon}{\phi} - \pi_{t+1} (1+\pi_{t+1}) \Big) (C_{t+1} - \gamma C_{t})^{-\sigma-1} e^{a_{t+1}} H_{t+1} \Big) \lambda_{2t+1} \Big],$$
(26)

where λ_{1t} is the Lagrange multiplier on the resource constraint, λ_{2t} is the Lagrange multiplier on the Phillips curve, and M_t represents the marginal utility of the goods lost through the adjustment costs generated by inflation. Equation (26) has the form of a generalized Euler equation because it depends on the derivative $M_c(a_{t+1}C_t)$.

We parameterize the model by setting $\beta = 0.99$, $\sigma = 1$, $\nu = 1$, $\chi = 1$, $\gamma = 0.6$, $\epsilon = 11$, $\phi = 60$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.01. Importantly, we have not introduced a production subsidy to offset the monopolistic distortion, nor a consumption tax to offset the consumption externality–the distortions caused by monopolistic competition and external consumption habits remain.

2.5. Model five - labor-search and discretionary monetary policy

The fifth model is an application of optimal discretionary monetary policy in a labor-search environment. It has the same key characteristic as the four models previously described–a generalized Euler equation associated with time-consistent behavior–but is somewhat more complex–and it has been parameterized so that Hosios' condition does not hold and to accentuate the non-linearity present through the search mechanism.

Households consist of a unit-continuum of worker-members, some of whom are employed and some of whom are unemployed. Employed members receive a wage income while unemployed members receive a benefit payment from the government, financed by a lump-sum tax. Complete insurance within the household ensures that each member receives the same consumption. Each period some workers lose their job according to an exogenous separation rate and some unemployed find work through the outcome of a matching technology. Firms produce according to a technology that is linear in employment and sell their goods in a monopolistically competitive market, choosing their price to maximize their value subject to a quadratic price-adjustment cost (Rotemberg, 1982). Firms that require additional employees post vacancies, paying a per-period fixed cost to do so. The real wage is determined through a Nash-bargain between hiring-firms and new workers. Households can save by purchasing one-period nominal bonds that are in zero-net supply and the central bank conducts policy optimally under discretion by setting the nominal interest rate on the bond. Labor-search models very much in the spirit of this one can be found in Krause et al. (2008), Blanchard and Galí (2010), or in Galí (2015).

We assume that household-utility depends only on consumption and is of the iso-elastic form:

$$U_t = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) \right],\tag{27}$$

 $\beta \in (0, 1)$ and $\sigma \in (0, \infty)$. With u_t denoting the unemployment rate and v_t denoting the vacancy rate, the matching technology is given by:

$$m_t = m u_t^{\alpha} v_t^{1-\alpha}, \tag{28}$$

where m_t is the number of matches, $m \in (0, \infty)$ denotes match-efficiency and $\alpha \in (0, 1)$ is the elasticity of matches with respect to the unemployment rate. We define labor-market tightness, θ_t , according to:

$$\theta_t = \frac{v_t}{u_t},\tag{29}$$

then the aggregate unemployment rate is given by:

$$u_t = 1 - (1 - \delta)n_{t-1},\tag{30}$$

where $\delta \in [0, 1]$ is the exogenous separation rate and n_t is the period-*t* employment rate, and the law-of-motion for aggregate employment obeys:

$$n_t = (1 - \delta)n_{t-1} + m(1 - (1 - \delta)n_{t-1})\theta_t^{1-\alpha}.$$
(31)

From the firm's problem we have (after aggregation) the production function:

$$y_t = e^{a_t} n_t, aga{32}$$

the Phillips curve:

$$\pi_t(1+\pi_t) = \frac{1-\epsilon+\epsilon\omega_t}{\phi} + \beta E_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \pi_{t+1}(1+\pi_{t+1}) \frac{y_{t+1}}{y_t} \right],$$
(33)

and the vacancy-creation equation:

$$\frac{\kappa c_t^{-\sigma}}{m\theta_t^{-\alpha}} = (1-\varsigma)(\omega_t e^{a_t} - b)c_t^{-\sigma} + \beta(1-\delta)E_t \left[c_{t+1}^{-\sigma} \left(1-\varsigma m\theta_{t+1}^{1-\alpha}\right)\frac{\kappa}{m\theta_{t+1}^{-\alpha}}\right].$$
(34)

where $\kappa \in [0, \infty]$ represents the cost of posting a vacancy, $b \in [0, \infty)$ represents real unemployment benefits, and $\varsigma \in (0, 1)$ denotes worker-bargaining power.

Finally, we have the resource constraint:

$$c_t = \left(1 - \frac{\phi}{2}\pi_t^2\right)e^{a_t}n_t - \kappa \left(1 - (1 - \delta)n_{t-1}\right)\theta_t.$$
(35)

After substituting the production function into the Phillips curve, the central bank's decision problem is to choose π_t to maximize household welfare (Eq. (27)), subject to equations (31), (33), (34) and (35). We introduce the auxiliary functions:

$$F(a_t, n_{t-1}) = c_t^{-\sigma} \pi_t (1 + \pi_t) e^{a_t} n_t,$$
(36)

$$G(a_t, n_{t-1}) = (1 - \delta)c_t^{-\sigma} \left(1 - \varsigma m\theta_t^{1-\alpha}\right) \frac{\kappa}{m\theta_t^{-\alpha}},\tag{37}$$

then, in addition to the four constraints specified above, the optimal discretionary policy leads to the first-order conditions:

$$\phi \pi_t \gamma_t = (1 + 2\pi_t) c_t^{-\sigma} \lambda_t, \tag{38}$$

$$\frac{\epsilon}{\phi}n_t\lambda_t = -(1-\varsigma)\chi_t,\tag{39}$$

$$\gamma_t - c_t^{-\sigma} = \sigma \left[\left(\frac{1 - \epsilon + \epsilon \omega_t}{\phi} - \pi_t (1 + \pi_t) \right) e^{a_t} n_t \lambda_t + \left((1 - \varsigma) (\omega_t e^{a_t} - b) - \frac{\kappa}{m \theta_t^{-\alpha}} \right) \chi_t \right] c_t^{-\sigma - 1}, \tag{40}$$

$$0 = -\kappa (1 - (1 - \delta)n_{t-1})\gamma_t - (1 - \alpha)m(1 - (1 - \delta)n_{t-1})\theta_t^{-\alpha}\mu_t + \alpha \frac{\kappa c_t^{-\alpha}}{m\theta_t^{1-\alpha}}\chi_t,$$
(41)

$$0 = \left(1 - \frac{\phi}{2}\pi_t^2\right)e^{a_t}\gamma_t - \left(\frac{1 - \epsilon + \epsilon\omega_t}{\phi} - \pi_t(1 + \pi_t)\right)e^{a_t}c_t^{-\sigma}\lambda_t + \mu_t - \beta(1 - \delta)E_t[Q_{t+1}]
: -\beta\lambda_t E_t[F_n(a_{t+1}, n_t)] - \beta\chi_t E_t[G_n(a_{t+1}, n_t)] = 0,$$
(42)

$$Q_t = \kappa \theta_t \gamma_t + \left(1 - m \theta_t^{1-\alpha} \mu_t\right),\tag{43}$$

where γ_t is the Lagrange multiplier on the resource constraint, λ_t is the Lagrange multiplier on the Phillips curve, μ_t is the Lagrange multiplier on the vacancy-creation equation. Equation (43) has the form of a generalized Euler equation because it depends on the expected derivatives $F_n(a_{t+1}, n_t)$ and $G_n(a_{t+1}, n_t)$.

We assume that aggregate technology shock obeys:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{44}$$

and parameterize the model by setting $\beta = 0.99$, $\sigma = 2.1$, $\kappa = 0.06$, $\epsilon = 11$, $\phi = 80$, $\delta = 0.12$, $\alpha = 0.52$, m = 0.35, b = 0.10, $\varsigma = 0.72$, $\rho = 0.95$, and the standard deviation of the technology innovation to 0.017. As with model four, we have not introduced a production subsidy to offset the monopolistic distortion.

3. A first-order perturbation solution

In this section we describe a procedure that uses perturbation to construct a first-order accurate solution to optimizationbased models involving time-inconsistency. Our focus on first-order solutions stems in large part from the fact that many non-linear models are well-approximated by linear solutions. However, should it be needed, we note that the extension from first-order accuracy to second-order accuracy is both straightforward and intuitive, and employed in Sections 5 and 6.

Our solution method is conceptually simple and easy to implement. To illustrate it, we enlist the help of model three. Recall that for this model the key equations are:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{45}$$

$$k_{t+1} = (1-\delta)k_t + e^{a_t}k_t^{\alpha} - c_t,$$
(46)

$$c_t^{-\sigma} = \beta E_t \Big[c_{t+1}^{-\sigma} \Big(\theta \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) + (1 - \theta) k_k (a_{t+1}, k_{t+1}) \Big) \Big].$$
(47)

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If $\theta = 1$, then the short-run discount rate and the long-run discount rate are equal, there is no time-inconsistency problem, Eq. () simplifies to:

$$c_t^{-\sigma} = \beta E_t \Big[c_{t+1}^{-\sigma} \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) \Big], \tag{48}$$

and we can easily solve for the model's steady state (the model's zeroth-order solution). This steady state provides a point to linearize around and solving the model for its first-order accurate equilibrium dynamics becomes straightforward using a method such as Klein (2000).

The difficulty arises when $\theta \neq 1$. When $\theta \neq 1$ we cannot solve for the model's steady state without knowing the derivative of the decision rule for capital, which is part of the model's first-order solution. So in order to solve for the steady state (the zeroth-order solution) we need to know the first-order solution. Although the decision rule for capital is a non-linear function, its linear approximation has the form:

$$k(a_t, k_t) \approx k_{ss} + \psi_a(a_t - a_{ss}) + \psi_k(k_t - k_{ss}), \tag{49}$$

where ψ_a and ψ_k are derivatives, $a_{ss} = 0$ is the steady state value for (log-) technology, and k_{ss} is the unknown steady state value for capital. From Eq. (49), the derivative of next period's capital with respect to capital is ψ_k . Suppose we know ψ_k . then the steady state of Eqs. (45)–(47) can be computed and is given by:

$$a_{\rm SS}=0, \tag{50}$$

$$k_{ss} = \left[\frac{1}{\alpha} \left(\frac{1 - \beta(1 - \theta)\psi_k}{\beta\theta}\right) - 1 + \delta\right]^{\frac{1}{\alpha - 1}},\tag{51}$$

$$c_{\rm ss} = k_{\rm ss}^{\alpha} - \delta k_{\rm ss},\tag{52}$$

which provides a point around which Eqs. (45)-(47) can be linearized.

Unfortunately, we cannot produce a first-order accurate solution from an iterative procedure that begins by guessing ψ_k , computing the steady state, linearizing around that steady state, solving the resulting linearized rational expectations model, extracting an update of ψ_k from the solution, and iterating to convergence. The reason this approach is incorrect is that in order to have a first-order accurate solution we require that the derivative $k_k(a_{t+1}, k_{t+1})$ in Eq. (47) be approximated to firstorder accuracy, and this requires that the decision rule for capital itself be approximated to second-order accuracy. Therefore, we require the model's first-order accurate solution to obtain its zeroth-order solution, we require its second-order accurate solution to obtain its first-order solution, we require its third-order accurate solution to obtain its second-order solution, etc.

Acknowledging this inconvenient recursion our solution procedure is as follows. Rather than assume that the model's third-order accurate solution is known, we simply assume that the terms in the third-order (and higher) accurate solution are sufficiently small that they can be safely ignored. Ignoring terms higher than second-order, the solution for next period's capital has the approximation:

$$k(a_t, k_t) \approx k_{ss} + \psi_a(a_t - a_{ss}) + \psi_k(k_t - k_{ss}) + \frac{\psi_{aa}}{2}(a_t - a_{ss})^2 + \psi_{ak}(a_t - a_{ss})(k_t - k_{ss}) + \frac{\psi_{kk}}{2}(k_t - k_{ss})^2,$$
(53)

and the partial derivative of future capital with respect to k_t is:

$$k_k(a_t, k_t) \approx \psi_k + \psi_{ak}(a_t - a_{ss}) + \psi_{kk}(k_t - k_{ss}).$$
(54)

With the building blocks established, our method for computing a first-order accurate solution to the model is as follows:

- 1. Set the loop-counter to zero, i = 0, set the convergence tolerance, *tol*, and initialize values for ψ_k^0 , ψ_{ak}^0 , ψ_{kk}^0 (that will be stored in the vector ψ^0).
- 2. With the derivative approximated by Eq. (54), solve Eqs. (45)-(47) using a second-order perturbation method. This solution delivers an estimate of the steady state and the first- and second-order equilibrium dynamics.
- Increment the loop-counter, i = i + 1, and extract ψⁱ_k, ψⁱ_{ak}, ψⁱ_{ak}, ψⁱ_{kk} (that will be stored in the vector ψⁱ) from the second-order solution for next period's capital, which takes the form of Eq. (53).
 While ||ψⁱ-ψⁱ⁻¹|| > tol, return to step 2.
- 5. Exit.

Having exited the algorithm we discard all second-order terms, retaining the steady state and the linear terms, which provide a solution that is first-order accurate. Clearly, aspects of the procedure described above are tailored to model three, but the modifications needed to apply it to other models are straightforward, and we describe a general procedure below.

Iddle I		
Projection	method	details

	Model 1	Model 2	Model 3	Model 4	Model 5
Quadrature nodes	21	21	21	21	21
Solution nodes (a_t, s_t)	(21, 51)	(21, 51)	(21, 51)	(21, 51)	(21, 51)
Polynomial orders (a_t, s_t)	(6,9)	(6, 14)	(6,9)	(5,6)	(6,9)
Technology (a_t) domain	$[\pm 0.096]$	$[\pm 0.288]$	$[\pm 0.096]$	$[\pm 0.096]$	$[\pm 0.163]$
Endog. state (s_t) domain	[27,42]	[4,17]	[2.6, 4.7]	[1.3, 1.8]	[0.92, 0.99]
$log_{10}(Euler error _{\infty})$	-11.2	-6.3	-9.5	-9.0	-7.1

3.1. A general environment

The solution method is described above in the context of a specific model, but its general application should be readily apparent. Suppose the model is described by the general non-linear form:

$$E_t \left[\widetilde{\mathbf{f}} \left(\mathbf{y}_{t+1}, \mathbf{x}_{t+1}, \frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}}, \mathbf{y}_t, \mathbf{x}_t, \varepsilon_{t+1} \right) \right] = \mathbf{0},$$
(55)

where \mathbf{x}_t is an $n_x \times 1$ vector of predetermined variables, \mathbf{y}_t is an $n_y \times 1$ vector of non-predetermined variables, ε_t is an $s \times 1$ vector of innovations, and $\tilde{\mathbf{f}}$ is an $(n_y + n_x) \times 1$ vector of "smooth" functions, whose solution takes the form:

$$\mathbf{x}_{t+1} = \mathbf{h}(\mathbf{x}_t, \vartheta) + \vartheta \varepsilon_{t+1}, \tag{56}$$

$$\mathbf{y}_t = \mathbf{g}(\mathbf{x}_t, \vartheta), \tag{57}$$

where ϑ is a perturbation parameter. Eq. (55) differs from the class of models considered in Schmitt-Grohé and Uribe (2004), Gomme and Klein (2011), Binning (2013) only through the presence of the Jacobian term: $\frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}}$.

Our iterative solution procedure estimates the Jacobian $\frac{\partial \mathbf{y}_{t+1}}{\partial \mathbf{x}_{t+1}}$ from a conjecture at equation (57) and substitutes the estimated Jacobian into Eq. (). After this substitution, Eq. (55) can be written as:

$$E_t[\mathbf{f}(\mathbf{y}_{t+1}, \mathbf{x}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \varepsilon_{t+1})] = \mathbf{0},$$
(58)

to which standard second-order perturbation methods can be applied. As Gomme and Klein (2011) (and others) show, the second-order solution can be expressed in the form:

$$\mathbf{x}_{t+1} = \mathbf{x}_{ss} + \frac{\mathbf{h}_{ss}}{2} + \mathbf{h}_{\mathbf{x}}(\mathbf{x}_t - \mathbf{x}_{ss}) + \left(l \otimes (\mathbf{x}_t - \mathbf{x}_{ss})^T \right) \frac{\mathbf{h}_{\mathbf{x}\mathbf{x}}}{2} (\mathbf{x}_t - \mathbf{x}_{ss}) + \mathbf{k}\varepsilon_{t+1},$$
(59)

$$\mathbf{y}_{t} = \mathbf{y}_{ss} + \frac{\mathbf{g}_{ss}}{2} + \mathbf{g}_{\mathbf{x}}(\mathbf{x}_{t} - \mathbf{x}_{ss}) + \left(I \otimes (\mathbf{x}_{t} - \mathbf{x}_{ss})^{T}\right) \frac{\mathbf{g}_{\mathbf{x}\mathbf{x}}}{2} (\mathbf{x}_{t} - \mathbf{x}_{ss}), \tag{60}$$

where \mathbf{x}_{ss} and \mathbf{y}_{ss} are vectors of steady state values, \mathbf{h}_{ss} and \mathbf{g}_{ss} are stochastic adjustments to the steady state, $\mathbf{h}_{\mathbf{x}}$ and $\mathbf{g}_{\mathbf{x}}$ are matrices containing information about the first-order dynamics, $\mathbf{h}_{\mathbf{xx}}$ and $\mathbf{g}_{\mathbf{xx}}$ are matrices containing information about the second-order dynamics, and \mathbf{k} is a loading matrix. Equation (60) is then differentiated with respect to \mathbf{x}_t to obtain a revised estimate of the Jacobian, and we then iterate to convergence.

In our application of this method to the five models above, we use the Gomme and Klein (2011) second-order perturbation method to solve each model and to impose saddle-point stability on the first order dynamics, and we linearize the models rather than log-linearize them. We linearize the models because the resulting solutions can be compared more directly to those obtained using the projection method. It is straightforward to modify the procedure to get a first-order accurate log-solution. Moreover, it is possible to extend this procedure to obtain solutions with higher-order accuracy. For example, by assuming that it is the fourth-order terms that can be safely ignored, conjecturing a second-order expression for the derivative, $g_x(x_t)$, using a third-order perturbation solver such as Binning (2013) to solve the model, updating the conjecture, and iterating to convergence.

4. Results

In this section we solve each of the five models described in Section 2 and present the results. First, we solve the models using a projection method based on Chebyshev polynomials and Gauss-Hermite quadrature that is described in Appendix B. We obtain a highly accurate solution from this projection method that we then use as the benchmark against which to assess the properties and accuracy of the first-order accurate solution procedure. For the projection method, details regarding the number of solution nodes, the order of the polynomials, the domain for each state variable, etc, are summarized in Table 1.

For all of the models the domain for the technology shock was chosen to be plus/minus three unconditional standard deviations while that for the endogenous state variable (capital for models one-three, lagged consumption for model four, and lagged employment for model five) was chosen to cover that variable's stationary distribution determined through a stochastic simulation (one million periods). We used 21 nodes for technology and 51 nodes for the endogenous state variable, with the order of the polynomials for these two variables varying according to the model. The nodes are constructed from the roots (zeros) of the corresponding Chebyshev polynomial. The last row of Table 1 reports on the Euler-equations errors for each model and speaks to the accuracy of the solution. These errors are based on the consumption-Euler equation for models one-three⁵, on the Phillips curve for model four, and on the vacancy-creation equation for model five. To compute the errors we sampled 100,000 draws from each model's stationary distribution.

In regard to the results, in addition to the solution itself, we compute and show for each model aspects of the solution that researchers and policy-makers are typically interested in: decision rules, stationary distributions, and impulse response functions.⁶ Furthermore, we evaluate and report the accuracy of each model's linear solution using the same approach employed to evaluate the accuracy of the projection-based solution. Clearly, because the linear solution has only local accuracy it will not match the accuracy of the projection method, but the statistic is still informative.

Beginning with the stochastic growth model, the linear solution expressed as deviations from steady state is:

$$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{k}_{t+1} \\ \hat{c}_{t} \\ \hat{y}_{t} \end{bmatrix} = \begin{bmatrix} 0.950 \ 0.000 \\ 2.216 \ 0.971 \\ 0.680 \ 0.039 \\ 2.896 \ 0.025 \end{bmatrix} \begin{bmatrix} \hat{a}_{t} \\ \hat{k}_{t} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} [\epsilon_{t+1}], \tag{61}$$

with the steady state given by: $a_{ss} = 0$, $k_{ss} = 34.609$, $c_{ss} = 2.377$, and $y_{ss} = 2.896$. Investment is computed from the solutions for output and consumption. From this solution we plot (slices through) decision rules, stationary distributions and impulse response function, see Fig. 1.

The panels in the top row of Fig. 1 display the solution for capital, consumption, output, and investment, respectively, as a function of capital, holding technology at its steady state value. The panels in the second row of Fig. 1 are similar, except the solutions are shown as functions of technology, holding capital at its steady state value. Stationary distributions for these variables are shown in the third row while the panels in the final row display response functions for a positive one standard deviation technology shock.

Recall that this model is not one in which decision-makers face a time-inconsistency problem. In this sense it represents a benchmark for accuracy against which the remaining models can be compared. What is suggested by the top two rows of panels in Fig. 1 is that the first-order approximation is really quite accurate over most of the domain used for the projection-based solution. Ideally, the solutions to the models with time-inconsistency will exhibit a similar level of accuracy. To more formally assess the accuracy of the linear approximation, we compute its Euler equation errors, just as we did for the projection method. Sampling 100,000 draws from the model's stationary distribution (the very same draws used for the projection-method) we obtain the log_{10} maximum absolute Euler-equation error of -3.5.

The solution results for model two, the model of time-consistent fiscal policy, are shown in Fig. 2. As previously, decision rules are shown in the top two rows of panels, stationary distributions are shown in the third row, and impulse response functions are shown in the fourth row.

Looking at the decision rules first, the panels in the top two rows of Fig. 2 show that the first-order solution is very accurate in the vicinity of the steady state, but that the accuracy deteriorates when capital (first row) or technology (second row) are far from steady state. This decrease in accuracy is to be expected from a first-order accurate perturbation solution. The panels in the third and fourth rows of Fig. 2 reveal that the stationary distributions and the impulses response functions (in particular) are accurately approximated, a consequence of the fact that the model spends very little time in the regions of the domain where the decision rules have less accuracy. Using 100,000 draws sampled from the model's stationary distribution, the accuracy of the linear solution in terms of the log_{10} maximum absolute Euler-equation error is -2.1.

The first-order solution for model two is:

\hat{a}_{t+1}		0.950	0.000		∟ 1_	
\hat{k}_{t+1}		1.206	0.929	Γâ.]	0	
\hat{c}_t	=	0.538	0.066	$\left \begin{array}{c} u_t \\ \hat{\iota} \end{array} \right +$	0	$[\epsilon_{t+1}]$
ĝ		0.158	0.022	$\lfloor \kappa_t \rfloor$	0	
\hat{y}_t		1.902	0.067_		[0]	

For this model investment is calculated by subtracting consumption and government spending from output. The steady state is: $a_{ss} = 0$, $k_{ss} = 8.531$, $c_{ss} = 1.150$, $g_{ss} = 0.326$, and $y_{ss} = 1.902$.

⁵ The $\log_{10}(||Euler error||_{\infty})$ from the government spending Euler equation in model two is -7.5.

⁶ Throughout the paper, the impulses responses are computed using the Monte-Carlo integration method described in Potter (2000), which are often called generalized impulses responses.



Fig. 1. Results for stochastic growth model.

Turning to the third model, which exhibits quasi-geometric discounting, the linear solution to this model is given by:

0.950	0.000		1	
0.755	0.906	$\begin{bmatrix} \hat{a}_t \end{bmatrix}_{t}$	0	[]
0.821	0.154	$ \hat{k}_t ^+$	0	$[\epsilon_{t+1}].$
1.576	0.160		0	
	0.950 0.755 0.821 1.576		$\begin{bmatrix} 0.950 & 0.000 \\ 0.755 & 0.906 \\ 0.821 & 0.154 \\ 1.576 & 0.160 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \end{bmatrix} +$	$\begin{bmatrix} 0.950 & 0.000 \\ 0.755 & 0.906 \\ 0.821 & 0.154 \\ 1.576 & 0.160 \end{bmatrix} \begin{bmatrix} \hat{a}_t \\ \hat{k}_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Again, investment is the residual between output and consumption. The steady state is: $a_{ss} = 0$, $k_{ss} = 3.538$, $c_{ss} = 1.222$, and $y_{ss} = 1.576$ and its accuracy assessed using Euler-equation errors is -2.9.

The results for the behavioral macro-model with quasi-geometric discounting are shown in Fig. 3, following the same layout as the two previous figures. From an accuracy perspective, the results are qualitatively similar to model two. The accuracy of the first-order approximation is clearly good in the vicinity of the steady state. When capital is far from steady state the linear decision rules tend to overstate consumption and output, which leads investment to be overstated too, but this inaccuracy is in the tail-region of the domain where the model spends very little time.



Fig. 2. Results for time-consistent fiscal policy.

The panels in the third row of Fig. 3 show that the stationary distributions are well-approximated even though the linear solution omits precautionary effects from uncertainty. The impulse responses displayed in the final row show little deviation from the projection-based solution, even in immediate aftermath of the shock when its effects are largest.

Our fourth set of results relate to the model with time-consistent monetary policy. Its solution requires solving a system with 12 non-predetermined variables, although only a few-consumption, labor, and inflation-are of primary interest.

The model's linear solution is:

$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{c}_t \\ \hat{h}_t \\ \hat{c}_t \end{bmatrix}$	=	0.950 0.860 -0.647	0.000 0.430 0.429	$\begin{bmatrix} \hat{a}_t \\ \hat{c}_{t-1} \end{bmatrix} +$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$[\epsilon_{t+1}]$
$\begin{bmatrix} \pi_t \\ \hat{\pi}_t \end{bmatrix}$		1.384	-0.849			

while the steady state equal to: $a_{ss} = 0$, $c_{ss} = 1.507$, $h_{ss} = 1.508$, and $\pi_{ss} = 1.886$. Using 100,000 draws sampled from the model's stationary distribution, the linear solution delivers an Euler-equation accuracy statistic of -2.7.



Fig. 3. Results for quasi-geometric discounting.

Figure 4 shows that the linear decision rules exhibit only small deviations from the nonlinear rules, with these deviation present when consumption and technology are far from the steady state. Recall that we did not introduce a production subsidy to offset the effects of monopolistic competition and nor did we introduce a tax to offset the consumption externality. For this reason, the model exhibits a discretionary inflation bias, which is shown in panel I, where the unconditional mean of inflation is just under 1.9 percent per annum for the chosen parameterization. The first-order solution accurately reflects this inflation bias, and indeed appears to capture the model's stationary distribution very well. The impulse responses shown in the final row of Fig. 4 well-approximate the decline in inflation that arises from a rise in technology and the transition dynamics that occur as the model returns to steady state.

One final note regarding model four is that if we introduce an optimal subsidy to offset the monopolistic distortion and an optimal consumption tax to offset the consumption externality, then the steady state is efficient, there is no discretionary inflation bias, and the steady state inflation rate is zero. In that case it is possible to form a LQ approximation around the zero-inflation efficient steady state (because the Phillips curve is not needed to derive a valid second-order approximation



Fig. 4. Results for time-consistent monetary policy.

to conditional welfare). In this special case, then, the model can be solved using a LQ solution technology, and the solution obtained would be identical to the first-order solution from our perturbation approach.

Finally we turn to model five-perhaps the most complex in terms of its underlying nonlinearity and overall size. For this model the main variables of interest are consumption, employment, labor-market tightness, and inflation.

Figure 5 displays the decision rules, the stationary distributions, and impulse response functions for these four variables. In this model, consumption, employment, and labor-market tightness are all increasing functions of technology and lagged employment, while inflation is increasing in lagged employment and decreasing in technology. From the impulse response functions, a shock to technology leads to an initial decline in inflation (a consequence of a decline in real marginal costs), while consumption, employment, and labor-market tightness all initially increase. The results in relation to differences between the linear and non-linear solutions are largely in line with the previous models, but there is a decline in accuracy. Looking at the Euler-equation errors, the linear solution yields a maximum error over the ergodic region of -1.5, which is lower than that for the previous four models. At the same time, for this model there is also a decline in the accuracy of



Fig. 5. Results for the labor-search model.

Table 2					
Statistical	moments	for	simulated	inflation	data.

Solution	Mean	Std. Dev.	Skewness	Kurtosis
Projection	1.803	0.519	0.834	1.125
First-order	1.724	0.492	-0.004	0.001
Second-order	1.804	0.503	0.821	0.892

the projection method (see Table 2), indicating that the non-linearities in this model present greater challenges for both methods.

The linear solution for this model is:

$$\begin{array}{c|c} \hat{a}_{t+1} \\ \hat{n}_{t} \\ \hat{c}_{t} \\ \hat{\theta}_{t} \\ \hat{\pi}_{t} \end{array} = \begin{bmatrix} 0.950 & 0.000 \\ 0.071 & 0.263 \\ 0.973 & 0.488 \\ 6.057 & 2.650 \\ -0.001 & -0.010 \end{bmatrix} \begin{bmatrix} \hat{a}_{t} \\ \hat{n}_{t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} [\epsilon_{t+1}]$$

where the steady state is given by $a_{ss} = 0$, $n_{ss} = 0.959$, $c_{ss} = 0.915$, $\theta_{ss} = 4.707$, and $\pi_{ss} = 0.501$.

5. Welfare

Our solution method works with a model's first-order conditions and does not compute conditional welfare as part of the solution process. In this section we discuss how conditional welfare can be computed.

We first note that when a model's non-linear first-order conditions are being solved using a projection method that computing conditional welfare is straightforward. One simply augments the system of equations being solved with a recursive representation of conditional welfare and then solves the expanded system of equations. To give a concrete example, for model two that involves time-consistent fiscal policy, one can obtain conditional welfare, U_t , by solving the expanded system:

$$a_{t+1} = \rho a_t + \varepsilon_{t+1},\tag{62}$$

$$k_{t+1} = (1-\delta)k_t + e^{a_t}k_t^{\alpha} - c_t - g_t,$$
(63)

$$c_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma} \left(1 + \left(1 - \frac{g_{t+1}}{e^{a_{t+1}} k_{t+1}^{\alpha} - \delta k_{t+1}} \right) \left(\alpha e^{a_{t+1}} k_{t+1}^{\alpha-1} - \delta \right) \right) \right], \tag{64}$$

$$\mu g_t^{-\eta} = \beta E_t \Big[\Big(c_{t+1}^{-\sigma} - \mu g_{t+1}^{-\eta} \Big) c_k(a_{t+1}, k_{t+1}) + \mu g_{t+1}^{-\eta} \Big(1 - \delta + \alpha e^{a_{t+1}} k_{t+1}^{\alpha - 1} \Big) \Big], \tag{65}$$

$$U_{t} = \frac{c_{t}^{1-\sigma}}{1-\sigma} + \mu \frac{g_{t}^{1-\eta}}{1-\eta} + \beta E_{t}[U_{t+1}].$$
(66)

If we were using our perturbation procedure to obtain a solution that was second-order accurate or higher, then we could mimic this approach and solve an expanded system, but with a first-order accurate solution we cannot. Nor can we take a direct quadratic approximation to conditional welfare, which gives the following for the fiscal policy model:

$$U_t \approx E_t \left[\sum_{t=0}^{\infty} \beta^t u_t \right], \tag{67}$$

where

$$u_{t} = u_{ss} + c_{ss}^{-\sigma} (c_{t} - c_{ss}) + \mu g_{ss}^{-\eta} (g_{t} - g_{ss}) - \frac{\sigma c_{ss}^{-\sigma-1}}{2} (c_{t} - c_{ss})^{2} - \frac{\mu \eta g_{ss}^{-\eta-1}}{2} (g_{t} - g_{ss})^{2},$$
(68)

and evaluate it using the model's first order solution. This fails because Eq. (68) contains linear terms.

As Kim and Kim (2006) emphasize, computing conditional welfare correctly to second-order requires that the model be solved to second-order accuracy. Accordingly, if one is interested in computing the conditional welfare associated with the time-consistent policy, then the solution procedure used would require using a third-order perturbation method to enable the derivative in the generalized-Euler equation to be approximated to second-order accuracy. The approach parallels that described in Section 3, but with a third-order perturbation method used and with iteration occurring over an expanded set of coefficients. Upon convergence, and after discarding the third-order terms, the resulting solution will be second-order accurate and can be used to compute conditional welfare using a quadratic approximation that retains linear terms (like Eqs. ()–(68) above), (Kim and Kim, 2006).

We implement the method described above and apply it to each of the five models. For this purpose we use the thirdorder perturbation method developed in Binning (2013). The absolute differences between the welfare computed using this second-order accurate solution and that computed using the projection method are displayed in Fig. 6.

Figure 6 shows that welfare is well-approximated in the vicinity of the steady state for all five models, with the absolute differences in welfare close to zero. Further, with the exception of model two, the perturbation solution delivers an estimate of welfare that is pretty accurate over the entire domain shown (which is the domain used by the projection method). For model two, welfare looks to be less well-approximated, most noticeably when capital is far from steady state.



Fig. 6. Absolute differences in computed welfare.

It is important to recognize that the welfare approximation that the perturbation method employs achieves second-order accuracy through the iterative use of a third-order perturbation solution to the model. Because a third-order solution is used, the resulting welfare approximation requires derivatives that are higher than second-order, at least for the models containing a generalized Euler equation (models two–five). As a consequence, the accuracy shown here is achieved through the use of first-, second-, and third-derivatives of the utility function and the constraints. As attractive as LQ methods are it seems difficult to apply them to models with time-inconsistency other than in special cases. One notable special case is that of discretionary monetary policy when the steady state is efficient and is characterized by zero inflation. Another special case is where the model does not contain any endogenous state variables (see Appendices A.1 and A.2, respectively).

6. A more sophisticated model

In this section we consider optimal discretionary monetary policy in a more sophisticated labor-search model, one that allows hours worked to vary in over the business cycle and that considers several different types of shocks. Specifically, we consider shocks to the elasticity of substitution among goods, which allows for a time-varying price mark-up, and to the separation rate in addition to the standard aggregate technology shock. In the spirit of Bernstein et al. (2020), an adverse shock to the separation rate can be used to model one aspect of the Covid-19 pandemic. In this model, non-linearities associated with the consumption-saving and labor-leisure choices interact with high-volatility mark-up and separation-rate shocks to create a numerically important response to risk. For this reason, we solve the model to both first- and second-order accuracy and present both sets of results. The model presented here is closely related to the one Dennis and Kirsanova (2021) used to study policy biases arising through supply shocks while the notation follows the model from Section 2.5. Readers interested in a fuller discussion/derivation the model are directed to Dennis and Kirsanova (2021).

The representative households consists of employed and unemployed members aggregating over which leads to the following expected discounted lifetime utility function:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} + \chi n_t \frac{(1-h_t)^{1-\nu} - 1}{1-\nu} \right) \right], \tag{69}$$

where $\beta \in (0, 1)$, $\sigma \in (0, \infty)$, $\chi \in (0, \infty)$, and $\nu \in (0, \infty)$. Complete insurance within the household means that the representative household chooses sequences for consumption and nominal bonds to carry into the next period, $\{c_t, B_{t+1}\}_{t=0}^{\infty}$, in order to maximize Eq. (69) subject to the budget constraint:

$$c_t + \frac{B_{t+1}}{P_t} + \tau_t = w_t h_t n_t + b(1 - n_t) + (1 + R_{t-1}) \frac{B_t}{P_t} + d_t,$$
(70)

where τ_t is a time-varying lump-sum tax (used to finance unemployment benefits), d_t , represents dividends earned through ownership of firms, P_t represents the aggregate price level, and R_t represents the period-t net nominal interest rate. The first-order conditions lead to the standard consumption-Euler equation:

$$c_t^{-\sigma} = \beta (1+R_t) E_t \left[\frac{c_{t+1}^{-\sigma}}{1+\pi_{t+1}} \right].$$
(71)

Now, with the production function taking the form:

$$y_t = e^{a_t} h_t n_t, (72)$$

from the firms' side we have the Phillips curve:

$$\phi\pi_t(1+\pi_t) = \left(1 - \epsilon_t + \epsilon_t \chi \frac{(1-h_t)^{-\nu}}{e^{a_t} c_t^{-\sigma}}\right) + \beta\phi E_t \left[\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma} \frac{e^{a_{t+1}} h_{t+1} n_{t+1}}{e^{a_t} h_t n_t} \pi_{t+1}(1+\pi_{t+1}) \right],\tag{73}$$

which assumes the goods market is monopolistically competitive and that prices are sticky as per Rotemberg (1982), and the vacancy-creation equation:

$$\frac{\kappa c_t^{-\sigma}}{(1-\varsigma)m\theta_t^{-\alpha}} = \chi \frac{(1-\nu h_t)(1-h_t)^{-\nu}-1}{1-\nu} - c_t^{-\sigma}b + \beta E_t \bigg[\kappa (1-\delta_{t+1})c_{t+1}^{-\sigma}\frac{1-\varsigma m\theta_{t+1}^{1-\alpha}}{(1-\varsigma)m\theta_{t+1}^{-\alpha}}\bigg],\tag{74}$$

which also reflects Nash-bargaining over real wages and hours worked.

Two remaining equations of importance are the aggregate resource constraint:

$$e^{a_t}h_t n_t = c_t + \kappa \left(1 - (1 - \delta_t)n_{t-1}\right)\theta_t + \frac{\phi}{2}\pi_t^2 e^{a_t}h_t n_t,$$
(75)

and the law-of-motion for employment:

$$n_t = (1 - \delta_t)n_{t-1} + m_t(1 - (1 - \delta_t)n_{t-1})\theta_t^{1-\alpha}.$$
(76)

For the discretionary central bank's decision problem, the relevant constraints are Eqs. (73)–(76), with the policy objective given by Eq. (69). With $\lambda_{1t} - \lambda_{4t}$ representing the Lagrange multipliers on Eqs. (73)–(76) respectively, the first-order conditions are:

$$\pi_t \lambda_{3t} = (1+2\pi_t)c_t^{-\sigma} \lambda_{1t},\tag{77}$$

$$0 = c_t + \sigma \left((\epsilon_t - 1) + \phi \pi_t (1 + \pi_t) \right) e^{a_t} h_t n_t \lambda_{1t} + \left(\sigma b + \frac{\sigma \kappa}{(1 - \varsigma) m \theta_t^{-\alpha}} \right) \lambda_{2t} + c_t^{1 + \sigma} \lambda_{3t},$$

$$(78)$$

$$\chi n_t (1 - h_t)^{-\nu} = \nu \chi h_t (1 - h_t)^{-\nu - 1} \lambda_{2t} + \left(\frac{\phi}{2} \pi_t^2 - 1\right) e^{a_t} n_t \lambda_{3t}$$

: + $\left((1 - \epsilon_t - \phi \pi_t (1 + \pi_t)) c_t^{-\sigma} e^{a_t} + \epsilon_t \chi (1 - (1 - \nu) h_t) (1 - h_t)^{-\nu - 1}\right) n_t \lambda_{1t},$ (79)

$$0 = \left((1 - \epsilon_t - \phi \pi_t (1 + \pi_t)) c_t^{-\sigma} h_t e^{a_t} + \epsilon_t \chi (1 - h_t)^{-\nu} h_t + \beta \phi E_t [F_1(n_t, \mathbf{s}_{t+1})] \right) \lambda_{1t}$$

$$: + \beta E_t [G_1(n_t, \mathbf{s}_{t+1})] \lambda_{2t} + \left(\frac{\phi}{2} \pi_t^2 - 1 \right) e^{a_t} h_t \lambda_{3t} - \lambda_{4t} + \chi \frac{(1 - h_t)^{1-\nu} - 1}{1 - \nu} + \beta E_t [Q_{t+1}],$$
(80)

$$\frac{\kappa c_t^{-\sigma} \alpha \theta_t^{\alpha-1}}{(1-\varsigma)m} \lambda_{2t} = \kappa (1 - (1-\delta_t)n_{t-1})\lambda_{3t} + (1-\alpha)m\theta_t^{-\alpha} (1 - (1-\delta_t)n_{t-1})\lambda_{4t},$$
(81)

where:

$$F(a_t, \epsilon_t, \delta_t, n_{t-1}) = c_t^{-\sigma} e^{a_t} h_t n_t \pi_t (1 + \pi_t),$$

$$\tag{82}$$

$$G(a_t, \epsilon_t, \delta_t, n_{t-1}) = \kappa (1 - \delta_t) c_t^{-\sigma} \frac{1 - \varsigma m \theta_t^{1-\alpha}}{(1 - \varsigma) m \theta_t^{-\alpha}},$$
(83)

$$Q_t = (1 - \delta_t) \left(\left(1 - m\theta_{t+1}^{1-\alpha} \right) \lambda_{4t+1} - \kappa \theta_{t+1} \lambda_{3t+1} \right).$$
(84)

The processes for the three shocks are assumed to be:

$$a_{t+1} = \rho_a a_t + \varepsilon_{t+1},\tag{85}$$

$$\log\left(\epsilon_{t+1}\right) = (1 - \rho_{\epsilon})\log\left(\epsilon\right) + \rho_{\epsilon}\log\left(\epsilon_{t}\right) + \eta_{t+1},\tag{86}$$

$$\log\left(\delta_{t+1}\right) = (1 - \rho_{\delta})\log\left(\delta\right) + \rho_{\delta}\log\left(\delta_{t}\right) + \zeta_{t+1}.$$
(87)

We parameterize the model by setting: $\beta = 0.99$, $\sigma = 1$, $\chi = 0.2$, $\nu = 5$, $\epsilon = 11$, $\phi = 80$, $\kappa = 0.06$, b = 0.07, $\delta = 0.12$, m = 0.66, $\alpha = 0.72$, $\varsigma = 0.72$, $\rho_a = 0.95$, $\rho_{\epsilon} = 0.8$, $\rho_{\delta} = 0.7$, and the standard deviations for the technology, mark-up (priceelasticity), and separation-rate shocks are set to 0.01, 0.10, and 0.03, respectively.

6.1. Results

We solve this four state variable model to both first- and second-order accuracy using the iterative procedure developed in this paper and using a projection method based on the Smolyak sparse grid.⁷ The results are reported in Figs. 7–11. As noted above, the volatility of the shocks–specifically those to the mark-up and the separation rate–interacting with the household's aversion to volatility produces some noteworthy differences between the three solutions.

Consider Fig. 7, which displays (slices through) decision rules constructed by allowing one state to vary while holding the remaining three states at their steady state values. Although quantitatively similar, these decision rules reveal some important qualitative differences. For example, while the second-order perturbation and the projection method produce decision rules that are similar, there are systematic differences between these and the first-order accurate decision rules, at least in regard to labor-market tightness (theta, column four) and inflation (column five), and sometimes also for hours worked

⁷ To solve the model, I use Smolyak polynomials based on a three layer isotropic grid to approximate the equilibrium functions, employing Gauss-Hermite quadrature with 11 nodes for each shock to compute expectations.



Fig. 7. Decision rules for different solutions.

(for example panel H). These systematic differences reflect the adjustment for risk that is captured by the second-order accurate solution and the projection solution, but not by the first-order accurate solution (which is certainty equivalent). Further, most notably for inflation (panels O and T), there are noticeable differences in curvature between the first-order solution and second-order solution; differences between the second-order solution and the projection solution can also be seen. These differences in curvature suggest that the perturbation solutions, especially the first-order perturbation solution, may have difficulty capturing accurately the volatility of inflation.⁸

Figure 8 displays the stationary distributions for each of the model's main variables based on 1,000,000 periods of simulated data. Where the first-order accurate solution produces distributions that are all symmetric (reflecting its certainty equivalence), the second-order accurate solution captures important asymmetry in the distribution for inflation. This asym-

⁸ We also note that evaluated over the ergodic region, the log_{10} largest absolute Euler-equation error for the first-order accurate solution is -2.1 while that for the second-order accurate solution is -2.8 and that for the projection method is -4.7. These errors are constructed from the vacancy-creation equation (Eq. (74)) and are based on 10,000 draws from the stationary distribution.



Fig. 8. Stationary distributions for different solutions.



Fig. 9. Responses to a positive three-standard-deviation technology shock.



Fig. 10. Responses to a positive three-standard-deviation price elasticity shock.



Fig. 11. Responses to a positive three-standard-deviation separation rate shock.

metry is revealed in the form of a right-skewed distribution, with this skewness primarily a consequence of the log-normally distributed shock to the mark-up (or price elasticity of substitution). Smaller differences between the distributions can also be seen in employment and hours worked. Relative to the projection solution, the distributions produced by the second-order accurate solution align well, except perhaps for inflation, where the perturbation solution has difficultly capturing the very extreme left-tail. To better summarize the differences between the distributions, Table 2 reports the first through fourth moments of inflation.

As expected, data produced by the first-order solution is unable to accurately capture the skewness and kurtosis present in the projection solution while the second-order solution captures the skewness, but understates the kurtosis. These inaccuracies in the distribution for inflation could be important in situations where interest focuses on tail events, or if the model were being estimated using some form of simulated method of moments.

Our final figures illustrate the model's behavior in response to each of the shocks, showing the generalized impulse responses produced by each of the three solutions. To tease out differences between the three solutions, and because sometimes one is interested in large shocks, we show here responses to three-standard-deviation shocks.⁹

In response in a positive three standard deviation aggregate technology shock, Fig. 9 shows consumption and employment rising (panels A and B) and labor-market tightness increasing (panel C). Greater labor market tightness combined with improved labor-productivity (due to the rise in technology) leads to a decline in real marginal costs and to a fall in inflation (panel E). Hours-worked decline slightly, but are largely unaffected by the shock (panel C). All three solutions produce very similar responses, even for this large shock.

Turning to the responses to a positive three-standard-deviation price-elasticity shock shown in Fig. 10, the effect of the shock is to temporarily increase market competition and to temporarily decrease the price mark-up. Greater competition in the goods market raises the demand for labor and lifts goods-production. The effect is to increase consumption, employment, hours-worked and labor-market tightness (panels A–D, respectively), while the decline in the price mark-up generates a decrease in inflation (panel E). For this large price elasticity shock it is apparent that the first-order solution tends to over-state the immediate impact of the shock, most noticeably for inflation, but displays accurate convergence behavior. Differences between the second-order solution and the projection solution are largely indiscernible, both in the initial period and subsequently.

Lastly, we consider the responses to a positive three-standard-deviation shock to the separation rate (Fig. 11). With workers and employers separating with greater frequency, the effect of the shock is to lower consumption (panel A) and employ-

⁹ For all three shocks, the three solutions produce responses that are essentially equivalent for one-standard-deviation shocks.

ment (panel B), leading to a decline in labor-market tightness (panel D). Firms respond to having fewer workers by raising the hours worked by their remaining workers, which causes hours-worked to rise (panel C) and, through an increase in the real wage (produced by the increased demand for hours), to a rise in inflation (panel E). Comparing the responses across the three solutions, in this case the first-order solution tends to understate the immediate impact of the large shock by a small magnitude, but otherwise captures the behavior in the projection solution. The responses from the second-order solution look to capture well the responses produced by the projection solution.

7. Conclusions

This paper has developed and illustrated a procedure for solving dynamic stochastic models containing generalized-Euler equations-models exhibiting time-inconsistency-using perturbation methods. The approach does not require the optimization problem being solved to be reformulated in terms of an approximate LQ problem, but works, instead, with systems of first-order conditions. For this reason, the procedure does not involve forming a second-order approximation to household welfare and, importantly, it can be applied without modification to models where the steady state is inefficient. Because it utilizes a perturbation around the steady state, the procedure inherits all the strengths and weaknesses of perturbation methods, including their local accuracy properties. We note how the solution method can be used to take log-linear approximations, as opposed to linear approximations, and illustrate in the context of computing conditional welfare and through the model in Section 6 how it can be adapted and used to construct second-order accurate approximations.

To demonstrate the solution procedure and to assess its behavior, we apply it to a range of models involving timeconsistent decision-making drawn from different areas of macroeconomics. To asses its accuracy, we compute Eulerequation errors and compare the results from the perturbation solution to those from a projection-based solution. Consistent with what one would expect from a local solution technology, we find that the method can be inaccurate in regions away from the steady state. Lastly, we show how perturbation can be employed to compute conditional welfare and note that the approximation requires third-order derivatives, which undermines the general applicability of LQ methods to models with time-inconsistency.

In principle, the method developed in this paper could be combined with the fifth-order approximation scheme developed by Levintal (2017) to achieve forth-order accuracy in models with time-inconsistency. Moreover, the method could potentially be applied to models with features such as occasionally binding constraints, large uncertainty shocks, and rare disasters, by combining it with the Taylor projection method developed in Levintal (2018). These are areas for future research.

Appendix A. LQ approximations

In this Appendix we consider two cases where a valid LQ approximation to a problem involving time-inconsistency can be formed. In Appendix A.1 we show that a valid LQ approximation is possible when the model's steady state is efficient; in Appendix A.2 we show that a valid LQ approximation is possible when the model does not contain any endogenous state variables. In Appendix A.3 we turn to the general case where the model's steady state is not efficient and the model contains endogenous state variables. For this general case we show that a valid LQ approximation is not possible because such an approximation requires knowing the equilibrium decision rules for the choice variables, along with their first- and second-derivatives with respect to the endogenous state variables.

A1. Efficient steady state

Here we briefly analyze the case where the model's steady state is efficient and show that it can be solved using LQ methods. Typically, but not always, the decision-maker facing the time-inconsistency problem will be a policy maker, such as the fiscal authority (model two) or the central bank (models four and five), but it could also be the representative household (model three). Before introducing the problem, we note that the case considered here gains traction when considering optimal discretionary monetary policy because in those models the steady state can often be rendered efficient through the simple introduction of a production subsidy or an employment subsidy to offset the distortionary effects of monopolistic competition. For other applications where time-inconsistency matters, such as fiscal policy or quasi-geometric preferences, this case is of less interest.

Efficient steady state

Let \mathbf{z}_t be an $s \times 1$ vector of shocks with innovations $\varepsilon_t \sim i.i.d[\mathbf{0}, \mathbf{\Omega}]$ and \mathbf{y}_t be an $n \times 1$ vector of non-predetermined/choice variables. We suppose that the constraints on the planner's problem are:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma} \mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0},\tag{A.1}$$

$$\mathbf{p}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) = \mathbf{0}, \tag{A.2}$$

and that the representative household's expected discounted lifetime utility takes the form:

$$U_t = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) \right].$$
(A.3)

The functions, **p** and *u*, are each assumed to be concave and smooth. Eqs. (A.1) and (A.2) distinguish between equations for the shocks processes, Eq. (A.1), and equations that bind on the endogenous variables, Eq. (A.2). We assume that there are *s* shocks and less than *n* equations in **p**. We further assume that Γ has spectral radius less than one and therefore that the steady state for \mathbf{z}_t found from Eq. (A.1) is the zero vector, **0**.

To determine the efficient steady state for \mathbf{y}_t we formulate the planner's problem, which is to choose $\{\mathbf{y}_t, \lambda_t\}_{t=0}^{\infty}$ to extremize the Lagrangian:

$$\mathcal{L} = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left(u(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) + \lambda_t^T \mathbf{p}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) \right) \right],$$
(A.4)

leading to the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial \mathbf{y}_{t}} : \mathbf{u}_{2} + \lambda^{T} \mathbf{p}_{2} + \beta E_{t} \Big[\mathbf{u}_{1}^{'} + (\lambda^{'})^{T} \mathbf{p}_{1}^{'} \Big] = \mathbf{0}, \tag{A.5}$$

$$\frac{\partial \mathcal{L}}{\partial \mathcal{L}}$$

$$\frac{\partial \lambda_t}{\partial \lambda_t} : \mathbf{p} = \mathbf{0}.$$
(A.b)

where we have suppressed the function arguments and time subscripts, introduced "primes" to signify next-period values, and used function-subscripts to denote the derivative with respect to the numbered argument. We denote by $\bar{\mathbf{y}}$ and $\bar{\lambda}$ the (efficient) steady state values that satisfy jointly Eqs. (A.5) and (A.6).

Time-consistent decision problem

Let the first-order conditions and constraints aggregated across households and firms that are not already included in Eq. (A.2) be denoted:

$$E_t[\mathbf{f}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t)] = \mathbf{0}.$$
(A.7)

Because the model's steady state is efficient, Eq. (A.7) holds at the values for $\overline{\mathbf{y}}$ and $\overline{\mathbf{z}}$ determined from the planner's problem. The problem for the decision-maker facing the time-consistency problem is to choose { \mathbf{y}_t , λ_t , μ_t } to solve the Bellman equation:

$$V(\mathbf{y}_{t-1}, \mathbf{z}_{t}) = \max_{\{\mathbf{y}_{t}, \lambda_{t}, \mu_{t}\}} \begin{bmatrix} u(\mathbf{y}_{t-1}, \mathbf{y}_{t}, \mathbf{z}_{t}) + \lambda_{t}^{T} [\mathbf{p}(\mathbf{y}_{t-1}, \mathbf{y}_{t}, \mathbf{z}_{t})] \\ + \mu_{t}^{T} E_{t} [\mathbf{f}(\mathbf{y}_{t-1}, \mathbf{y}_{t}, \mathbf{g}(\mathbf{y}_{t}, \mathbf{z}_{t+1}), \mathbf{z}_{t})] + \beta E_{t} [V(\mathbf{y}_{t}, \mathbf{z}_{t+1})] \end{bmatrix},$$
(A.8)

where $\mathbf{y}_t = \mathbf{g}(\mathbf{y}_{t-1}, \mathbf{z}_t)$ is the unknown equilibrium law-of-motion relating \mathbf{y}_t to the state variables. From the Bellman equation, we recover Eqs. (A.2) and (A.7) and obtain the following first-order and envelope conditions:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T \mathbf{p}_2 + \mu^T (\mathbf{f}_3 \mathbf{g}_1 + \mathbf{f}_2) + \beta E_t \left[\mathbf{V}_1 \right]$$
(A.9)

$$\mathbf{V}_1 = \mathbf{u}_1 + \lambda^T \mathbf{p}_1 + \mu^T \mathbf{f}_1, \tag{A.10}$$

$$\mathbf{V}_2 = \mathbf{u}_3 + \lambda^T \mathbf{p}_3 + \mu^T \mathbf{f}_4, \tag{A.11}$$

respectively; again we have suppressed the function arguments and time-subscripts for clarity. Combining Eqs. (A.9) and (A.10) we get:

$$\mathbf{0} = \mathbf{u}_{2} + \lambda^{T} \mathbf{p}_{2} + \mu^{T} (\mathbf{f}_{3} \mathbf{g}_{1} + \mathbf{f}_{2}) + \beta E_{t} \Big[\mathbf{u}_{1}^{'} + (\lambda^{'})^{T} \mathbf{p}_{1}^{'} + (\mu^{'})^{T} \mathbf{f}_{1}^{'} \Big],$$
(A.12)

which is in the form of a generalized Euler equation (note Eq. (A.12) can be simplified using Eq. (A.5)).

The next step is to form a second-order approximation to the Bellman equation around the efficient steady state. Because the steady state is efficient, this approximation employs a second-order approximation to the constraints in Eq. (A.2), but only a first-order approximation to the equations in Eq. (A.7). The second-order approximation to the Bellman equation is:

$$\begin{split} \begin{bmatrix} \widehat{\mathbf{y}}_{t-1}^T & \widehat{\mathbf{z}}_t^T \end{bmatrix} \begin{bmatrix} \overline{\mathbf{v}}_{11} & \overline{\mathbf{v}}_{12} \\ \overline{\mathbf{v}}_{21} & \overline{\mathbf{v}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{y}}_{t-1} \\ \widehat{\mathbf{z}}_t \end{bmatrix} + \nu &= f.o.t. + s.o.t. \\ + \lambda_t^T (\overline{\mathbf{p}}_1 \widehat{\mathbf{y}}_{t-1} + \overline{\mathbf{p}}_2 \widehat{\mathbf{y}}_t + \overline{\mathbf{p}}_3 \widehat{\mathbf{z}}_t) \\ &+ \mu_t^T (\overline{\mathbf{f}}_1 \widehat{\mathbf{y}}_{t-1} + (\overline{\mathbf{f}}_2 + \overline{\mathbf{f}}_3 \overline{\mathbf{g}}_1) \widehat{\mathbf{y}}_t + \overline{\mathbf{f}}_4 \widehat{\mathbf{z}}_t) \end{split}$$

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$$+\beta E_t \left[\begin{bmatrix} \widehat{\mathbf{y}}_t^T & \widehat{\mathbf{z}}_{t+1}^T \end{bmatrix} \begin{bmatrix} \overline{\mathbf{V}}_{11} & \overline{\mathbf{V}}_{12} \\ \overline{\mathbf{V}}_{21} & \overline{\mathbf{V}}_{22} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{y}}_t \\ \widehat{\mathbf{z}}_{t+1} \end{bmatrix} + \nu \right]$$
(A.13)

where f.o.t. and s.o.t. refer to first order terms and second order terms, respectively, and the first order terms are:

$$f.o.t. = \left(\overline{\mathbf{u}}_{1} + \overline{\lambda}^{T}\overline{\mathbf{p}}_{1} + \overline{\mu}^{T}\overline{\mathbf{f}}_{1} - \overline{\mathbf{V}}_{1}\right)\widehat{\mathbf{y}}_{t-1} + \left(\overline{\mathbf{u}}_{2} + \overline{\lambda}^{T}\overline{\mathbf{p}}_{2} + \overline{\mu}^{T}\left(\overline{\mathbf{f}}_{3}\overline{\mathbf{g}}_{1} + \overline{\mathbf{f}}_{2} + \beta\overline{\mathbf{V}}_{1}\right)\right)\widehat{\mathbf{y}}_{t} + \left(\overline{\mathbf{u}}_{3} + \overline{\lambda}^{T}\overline{\mathbf{p}}_{3} + \overline{\mu}^{T}\overline{\mathbf{f}}_{4} - \overline{\mathbf{V}}_{2}\right)\widehat{\mathbf{z}}_{t},$$
(A.14)

Looking at the expression for the first-order terms in the welfare approximation, Eqs. (A.9)-(A.11) imply that the first order terms equal zero, that the welfare approximation contains only second-order terms and is a quadratic, and that the approximated Bellman equation is LQ.

A2. No endogenous state variables

In this appendix we treat that case where the underlying model does not have any endogenous state variables. We show that it is possible in this case to formulate the time-inconsistent decision problem in terms of an approximate LQ problem. Let the model be described by the system:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma} \mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0},\tag{A.15}$$

$$E_t[\mathbf{h}(\mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t)] = \mathbf{0}, \tag{A.16}$$

where Eq. (A.16) represents the constraints and first-order conditions associated with the decision-maker that does not face a time-inconsistency problem, and let the representative household's expected discounted lifetime utility take the form:

$$U_t = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\mathbf{y}_t, \mathbf{z}_t) \right].$$
(A.17)

The problem for the decision-maker facing time-inconsistency is to choose $\{\mathbf{y}_t, \lambda_t\}$ to solve the Bellman equation:

$$V(\mathbf{z}_{t}) = \max_{\{\mathbf{y}_{t}, \lambda_{t}\}} \left[u(\mathbf{y}_{t}, \mathbf{z}_{t}) + \lambda_{t}^{T} E_{t}[\mathbf{h}(\mathbf{y}_{t}, \mathbf{g}(\mathbf{z}_{t+1}), \mathbf{z}_{t})] + \beta E_{t}[V(\mathbf{z}_{t+1})] \right],$$
(A.18)

where $\mathbf{y}_{t+1} = \mathbf{g}(\mathbf{z}_{t+1})$ is the unknown equilibrium law-of-motion relating \mathbf{y}_{t+1} to the state variables. From the Bellman equation, the first-order conditions are:

$$\frac{\partial V(\mathbf{z})}{\partial \mathbf{y}} : \mathbf{u}_1 + \lambda^T \mathbf{h}_1 = \mathbf{0}, \tag{A.19}$$

along with Eq. (A.16), while the derivative with respect to \mathbf{z}_t (which will be useful later) is:

$$\frac{\partial V(\mathbf{z})}{\partial \mathbf{z}} : \mathbf{u}_2 + \lambda^T \mathbf{h}_3 = \mathbf{0}.$$
(A.20)

From Eqs. (A.15)–(A.16) and (A.19) we can determine the steady state values for \bar{z} , \bar{y} , and $\bar{\lambda}$, which need not be efficient.

The next step is to form a second-order approximation to the Bellman equation around the steady state. Using *s.o.t* to denote second order terms the approximation gives:

$$\widehat{\mathbf{z}}_{t}^{T}\mathbf{V}\widehat{\mathbf{z}}_{t} + \nu = \max\left[f.o.t. + s.o.t + \lambda_{t}^{T}\left(\overline{\mathbf{h}}_{1}\widehat{\mathbf{y}}_{t} + \overline{\mathbf{h}}_{3}\widehat{\mathbf{z}}_{t}\right) + \beta E_{t}\left[\widehat{\mathbf{z}}_{t+1}^{T}\mathbf{V}\widehat{\mathbf{z}}_{t+1} + \nu\right]\right],\tag{A.21}$$

where the linear terms, *f.o.t.*, are given by:

$$f.o.t = \left(\overline{\mathbf{u}}_1 + \overline{\lambda}^T \overline{\mathbf{h}}_1\right) \widehat{\mathbf{y}}_t + \left(\overline{\mathbf{u}}_2 + \overline{\lambda}^T \overline{\mathbf{h}}_3\right) \widehat{\mathbf{z}}_t.$$
(A.22)

Looking at the expression for f.o.t., Eqs. (A.19) and (A.20) imply that the first order terms equal zero, that the welfare approximation is a quadratic, and that the approximated Bellman equation is LQ.

A3. Endogenous state variables and distorted steady state

Here we consider models for which the steady state is not efficient and in which there are endogenous state variables. Models two, three, and four and five from the main text fall into this category. For this category of models we illustrate the difficulties associated with trying to form a valid LQ approximation to the problem and show why such an approximation is not possible.

Let the model be described by the system:

$$\mathbf{z}_{t+1} - \mathbf{\Gamma} \mathbf{z}_t - \varepsilon_{t+1} = \mathbf{0},$$

(A.23)

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$$E_t[\mathbf{h}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}, \mathbf{z}_t)] = \mathbf{0}, \tag{A.24}$$

with the representative household's expected discounted lifetime utility given by:

$$U_t = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) \right].$$
(A.25)

The problem for the decision-maker in the decentralized model is to choose $\{\mathbf{y}_t, \lambda_t\}$ to solve the Bellman equation:

$$V(\mathbf{y}_{t-1}, \mathbf{z}_t) = \max_{\{\mathbf{y}_t, \lambda_t\}} \left[u(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{z}_t) + \lambda_t^T [E_t[\mathbf{h}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{g}(\mathbf{y}_t, \mathbf{z}_{t+1}), \mathbf{z}_t)]] + \beta E_t[V(\mathbf{y}_t, \mathbf{z}_{t+1})] \right],$$
(A.26)

where $\mathbf{y}_t = \mathbf{g}(\mathbf{y}_{t-1}, \mathbf{z}_t)$ is the unknown equilibrium law-of-motion relating \mathbf{y}_t to the state variables. From the Bellman equation, the first-order conditions and the envelope conditions are:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T (\mathbf{h}_3 \mathbf{g}_1 + \mathbf{h}_2) + \beta E_t [\mathbf{V}_1], \tag{A.27}$$

$$\mathbf{V}_1 = \mathbf{u}_1 + \lambda^T \mathbf{h}_1, \tag{A.28}$$

$$\mathbf{V}_2 = \mathbf{u}_3 + \lambda^T \mathbf{h}_4,\tag{A.29}$$

along with Eq. (A.24). Equations (A.27) and (A.28) can be combined to give:

$$\mathbf{0} = \mathbf{u}_2 + \lambda^T (\mathbf{h}_3 \mathbf{g}_1 + \mathbf{h}_2) + \beta E_t \left[\mathbf{u}_1' + \left(\lambda' \right)^T \mathbf{h}_1', \right].$$
(A.30)

Ordinarily, Eqs. (A.23), (A.24) and (A.30) would be solved to obtain the steady state values for $\overline{z}, \overline{y}$, and $\overline{\lambda}$. However, in this case, this is not possible because Eq. (A.30) is a generalized Euler equation that contains the unknown derivative, g_1 .

Suppose that the steady state values \bar{z},\bar{y} , and $\bar{\lambda}$ were somehow known. To produce a valid LQ approximation of the Bellman equation we need to form a second-order Taylor approximation to both period-*t* utility, $u(y_{t-1}, y_t, z_t)$, and Eq. (A.24). Approximating the former is not difficult. However, Eq. (A.24) contains **g** so a second-order approximation to Eq. (A.24) requires taking first- and second-derivatives of **g**, but **g** is unknown. Even if one were to guess values for these derivatives in an attempt to implement at an iterative solution (successive approximation) the approach would fail because the linear solution obtained would not allow a correct update of **g**'s second derivatives.

Appendix B. The projection algorithm

The projection method used to solve each model is relatively simple. All of the "heavy lifting" involving Chebyshev polynomials and their derivatives, computing weights and evaluating the polynomials themselves is done by the Julia package ChebyshevApprox.jl, which I have released under the MIT license and is available for public use provided the terms of the MIT license are met. The details of the Chebyshev polynomials, number of approximating points, domains for each state variable, and the order of the polynomials is described in the text and reported in Table 1. The weights in the Chebyshev polynomials are computed using Chebyshev regression. At each point on the approximating grid (tensor-product grid) we perform Gauss-Hermite quadrature to compute conditional expectations and solve for the fix-point of the model's equilibrium conditions using a trust-region method (the Julia package NLsolve.jl was used).

Let *S* represent the number of functions that are being approximated. In pseudo-code, the structure of the solution algorithm is as follows:

- 1. Set the loop-counter to zero, i = 0, set the convergence tolerance, *tol*, and initialize arrays for each function to be approximated, \mathbf{s}_{i}^{0} , j = 1, .., S.
- 2. Use Chebyshev regression to compute from s_j^i the corresponding Chebyshev weights, ω_j , j = 1, ..., S, for each approximating polynomial and update the loop counter, i = i + 1.
- 3. At each node in the approximating grid:
 - (a) Use Gauss-Hermite quadrature to compute conditional expectations.
 - (b) Solve for the fix-point of the model's system of nonlinear equilibrium conditions, storing the solutions in the arrays s_i^i , j = 1, ..., S.
- 4. While $\|\mathbf{s}_{j}^{i} \mathbf{s}_{j}^{i-1}\| > tol$, for some j = 1, ..., S. return to step 2. 5. Exit.

For the models solved in section four, the number of functions that need to be approximated was small-usually just two (S = 2).

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