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Reduced-complexity interpolating control with periodic invariant sets

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ABSTRACT
This paper deals with the constrained control of linear systems and presents a low-complexity interpolating control scheme based on the concept of *periodic invariance*. Periodic invariance avoids the expensive computation of the control invariant set and allows the state trajectory to leave the set temporarily but return into the set in a finite number of steps, i.e. period of the invariant set, all by satisfying the constraints. Furthermore, periodic invariant sets provide a reduction of complexity representation by replacing the classical control invariant set. In practice, a reachability problem is solved off-line for each vertex of the set and provides a contractive control sequence that steers the system state back into the original set (*vertex reachability of target sets*). Online, the periodic interpolating control (pIC) scheme allows to transition between such periodic invariant sets and a inner set endorsed with positive invariance properties. Proofs of recursive feasibility and asymptotic stability of the pIC are given. A numerical example demonstrates that pIC provides similar performance compared to more expensive optimization-based schemes.

KEYWORDS
interpolating control; invariant sets; periodic invariance; constrained systems

1. Introduction

Interpolating control (IC) is a design principle which merges the output of two or more feedback functions in order to obtain the control input for a dynamical systems. It was shown to be effective in particular for handling state and control constraints (Nguyen, 2014; Nguyen, Gutman, Olaru, & Hovd, 2013). Interpolation-based technique can be seen as an alternative to model predictive control Rossiter and Ding (2010); Rubin, Mercader, Gutman, Nguyen, and Bemporad (2020). IC provides an enlarged region of attraction and a better (smoothness, complexity) exploitation of the control degrees of freedom within the admissible regions of the state space.

One of the first formulation which can be considered within the IC class of control design methodologies, relies on the vertex control proposed in the early work (Gutman & Cwikel, 1986) for linear time-invariant discrete-time systems with polytopic state and control constraints, and later extended to uncertain plants in Blanchini (1995). Vertex control is based on the existence of an admissible controllable set and considers a decomposition of the invariant set as a collection of simplices obtained from the solution of a convex decomposition which takes the form of a multi-parametric programming problem in the state-feedback control context. At each vertex, an admissible control action pushes the state away from the boundary of the set in a contractive way. A limitation of the vertex approach lies in the control action that only exploits the border of the invariant set and would lead to a slower convergence compared to a time-optimal control

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action (Nguyen et al., 2013). To overcome this limitation, a switching control scheme between a vertex controller and a high gain stabilizing feedback controller when the state approaches the origin can improve the rate of convergence. However, it comes with the drawback of a non-smooth behaviour (Nguyen, Olaru, & Hovd, 2012). Furthermore, vertex control requires a vertex representation of the invariant set associated to the constrained system that could lead to expensive computations associated to the high number of generators that describe the polytope.

IC in its modern formulations applies a control action with smooth transition from a low-gain/vertex control and a high-gain feedback control, that is, an interpolation between the two control actions. This provides an enlarged stabilizing set and faster convergence to the origin (Nguyen et al., 2013). The explicit version of the resulting controller has been characterized together with the geometrical properties and calls for the solution of a multi-parametric optimization problem (Nguyen, Olaru, & Hovd, 2016). Furthermore, it has been extended to several interpolation factors and robustness (Nguyen, Olaru, Gutman, & Hovd, 2014). A similar design philosophy was adopted in works dedicated to control sharing and merging (Grammatico, Blanchini, & Caiti, 2013) as well as in extensions to different classes of control Lyapunov functions (Kheawhom & Bumroongsri, 2015; Nguyen, Lazar, & Spinu, 2014). Recently, applications have been reported in automotive industry (Ballesteros-Tolosana, Olaru, Rodriguez-Ayerbe, Pita-Gil, & Deborne, 2016), transportation (Tuchner & Haddad, 2017), and interconnected systems (Scialanga & Ampountolas, 2018) .

Past work on interpolation-based control schemes relies on the availability of the invariant sets that can be computationally prohibitive, and whose complexity is in direct relationship with the computational complexity of the vertex controller. In general, the approximation of the maximal controllable set is a tedious task from the computation to the representation point of view.

The present work introduces a novel interpolating control scheme based on periodic invariance that provides a relaxation of the existing strict set invariance design. The role of periodic invariant sets is to reduce the complexity of the representation of traditional invariant sets and avoid their expensive computation.

Periodic invariant sets have been used in previous works in order to enlarge the stabilizable region and allow the state to leave the set and return after a finite number of steps (Lee, 2004; Lee & Kouvaritakis, 2006). With a similar philosophy, the proposed periodic interpolating control (pIC) considers an initial region that guarantees constraints satisfaction while a sequence of a priori computed control actions keeps the system trajectories within the admissible controllable region.

Periodic invariance, as main contribution to the IC design, avoids the expensive computation of the maximal control invariant set. It will be shown that a reachability procedure can provide a control sequence that allows the state to leave the periodic set for the constrained discrete-time system for a finite number of time steps. Reachability of state-space regions has been largely investigated in the past (Bertsekas, 1972; Bertsekas & Rhodes, 1971) and recently in computational geometry (Kerrigan, 2000; Kerrigan, Lygeros, & Maciejowski, 2002; Raković, Kerrigan, Mayne, & Lygeros, 2006). In the present work, an *offline* reachability problem is solved based on optimization and provides a sequence of control actions for each vertex of the set that will steer the state of the system back into the target set after a finite number of time-steps in a contractive way. As a direct consequence, the computational complexity of pIC is concentrated in the off-line characterization of the reachable sets. The interpolation procedure is reduced to an inexpensive linear programming (LP) problem to be solved at the beginning of each periodic cycle. This work is completed with remarks on the evolution of the state with periodic interpolating control that leads to an improved formulation of the proposed method. Proofs of recursive feasibility and stability of the proposed pIC scheme and its improved version are given.

The rest of the paper is organised as follows. Section 2 formulates the mathematical framework for the problem under study, outlines some required definitions from the invariant set theory and summarises the main points behind the previous introduced interpolating control. Section 3 presents the main results of this work including the proposed pIC scheme with constrained vertex reachability of target sets, its improved version, and required proofs of recursive feasibility and stability. Section 4 demonstrates the efficiency of pIC via a numerical example. Section 5 concludes the paper.

2. Preliminaries

2.1. Problem formulation and Set Invariance

2.1.1. System Dynamics

Consider the discrete-time linear system with external control inputs,

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and where $u \in \mathbb{R}^m$ is the control vector. The state and control vectors of (1) are subject to polyhedral constraints:

$$\begin{cases} x(k) \in \mathcal{X},\, \mathcal{X} = \{x \in \mathbb{R}^n :\ F_x x \leq g_x\}, \\ u(k) \in \mathcal{U},\, \mathcal{U} = \{u \in \mathbb{R}^m :\ F_u u \leq g_u\}, \end{cases} \tag{2}$$

$\forall\, k \geq 0$, where $\mathcal{X}$ and $\mathcal{U}$ are written in their half-space representation with F_x, F_u constant matrices and g_x, g_u constant vectors of appropriate dimension and with positive elements. The sets $\mathcal{X}$ and $\mathcal{U}$ are convex and compact and contain the origin as an interior point. The inequalities are considered component-wise.

The goal in this work is to devise an interpolating control scheme that regulates (1) to the origin subject to the state and control constraints (2). Assume that the pair (A, B) in (1) is controllable and consequently a state-feedback controller $u(k) = Kx(k)$, with $K \in \mathbb{R}^{m\times n}$ as a gain matrix, can be designed for unconstrained stabilization. The state-feedback controller can be enhanced for unconstrained stabilization with some user-desired performance specifications. Then it can be used to determine an invariant and constrained-admissible set with respect to the unconstrained closed-loop dynamics. The next section introduces formal definition of traditional (one-step) invariant sets and subsequently presents as novelty, the concept of periodic invariance that will be used to enlarge the stabilizing region of the constrained system.

2.1.2. Invariant sets

Invariant sets theory characterises the behaviour of dynamical systems and offers the tools for the analysis of the set of initial states that guarantees no violation of system constraints along the time-evolution. This section briefly reviews the set invariance theory with basic definitions of set invariance and periodic invariance (Blanchini & Miani, 2015; Borrelli, Bemporad, & Morari, 2017; Lee & Kouvaritakis, 2006; Olaru, Stanković, Bitsoris, & Niculescu, 2014) that will be used in the rest of the paper.

Definition 2.1 (Constraint-admissible Invariant Set)**.** The set $\Omega \subseteq \mathcal{X}$ is a positively constraint-admissible invariant set with respect to $x(k+1) = A_c x(k)$, where $A_c = A + BK$ is a closed-loop state matrix related to (1), subject to the constraints (2), if $\forall\, x(k) \in \Omega$, the system evolution satisfies $x(k+1) \in \Omega$ and $Kx(k) \in \mathcal{U}$, $\forall\, k \geq 0$.

The largest positively invariant set for the system (1) in closed-loop with a static feedback control $u(k) = Kx(k)$ that respects constraints (2) is called *maximal admissible set* (MAS) (Gilbert & Tan, 1991). Under stability and mild structural assumptions on the topology of the constraints (2) (Blanchini & Miani, 2015; Gutman & Cwikel, 1986; Kerrigan, 2000), MAS exists, it is finitely determined and can be defined in polyhedral form as $\Omega = \{x \in \mathbb{R}^n : F_\Omega x \leq g_\Omega\}$, where F_Ω is a constant matrix and g_Ω is a constant vector of appropriate dimensions.

Definition 2.2 (Control Invariant Set)**.** Given the system (1) and the constraints (2), the set $\Psi \subseteq \mathcal{X}$ is control invariant, if $\forall x(k) \in \Psi$, there exists an admissible control sequence $u(k) \in \mathcal{U}$ such that $x(k+1) \in \Psi$, $\forall\, k \geq 0$.

The maximal control invariant set might not be finitely determined within the class of polyhedral sets (Borrelli et al., 2017). However, in the sequel, a polyhedral approximation will be considered with the half-space representation given by $\Psi = \{x \in \mathbb{R}^n : F_\Psi x \leq g_\Psi\}$ where F_Ψ is a constant matrix and g_Ψ is a constant vector of appropriate dimensions.

For any scaling factor $\lambda > 0$, λS is understood as $\lambda S := \{\lambda x \,|\, x \in S\}$ for any set $S \subset \mathbb{R}^n$. Set invariance is a limit case of λ-contractiveness as indicated by the next definition.

Definition 2.3 (Control λ-contractive Set)**.** Given a scalar $\lambda \in (0, 1]$, a set $\Psi \subseteq \mathcal{X}$ containing the origin is called control λ-contractive for (1) with respect to (2), if for any $x(k) \in \Psi$ there exists $u \in \mathcal{U}$ such that $x(k+1) \in \lambda\Psi$, for all $k > 0$.

Definition 2.4 (Periodic invariance (Lee & Kouvaritakis, 2006))**.** For a given $\lambda \in (0, 1]$ the set $\mathcal{S} \subset \mathbb{R}^n$ containing the origin is called *periodic λ-contractive* for $x(k+1) = A_c x(k)$, where $A_c = A + BK$ is a closed-loop state matrix related to (1), if there exists a positive number $p \in \mathbb{Z}_+$ such that for any $x(k) \in \mathcal{S}$ it holds that $x(k+p) \in \lambda\mathcal{S}$. If $\lambda = 1$ the set is called *periodic invariant.*

This notion should be understood as a 'strong periodic invariance' (Soyer, Olaru, & Fang, 2020) and to be distinguished from the 'weak' version which impose the satisfaction of constraints $x(k+i) \in \lambda\mathcal{S}$ for an index $0 < i \leq p$.

Definition 2.5 (Control Periodic Invariant Set (Lee & Kouvaritakis, 2006))**.** For a given $\lambda \in \mathbb{R}_{(0,1]}$ the set $S \subset \mathbb{R}^n$ containing the origin is called *control periodic λ-contractive* with respect to the system (1) and constraints (2) if there exists a positive number $p \in \mathbb{Z}_+$ such that for any $x(k) \in S$ there exists an admissible control sequence $u(k+i) \in \mathcal{U}$, $i = 0, \ldots, p-1$, such that $x(k+p) \in \lambda S$ holds. If $\lambda = 1$ the set is called *control periodic invariant.*

Periodic invariant sets remove the consideration of strict invariance in the definition of a positively invariant set, i.e. allow the state vector to leave the invariant set temporarily but return into the set in a finite number of time steps. Intuitively, controllers designed with the use of periodic invariant sets can provide flexibility since periodic invariant sets are formally a strictly larger class than the one of ordinary positively invariant sets. Periodic invariant sets associated with a family of different state feedback control gains can be also used to better explore the signal space and extend the domain of attraction (see Section 3.2). The interest in periodic invariance stems from real applications where violation of constraints is allowed for certain time periods, as well as their applications to periodic, distributed, and time-delayed systems.

The next section presents the interpolating control technique introduced in (Nguyen, 2014), that is adopted later in this work and paired with the periodic invariance theory.

3. Periodic interpolating control

In this section we introduce the interpolating control and its periodic formulation. Interpolating Control (IC) is a recent control approach for linear (time-varying) systems that incorporates the state and control constraints into the problem formulation. It relies on the (smooth) interpolation between a vertex controller (defined on the outer set) and a high-gain feedback controller (defined within the inner set). The latter can be obtained as solution of a linear-quadratic control problem, where the performance is measured by a quadratic cost with a relatively large weighting on the state error in comparison with the weight on the norm of the control signal. The only assumption necessary here is that its associated maximal admissible set (see Definition 2.1) is contained in the outer set. Standard interpolating control can face implementation limi-

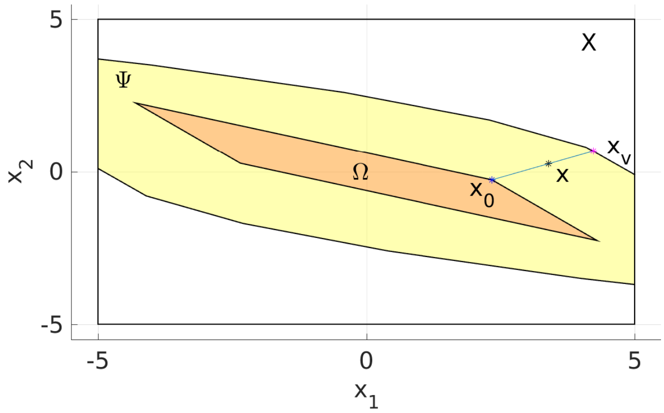


Figure 1. Interpolating control concept. The white region indicates the set $\mathcal{X}$ of state constraints; The light yellow region indicates the outer set Ψ (maximal control invariant set); The orange region depicts the inner set Ω (MAS associated with feedback controller $u = Kx$). The state x is decomposed as convex combination of the outer state $x_v \in \Psi$ and the inner state $x_0 \in \Omega$.

tations due to the high computational time of the algorithms that compute invariant sets, and the complexity of vertex control that is proportional to the number of vertices in this set representation. Periodic invariant sets introduces a reduced-complexity framework for interpolating control and provides a control design that stabilizes the constrained system (1)-(2) while avoids the expensive computations of controllable sets. In the present work, the use of a set with relative simple representation is advocated. Such a candidate set will be defined in the controllable region of the system (admissible with respect to the state and control constraints).[1]

3.1. Interpolating control with vertex representation

Vertex control is a control technique proposed in Gutman and Cwikel (1986) for constrained linear time-invariant systems and extended to linear time-varying systems in Blanchini (1994). It relies on the availability of control invariant sets (see Section 2.1.2). That is, there exists an admissible control action applied to each vertex of the invariant polytope $\mathcal{P}$ that pushes the state away from the boundary of the set and brings the state to the interior of the set in finite time. Then, a stabilizing control action for every point within the invariant set is a convex combination of the vertex control inputs. Interpolating control (IC) (Nguyen, 2014; Nguyen et al., 2013) was introduced as a more effective alternative to vertex control. Figure 1 shows the idea behind interpolating control using two controlled invariant sets. The set Ψ, depicted in light yellow colour, is the outer controlled invariant set and the MAS Ω, depicted in orange colour, is the inner controlled invariant set. The convex (polyhedral) outer and inner sets are to be understood by the relationship $\Omega \subseteq \Psi \subseteq \mathcal{X}$. Any $x(k) \in \Psi$ can be decomposed as follows,

$$x(k) = s(k)\, x_v(k) + (1 - s(k))\; x_0(k), \tag{3}$$

where $x_v(k) \in \Psi$ and $x_0(k) \in \Omega$, and $s(k) \in [0, 1]$ is the interpolating coefficient. Similarly, a control action is computed at each time step as convex combination of an inner stabilizing controller $u_0(k) = K\, x_0(k)$ associated with the MAS and $u_v(k)$, the vertex control applied to $x_v(k)$:

$$u(k) = s(k)u_v(k) + (1 - s(k))u_0(k). \tag{4}$$

In order to steer the state $x(k)$ as close as possible to the positively constraint-admissible invariant set Ω, one would like to minimise the interpolating coefficient s. To obtain it, consider

[1]The proposed approach is generic and any results presented here in $\mathbb{R}^2$ are applicable in a straightforward way to $\mathbb{R}^n$.

the change of variables $r_0 = (1-s)x_0$ and $r_v = sx_v$, where r_0, r_v are vectors of appropriate dimensions. It follows that $r_0 \in (1-s)\,\Omega$ and $r_v \in s\,\Psi$. The state decomposition (3) can be rewritten as $r_0 = x - r_v$ and the following LP problem is formulated (index k is omitted for clarity):

$$
\begin{aligned}
&s^*(x) = \arg\min_{s, r_v} \; s \\
&\text{subject to:} \\
&\begin{cases}
F_\Psi r_v \le s\, g_\Psi, \\
F_\Omega (x - r_v) \le (1-s) g_\Omega, \\
0 \le s \le 1.
\end{cases}
\end{aligned}
\tag{5}
$$

The solution of the LP problem is the interpolating coefficient s^* and the variable previously defined r_v^*. The original state variables can be recovered from $r_0^* = x - r_v^*$ with change of variables introduced previously. The LP problem (5) produces an admissible control action (4) that provides a smooth transition between the two controllers and a fast convergence to the origin of the state space. Once the state enters the MAS – Ω, the interpolation control is equivalent to the stabilizing high-gain feedback controller $u_0(k) = K\, x_0(k)$.

3.2. Periodic Invariance and State Feedback control

Consider a simple set $\mathcal{B}$ in the controllable region that includes the MAS Ω. Obviously, the outer set defined in this way cannot guarantee invariance in the closed-loop evolution of the state based on interpolation control, since it is not an invariant set itself, though it is contained in the maximal controllable set. In order to guarantee that the state will evolve toward the origin, we consider a periodic invariant set and a sequence of controls such that the for every state $x \in \mathcal{B}$, x will return into the initial set $\mathcal{B}$ in a finite number of steps.

Consider the constrained system (1)-(2) associated with a low-gain state-feedback controller $u(k) = K_1 x(k)$, where $K_1 \in \mathbb{R}^{m \times n}$ is a gain matrix, which asymptotically stabilizes the system. Take, for simplicity of the exposition, a box $\mathcal{B}$ with edges parallel to the axes such that $\Omega \subseteq \mathcal{B} \subseteq \mathcal{X}$ and the closed-loop system,

$$
x(k+1) = (A + BK_1)\, x(k). \tag{6}
$$

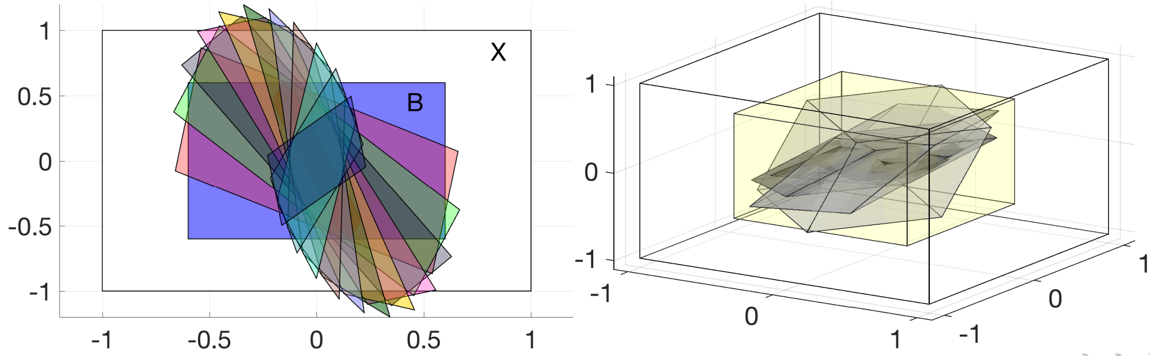
Assume that $\mathcal{B}$ is a periodic invariant set according to Definition 2.4 with a periodicity index p. Set $\mathcal{B}$ to be the starting and target set of a periodic sequence with period $p \in \mathbb{Z}_+$. If $\mathcal{B}$ is periodic invariant with respect to the closed-loop system (6), a sequence of p sets S_j, $j = 1, \ldots, p$ can be computed as reachable sets starting from $S_0 = \mathcal{B}$; So that for any initial state $x(\bar{k}) \in S_0$, the future states $x(\bar{k} + k) \in S_k$, $k = 1, \ldots, p$, with $S_p = \lambda \mathcal{B}$, $0 < \lambda \le 1$, where $\bar{k}$ is the first time step of the periodic sequence.

The sequence of periodic invariant sets can be, therefore, computed iteratively as the reachable set of the system, (6), with $S_j = \mathrm{Reach}(S_{j-1})$ based on:

$$
\mathrm{Reach}(S_{j-1}) = \{x \in \mathbb{R}^n \colon \; \exists x^- \in S_{j-1} \text{ s.t. } x = (A + B\,K_1)\, x^-\}, \tag{7}
$$

$\forall j = 1, \ldots, p$, where x^- denotes a state in the $(j-1)$-set of the periodic sequence. Note that the period p is under determination and its construction is related to the shape of the set $\mathcal{B}$ and the dynamics of the system. Additionally, $\mathcal{B}$ must be a subset of the maximal controllable (admissible) set. A practical way to construct this index is provided in Section 3.3.1.

Figure 2(a) and Figure 2(b) illustrate the concept with an initial set $\mathcal{B}$ in $\mathbb{R}^2$ and $\mathcal{B}$ in $\mathbb{R}^3$, respectively, that verifies the state constraints (in white) and the sequence of sets that starts from $\mathcal{B}$ and re-enters the target set in $p = 11$ and $p = 5$ steps, respectively. The sequence of sets is not necessary to be stored and it is plotted to show the periodic invariance idea and how the



(a) (b)

Figure 2. Periodic invariant sets for the closed-loop equation $x(k+1) = (A + BK_1)x$ with low-gain state feedback control $u(k) = K_1x(k)$. The set $\mathcal{B}$ is the starting and ending set. (a) The evolution of the states in $\mathcal{B}$ (in violet color) in the two-dimensional state-space. Any initial state $x \in \mathcal{B}$ returns into the box after $p = 11$ steps. (b) The evolution of the states in $\mathcal{B}$ (in yellow color) in the three-dimensional state-space. Any initial state $x \in \mathcal{B}$ returns into the box after $p = 5$ steps.

period length is determined. Periodic invariant sets computed as in (7) do not verify the state constraints necessarily. Periodic invariance allows for the state vector to leave the invariant set temporarily but return into the set in a finite number of time steps, i.e., to leave the set for $k < p$, where p is the length of the period, and converge to $\mathcal{B}$ at $k = p$.

The ultimate goal in the next section is to devise an interpolating control scheme that will blend a constrained controller with the periodic invariant set. This set will play the role of target set in a reachability problem. The initial state $x(0)$ will belong inside the periodic invariant set, while an admissible control sequence will steer back the state into the original target set in a finite number of steps.

3.3. Periodic interpolating control with vertex reachability of target sets

This section presents a novel scheme of interpolating control with periodic invariance and constrained vertex reachability. Reachability of state-space regions or target sets for constrained discrete-time systems have been largely investigated in the past decades and have produced new interest in the control community thanks to improvements in computational geometry (Bertsekas, 1972; Bertsekas & Rhodes, 1971; Kerrigan, 2000; Kerrigan et al., 2002; Raković et al., 2006).

The main idea is to use an easy representation of the outer control invariant set (e.g., a rectangle or hexagon or octagon in $\mathbb{R}^2$ case and their corresponding complexity counterpart in higher dimensions), and then to solve a reachability problem for each vertex of the outer set for the constrained discrete-time system. Since for a particular outer set the vertices are known beforehand, the constrained reachability problem determines for each vertex of the outer set a sequence of admissible controls that steer the state of the system back into the original target set after a finite number of time steps. Note that the solvability of the reachability problem is guaranteed by the inclusion of the outer control invariant set in the maximal control invariant set. A formal proof is provided in Section 3.5 with Theorem 3.1. For the interpolation, an inexpensive LP problem is solved at the beginning of each periodic cycle.

The proposed periodic interpolating control scheme with constrained vertex reachability of a target set involves *off-line* and *on-line* procedures. The *off-line* procedure involves for each vertex of the outer set (with known shape) the solution of a constrained reachability problem. The result is a sequence of admissible controls that steer the state of the system back into a target set after a finite number of time steps. The obtained sequence of admissible controls is then stored in high-speed access storage to be available during the on-line procedure. The *on-line* procedure involves the interpolation between the MAS Ω and the simple outer set via the solution of an inexpensive LP problem at the beginning of each periodic cycle. The control action consists in a convex combination of the inner stabilizing control associated to the positive admissible set and the sequence of controllers obtained from the off-line computations.

3.3.1. Off-line: p-step vertex reachability problem for constrained discrete-time systems

Consider the linear time invariant system (1) subject to state and control constraints (2). Assume that a state feedback controller $u(k) = Kx(k)$ exists, which satisfies some user-desired performance specifications and compute the maximal admissible set Ω associated to it. Ω plays the role of inner controlled invariant set in the proposed interpolating control scheme. Assume an outer candidate set $\mathcal{B} \subseteq X$ with parallel (symmetric) faces and n_v vertices (e.g. for a rectangle $n_v = 4$, hexagon $n_v = 6$, octagon $n_v = 8$, ... in $\mathbb{R}^2$). Let v_i, $i = 1, \ldots, n_v$ be the vertices of the relevant outer set. The objective of the reachability problem is to compute a sequence of admissible controls u_{v_i} for each vertex v_i that steers the state of the system back into the target set $\mathcal{B}$ after a finite number of p_i steps. The constrained reachability problem allow us to satisfy the constraints, since the outer set representation (e.g. rectangle) is not an invariant set. In other words, it is not guaranteed that the state will remain inside the outer set at each time step without solving the constrained reachability problem. However, from its periodic invariance property, it will return into the target set at the end of the periodic sequence.

First, the common period length for the constrained system (1)–(2) and set $\mathcal{B}$ can be then defined as the index $p \in \mathbb{N}$ guaranteeing the feasibility of the following set of linear inequalities (constrained reachability problem):

$$\begin{cases} Av_i + Bu_{v_i}(0) \in \mathcal{X}, \forall i = 1, \ldots, n_v \\ A^2 v_i + ABu_{v_i}(0) + Bu_{v_i}(1) \in \mathcal{X}, \forall i = 1, \ldots, n_v \\ \vdots \\ A^{p-1} v_i + A^{p-2} Bu_{v_i}(0) + \cdots + Bu_{v_i}(p-2) \in \mathcal{X}, \forall i = 1, \ldots, n_v \\ A^p v_i + A^{p-1} Bu_{v_i}(0) + \cdots + Bu_{v_i}(p-1) \in \mathcal{B}, \forall i = 1, \ldots, n_v \\ u_{v_i}(k) \in \mathcal{U}, \quad \forall k = 0, \ldots, p-1, \forall i = 1, \ldots, n_v \end{cases} \tag{8}$$

Where $u_{v_i} = \{u_{v_i}(0), \ldots, u_{v_i}(p-1)\}$ denotes, with a slight abuse of notation, the p admissible control sequence for each vertex v_i, $i = 1, \ldots, n_v$. A common period length for the constrained system (1)–(2) and set $\mathcal{B}$ can be then defined as the least common multiple between all p_i, $i = 1, \ldots, 2n$, i.e., $p = \text{l.c.m. } p_i$, $i = 1, \ldots, 2n$.

Remark 1. *The user-defined outer set $\mathcal{B}$ should be inside the control invariant set $\Psi \subseteq \mathcal{X}$ (which is assumed unknown), otherwise the reachability problem is not feasible for a finite $p \in \mathbb{N}$. Note however that under the assumption that the origin is strictly inside the admissible set, the feasibility can be retrieved by scaling for any set $\mathcal{B}$, i.e. there exist $0 \leq \alpha \leq 1$ such that $\alpha\mathcal{B} \subseteq \Psi \subseteq \mathcal{X}$.*

Once the periodicity index p is available, the 'optimal' controls u_{v_i} are obtained by the optimal argument for the following problem for each vertex v_i:

$$\lambda\left(u_{v_i}(0),\ldots,u_{v_i}(p-1)\right) = \arg \min_{u_{v_i}(0),\ldots,u_{v_i}(p-1),\lambda_i} \lambda_i$$
$$\text{subject to:}$$
$$\begin{cases} Av_i + Bu_{v_i}(0) \in \mathcal{X}, \\ A^2 v_i + ABu_{v_i}(0) + Bu_{v_i}(1) \in \mathcal{X}, \\ \vdots \\ A^{p-1} v_i + A^{p-2}Bu_{v_i}(0) + \cdots + Bu_{v_i}(p-2) \in \mathcal{X} \\ A^p v_i + A^{p-1}Bu_{v_i}(0) + \cdots + Bu_{v_i}(p-1) \in \lambda_i \mathcal{B} \\ u_{v_i}(k) \in \mathcal{U}, \quad k = 0,\ldots,p-1, \\ 0 \le \lambda_i < 1. \end{cases} \tag{9}$$

Any point x_v in the boundary of the outer set $\mathcal{B}$ can be written as a convex combination of its vertices v_i. Then, there exists a sequence of p admissible control actions that steer the state of the system back into the target set in p steps. Note that the sequence of admissible controls $\{u_{v_i}(0),\ldots,u_{v_i}(p_i-1)\}$, $i = 1,\ldots,n_v$ is stored to be accessed later on to steer the initial state $x \in \mathcal{B}$ back into the target set $\mathcal{B}$ via periodic interpolation.

3.3.2. On-line: Periodic interpolating control

3.3.2.1. Periodic interpolating control with constant interpolating coefficient. Consider an arbitrary initial state $x(0)$ inside the outer set $\mathcal{B}$ (and target set of periodic control). A scaling factor $\mu_1 \in [0,1]$ can be computed such that the initial state is contained on the frontier of the scaled set $\mu_1 \mathcal{B}$. μ_1 is the minimal factor such that $x(0) \in \mu_1\mathcal{B}$, and it can be obtained by solving the following LP problem:

$$\mu_1 = \arg\min_{\mu} \mu$$
$$\text{subject to:}$$
$$\begin{cases} F_{\mathcal{B}}\, x \le \mu\, g_{\mathcal{B}}, \\ 0 \le \mu \le 1, \end{cases} \tag{10}$$

where $F_{\mathcal{B}}$ and $g_{\mathcal{B}}$ are the matrix and the vector that defines the half-space representation of $\mathcal{B}$. Then, $\mu_1\mathcal{B}$ can be set as the target set for our periodic control sequence[2] ($\mathcal{B} \longleftarrow \mu_1\mathcal{B}$).

The state $x(0)$ can be decomposed as $x(0) = s(0)\, x_v(0) + (1-s(0))\, x_0(0)$ by solving the LP problem (5) with $\Psi \longleftarrow \mathcal{B}$. The states x_v and x_0 lie on the border of $\mathcal{B}$ and Ω, respectively. Then, $x_v(0)$ can be written as a convex combination of the vertices of the outer set $\mathcal{B}$, i.e.,

$$x_v(0) = \sum_{i=1}^{n_v} \alpha_i(0)\, v_i, \quad \alpha_i \ge 0, \quad \sum_{i=1}^{n_v} \alpha_i = 1, \tag{11}$$

where α_i, $i = 1,\ldots,n_v$ are convexity coefficients in the unit simplex. The control action at $k = 0$ is a convex combination of the state feedback control applied to the state $x_0(0)$ and the combination of the controls applied to the vertices v_i, as in the decomposition (11), i.e.,

$$u(0) = s(0) \sum_{i=1}^{n_v} \alpha_i(0)\, u_{v_i}(0) + (1-s(0))Kx_0(0),$$

[2]Note that periodic invariance properties are preserved by scaling

where $u_{v_i}(0)$ is the first element of the control sequence (9) applied to the vertex v_i. For the next $p-1$ steps, consider the p-sequence of interpolating controls, which are available from the solution of the problem (9), to obtain the control action,

$$u(k) = s(0) \sum_{i=1}^{n_v} \alpha_i(0)\, u_{v_i}(k) + (1 - s(0))K(A + BK)^k x_0(0), \quad k = 0, \ldots, p-1. \tag{12}$$

The control action (12) is applied to the system (1) for up to p steps or until the state reaches one of its target sets, i.e., either the scaled set $x(\bar{k}) \in \mu_1\mathcal{B}$ or the admissible set $x(\bar{k}) \in \Omega$. It guarantees that the initial state $x(0)$ enters the contractive set $\mu_1\mathcal{B}$ in p steps maximum. After the state returns into the set $\mu_1\mathcal{B}$, a new periodic sequence is computed. Note that in (12), the interpolating coefficient s and the coefficients α_i, $i = 1, \ldots, n_v$, in the convex combination (11) are kept constant, i.e. $s(k) = s(0)$ and $\alpha_i(k) = \alpha_i(0)$, $k = 1, \ldots, p$, $i = 1, \ldots, n_v$.

The scaling factor μ_1 associated to the target set $\mathcal{B}$ is updated for the new state $x(\bar{k})$ by solving the LP problem (10), where $\bar{k}$ becomes the first time step of the periodic sequence $x(0) \longleftarrow x(\bar{k})$. The current state would be inside $\mu_2\mathcal{B}$, $\mu_2 < \mu_1$, where $\mathcal{B}$ is the outer set of the periodic IC. After a new solution of the problem (9) is obtained, a new interpolating decomposition $(s(\bar{k}), x_v(\bar{k}), x_0(\bar{k}))$ is computed between the outer set $\mu_2\mathcal{B}$ and the inner set Ω with (5). The outer state is defined as convex combination of the vertices of the outer set as in (11) with coefficients $\alpha_i(\bar{k})$, $i = 1, \ldots, n_v$. Similar to the control action (12) applied to the initial set, a sequence of pIC associated to the new state is applied to the system, i.e.,

$$u(\bar{k} + k) = s(\bar{k}) \sum_{i=1}^{n_v} \alpha_i(\bar{k})\, u_{v_i}(\bar{k} + k) + (1 - s(\bar{k}))K(A + BK)^k x_0(\bar{k}),\ k = 0, \ldots, p-1, \tag{13}$$

where $s(\bar{k})$ is the new interpolating coefficient to be kept constant throughout the new periodic sequence.

Some remarks on the evolution of the state with periodic interpolating control Interpolating control approaches introduced in literature decompose the current state as $x(k) = s(k)\, x_v(k) + (1 - s(k))\, x_0(k)$, i.e., as the convex combination of two states x_0 and x_v. The decomposition is obtained as solution of the LP problem (5) and it is computed at each time step k. Then, the convex combination between two controls $u(k) = s(k)\, u_v(k) + (1 - s(k))\, u_0(k)$, with u_0 applied to the inner state x_0 and u_v applied to the outer state x_v, is computed in order to steer the state x to the origin. The solution of the LP problem provides the interpolating coefficient s and the state variables x_v and x_0 that lie in the border of $\mathcal{B}$ and Ω, respectively.

The periodic interpolating control approach presented in the previous section solves the LP problem (5) at the beginning of the periodic cycle and then applies the control (13) for the next p steps until the state enters the target set. During a periodic sequence, the proportion between the outer and inner state keeps the same, that is, the interpolating coefficient is $s(\bar{k}+k) = s(\bar{k})$, where $\bar{k}$ is the first time step of a periodic sequence. A nice feature of this approach is that, while the outer state x_v is traveling outside $\mathcal{B}$ with control action (9) and returning inside after p steps, the inner state x_0 evolves towards the origin with control action $u = Kx$. Figure 3 shows this behaviour. As can be seen, the evolution of the inner state x_0 (indicated with blue stars) can go back to the border at the end of the periodic cycle ($t = 9$) while it is already closer to the origin. This is the result of the LP problem (5) that is solved at the beginning of a new periodic sequence. On the other hand, it would be preferable to keep the inner state on the boundary of the inner MAS Ω, hence closer to the system state x, in order to provide a smooth transition between periodic sequences. This extension is presented in the next section.

3.3.2.2. Improved periodic interpolating control using vertex reachability and updated interpolating coefficient. The analysis presented in the previous section showed that

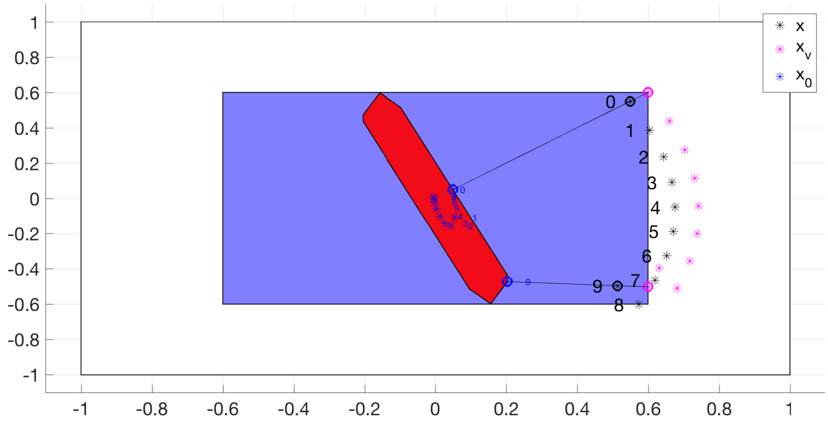


Figure 3. Decomposition of the state $x(k)$ (black) and evolution of the outer state x_v (magenta) and the inner state x_0 (blue) with the proposed pIC in Section 3.3.2.1. The states with a circle mark indicate the states at the beginning of a periodic cycle

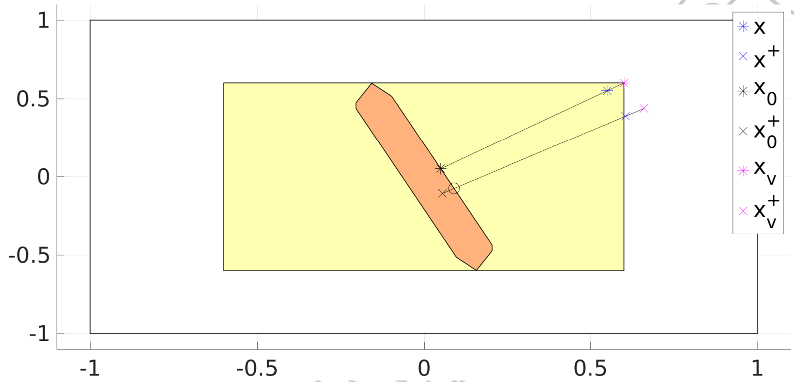


Figure 4. IpIC: The inner state x_0 and interpolating coefficient s are updated at each time step of a periodic sequence. The state at the previous time step is marked with stars and at the next time step with crosses. The updated x_0 obtained from (15) is marked with a black circle.

the outer state x_v and the inner state x_0 of the state decomposition during the periodic sequence evolve independently with outer controller u_v and local controller u_0, respectively, while the interpolating coefficient balances their control action to the current state in the sequence. This section provides an improved periodic interpolating control scheme that provides a decreasing interpolating coefficient at each time step during a periodic sequence.

The proposed improved periodic interpolating (IpIC) control works as follows. At the first time step $\bar{k}$ of the periodic sequence the control action (13) at $k = 0$ is applied to the state $x(\bar{k}+k) = x(\bar{k})$ and the next state $x(\bar{k}+1)$ at $k = 1$ is calculated. The evolution of the outer state $x_v(\bar{k})$ depends on the outer control $u_v(\bar{k}) = \sum_{i=1}^{n_v} \alpha_i(\bar{k})\, u_{v_i}(\bar{k})$, i.e., $x_v(\bar{k}+1) = A\, x_v(\bar{k}) + B\, u_v(\bar{k})$. However, the inner state does not follow the evolution associated with the system (1) with $u_0(\bar{k}) = Kx_0(\bar{k})$ as in the previous section, but it is updated at each time step $k = 1, \ldots, p-1$. To this end, the interpolating coefficient s is updated too. Figure 4 displays the improved periodic interpolating control scheme. Figure 4 shows the evolution of the initial state $x(\bar{k})$ ($\mathtt{x}$) into $x(\bar{k}+1)$ ($\mathtt{x}^+$), the outer state $x_v(\bar{k})$ ($\mathtt{x_v}$) into $x_v(\bar{k}+1)$ ($\mathtt{x_v^+}$) at the first time step of a periodic sequence. It also shows the inner state $x_0(\bar{k})$ ($\mathtt{x_0}$) and how it evolves ($\mathtt{x_0^+}$) if the inner controller associated to the state is applied. At each step $k = 1, \ldots, p-1$, the updated $x_0(\bar{k}+k)$ is computed such that it is contained in the boundary of the admissible set Ω as the solution of the following optimization problem,

$$\begin{aligned}
&\min_{s(\bar{k}+k),x_0(\bar{k}+k)} \quad s(\bar{k}+k)\\
&\text{subject to:}\\
&\begin{cases} x(\bar{k}+k) = s(\bar{k}+k)\,x_v(\bar{k}+k) + (1-s(\bar{k}+k))\,x_0(\bar{k}+k)\\ F_\Omega\, x_0(\bar{k}+k) \le g_\Omega,\\ 0 \le s(\bar{k}+k) \le 1. \end{cases}
\end{aligned} \tag{14}$$

This is a bilinear optimization problem that can be transformed into an LP problem with the change of variable $r_0(\bar{k}+k) = (1-s(\bar{k}+k))\,x_0(\bar{k}+k)$. The equality constraint can be re-written as $r_0(\bar{k}+k) = x(\bar{k}+k) - s(\bar{k}+k)\,x_v(\bar{k}+k)$. Then, the optimization problem becomes:

$$\begin{aligned}
&\min_{s(\bar{k}+k)} \quad s(\bar{k}+k)\\
&\text{subject to:}\\
&\begin{cases} (g_\Omega - F_\Omega\, x_v(\bar{k}+k))\, s(\bar{k}+k) \le g_\Omega - F_\Omega\, x(\bar{k}+k),\\ 0 \le s(\bar{k}+k) \le 1, \end{cases}
\end{aligned} \tag{15}$$

where F_Ω and g_Ω are the matrix and the vector of the half-space representation of the inner set Ω; $x(\bar{k}+k)$ and $x_v(\bar{k}+k)$ are the known updated states. The optimization problem (15) provides the updated inner state $x_0(\bar{k}+k)$ subject to a pre-imposed periodic invariance. The solution of (15) at each time step is the new interpolating coefficient $0 \le s(\bar{k}+k)$ and verifies the definition of Lyapunov function for the closed loop system , i.e., $s(\cdot) \ge 0$, $s(\bar{k}+k) < s(\bar{k}+k-1)$ (see Theorem 3.3). The inner state $x_0(\bar{k}+k)$ can be recovered from the equality constraint. Note that the updated value of the inner state x_0 is contained in the line between the states x and x_v and it is closer to the state x compared to the x_0 obtained in the pIC presented in the previous section. Concluding the updated interpolating coefficient is monotonically decreasing within each periodic sequence in the innovative IpIC presented. For a formal proof see Theorem 3.3.

3.4. Algorithm of improved periodic interpolating control

Algorithm 1 summarises the procedure of the improved pIC. It applies a control that is *closer* (as measured by the interpolation factor) to the stabilizing high-gain feedback controller $u = K\,x$ and steers the state system faster to the origin compared to the control procedure (13). The proposed IpIC computes the state decomposition (3) at the first time step $\bar{k}$ of each periodic sequence and then applies the control,

$$u(\bar{k}+k) = s(k+\bar{k}) \sum_{i=1}^{n_v} \alpha_i(\bar{k})\, u_{v_i}(\bar{k}+k) + (1 - s(k+\bar{k}))K x_0(\bar{k}+k), \tag{16}$$

where $u_{v_i}(\bar{k}+k)$ is the k-th control action of the control sequence associated to the vertex v_i and $x_0(\bar{k}+k)$ is the updated inner state obtained as solution of the LP problem (15). The IpIC is applied for $k = 1, \ldots, p-1$, or until the state reaches one of its targets i.e., either $\mu_j \mathcal{B}$, where j is the number of the (pseudo-)cycle[3], or the MAS Ω paired with the stabilizing feedback controller. If the state enters the scaled target set, a new periodic sequence starts afterwards. On the other side, if the state enters the MAS in less than p steps, the control action changes to the state-feedback controller $u = K\,x$ and the interpolating coefficient is set $s = 0$.

[3]If the trajectory outside the state $\mu_j \mathcal{B}$ has p steps then this completes the cycle, otherwise the trajectory enters $\mu_j \mathcal{B}$ at an earlier stage.

Algorithm 1: IpIC: Improved periodic interpolating control using vertex reachability updated interpolating coefficient as in Section 3.3.2.2.

input : System matrices A, B; High-gain feedback matrix K; Sets $\mathcal{X}$, $\mathcal{U}$; Outer/target set $\mathcal{B}$; Number of steps N.

output: x, u, s.

Solve the reachability problem (9) for each vertex v_i, $i = 1, \ldots, n_v$:
- **Store** the control actions u_{v_i};
- **Determine** a common period p of the overall system as $l.c.m.(p_{v_i})$;

Define the initial state $x(0)$ that belongs to the set $\mathcal{B}$;
Define $x \leftarrow x(0)$, $l \leftarrow 0$, $\bar{k} \leftarrow 0$, $\mu \leftarrow 1$;
for $i = 0$ **to** N **do**
  if $x \notin \Omega$ **then**
    if *($l = p$) or ($x \in \mu\mathcal{B}$)* **then**
      Compute the minimum scaling factor μ, $\mu \in [0, 1]$, such that $x \in \mu\mathcal{B}$;
      Compute (s, x_v, x_0) solving the LP problem (5);
      Set $\bar{k} \leftarrow \bar{k} + l$;
      Apply the control u (16) with $k = 0$;
      Set $l \leftarrow 1$;
    else
      Update the outer state $x_v = A\, x_v^- + B\, u_v$, ($x_v^-$ is the outer state at the previous time step);
      Update (s, x_0) by solving (15) such that:
      - $x_0 \in \partial\Omega$,
      - x_0 is in the line that contains x and x_v,
      - s: $x = s\, x_v^- + (1 - s)\, x_0$;

      Apply the control u (16) with $k = l$;
      Set $l \leftarrow l + 1$;
    end
  else
    Apply the control $u = Kx$;
    Set $s \leftarrow 0$;
  end
  Update $x = Ax + Bu$
end

3.5. Feasibility and Asymptotic Stability

This section provides proofs of feasibility and asymptotic stability for the proposed periodic interpolating control using constrained vertex reachability with (or without) updated interpolating coefficients, presented in Sections 3.3.2.1 and 3.3.2.2.

3.5.1. p-step feasibility

For p-step feasibility of the pIC, we have to prove that $u(k) \in \mathcal{U}$ and that if the state $x(k)$ is feasible at time k, it will be also feasible at time $k+p$. In other words there exists an admissible control sequence $u(k) \in \mathcal{U}$ that steers the state in the feasible set in p steps. Let u_{v_i} be the vector of admissible control sequence $\{u_{v_i}(0), \dots, u_{v_i}(p-1)\}$ that steers each vertex v_i, $i = 1, \dots, n_v$ into the set $\mathcal{B}$ as solution of the reachability problem (9), and let p be the number of time steps necessary in order to bring the states contained in the outer set $\mathcal{B}$ back into the target set. The next theorem provides a proof of the p-step reachability problem presented in Section 3.3.1.

Theorem 3.1. *The periodic interpolating control* (3), (4), (5), (13) *is p-step feasible for the linear time invariant system* (1) *with state and control constraints* (2) *and for all states inside the feasible region $\mathcal{B}$. That is, the state will return inside the feasible set after p steps, i.e.,*

$$\forall x(k) \in \mathcal{B} \implies x(k+p) \in \mathcal{B}, \quad k \geq 0.$$

Proof. We want to prove that $u(k) \in \mathcal{U}$ for all $k \geq 0$. The control actions needs to verify $F_u u(k) \leq g_u$, with $u(k) = s(k)\, u_v(k) + (1 - s(k))\; u_0(k)$, $\forall k \geq 0$. Firstly, we prove that the outer control u_v verifies the control constraints (the index k is omitted for clarity):

$$u_v = \sum_{i=1}^{n_v} \alpha_i\, u_{v_i}, \qquad \alpha_i \geq 0, \qquad \sum_{i=1}^{n_v} \alpha_i = 1 \tag{17}$$

$$F_u\, u_v = F_u \sum_{i=1}^{n_v} \alpha_i\, u_{v_i} = \sum_{i=1}^{n_v} \alpha_i F_u\, u_{v_i} \leq \sum_{i=1}^{n_v} \alpha_i\, g_u = g_u. \tag{18}$$

The last inequality in (18) holds because u_{v_i} is one of the control actions and solutions of the reachability problem (9). We now prove that the control action (13) is admissible:

$$\begin{aligned} F_u u(k) &= F_u\left(s(k)\, u_v(k) + (1 - s(k))\; u_0(k)\right) \\ &= s(k) F_u u_v(k) + (1 - s(k)) F_u u_0(k) \\ &\leq s(k) g_u + (1 - s(k))\, g_u \\ &= g_u, \end{aligned} \tag{19}$$

where the last inequality hold from (18), where u_0 is control action within the MAS Ω.

Now we go back to prove that the set $\mathcal{B}$ is p-step feasible set, that is, for all $x(\bar{k}) \in \mathcal{B}$, then $x(\bar{k}+p) \in \mathcal{B}$. Consider the state decomposition,

$$x(\bar{k}) = s(\bar{k})\, x_v(\bar{k}) + \left(1 - s(\bar{k})\right)\; x_0(\bar{k}),$$

obtained from the solution of the LP problem (5). The inner state $x_0 \in \Omega$ evolves with the stabilizing high-gain feedback controller as the closed-loop system $x_0(\bar{k}+1) = (A + B\,K)\, x_0(\bar{k})$, while the outer state $x_v \in \mathcal{B}$ is contained in the boundary of the set $\mathcal{B}$ and evolves with the outer controller u_v. During the p-step periodic sequence, x_v evolves according to: $x_v(\bar{k}+k) = A\, x_v(\bar{k}+k-1) + B\, u_v(\bar{k}+k-1)$ for $k = 1, \dots, p$. After p steps, the state $x_v(k)$ returns into the target set $\mathcal{B}$ because of the construction of the periodic invariant set sequence, i.e., $x_v(\bar{k}+p) \in \mathcal{B}$. The inner state evolves according to: $x_0(\bar{k}+k) = (A + B\,K)^k\, x_0(k)$ for $k = 1, \dots, p$. Since Ω is computed as the MAS of the system with control $u = Kx$, after p steps the state $x_0(\bar{k})$ is inside the MAS, i.e. $x_0(\bar{k}+p) \in \Omega$.

From the state decomposition (3) and the interpolating coefficient $s(\bar{k})$ computed for $x(\bar{k})$, we obtain that the outer state evolves according to:

$$\begin{cases} s(\bar{k})x_v(\bar{k}+k) = s(\bar{k})\left(A\,x_v(\bar{k}+k-1) + B\,u_v(\bar{k}+k-1)\right), & k = 1,\dots,p-1, \\ s(\bar{k})x_v(k+p) \in s(\bar{k})\mathcal{B}, \end{cases} \tag{20}$$

while the inner state $x_0(k)$ evolves according to:

$$\begin{cases} (1-s(\bar{k}))x_0(\bar{k}+k) = (1-s(\bar{k}))\,(A+B\,K)^k\, x_0(\bar{k}), & k = 1,\dots,p-1, \\ \left(1-s(\bar{k})\right) x_0(\bar{k}+p) \in (1-s(\bar{k}))\Omega. \end{cases} \tag{21}$$

The initial state $x(\bar{k})$ after p steps is decomposed as,

$$x(\bar{k}+p) = s(\bar{k})x_v(\bar{k}+p) + (1-s(\bar{k}))x_0(\bar{k}+p),$$

where $s(\bar{k})x_v(\bar{k}+p) \in s(\bar{k})\mathcal{B}$ and $(1-s(\bar{k}))x_0(\bar{k}+p) \in (1-s(\bar{k}))\Omega$. Since $\Omega \subseteq \mathcal{B}$, it follows that $(1-s(\bar{k}))\Omega \subseteq (1-s(\bar{k}))\mathcal{B}$ and $x(\bar{k}+p)$ is convex combination of points in $\mathcal{B}$, i.e. $x(\bar{k}+p) \in \mathcal{B}$, which concludes the proof. □

Corollary 1. *The improved periodic interpolating control* (3), (4), (5), (16) *is p-step feasible for the linear time invariant system* (1) *with state and control constraints* (2) *and for all states inside the feasible region $\mathcal{B}$. That is, the state will return inside the feasible set after p steps, i.e.,*

$$\forall x(k) \in \mathcal{B} \implies x(k+p) \in \mathcal{B}, \quad k \geq 0.$$

Proof. The proof of recursive feasibility of the improved periodic interpolating control with updated interpolating coefficient (3), (4), (5), (16) follows from the observation that the updated inner state $x_0 \in \Omega$. Then similar arguments as in Theorem 3.1 can be applied. □

3.5.2. Asymptotic stability

Theorem 3.2. *The periodic interpolating control* (3), (4), (5), (13) *guarantees asymptotic stability of the linear time invariant system* (1) *with state and control constraints* (2) *for any initial point* $x(0) \in \mathcal{B}$*.*

Proof. We want to prove that for each initial state in the feasible set, x converges to Ω in finite time. Consider $V(x(k)) = s^*(x(k))$, $\forall x \in \mathcal{B} \setminus \Omega$ as candidate Lyapunov function. After solving the LP problem (5) for the state $x(k)$ and applying the control (13) for p steps, one obtains,

$$x(k+p) = s^*(x(k))\,x_v(k+p) + (1-s^*(x(k)))\,x_0(k+p),$$

where $x_v(k+p) \in \mathcal{B}$ is the outer state obtained after applying the outer controller for p steps and $x_0(k+p) = (A+B\,K)^p\,x_0(k) \in \Omega$. Hence, $s^*(x(k))$ is feasible solution of the LP problem (5) at time $k+p$. Solving the LP (5) again, one gets $s^*(x(k+p))$ and state decomposition $x(k+p) = s^*(x(k+p))\,x_v^*(k+p) + (1-s^*(x(k+p)))\,x_0^*(k+p)$ with $x_v^*(k+p) \in \mathcal{B}$ and $x_0^*(k+p) \in \Omega$. Since $s^*(x(k+p)) \leq s^*(x(k))$, the candidate Lyapunov function $V(x)$ is non-increasing. Furthermore, since the outer controller u_v computed by solving the optimization problem (9) is contractive over the periodic cycle, it guarantees convergence to Ω in finite time. Inside Ω, the interpolating coefficient $s(x)$ is null and the control action is the stabilizing state feedback control $u = Kx$. Finally, since the local feedback controller is contractive, asymptotic stability is guaranteed for all $x \in \mathcal{B}$ with control action (13). □

The following theorem proofs the asymptotic stability of the improved IpIC with constrained vertex feasibility and updated interpolating coefficient. The asymptotic stability assumes that the state converges toward the origin at each time step when it is inside the outer set $\mathcal{B}$ and converges to the target set $\mathcal{B}$ within each periodic sequence when the state is traveling outside the set. That is, a 2-level Lyapunov function can be defined. One function is applied to the system (1) that takes values at the beginning of each periodic sequence, and another Lyapunov function guarantees stability within the periodic sequence.

Theorem 3.3. *The improved periodic interpolating control* (3), (4), (5), (16) *guarantees asymptotic stability of the linear time invariant system* (1) *with state and control constraints* (2) *for any initial point* $x(0) \in \mathcal{B}$.

Proof. We want to prove that for each initial state in the feasible set, x converges to Ω in finite time with control action (16). Consider $V_{\mathcal{B}}(x(k)) = s^*_{\mathcal{B}}(x(k))$, as candidate Lyapunov function for $\forall x \in \mathcal{B} \setminus \Omega$ obtained by solving the LP problem (5) in order to determine the state decomposition, and $V_p(x(k)) = s^*_p(x(k))$, $\forall x \in \mathcal{X}$ as local candidate Lyapunov function defined at each periodic cycle and obtained by solving the LP problem (15) when a new local state x_0 is defined. A new $V_p(x(k))$ function is considered at each periodic sequence. The motivation to consider two different Lyapunov functions is that the interpolating coefficient (candidate Lyapunov function) is updated in two different ways: (a) for the initial state of the periodic sequence; (b) during the periodic cycle of period p. For this reason two Lyapunov functions are required. We first prove that the function $V_p(x(k))$ defines a local Lyapunov function within a periodic sequence, i.e., for $k \leq p$ steps. Figure 4 depicts the state decomposition at time $\bar{k}$ and $\bar{k}+1$. If the local state is not updated, the proportion between the segments that connect the inner and outer states to current state do not change. That is, the interpolating coefficient s is constant and equal to $s^*_p(x(\bar{k})) = s^*_{\mathcal{B}}(x(\bar{k}))$ and,

$$x(\bar{k}+1) = s^*_p(x(\bar{k}))\, x_v(\bar{k}+1) + (1 - s^*_p(x(\bar{k})))\, x_0(\bar{k}+1).$$

Consider now the updated decomposition,

$$x(\bar{k}+1) = s^*_p(x(\bar{k}+1))\, x_v(\bar{k}+1) + (1 - s^*_p(x(\bar{k}+1)))\, x^*_0(\bar{k}+1),$$

which follows from the solution of the LP problem (15). Note that the updated inner state $x^*_0(\bar{k}+1)$ is closer to the current state $x(\bar{k}+1)$ compared to its previous value while the outer state is kept constant. Then the updated interpolating coefficient provides a smaller value compared to the interpolating coefficient in the previous time step $s^*(x(\bar{k}+1)) \leq s^*(x(\bar{k}))$. This argument can be applied at each time step of the periodic sequence, and thus it verifies a Lyapunov function that guarantees stability to the system within a periodic sequence.

To conclude the proof, we now show that $V_{\mathcal{B}}(x(k))$ is as global candidate Lyapunov function. Consider the initial state $x(\bar{k}) \in \mathcal{B}$ and its decomposition $(x_v(\bar{k}), x_0(\bar{k}))$ with interpolating coefficient $s^*_{\mathcal{B}}(x(\bar{k}))$, where $x_v(\bar{k}) \in \partial\mathcal{B}$ and $x_0(\bar{k}) \in \partial\Omega$, obtained as solution of (5). When the state re-enters the target set, the state variable $x(\bar{k}+k)$, $k \leq p$ can be decomposed as,

$$x(\bar{k}+k) = s^*_{\mathcal{B}}(x(\bar{k}))\, x_v(\bar{k}+k) + (1 - s^*_{\mathcal{B}}(x(\bar{k})))\, x_0(\bar{k}+k),$$

where $x_v(\bar{k}+k) \in \mathcal{B}$ and $x_0(\bar{k}+k) \in \Omega$. Then, $s^*_{\mathcal{B}}(x(\bar{k}))$ is a feasible solution of the LP problem (5) and state $x(\bar{k}+k)$. The optimal state decomposition is $x(\bar{k}+k) = s^*_{\mathcal{B}}(x(\bar{k}+k))\, x^*_v(\bar{k}+k) + (1 - s^*_{\mathcal{B}}(x(\bar{k}+k)))\, x^*_0(\bar{k}+k)$. Since $s^*_{\mathcal{B}}(x(\bar{k}+k)) \leq s^*_{\mathcal{B}}(x(\bar{k}))$, the Lyapunov function $V_{\mathcal{B}}(x(k))$ is not increasing. Similar to Theorem 3.2, we note that the contractive outer controller u_v guarantees convergence into the set Ω in finite time and the stabilizing local control action $u = Kx$ guarantees asymptotic stability to the origin. □

Remark 2. *The improved periodic interpolating control* (3), (4), (5), (16) *requires an asymptotic stability with* 2*-level Lyapunov function since the interpolating coefficient obtained via* (5) *is computed after the state re-enters the target set $\mathcal{B}$, and thus might not be decreasing compared to the updated interpolating coefficient obtained via* (15) *within the periodic sequence. However, convergence is guaranteed in each periodic sequence, and thus the control action* (16) *guarantees asymptotic stability of the constrained linear time invariant system.*

4. Numerical Example

This section demonstrates the effectiveness of the proposed periodic interpolating scheme using reachability of target sets for constrained control presented in Section 3. In particular, periodic interpolating control will be applied to a linear time-invariant system using both pIC (see Section 3.3.2.1) and IpIC methods (see Section 3.3.2.2 and Algorithm 1)

Consider the discrete-time linear system with two state variables and one control variable in Kothare, Balakrishnan, and Morari (1996). The state and control matrices are as follows,

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}. \tag{22}$$

State and control variables are subject to symmetric constraints,

$$|x_i| \leq 1 \text{ with } x = [x_1\ x_2]^\mathsf{T}, i = 1, 2 \text{ and } |u| \leq 2. \tag{23}$$

The system (22) is controllable. A state feedback controller that stabilizes the system can be defined as $u = Kx$ with

$$K = -\,[30.3781\ 9.6139]\,. \tag{24}$$

The closed-loop system $A + B\,K$ has complex eigenvalues $\lambda_{1,2} = 0.6167 \pm 0.3036\imath$ with module $0.6874 < 1$. Given that the module of the eigenvalues is smaller than the unit, the system is stable. Ω is the maximal admissible set that verifies the system constraints (23) with a high-gain state feedback control input $u = K\,x$ with gain matrix (24). The set Ω is computed with the Invariant Set toolbox (Kerrigan, 2000). Its minimal half-space representation $\Omega = \left\{x \in \mathbb{R}^2 \colon F_\Omega\, x \leq g_\Omega\right\}$ is given by,

$$F_\Omega = \begin{bmatrix} -30.3781 & -9.6139 \\ 30.3781 & 9.6139 \\ -7.3936 & -5.2816 \\ 7.3936 & 5.2816 \\ 5.2333 & -1.9720 \\ -5.2333 & 1.9720 \\ 9.9479 & 0.0631 \\ -9.9479 & -0.0631 \end{bmatrix} \quad \text{and} \quad g_\Omega = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

To apply the developed interpolating control strategy in Section 3, consider a box with edges parallel to the axis $\mathcal{B} = \{x \colon F_\mathcal{B} x \leq g_\mathcal{B}\}$ with,

$$F_\mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad g_\mathcal{B} = \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}, \tag{25}$$

that contains the maximal admissible set Ω. $\mathcal{B}$ plays the role of outer set in the computation of periodic interpolating control and avoids the expensive computation and complex representation of the maximal controllable set Ψ,

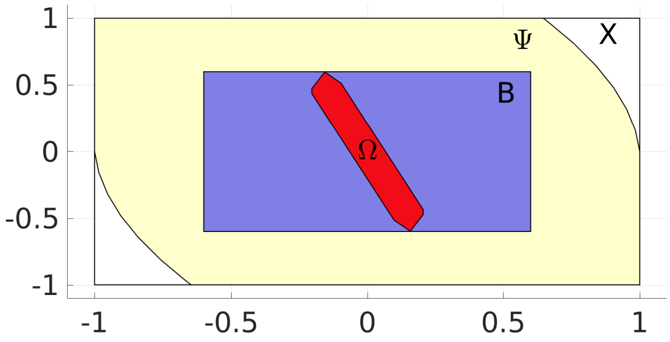


Figure 5. Numerical Example (Reachability problem with target): Invariant sets.

Figure 5 depicts the MAS Ω in red color, the set $\mathcal{B}$ in blue colour, and the state space in white colour. An approximation of the maximal controllable set Ψ is depicted in yellow colour and will be used to implement standard interpolating control (Nguyen, Gutman, & Bourdais, 2014). Ψ is computed with the Invariant Set toolbox (Kerrigan, 2000). Its half-space representation $\Psi = \left\{x \in \mathbb{R}^2 \colon F_\Psi\, x \le g_\Psi\right\}$ is given by,

$$F_\Psi = \begin{bmatrix} -1.0000 & -0.6793 \\ 1.0000 & 0.6793 \\ -1.0000 & -0.5852 \\ 1.0000 & 0.5852 \\ -1.0000 & -0.4901 \\ 1.0000 & 0.4901 \\ 1.0000 & 0.3940 \\ -1.0000 & -0.3940 \\ 1.0000 & 0.2970 \\ -1.0000 & -0.2970 \\ -1.0000 & -0.1990 \\ 1.0000 & 0.1990 \\ -1.0000 & -0.1000 \\ 1.0000 & 0.1000 \\ 1.0000 & 0 \\ 0 & 1.0000 \\ -1.0000 & 0 \\ 0 & -1.0000 \end{bmatrix} \quad \text{and} \quad g_\Psi = \begin{bmatrix} 1.3251 \\ 1.3251 \\ 1.2330 \\ 1.2330 \\ 1.1558 \\ 1.1558 \\ 1.0938 \\ 1.0938 \\ 1.0471 \\ 1.0471 \\ 1.0157 \\ 1.0157 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}.$$

The rectangle $\mathcal{B}$ will play the role of outer set in the computation of pIC and IpIC. The set $\mathcal{B}$ is not control invariant. It is chosen as a subset of the maximal controllable set Ψ, and thus for every state $x \in \mathcal{B}$ a control action can be computed that verifies the control constraints $u \in \mathcal{U}$ while the evolution of the state may not be within the rectangle $\mathcal{B}$. However, convergence into the target rectangle is guaranteed in a finite number of steps with a sequence of admissible controls that can be obtained using the reachability problem (9).

As first step in order to implement the pIC for the constrained system (22)–(23), four reachability problems (9) for the four vertices of the rectangle $\mathcal{B}$ are solved. The four vertices of $\mathcal{B}$ read: $v_1 = [0.6\ 0.6]^\top$, $v_2 = [-0.6\ 0.6]^\top$, $v_3 = [-0.6\ \ -0.6]^\top$, and $v_4 = [0.6\ \ -0.6]^\top$. From the solution of the reachability problem (9) for the fours vertices v_i, $i = 1, \ldots, 4$, we obtain the periodicity of each vertex with $p_1 = p_3 = 9$ steps and $p_2 = p_4 = 1$ steps. Thus the least common period length equals to $p = 9$ of the LTI system.

The proposed interpolating control computes an admissible control sequence in each periodic sequence over the period p until the system state enters the MAS Ω or reaches a target set defined

Control method	pIC1	pIC2	IC
CPU-seconds	0.41527	0.35888	3.304
Improvement (%)	87	89	-

Table 1. Computation time. Processor: 2.3GHz Dual-core Intel i5.

as $\mu\mathcal{B}$. The constant μ is computed at the beginning of each periodic sequence by solving the LP problem (10). This is the smallest scaling factor of the target set $\mathcal{B}$ such that the current state $x \in \mu\mathcal{B}$. After the state enters the target set, a new periodic sequence is calculated and a new admissible control sequence is applied to the system. This process is then iterated until the state evolution enters the MAS, and thus the stabilizing state-feedback controller can be applied with $s = 0$.

As discussed in Section 3.3.2, the proposed periodic control with target $\mu\mathcal{B}$ can determine the interpolating coefficient $s(k)$ by two different approaches, namely pIC (Section 3.3.2.1) and IpIC (Section 3.3.2.2):

pIC: $s(k)$ is updated at the beginning of a new periodic sequence but is kept constant within a periodic sequence;

IpIC: $s(k)$ is updated both at the beginning of a new periodic sequence and within a periodic sequence via (5) and (15), respectively (Algorithm 1).

Figure 6 shows the state and control trajectories for the initial state $x(0) = [0.55\ 0.55]^\mathsf{T}$ under pIC (denoted as pIC1), IpIC (denoted as pIC2), and traditional IC (Nguyen, Gutman, & Bourdais, 2014) (where Ψ is used). As can be seen, all three approaches exhibit similar state and control trajectories (see Figures 6(a)–6(c)), albeit with different interpolating coefficients (see Figure 6(d)). The scaling factors μ are $\mu_{pIC1} = \{0.9167, \quad 0.8560\}$ and $\mu_{pIC2} = \{0.9167, \quad 0.9109\}$ for pIC1 and pIC2, respectively. The control effort of the three approaches is: $\|u_{\text{pIC1}}\|_2 = 7.1749$, $\|u_{\text{pIC2}}\|_2 = 7.2015$ and $\|u_{\text{IC}}\|_2 = 6.3172$ for pIC1, pIC2 and IC, respectively. Table 4 benchmarks the performance (in CPU-seconds) of the proposed pIC1 approach and its improved version pIC2 over the IC. As can be seen, both pIC1 and pIC2 are around 85% faster than standard IC.

Figure 6(d) depicts the interpolating coefficients for the three methods. As can be seen, the interpolating coefficient of pIC1 takes the value $s = 0.9$ and remains constant over $p = 9$ steps. Then, it decreases to $s = 0.8$ for the new periodic sequence with $p = 5$ steps. Finally, it takes the value $s(15) = 0$ at $k = 15$ because the system state has entered the MAS, and thus the high-gain state-feedback controller is applied. In this example the periods for each periodic sequence are the same under pIC1 and pIC2. On the other hand, the interpolating coefficient for pIC2 is decreasing at every time step until it reaches $s = 0$, that is, the state x is in the MAS. In both approaches, the interpolating coefficient plays the role of Lyapunov function and guarantees convergence of our approach. It should be noted that all three approaches converge to the MAS at $k = 15$, see Figure 6(d) (i.e., their interpolating coefficients $s(15) = 0$ at $k = 15$). Figure 7 shows the evolution of system state in the $\mathbb{R}^2$-space under all three methods for the initial point $x(0) = [0.55 \quad 0.55]^\mathsf{T}$.

Concluding, the proposed approach although employed a naive rectangular inner approximation of the control invariant set Ψ, it provided similar performance to the more expensive IC while it guaranteed convergence and satisfaction of the state and control constraints. This is a promising result. Intuitively, more complex but easy representations of the outer control invariant set (e.g., hexagons or octagons with more than 4 vertices) can provide adjustable complexity alternatives with similar performance all by preserving the essential feasibility and constrained stability guarantees.

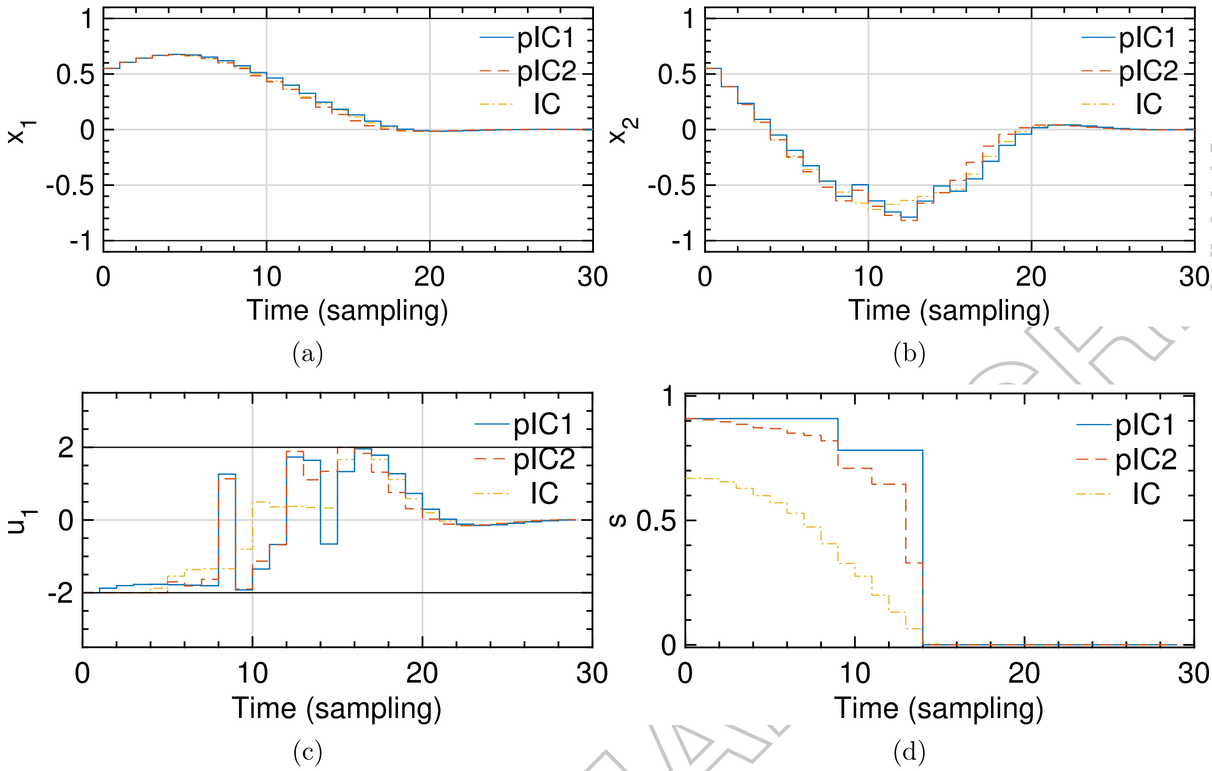


Figure 6. Example: (a), (b): State trajectories for the initial state $[0.55\ 0.55]^\mathsf{T}$; (c): Control trajectories; (d): Interpolating coefficient.

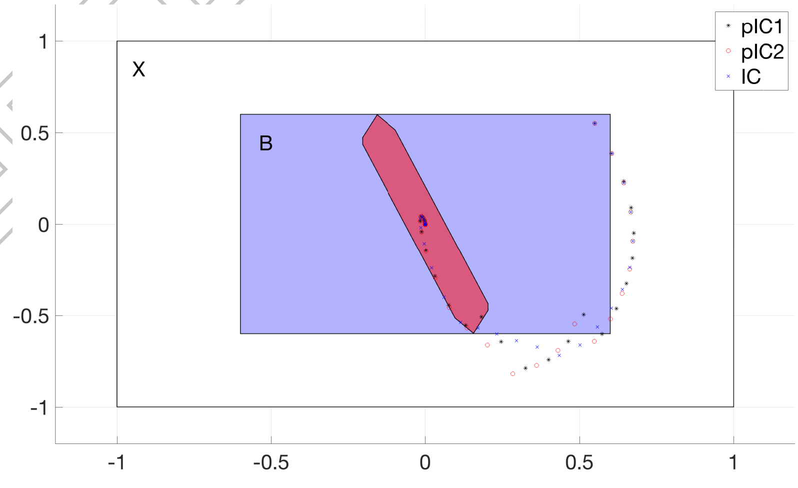


Figure 7. Example: Evolution of the initial state $[0.55\ 0.55]^\mathsf{T}$. pIC1 is in black (interpolating coefficient $s(k)$ constant throughout each period); pIC2 is in red ($s(k)$ updated at each time step) and standard IC is in blue.

5. Conclusions

This work presented a novel interpolating control scheme with periodic invariance and constrained vertex reachability for state-space regions or target sets of linear systems with state and control constraints. It proposed to use an easy representation of the outer control invariant set (e.g., a rectangle or hexagon or octagon or zonotopes in the general case), and then to solve a reachability problem for the outer set for the constrained discrete-time system. Since for a particular outer set the vertices are known beforehand, the constrained reachability problem determines for each vertex of the outer set a sequence of admissible controls that steer the state of the system back into the original target set after a finite number of time steps. For the interpolation, an inexpensive LP problem (or two LP problems in the case of variable s) is solved at the beginning of each periodic cycle. Proofs of p- step recursive feasibility and asymptotic stability of the periodic IC approach are given. The numerical example demonstrated that the proposed scheme provides similar performance compared to the traditional IC while it considers low-complexity representation of the control (periodic) invariant set.

We presented the outcomes of this paper in the $\mathbb{R}^2$-space, though their extension to high-dimensional spaces is straightforward (e.g., a hyper-box or hyper-cube can be used in $\mathbb{R}^n$-space). Results can be further extended to different research directions. Periodic invariant sets remove the consideration of strict invariance in the definition of a positively invariant set, and thus can be applied to real applications where violation of constraints is allowed for certain time periods, as well as their applications to periodic, distributed, and time-delayed systems.

Disclosure statement

No potential conflict of interest was reported by the authors.

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