Penrose Superradiance in a Nonlinear Optics Superfluid: supplementary information

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I. LINEARIZED EQUATIONS AND THEORY

In this section we derive the linearized Eqs. (3-4) of the main text that govern the evolution of the signal and idler fields in the photon fluid in the presence of the strong pump field. We start from the nonlinear Schrödinger equation for a monochromatic field of frequency ω_0 and a local nonlinearity describing propagation in the photon fluid

$$\frac{\partial E}{\partial z} = \frac{i}{2k}\nabla_\perp^2 E + ik_0 n_2|E|^2 E, \tag{S1}$$

where E is a monochromatic light field with wavelength λ, n_0 the linear refractive index of the medium, $k = 2\pi n_0/\lambda = n_0 k_0 = n_0\omega_0/c$ is the wavenumber, and $\nabla_\perp^2$ is the transverse Laplacian, accounting for optical diffraction. We observe that optical fluids are two-dimensional , that is, they live in the plane (x, y) orthogonal to the direction of propagation z. We consider a defocusing nonlinearity $(n_2 < 0)$ and a vortex pump solution of the form

$$\begin{aligned} E(r,\theta,z) &= \mathcal{E}_0(r)e^{i(\beta_\ell z+\ell\theta)} \\ &= \sqrt{I_\ell}\; u_\ell(r)e^{i(\beta_\ell z+\ell\theta)}, \end{aligned} \tag{S2}$$

where I_ℓ is the background intensity of the vortex of OAM ℓ, $u_\ell(r)$ is the corresponding vortex profile which has a core size denoted r_ℓ, and $\beta_\ell = k_0 n_2 I_\ell < 0$. The vortex profile, which we take as real without loss of generality, obeys the equation

$$\beta_\ell u_\ell = \frac{1}{2k}\nabla_\ell^2 u_\ell + k_0 n_2 I_\ell u_\ell^3, \tag{S3}$$

where $u_\ell(r) \to 1$ for $r >> r_\ell$, and we have defined $\nabla_p^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{p^2}{r^2}$.

To proceed we make use of the fact that in the presence of the strong pump $\mathcal{E}_0$ of OAM ℓ and a weak externally applied signal field $\mathcal{E}_s$ of OAM n, the total field may be written as

$$E(r,\theta,z) = \left[\mathcal{E}_0(r)e^{i\ell\theta} + \mathcal{E}_s(r,z)e^{in\theta} + \mathcal{E}_i(r,z)e^{iq\theta}\right]e^{i\beta_\ell z}, \tag{S4}$$

$$= \left[\mathcal{E}_0(r) + \mathcal{E}_s(r,z)e^{i(n-\ell)\theta} + \mathcal{E}_i(r,z)e^{-i(n-\ell)\theta}\right]e^{i(\beta_\ell z+\ell\theta)}, \tag{S5}$$

with $\mathcal{E}_i$ the generated idler field with OAM $q = (2\ell - n)$.

Then substituting the expansion (S4) into the starting Eq. (S1), linearizing in the signal and idler fields, and separating the signal and idler equations on the basis of their differing OAM yields

$$\begin{aligned} \frac{\partial \mathcal{E}_s}{\partial z} &= \frac{i}{2k}\nabla_n^2\mathcal{E}_s + ik_0 n_2\left[2|\mathcal{E}_0|^2\mathcal{E}_s + \mathcal{E}_0^2\mathcal{E}_i^*\right] - i\beta_\ell\mathcal{E}_s, \\ \frac{\partial \mathcal{E}_i}{\partial z} &= \frac{i}{2k}\nabla_q^2\mathcal{E}_i + ik_0 n_2\left[2|\mathcal{E}_0|^2\mathcal{E}_i + \mathcal{E}_0^2\mathcal{E}_s^*\right] - i\beta_\ell\mathcal{E}_i. \end{aligned} \tag{S6}$$

Finally we substitute the strong vortex pump in Eq. (S2) into the above equations to obtain

$$\begin{aligned} \frac{\partial \mathcal{E}_s}{\partial z} &= \frac{i}{2k}\nabla_n^2\mathcal{E}_s + i\beta_\ell u_\ell^2(r)\left[2\mathcal{E}_s + \mathcal{E}_i^*\right] - i\beta_\ell\mathcal{E}_s, \\ \frac{\partial \mathcal{E}_i}{\partial z} &= \frac{i}{2k}\nabla_q^2\mathcal{E}_i + i\beta_\ell u_\ell^2(r)\left[2\mathcal{E}_i + \mathcal{E}_s^*\right] - i\beta_\ell\mathcal{E}_i. \end{aligned} \tag{S7}$$

These equations are the basis for the manuscript discussion and describe the parametric interaction between the signal and idler fields in the presence of the pump, this parametric interaction arising from Four Wave Mixing (FWM).

II. TRAPPING OF THE IDLER WAVE

A. Idler propagation

The idler propagation equation (S7) can be rearranged as

$$\frac{\partial \mathcal{E}_i}{\partial z} = \frac{i}{2k}\nabla_q^2 \mathcal{E}_i + i\underbrace{2\beta_\ell[u_\ell^2(r)-1]}_{waveguide}\mathcal{E}_i + i\beta_\ell\mathcal{E}_i + \underbrace{i\beta_\ell u_\ell^2(r)\mathcal{E}_s^*}_{source}. \tag{S8}$$

Since the nonlinear parameter β_ℓ is negative, the underbraced term $2|\beta_\ell|[1-u_\ell^2(r)] = k_0\Delta n(r)$ defines a two-dimensional refractive-index profile which is guiding since $u_\ell^2(r)$ is zero at the pump vortex center $r=0$, so $\Delta n(r) = 2|\beta_\ell|$ is maximum there, and is unity for $r >> r_\ell$ away from the vortex core, so that $\Delta n(r)$ goes to zero. The pump vortex therefore creates a cross-phase-modulation (XPM) induced waveguide that is experienced by the idler wave.

The underbraced source term in the above equation describes how the idler wave, that is absent at the input, is driven by the signal beam via the parametric interaction. As shown in the manuscript (see Eq. (6) of the manuscript) the signal field can be approximated as

$$\mathcal{E}_s(r,z) \approx c_s V_n(r,z) e^{-i(1+|n|)\phi_G(z)} e^{2i\beta_\ell\Gamma_n(z)z - i\beta_\ell z}, \tag{S9}$$

where $V_n(r,z)$ is the normalized z-dependent Laguerre-Gauss mode profile, $\phi_G(z) = \tan^{-1}(z/z_0)$ is the Gouy phase-shift at the focus with Rayleigh range defined as $z_0 = kw_0^2/2$ and $\Gamma_n(z) = \int_0^\infty 2\pi r dr\ |V_n(r,z)|^2 u_\ell^2(r)$ being the signal phase variation induced by the pump core on the signal.
In the following section we compute the guided idler modes spectrum.

B. Guided idler modes

The spectrum of guided idler waves with OAM q can be found by solving the wave equation (S8) combining beam diffraction and the XPM-induced refractive-index profile. Neglecting the source for the time being, and for idler fields of the form

$$\mathcal{E}_i(r,z) = c_i U_{pq}(r) e^{i(\beta_\ell + \Lambda_{pq})z}, \tag{S10}$$

with radial mode-index p, this leads to the equation for the modes ($p = 0, 1, 2\ldots$)

$$\left(\frac{1}{2k}\nabla_q^2 + 2\beta_\ell[u_\ell^2(r)-1]\right)U_{pq}(r) = \Lambda_{pq}U_{pq}(r). \tag{S11}$$

This eigenproblem can be solved for the guided idler modes for a given pump vortex profile $u_\ell(r)$ and value of the nonlinear parameter β_ℓ. Note that it is possible that no guided idler modes exist in which case the Penrose process cannot occur. The eigenvalues Λ_{pq} are positive and decrease with increasing p, and for the present purposes the lowest radial mode $p=0$ is the relevant one. We hereafter drop the radial mode index for simplicity in notation, and assume $U_q(r)$ and Λ_q exist and we obtain them numerically.

The modal solution $U_q(r)$ allows us to evaluate the ergosphere radius in a more systematic way: this mode has a single-ringed intensity profile and one can find the radius r_q of the peak intensity. Physically any idler energy excited this guided mode will effectively be confined or trapped within the radius r_q, so we identify r_q with a viable measure of the radius of the ergosphere r_e. The approach $r_e = r_q$ agrees quite well with the previous approximation, particularly for larger q.