



Nimmo, J.J.C. and Ruijsenaars, S.N.M. (2009) *Tzitzeica solitons versus relativistic Calogero–Moser three-body clusters*. *Journal of Mathematical Physics*, 50 (4). 043511. ISSN 0022-2488

<http://eprints.gla.ac.uk/25264/>

Deposited on: 26 March 2010

Tzitzeica solitons versus relativistic Calogero–Moser three-body clusters

J. J. C. Nimmo^{1,a)} and S. N. M. Ruijsenaars^{2,b)}

¹*Department of Mathematics, University of Glasgow, Glasgow G12 8QW, United Kingdom*

²*Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom and Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, United Kingdom*

(Received 27 October 2008; accepted 5 March 2009; published online 21 April 2009)

We establish a connection between the hyperbolic relativistic Calogero–Moser systems and a class of soliton solutions to the Tzitzeica equation (also called the Dodd–Bullough–Zhiber–Shabat–Mikhailov equation). In the $6N$ -dimensional phase space Ω of the relativistic systems with $2N$ particles and N antiparticles, there exists a $2N$ -dimensional Poincaré-invariant submanifold Ω_p corresponding to N free particles and N bound particle-antiparticle pairs in their ground state. The Tzitzeica N -soliton tau functions under consideration are real valued and obtained via the dual Lax matrix evaluated in points of Ω_p . This correspondence leads to a picture of the soliton as a cluster of two particles and one antiparticle in their lowest internal energy state. © 2009 American Institute of Physics.

[DOI: [10.1063/1.3110012](https://doi.org/10.1063/1.3110012)]

I. INTRODUCTION

The equation

$$\Psi_{uv} = e^\Psi - e^{-2\Psi} \quad (1.1)$$

has a curious history. It first arose a century ago in the work of the Rumanian mathematician Tzitzeica.^{1,2} He arrived at it from the viewpoint of the geometry of surfaces, obtaining an associated linear representation and a Bäcklund transformation.

For many decades after Tzitzeica’s work, Eq. (1.1) was not studied, two papers by Jonas^{3,4} being a notable exception. Thirty years ago, it was reintroduced within the area of soliton theory, independently by Dodd and Bullough⁵ and Zhiber and Shabat,⁶ see also Mikhailov’s paper.⁷ In this setting, (1.1) is viewed as an integrable relativistic theory for a field $\Psi(t, y)$ in two space-time dimensions, written in terms of light cone (characteristic) coordinates,

$$t = u - v, \quad y = u + v. \quad (1.2)$$

Accordingly, the partial differential equation (1.1) is known under various names and has been studied from several perspectives, including geometry,^{1–4} classical soliton theory,^{5–20} and quantum soliton theory.^{21–26} Moreover, it has shown up within the context of gas dynamics.^{27,28}

The principal aim of this paper is to tie in a class of soliton solutions to the Tzitzeica equation (1.1) with integrable particle dynamics of relativistic Calogero–Moser type. (A survey covering both relativistic and nonrelativistic Calogero–Moser systems can be found in Ref. 29.) The intimate relation of the latter integrable particle systems to soliton solutions of various evolution equations (including the sine-Gordon, Toda lattice, Korteweg–de Vries (KdV), and modified KdV equations) was already revealed in the paper in which they were introduced³⁰ and was elaborated

^{a)}Electronic mail: j.nimmo@maths.gla.ac.uk.

^{b)}Electronic mail: siru@maths.leeds.ac.uk.

on in Ref. 31. Later on, the list of equations whose soliton solutions are connected to the relativistic Calogero–Moser systems was considerably enlarged.^{32–34} In all of these cases, the N solitons correspond to N point particles.

The novelty of the present soliton-particle correspondence is that the Tzitzeica N -soliton solutions at issue correspond to an integrable reduction of the $3N$ -body relativistic Calogero–Moser dynamics. Physically speaking, a Tzitzeica soliton may be viewed as a lowest energy bound state of three Calogero–Moser “quarks,” one of which has negative charge, whereas the other two have positive charge.

A crucial ingredient for establishing the correspondence is the relation between an extensive class of two-dimensional (2D) Toda solitons and the relativistic Calogero–Moser systems, already studied in Ref. 33. Indeed, the relation can be combined with the link between the Tzitzeica equation and the 2D Toda equation. The latter link has been known for quite a while, and we proceed to sketch it in a form that suits our later requirements.

Assume that ϕ_n is a solution to the 2D Toda equation in the form³⁵

$$\phi_{n,uv} = \exp(\phi_n - \phi_{n-1}) - \exp(\phi_{n+1} - \phi_n), \quad n \in \mathbb{Z}, \quad (1.3)$$

which has the symmetry property

$$\phi_{-n} = -\phi_n \quad (1.4)$$

and which is moreover 3-periodic, i.e.,

$$\phi_{n+3} = \phi_n. \quad (1.5)$$

Then one has, in particular,

$$\phi_0 = 0, \quad \phi_2 = -\phi_1, \quad (1.6)$$

so that

$$\Psi = \phi_1 \quad (1.7)$$

satisfies (1.1). Conversely, a solution Ψ to (1.1) yields a solution ϕ_n to (1.3) satisfying (1.4) and (1.5) when one sets

$$\phi_{3k} = 0, \quad \phi_{1+3k} = -\phi_{2+3k} = \Psi, \quad k \in \mathbb{Z}. \quad (1.8)$$

The point is now that there exist soliton solutions to (1.3) that can be made to satisfy the extra requirements (1.4) and (1.5), hence yielding soliton solutions to (1.1). The relevant 2D Toda solitons are those found by the Kyoto school.^{36,37} These solitons also formed the starting point for Ref. 33. They are most easily expressed in tau function form, the relation of τ_n to ϕ_n being given by

$$\phi_n = \ln(\tau_{n+1}/\tau_n), \quad n \in \mathbb{Z}. \quad (1.9)$$

In terms of τ_n , the evolution equation becomes

$$\partial_u \partial_v \ln \tau_n = 1 - \tau_{n-1} \tau_{n+1} / \tau_n^2, \quad (1.10)$$

and the extra features (1.4) and (1.5) amount to

$$\tau_{-n+1} = \tau_n \quad (1.11)$$

and

$$\tau_{n+3} = \tau_n. \quad (1.12)$$

As we show in Sec. II, one can make special parameter choices in the 2D Toda $2N$ -soliton solutions $\tau_n(u, v)$ so that they satisfy (1.11) and (1.12). The function

$$\Psi = \ln(\tau_2/\tau_1) \quad (1.13)$$

then satisfies (1.1) and can be viewed as a Tzitzeica N -soliton solution. (In Lie algebraic terms, the successive requirements (1.11) and (1.12) amount to reductions $A_\infty \rightarrow B_\infty \rightarrow A_2^{(2)}$.)

More specifically, the 2D Toda $2N$ -solitons of Sec. II are of the form

$$\tau_n(u, v) = \det(1_{2N} + D(n, u, v)C), \quad (1.14)$$

where the dependence of the $2N \times 2N$ (Cauchy-type) matrix C and diagonal matrix D on the parameters $a, b, \xi^0 \in \mathbb{C}^{2N}$ is suppressed. To satisfy the B_∞ restriction (1.11) and to prepare for the 3-periodicity restriction (1.12), these $6N$ parameters are expressed in terms of $2N$ parameters $\phi, \theta \in \mathbb{C}^N$ and a coupling parameter c . We then show that the tau functions have period l for c equal to π/l , so that the Tzitzeica restrictions are satisfied for $c = \pi/3$.

In Sec. III we make a further parameter change, trading ϕ_1, \dots, ϕ_N for “positions” q_1, \dots, q_N . This reparametrization ensures, in particular, that the summand involving all exponentials has coefficient 1. Restricting attention to the parameter set

$$\mathcal{P} = \{(q, \theta) \in \mathbb{R}^{2N} | \theta_N < \dots < \theta_1\}, \quad (1.15)$$

the tau functions take their simplest and most natural form. In particular, for parameters in \mathcal{P} the tau functions are real valued. Furthermore, their space-time dependence is such that the q_j and θ_j can be interpreted as relativistic positions and rapidities. Last but not least, it is in this form that the $c = \pi/3$ Tzitzeica N -soliton tau functions can be most easily compared to the tau functions arising in the framework of the $3N$ -body relativistic Calogero–Moser systems.

Section IV is devoted to this comparison. The choice of regime for the Calogero–Moser systems is the same as for almost all other soliton equations. Specifically, the regime is the hyperbolic one, with the Poincaré group generators given by

$$H = \frac{M_0}{2}(S_+ + S_-), \quad P = \frac{M_0}{2}(S_+ - S_-), \quad B = \sum_{i=1}^{N_+} x_i^+ + \sum_{j=1}^{N_-} x_j^-, \quad (1.16)$$

$$S_\pm = \sum_{1 \leq i \leq N_+} \exp(\pm p_i^+) V_i^+ + \sum_{1 \leq j \leq N_-} \exp(\pm p_j^-) V_j^-, \quad (1.17)$$

$$(V_i^+)^2 = \prod_{1 \leq k \leq N_+, k \neq i} \left(1 + \frac{\sin^2 c}{\sinh^2(x_i^+ - x_k^+)/2} \right) \prod_{1 \leq j \leq N_-} \left(1 - \frac{\sin^2 c}{\cosh^2(x_i^+ - x_j^-)/2} \right), \quad (1.18)$$

$$(V_j^-)^2 = \prod_{1 \leq l \leq N_-, l \neq j} \left(1 + \frac{\sin^2 c}{\sinh^2(x_j^- - x_l^-)/2} \right) \prod_{1 \leq i \leq N_+} \left(1 - \frac{\sin^2 c}{\cosh^2(x_j^- - x_i^+)/2} \right). \quad (1.19)$$

Here, the generalized positions and momenta vary over the phase space

$$\Omega = \{(x^+, x^-, p^+, p^-) \in \mathbb{R}^{2(N_+ + N_-)} | x_{N_+}^+ < \dots < x_1^+, x_{N_-}^- < \dots < x_1^-\}, \quad (1.20)$$

equipped with the symplectic form

$$\omega = \sum_{1 \leq i \leq N_+} dx_i^+ \wedge dp_i^+ + \sum_{1 \leq j \leq N_-} dx_j^- \wedge dp_j^-. \quad (1.21)$$

(Only the Landau–Lifshitz solitons of Ref. 38 involve the more general elliptic regime.³²) The coupling c that is needed, however, differs from the value $\pi/2$ relevant for most soliton equations (which correspond to $A_1^{(1)}$). Indeed, as already mentioned, the connection of the space-time dynamics (1.16)–(1.19) to the Tzitzeica tau functions arises for the A_2 -value,

$$c = \pi/3. \quad (1.22)$$

In order to clarify further restrictions and to describe our main result in some detail, a few more ingredients need to be introduced. First, the Poisson commuting “light-cone” Hamiltonians S_+ and S_- can be expressed in terms of an $(N_+ + N_-) \times (N_+ + N_-)$ Lax matrix on Ω via

$$S_+ = \text{Tr}(L), \quad S_- = \text{Tr}(L^{-1}). \quad (1.23)$$

This matrix has a product structure $D\mathcal{M}D$, with a diagonal matrix D and a matrix \mathcal{M} that arises by suitable substitutions in Cauchy’s matrix $\mathcal{C}(x, y)$, i.e., a matrix with elements $(x_i - y_j)^{-1}$, whose determinant is given by Cauchy’s identity

$$|\mathcal{C}(x, y)| = \frac{\prod_{i < j} (x_i - x_j)(y_j - y_i)}{\prod_{i, j} (x_i - y_j)}. \quad (1.24)$$

Hence one can explicitly determine the symmetric functions of L and its inverse, and all of these functions Poisson commute.

Specifically, the Lax matrix

$$L = D\mathcal{M}D \quad (1.25)$$

is given by

$$D = \text{diag}(\exp(p_1^+/2)(V_1^+)^{1/2}, \dots, \exp(p_{N_+}^+/2)(V_{N_+}^+)^{1/2}, \\ \exp(p_1^-/2)(V_1^-)^{1/2}, \dots, \exp(p_{N_-}^-/2)(V_{N_-}^-)^{1/2}), \quad (1.26)$$

$$\mathcal{M}_{ik} = \frac{i \sin c}{\sinh((x_i^+ - x_k^+)/2 + ic)},$$

$$\mathcal{M}_{N_+ + j, N_+ + l} = \frac{i \sin c}{\sinh((x_j^- - x_l^-)/2 + ic)},$$

$$\mathcal{M}_{N_+ + j, k} = \frac{\sin c}{\cosh((x_j^- - x_k^+)/2 + ic)},$$

$$\mathcal{M}_{i, N_+ + l} = -\overline{\mathcal{M}_{N_+ + l, i}}, \quad (1.27)$$

where $i, k \in \{1, \dots, N_+\}$ and $j, l \in \{1, \dots, N_-\}$. It is clear from this that L is self-adjoint when N_+ or N_- vanish. Physically speaking, this is the case in which only particles or antiparticles are present. For $N_+ N_- > 0$, however, L is not self-adjoint. Instead, L is J self-adjoint, that is, the adjoint L^* equals JLJ , where

$$J = \text{diag}(1_{N_+}, -1_{N_-}). \quad (1.28)$$

When one takes the interparticle distances to infinity, it becomes clear that there is a subset of Ω on which L is diagonalizable with real eigenvalues. (Of course, for the one-charge case this is true on all of Ω .) However, already for the simplest non-self-adjoint case $N_+ = N_- = 1$, complex-conjugate eigenvalues are also present, reflecting the existence of bound states of a particle and antiparticle. More precisely, choosing from now on

$$c \in (0, \pi/2), \quad (1.29)$$

these bound states are encoded by eigenvalues

$$\exp(\hat{\theta}_+), \quad \exp(\hat{\theta}_-), \quad \hat{\theta}_\pm = \theta \pm i\kappa, \quad \kappa \in (0, c]. \quad (1.30)$$

Moreover, the case of an eigenvalue pair on the boundary of this sector corresponds to phase space points

$$x_+ = x_- = x, \quad p_+ = p_- = p. \quad (1.31)$$

For $p=0$ this yields a one-parameter set with minimal energy $H=2M_0 \cos c$, see (1.16)–(1.19), and, more generally, no oscillation takes place for the two-parameter set (1.31).

The special case just discussed is the only one that can be explicitly understood in an elementary way. Already for $N_+ + N_- = 3$ (the simplest case relevant for this paper) a complete account involves considerable analysis. The general case has been elucidated in great detail in Ref. 31, and we need to make extensive use of Secs. 2 B–6 B of that paper, which pertain to the c -range (1.29). The action-angle map constructed there makes it possible to understand all of the Poisson commuting dynamics at once, including their soliton-type long-time asymptotics (conservation of momenta and factorized position shifts).

It is beyond our scope to recapitulate even the case $N_+ + N_- = 3$, but we do mention the key starting point for the analysis performed in Ref. 31. This is because it clarifies why the diagonal matrix

$$A(x) = \text{diag}(\exp(x_1^+), \dots, \exp(x_{N_+}^+), -\exp(x_1^-), \dots, -\exp(x_{N_-}^-)) \quad (1.32)$$

is of pivotal importance. This matrix is referred to as the dual Lax matrix, and its relation to L is encoded in the commutation relation

$$i \cot(c)(LA - AL) = LA + AL - 2e \otimes e. \quad (1.33)$$

(This relation is readily verified; we do not need the dyadic in the sequel.)

One need only inspect (1.33) to see that A and L play symmetric roles. In the one-charge case it is possible to diagonalize the self-adjoint matrix L in such a way that A takes essentially the same form in terms of suitable variables (which are just the action-angle variables). This self-duality is no longer present for $N_+ N_- > 0$, however. Indeed, A is then still self-adjoint, whereas L is not. Even so, (1.33) can again be used to great advantage in the construction of the action-angle map, as detailed in Ref. 31.

We are now prepared to specialize the above to the arena with which the present paper is concerned. First of all, we choose

$$N_+ = 2N, \quad N_- = N. \quad (1.34)$$

This choice ensures that there exists a $2N$ -dimensional Poincaré-invariant submanifold

$$\Omega_P \simeq \mathcal{P}, \quad (1.35)$$

with \mathcal{P} given by (1.15), of the $6N$ -dimensional phase space $\langle \Omega, \omega \rangle$. This submanifold will be described in detail in Sec. IV, and we shall presently add a brief qualitative description.

Consider now the solution

$$\exp(tH - yP)Q, \quad Q = (x, p) \in \Omega, \quad (1.36)$$

to the joint Hamilton equations for H and P . Such a joint solution exists and is given by (1.36), since the H -flow and P -flow commute. Using (1.2) and (1.16), we can rewrite (1.36) as

$$\exp(M_0(uS_- - vS_+))Q = (x(u,v), p(u,v)). \quad (1.37)$$

Next, specializing to

$$M_0 = \sqrt{3}, \quad c = \pi/3, \quad (1.38)$$

we introduce

$$\tau_n(u,v) = \det(1_{3N} + \exp(i\pi(1-2n)/3)A(x(u,v))). \quad (1.39)$$

For a given point Q in the phase space Ω , this yields a function depending on $n \in \mathbb{Z}$ and $(u,v) \in \mathbb{R}^2$.

We are finally in the position to state the principal result of this paper: When the tau function (1.39) is restricted to the $2N$ -dimensional Poincaré-invariant subspace Ω_P of Ω , it coincides with the Tzitzeica tau function (1.14) evaluated on \mathcal{P} (1.15). Our demonstration of this equality uses, in particular, special cases of the fusion identities obtained in Ref. 33. (For completeness we add the proof of these specializations in Appendix A.) The reduction in the matrix size from $3N \times 3N$ to $2N \times 2N$ hinges on all points in Ω_P yielding pairs of complex-conjugate eigenvalues $\exp(\theta_j \pm i\pi/3)$ for the Lax matrix L on the boundary of the allowed angular sector. For each of these pairs there is an additional eigenvalue $\exp(\theta_j)$, so that the spectrum of L on Ω_P involves only N degrees of freedom $\theta_1, \dots, \theta_N \in \mathbb{R}$ (and not the $3N$ of the full phase space). These variables may be viewed as action variables, and there are N canonically conjugate “angle” variables $q_1, \dots, q_N \in \mathbb{R}$. As a consequence, Ω_P can be identified with \mathcal{P} , see (1.35) and (1.15).

A better understanding of how Ω_P arises as a submanifold of the $6N$ -dimensional phase space can only be achieved by invoking a great many details concerning the action-angle map constructed in Ref. 31, which are summarized in Sec. IV. At this point we only add a few qualitative remarks, so as to render these details more accessible.

First, the above coordinates are not quite action-angle coordinates. Rather, the precise definition of Ω_P involves the harmonic oscillator transform. This transform is an extension of the action-angle transform, which takes into account that Ω contains an open dense subset that is the union of submanifolds for which the angles vary over $\mathbb{R}^{3N-l} \times \mathbb{T}^l$, where $\mathbb{T} \simeq (-\pi, \pi]$ and l takes all values in $\{0, 1, \dots, N\}$. From a physical point of view, these submanifolds can be regarded as the subsets of Ω on which l particle-antiparticle bound states are present. Now when the bound state internal actions converge to their minima, the l -torus collapses to lower-dimensional tori in precisely the same way as for a harmonic oscillator Hamiltonian $\sum_{j=1}^l (p_j^2 + x_j^2)$, which motivates the terminology.

In these terms, Ω_P amounts to the subset that arises from the submanifold with N bound states by taking the torus \mathbb{T}^N to a point, so that each of the pairs is in its ground state; moreover, each of the remaining particles has action and angle variables that are paired with those of the bound states, in such a way that the asymptotic space-time dependence of Ψ is that of N three-body clusters moving apart, each of the clusters staying together as in the $N=1$ case.

In Sec. V we present a close-up of the latter case. For $N=1$, various questions can be rather easily answered, and this special case is also useful as an illustration of the general case. In particular, we obtain some explicit information on the two-dimensional space Ω_P for $c \in (0, \pi/2)$ and study the Tzitzeica one-soliton solution corresponding to $c = \pi/3$.

In Sec. VI we compare the Tzitzeica solitons under consideration to the ones obtained by Kaptsov and Shan’ko¹⁶ and to the solitons arising by the above two-step reduction (1.11) and (1.12) from a class of 2D Toda solitons constructed via Darboux transformations. At face value, these two types of Tzitzeica solitons seem different from the ones obtained from the Kyoto solitons in Sec. II. As we show, however, the latter are a subclass of the former.

We have added Sec. VI primarily because it yields an affirmative answer to the two natural equality questions at issue, but as a bonus it yields a new insight on the Tzitzeica tau function τ_0 (given both by (1.39) and (1.14)): It is in fact positive and equal to the square of a simpler tau function occurring in the Kaptsov–Shan’ko work.¹⁶

In Sec. VII we consider further aspects of our results, in the form of several remarks. Specifically, we show that the space-time translation generators restricted to Ω_p give rise to a reduced integrable system, introduce and comment on space-time trajectories for the Tzitzeica solitons, isolate the difference with the setup of Ref. 33, and comment on an eventual quantum analog of the correspondence between three-body clusters and solitons.

A final remark concerns the solitons of the Demoulin system of equations.^{39,40} As it turns out, their tau function form is obtained by taking $c = \pi/6$ in Secs. II and III. It is a challenging question whether they can also be tied in with the relativistic Calogero–Moser systems.

We have relegated a proof of the fusion identities used in Sec. IV to Appendix A. In Appendix B we collect some auxiliary results concerning Pfaffians, which we need in Sec. VI. Finally, Appendix C contains a sketch of the Darboux-type construction of explicit tau function solutions to the 2D Toda equation (1.10).

II. THE SOLITON REDUCTION 2D Toda \rightarrow Tzitzeica

Our starting point is the 2D Toda M -soliton solutions introduced by the Kyoto school^{36,37} in the form

$$\tau_n = \sum_{\mu_1, \dots, \mu_M=0,1} \prod_{1 \leq j \leq M} \exp(\mu_j \xi_{j,n}) \cdot \prod_{1 \leq j < k \leq M} f_{jk}^{\mu_j \mu_k}, \quad (2.1)$$

$$f_{jk} = \frac{(a_j - a_k)(b_j - b_k)}{(a_j - b_k)(b_j - a_k)}, \quad j, k = 1, \dots, M, \quad (2.2)$$

$$\xi_{j,n} = \xi_j^0 + n \ln(a_j/b_j) + i \sum_{l=1}^{\infty} (t_{l,+}(a_j^l - b_j^l) + t_{l,-}(a_j^{-l} - b_j^{-l})). \quad (2.3)$$

The sequence of “times” $t_{l,\pm}$ encodes the hierarchy of commuting flows. Restricting attention to the 2D Toda equation (1.10), one should set

$$t_{l,+} = t_{l,-} = 0, \quad l > 1, \quad t_{1,+} = -v, \quad t_{1,-} = u, \quad (2.4)$$

to obtain a solution.

We proceed in several steps to show how a suitable specialization of the $3M$ parameters a , b , and ξ^0 gives rise to a tau function with the B_∞ -symmetry (1.11). This also involves the choice

$$M = 2N, \quad N \in \mathbb{N}^*, \quad (2.5)$$

which will be in force from now on.

Proposition 2.1: *Define a Cauchy-type matrix C with elements*

$$C_{jk} = \frac{a_j - b_j}{a_j - b_k}, \quad j, k = 1, \dots, M, \quad (2.6)$$

and a diagonal matrix

$$D_n = \text{diag}(\exp(\xi_{1,n}), \dots, \exp(\xi_{M,n})). \quad (2.7)$$

Then τ_n may be rewritten as

$$\tau_n = |\mathbf{1}_M + D_n C|. \quad (2.8)$$

Proof: Using Cauchy’s identity (1.24), the principal minor expansion of the right hand side of (2.8) yields

$$\tau_n = \sum_{l=0}^M \sum_{I \subset \{1, \dots, M\}, |I|=l} \prod_{j \in I} \exp(\xi_{j,n}) \cdot \prod_{j,k \in I, j < k} f_{jk}, \tag{2.9}$$

which amounts to (2.1). □

Next, we specialize the parameters so as to achieve the B_∞ reduction. Specifically, we choose

$$b_{M-j+1} = -a_j, \quad j = 1, \dots, M, \tag{2.10}$$

$$\exp(\xi_j^0) = \exp(\varphi_j) a_{M-j+1} (a_j + a_{M-j+1})^{-1}, \quad j = 1, \dots, N, \tag{2.11}$$

$$\exp(\xi_{M-j+1}^0) = -\exp(\varphi_j) a_j (a_j + a_{M-j+1})^{-1}, \quad j = 1, \dots, N. \tag{2.12}$$

Proposition 2.2: *With (2.10)–(2.12) in effect, τ_n has the B_∞ symmetry (1.11).*

Proof: Setting

$$\chi_j = \exp((\varphi_j + \psi_j)/2), \quad j = 1, \dots, M, \tag{2.13}$$

with

$$\varphi_{M-j+1} = \varphi_j, \quad j = 1, \dots, N, \tag{2.14}$$

$$\psi_j = -iv(a_j + a_{M-j+1}) + iu(a_j^{-1} + a_{M-j+1}^{-1}), \quad j = 1, \dots, M, \tag{2.15}$$

we clearly have

$$\chi_{M-j+1} = \chi_j, \quad j = 1, \dots, M. \tag{2.16}$$

Also, the reparametrized matrix $D_n C$ is the product of the matrix

$$\mathcal{J} = \text{diag}(\mathbf{1}_N, -\mathbf{1}_N), \tag{2.17}$$

and the matrix with jk -element

$$\chi_j^2 \left(-\frac{a_j}{a_{M-j+1}} \right)^n \frac{a_{M-j+1}}{a_j + a_{M-k+1}}, \quad j, k = 1, \dots, M. \tag{2.18}$$

After a similarity transformation τ_n may be rewritten as

$$\tau_n = |\mathbf{1}_M + \mathcal{D} \mathcal{A}_n \mathcal{D} \mathcal{J}|, \tag{2.19}$$

where

$$\mathcal{D} = \text{diag}(\chi_1, \dots, \chi_N, \chi_N, \dots, \chi_1), \tag{2.20}$$

$$\mathcal{A}_{n,jk} = \frac{(-a_j)^n (a_{M-k+1})^{-n+1}}{a_j + a_{M-k+1}}, \quad j, k = 1, \dots, M. \tag{2.21}$$

From this definition of \mathcal{A}_n we readily infer that

$$\mathcal{A}_{-n+1,jk} = -\mathcal{A}_{n,M-k+1,M-j+1}. \tag{2.22}$$

Hence we obtain

$$\mathcal{A}_{-n+1} = -\mathcal{R}_M \mathcal{A}_n^t \mathcal{R}_M, \tag{2.23}$$

where the superscript denotes the transpose, and \mathcal{R}_M denotes the $M \times M$ reversal permutation matrix. Now we have

$$\mathcal{R}_M \mathcal{D} \mathcal{R}_M = \mathcal{D}, \quad \mathcal{R}_M \mathcal{J} \mathcal{R}_M = -\mathcal{J}. \tag{2.24}$$

From these formulas we deduce

$$\tau_{-n+1} = |\mathbf{1}_M - \mathcal{D} \mathcal{R}_M \mathcal{A}_n^t \mathcal{R}_M \mathcal{D} \mathcal{J}| = |\mathbf{1}_M + \mathcal{R}_M \mathcal{D} \mathcal{A}_n^t \mathcal{D} \mathcal{J} \mathcal{R}_M| = |\mathbf{1}_M + \mathcal{D} \mathcal{A}_n^t \mathcal{D} \mathcal{J}| = |\mathbf{1}_M + \mathcal{J} \mathcal{D} \mathcal{A}_n \mathcal{D}| = \tau_n, \tag{2.25}$$

as required. □

Finally, we show how by a further specialization of parameters periodic reductions of the solitons of the B_∞ Toda lattice arise. We trade the $2N$ parameters a_1, \dots, a_M for N parameters $\theta_1, \dots, \theta_N$ and another parameter c , which corresponds to the coupling parameter in (1.16)–(1.19):

$$a_j = \exp(\theta_j - ic), \quad j = 1, \dots, N, \tag{2.26}$$

$$a_{M-j+1} = -\exp(\theta_j + ic), \quad j = 1, \dots, N. \tag{2.27}$$

Proposition 2.3: *With (2.26) and (2.27) in force, we have an implication*

$$c = \pi/l \Rightarrow \tau_{n+l} = \tau_n, l = 1, 2, \dots. \tag{2.28}$$

In particular, the tau function is 3-periodic for $c = \pi/3$, and hence is a tau function form of the Tzitzeica N -soliton solutions.

Proof: Using also (2.11)–(2.15), we obtain

$$\exp(\xi_{j,n}) = \frac{e^{ic}}{2i \sin c} \chi_j^2 e^{-2inc}, \tag{2.29}$$

$$\exp(\xi_{M-j+1,n}) = \frac{e^{-ic}}{2i \sin c} \chi_j^2 e^{2inc}, \tag{2.30}$$

$$\chi_j^2 = \exp(\varphi_j - 2 \sin(c)[v e^{\theta_j} + u e^{-\theta_j}]), \tag{2.31}$$

where $j = 1, \dots, N$. From this (2.28) is obvious. □

III. EXPLICIT FORM OF THE TAU FUNCTIONS

In this section we first make the soliton formulas described in Sec. II more explicit, exemplifying them for the simplest case. A suitable parametrization of the ϕ_j in (2.11) and (2.12) then yields a sum formula for the arbitrary- c tau function that is not only compact and informative but which can also be tied in with the relativistic Calogero–Moser tau function (1.39) for the special c -value $\pi/3$.

First, from (2.1)–(2.4), the two-soliton solution of the 2D Toda lattice is

$$\tau_n = 1 + e^{\xi_{1,n}} + e^{\xi_{2,n}} + \frac{(a_1 - a_2)(b_1 - b_2)}{(a_1 - b_2)(b_1 - a_2)} e^{\xi_{1,n} + \xi_{2,n}}. \tag{3.1}$$

After the reduction to the B_∞ Toda lattice described in Proposition 2.2, (2.3) and (2.4) yield

$$\exp(\xi_{j,n}) = \frac{(-1)^n a_j^n a_{M-j+1}^{1-n}}{a_j + a_{M-j+1}} e^{\varphi_j(u,v)}, \quad \exp(\xi_{M-j+1,n}) = \frac{(-1)^{n+1} a_j^{1-n} a_{M-j+1}^n}{a_j + a_{M-j+1}} e^{\varphi_j(u,v)}, \tag{3.2}$$

where

$$\varphi_j(u,v) = \varphi_j + i[(a_j^{-1} + a_{M-j+1}^{-1})u - (a_j + a_{M-j+1})v], \tag{3.3}$$

and $j = 1, \dots, N$, and (2.2) becomes

$$f_{jk} = \frac{(a_j - a_k)(a_{M-j+1} - a_{M-k+1})}{(a_j + a_{M-k+1})(a_{M-j+1} + a_k)}, \quad (3.4)$$

where $j, k = 1, \dots, M$. The two-soliton solution of 2D Toda (3.1) becomes the one-soliton solution of the B_∞ Toda equation, given by

$$\tau_n = 1 + \frac{(-1)^n (a_1^n a_2^{1-n} - a_1^{1-n} a_2^n)}{a_1 + a_2} e^{\varphi_1(u,v)} + \frac{a_1 a_2 (a_1 - a_2)^2}{(a_1 + a_2)^4} e^{2\varphi_1(u,v)}. \quad (3.5)$$

The result of the specialization (2.26) and (2.27) is that (3.2) and (3.3) become

$$\exp(\xi_{j,n}) = \frac{e^{(1-2n)ic}}{2i \sin c} e^{\varphi_j(u,v)}, \quad \exp(\xi_{M-j+1,n}) = \frac{e^{-(1-2n)ic}}{2i \sin c} e^{\varphi_j(u,v)}, \quad (3.6)$$

$$\varphi_j(u,v) = \varphi_j - 2 \sin(c) [ue^{-\theta_j} + ve^{\theta_j}], \quad (3.7)$$

where $j = 1, \dots, N$. Using (2.26) and (2.27) in (3.4), we finally obtain the following key expressions for the functions f_{jk} :

$$f_{jk} = \frac{\sinh^2((\theta_j - \theta_k)/2)}{\sinh^2((\theta_j - \theta_k)/2) + \sin^2 c}, \quad f_{j,M-k+1} = \frac{\cosh^2((\theta_j - \theta_k)/2) - \sin^2 c}{\cosh^2((\theta_j - \theta_k)/2)} \quad (3.8)$$

for $j, k = 1, \dots, N$ and

$$f_{jk} = f_{M-j+1, M-k+1}, \quad (3.9)$$

for $j, k = 1, \dots, M$. Furthermore, the specialization of the one-soliton solution of B_∞ Toda (3.5) gives

$$\tau_n = 1 + \frac{\cos((2n-1)c)}{i \sin c} e^{\varphi_1(u,v)} - \frac{\cos^2 c}{4 \sin^2 c} e^{2\varphi_1(u,v)}. \quad (3.10)$$

So far in this section we have simply restated various formulas in Sec. II in more explicit form. Now we make a final reparametrization of the phase constants φ_j . Specifically, choosing from now on $(q, \theta) \in \mathcal{P}$ (see (1.15)), we set

$$\exp(\varphi_j) = 2i \sin(c) \exp(q_j/2) F_j(\theta), \quad j = 1, \dots, N, \quad (3.11)$$

$$F_j(\theta) = \prod_{k=1, \dots, M, k \neq j} f_{jk}^{-1/2}, \quad j = 1, \dots, N. \quad (3.12)$$

Substituting this in (3.6) and (3.7), we obtain

$$\exp(\xi_{j,n}) = \exp((1-2n)ic + q_j(u,v)/2) F_j(\theta), \quad (3.13)$$

$$\exp(\xi_{M-j+1,n}) = \exp((2n-1)ic + q_j(u,v)/2) F_j(\theta), \quad (3.14)$$

$$q_j(u,v) = q_j - 4 \sin(c) [ve^{\theta_j} + ue^{-\theta_j}], \quad (3.15)$$

where $j = 1, \dots, N$. Note from (3.8) that the functions f_{jk} are positive and hence so are F_1, \dots, F_N .

We have now arrived at the final form of the tau function. In the proof of the next proposition we obtain an explicit sum formula to show that it is real valued.

Proposition 3.1: *The tau function τ_n given by (2.9) with the factors defined by (3.8) and (3.9) and (3.12)–(3.15) is real valued. In particular, for $c = \pi/3$ it yields a real valued N -soliton solution of the Tzitzeica equation.*

Proof: It remains to prove that τ_n is real. It is clear from (2.9) that τ_n is equal to a sum of terms

T_S associated with all subsets S of $\{1, \dots, M\}$. To obtain an explicit expression for T_S , we write the subsets as

$$S = \{i_1, \dots, i_s, M - j_b + 1, \dots, M - j_1 + 1\}, \quad (3.16)$$

where

$$1 \leq i_1 < \dots < i_s \leq N, \quad 1 \leq j_1 < \dots < j_b \leq N, \quad s, b \in \{0, 1, \dots, N\}. \quad (3.17)$$

Then we deduce from (2.9) that the contribution to τ_n corresponding to S is given by

$$T_S = \exp\left((b-s)(2n-1)ic + \sum_{\sigma=1}^s q_{i_\sigma}(u,v)/2 + \sum_{\beta=1}^b q_{j_\beta}(u,v)/2\right) F_S(\theta), \quad (3.18)$$

where

$$F_S(\theta) = \prod_{j \in S, k \notin S} f_{jk}^{-1/2}. \quad (3.19)$$

Next, defining the reversal permutation r of $\{1, 2, \dots, M\}$ by $r(j) = M - j + 1$, it follows from (3.9) that

$$F_{r(S)} = F_S \quad (3.20)$$

and then from (3.18) that

$$T_{r(S)} = \overline{T_S}, \quad (3.21)$$

where the overbar denotes the complex conjugate. Hence, if $r(S)$ is equal to S then T_S is real, and otherwise $T_S + T_{r(S)}$ is real. Thus τ_n can be written as a sum of real terms. \square

To illustrate this result, we write the one-soliton tau function in the explicit form described in the above proof. We have

$$T_\emptyset = 1, \quad T_{\{1,2\}} = \exp(q_1(u,v)) F_{\{1,2\}}, \quad (3.22)$$

$$T_{\{1\}} = \exp((1-2n)ic + q_1(u,v)/2) F_{\{1\}}, \quad (3.23)$$

$$T_{\{2\}} = \exp((2n-1)ic + q_1(u,v)/2) F_{\{2\}}, \quad (3.24)$$

where (recall (3.8) and $c \in (0, \pi/2)$)

$$F_{\{2\}} = F_{\{1\}} = f_{12}^{-1/2} = 1/\cos c, \quad F_{\{1,2\}} = 1. \quad (3.25)$$

Clearly, the terms T_\emptyset and $T_{\{1,2\}}$ are real, and so is the sum

$$T_{\{1\}} + T_{\{2\}} = \frac{2 \cos((2n-1)c)}{\cos c} \exp(q_1(u,v)/2). \quad (3.26)$$

The one-soliton tau function is given by

$$\tau_n = 1 + \frac{2 \cos((2n-1)c)}{\cos c} e^{q_1(u,v)/2} + e^{q_1(u,v)}. \quad (3.27)$$

In particular, for the Tzitzeica case $c = \pi/3$ we have

$$\tau_0 = \tau_1 = 1 + 2e^{q_1(u,v)/2} + e^{q_1(u,v)} = (1 + e^{q_1(u,v)/2})^2 \quad (3.28)$$

and

$$\tau_2 = 1 - 4e^{q_1(u,v)/2} + e^{q_1(u,v)}. \quad (3.29)$$

Finally, we sketch how to obtain the two-soliton tau function in a similar way. In this case M equals 4 and for the 16 subsets of $\{1,2,3,4\}$ there are ten real valued combinations

$$\begin{aligned} T_\emptyset, \quad T_{\{1,4\}}, \quad T_{\{2,3\}}, \quad T_{\{1,2,3,4\}}, \quad T_{\{1\}} + T_{\{4\}}, \quad T_{\{2\}} + T_{\{3\}}, \\ T_{\{1,2\}} + T_{\{3,4\}}, \quad T_{\{1,3\}} + T_{\{2,4\}}, \quad T_{\{1,2,3\}} + T_{\{2,3,4\}}, \quad T_{\{1,2,4\}} + T_{\{1,3,4\}}, \end{aligned} \quad (3.30)$$

whose sum gives the two-soliton tau function.

IV. TZITZEICA SOLITONS VERSUS RELATIVISTIC CALOGERO–MOSER DYNAMICS

In Sec. I we have already sketched in general terms how the $2N$ -dimensional Poincaré-invariant subset Ω_P of the phase space $\langle \Omega, \omega \rangle$ arises. In order to fill in the details of this qualitative picture, we begin by specifying the subspace of Ω on which the maximal number N of bound states is present, choosing the internal actions at first larger than their minimum, so that there are N angles varying over \mathbb{T}^N . Moreover, until further notice we work with $c \in (0, \pi/2)$, since this eases the notation and adds insight on the Tzitzeica case $c = \pi/3$. With this starting point understood, the variables \hat{q} , $\hat{\theta}$ in Eqs. (4.20)–(4.24) of Ref. 31 should be specialized as follows.

First, as already mentioned (see (1.34)), the particle and antiparticle numbers N_+ and N_- of Ref. 31 must be chosen equal to $2N$ and N (so that the number N used in Ref. 31 becomes $3N$), and the parameters of Ref. 31 should be specialized as

$$\mu = \beta = 1, \quad \tau = g/2 = c. \quad (4.1)$$

Second, the bound state number l of Ref. 31 should be taken equal to N . Thus the numbers k_+ and k_- in (4.20)–(4.24) are equal to N and 0, respectively, and we obtain $2N$ particle variables

$$\hat{q}_1, \dots, \hat{q}_N \in \mathbb{R}, \quad \hat{\theta}_N < \dots < \hat{\theta}_1 \quad (4.2)$$

and $4N$ bound state variables

$$\hat{q}_{N+1}, \dots, \hat{q}_{2N}, \hat{\theta}_{N+1}, \dots, \hat{\theta}_{2N} \in \mathbb{C}, \quad \Im(\hat{\theta}_{N+j}) \in (0, c), \quad \hat{\theta}_{N+j} \neq \hat{\theta}_{N+k}, \quad (4.3)$$

$$\hat{q}_{2N+j} = \bar{q}_{N+j} + \frac{i\pi}{2}(1 - (-)^N), \quad \hat{\theta}_{2N+j} = \bar{\theta}_{N+j}, \quad (4.4)$$

where $j, k = 1, \dots, N$ and $k \neq j$.

One advantage of these variables is that they give rise to compact expressions for the symmetric functions of the dual Lax matrix A . Specifically, they are given by

$$S_l(A) = \sum_{|I|=l} \exp\left(\sum_{i \in I} \hat{q}_i\right) p_l(\hat{\theta}), \quad l = 0, \dots, 3N, \quad (4.5)$$

with

$$p_l(\hat{\theta}) = \prod_{i \in I, j \notin I} \frac{[\sinh^2((\hat{\theta}_i - \hat{\theta}_j)/2) + \sin^2 c]^{1/2}}{\sinh((\hat{\theta}_{\min(i,j)} - \hat{\theta}_{\max(i,j)})/2)}, \quad (4.6)$$

see (5.68) in Ref. 31 with parameters (4.1).

The action-angle variables are now given by

$$x_j^s = \hat{q}_j, \quad p_j^s = \hat{\theta}_j, \quad (4.7)$$

$$x_j = \mathfrak{R}(\hat{q}_{N+j}), \quad p_j = 2\mathfrak{R}(\hat{\theta}_{N+j}), \tag{4.8}$$

$$\gamma_j = \mathfrak{I}(\hat{q}_{N+j}) + (N + j - 1)\pi, \quad \delta_j = -2\mathfrak{I}(\hat{\theta}_{N+j}), \tag{4.9}$$

where $j=1, \dots, N$ and $\gamma_j \in (-\pi, \pi] \pmod{2\pi}$, see (4.26) in Ref. 31. To handle the limits $\delta_j \downarrow -2c$, for which the γ_j -torus reduces to a point, we should switch from γ_j, δ_j to the ‘‘harmonic oscillator variables’’ u_j, v_j , the minimum $\delta_j = -2c$ corresponding to the origin $u_j = v_j = 0$, see Chap. 5 in Ref. 31. We shall presently analyze the behavior of $S_j(A)$ under this limit. Before embarking on this rather technical issue, however, we clarify how the limit gives rise to a $2N$ -dimensional submanifold that is invariant under the Poincaré (inhomogeneous Lorentz) group.

To this end we use the explicit description of the commuting flows generated by Hamiltonians of the form

$$H_h = \text{Tr } h(\ln L), \tag{4.10}$$

with $h(z)$ an arbitrary entire function (see (6.79) in Ref. 31), which follows from (6.6) in Ref. 31. It reads

$$\begin{aligned} \exp(tH_h)(x^s, p^s; x, p, 0, 0) &= (x_1^s + th'(p_1^s), \dots, x_N^s + th'(p_N^s), p^s; x_1 + t\mathfrak{R}(h'(p_1/2 - ic)), \\ &\dots, x_N + t\mathfrak{R}(h'(p_N/2 - ic)), p, 0, 0). \end{aligned} \tag{4.11}$$

The time and space translation generators H and P (see (1.16)), for which $h(z)$ equals $M_0 \cosh(z)$ and $M_0 \sinh(z)$, respectively, reduce to

$$H = M_0 \sum_{j=1}^N (\cosh(p_j^s) + 2 \cos(c) \cosh(p_j/2)), \quad P = M_0 \sum_{j=1}^N (\sinh(p_j^s) + 2 \cos(c) \sinh(p_j/2)) \tag{4.12}$$

on the submanifold at issue, yielding flows

$$\begin{aligned} e^{tHM_0}(x^s, p^s; x, p, 0, 0) &= (x_1^s + t \sinh(p_1^s), \dots, x_N^s + t \sinh(p_N^s), p^s; x_1 + t \cos(c) \sinh(p_1/2), \\ &\dots, x_N + t \cos(c) \sinh(p_N/2), p, 0, 0), \end{aligned} \tag{4.13}$$

$$\begin{aligned} e^{-yPM_0}(x^s, p^s; x, p, 0, 0) &= (x_1^s - y \cosh(p_1^s), \dots, x_N^s - y \cosh(p_N^s), p^s; x_1 \\ &- y \cos(c) \cosh(p_1/2), \dots, x_N - y \cos(c) \cosh(p_N/2), p, 0, 0). \end{aligned} \tag{4.14}$$

The boost generator is given by (see (1.16))

$$B = \sum_{j=1}^N (x_j^s + 2x_j), \tag{4.15}$$

whence we have

$$e^{\alpha B}(x^s, p^s; x, p, 0, 0) = (x^s, p_1^s + \alpha, \dots, p_N^s + \alpha; x, p_1 + 2\alpha, \dots, p_N + 2\alpha, 0, 0). \tag{4.16}$$

From these formulas it is obvious that the $2N$ -dimensional submanifold Ω_p obtained by requiring

$$x_j = \cos(c)x_j^s, \quad p_j = 2p_j^s, \quad j = 1, \dots, N, \tag{4.17}$$

is Poincaré invariant. On Ω_p we have

$$H = p(c)M_0 \sum_{j=1}^N \cosh p_j^s, \quad P = p(c)M_0 \sum_{j=1}^N \sinh p_j^s, \quad B = p(c) \sum_{j=1}^N x_j^s, \quad (4.18)$$

where the prefactor $p(c)$ is given by

$$p(c) = 1 + 2 \cos(c). \quad (4.19)$$

Also, the symplectic form ω reduces to

$$\omega = p(c) \sum_{j=1}^N dx_j^s \wedge dp_j^s. \quad (4.20)$$

Thus, when we reparametrize Ω_P with variables

$$q_j = p(c)x_j^s, \quad \theta_j = p_j^s, \quad j = 1, \dots, N, \quad (4.21)$$

then q, θ are canonical coordinates. Moreover the space-time flow reads

$$e^{tH - yP}(q, \theta) = (q_1 + p(c)M_0(t \sinh \theta_1 - y \cosh \theta_1), \dots, q_N + p(c)M_0(t \sinh \theta_N - y \cosh \theta_N), \theta), \quad (4.22)$$

or, equivalently,

$$e^{uS_- - vS_+}(q, \theta) = (q_1(u, v), \dots, q_N(u, v), \theta), \quad (4.23)$$

where

$$q_j(u, v) = q_j - p(c)M_0(ue^{-\theta_j} + ve^{\theta_j}), \quad j = 1, \dots, N. \quad (4.24)$$

With (1.38) in force, this coincides with the Tzitzeica soliton space-time dependence (3.15). Of course, this state of affairs is still far from a proof that the tau function defined at the end of Sec. III and the tau function (1.39) evaluated on Ω_P are equal.

In order to demonstrate this equality, we return to (4.5) and (4.6) and the variables $\hat{q}, \hat{\theta}$. On Ω_P the latter specialize as

$$\hat{q}_j = q_j/p(c), \quad \Re(\hat{q}_{N+j}) = \cos(c)q_j/p(c), \quad j = 1, \dots, N, \quad (4.25)$$

$$\hat{\theta}_j = \theta_j, \quad \hat{\theta}_{N+j} = \theta_j + ic, \quad j = 1, \dots, N, \quad (4.26)$$

and (4.4) still holds. More precisely, we have

$$\hat{\theta}_{2N+j} = \theta_j - ic, \quad j = 1, \dots, N, \quad (4.27)$$

but we can only infer

$$\hat{q}_{N+j} + \hat{q}_{2N+j} = 2 \cos(c)q_j/p(c) + \frac{i\pi}{2}(1 - (-)^N), \quad (4.28)$$

since the imaginary parts are undetermined in the collapsing torus limit.

We proceed to analyze this limit for (4.5) and (4.6). First, whenever I contains the index $N+j$ but not the index $2N+j$, with $j=1, \dots, N$, or vice versa, then it is clear from (4.6) that the limit of $p_j(\hat{\theta})$ vanishes. Thus we need only consider I that contain either both indices or neither. As a consequence, we only encounter \hat{q}_{N+j} and \hat{q}_{2N+j} in the combination (4.28), which confirms that the limit is well defined.

Introducing the breather sets

$$B_j = \{N + j, 2N + j\}, \quad j = 1, \dots, N, \tag{4.29}$$

we proceed to simplify the limit of $p_I(\hat{\theta})^2$ for I of the form

$$I = \{i_1, \dots, i_s\} \cup B_{j_1} \cup \dots \cup B_{j_b}, \tag{4.30}$$

with (3.17) in effect. This boils down to a quite special case of the fusion procedure detailed on pp. 237–238 of Ref. 33. Thus we put

$$J = \{i_1, \dots, i_s, j_1 + N, \dots, j_b + N\} \subset \{1, \dots, 2N\} \tag{4.31}$$

and

$$\eta_j = \theta_j, \quad \eta_{j+N} = \theta_j, \quad c_j = c, \quad c_{j+N} = 2c, \quad j = 1, \dots, N, \tag{4.32}$$

to obtain

$$\lim p_I(\hat{\theta})^2 = \prod_{j \in J, k \notin J} \frac{\sinh^2((\eta_j - \eta_k)/2) + \sin^2((c_j + c_k)/2)}{\sinh^2((\eta_j - \eta_k)/2) + \sin^2((c_j - c_k)/2)}, \tag{4.33}$$

see (2.27) in Ref. 33. To render this step in our reasoning self-contained, we present a proof of (4.33) in Appendix A.

Requiring (1.38) from now on, the following two cases arise for the product factor p_{jk} on the right hand side of (4.33):

$$j, k \leq N \quad \text{or} \quad j, k > N \quad \text{with} \quad j \neq k \Rightarrow p_{jk} = \frac{\sinh^2((\theta_j - \theta_k)/2) + 3/4}{\sinh^2((\theta_j - \theta_k)/2)}, \tag{4.34}$$

$$j \leq N, \quad k > N \quad \text{or} \quad j > N, \quad k \leq N \Rightarrow p_{jk} = \frac{\sinh^2((\theta_j - \theta_k)/2) + 1}{\sinh^2((\theta_j - \theta_k)/2) + 1/4}. \tag{4.35}$$

Taking positive square roots of the p_{jk} , we deduce from (4.33) that we have

$$\lim p_I(\hat{\theta}) = s_J \prod_{j \in J, k \notin J} p_{jk}^{1/2} \tag{4.36}$$

for a certain sign s_J . We shall determine this sign later on, see (4.45).

Next, we focus on the principal minor expansion of the tau function (1.39), restricted to the submanifold of Ω with N bound states present, with internal actions $\delta_1, \dots, \delta_N$ in $(-2\pi/3, 0)$. It reads

$$\tau_n(u, v) = \sum_{l=0}^{3N} \exp(i\pi l(1 - 2n)/3) S_l(A(x(u, v))) = \sum_{l=0}^{3N} \exp(i\pi l(1 - 2n)/3) \sum_{|I|=l} \exp\left(\sum_{i \in I} \hat{q}_i(u, v)\right) p_I(\hat{\theta}). \tag{4.37}$$

Taking the actions to their minima $-2\pi/3$, we deduce from the above that the limiting tau function is a sum of nonzero contributions C_J for all J of the form (4.31), with

$$C_J = \exp(i\pi(s + 2b)(1 - 2n)/3) \prod_{\sigma=1}^s \exp(q_{i_\sigma}(u, v)/2) \times \prod_{\beta=1}^b \exp\left(q_{j_\beta}(u, v)/2 + \frac{i\pi}{2}(1 - (-)^{N_j})\right) \cdot s_J \prod_{j \in J, k \notin J} p_{jk}^{1/2}. \tag{4.38}$$

We are now prepared to state and prove the principal result of this paper.

Theorem 4.1: *Let $c = \pi/3$. Then the relativistic Calogero–Moser tau function (1.39) restricted*

to the subspace $\Omega_{\mathcal{P}} \approx \mathcal{P}$ is equal to the Tzitzeica tau function (2.9) with the factors given by (3.8), (3.9) and (3.12)–(3.15), and with parameters in \mathcal{P} .

Proof: We need only show equality of the general contribution C_J to the general term T_S in (2.9), see (3.18). First, we compare f_{jk} , given by (3.8) and (3.9) with $c = \pi/3$, to p_{jk} given by (4.34) and (4.35). From this we easily deduce

$$F_S(\theta) = \prod_{j \in J, k \notin J} p_{jk}^{1/2}. \tag{4.39}$$

Next, we note that the factors involving $q_j(u, v)$ are in agreement. Hence the asserted equality comes down to an equality of the remaining numerical factors. Specifically, it remains to show

$$(-)^b \exp\left(\frac{i\pi b}{2}(1 - (-)^N)\right) s_J = 1. \tag{4.40}$$

To this end we now calculate the sign s_J in (4.36). We begin by noting that for $\hat{\theta} \in \mathbb{C}^{3N}$ given by (4.2) and (4.3), the product $p_I(\hat{\theta})$ (4.6) has a positive numerator. (Indeed, each radicand is either positive or has a nonzero imaginary part; in the latter case it is matched by a factor with the complex-conjugate radicand.) We are therefore reduced to analyzing the phase of

$$\Pi_I = \prod_{i \in I, j \notin I} s_{ij}, \quad s_{ij} = \sinh((\hat{\theta}_{\min(i,j)} - \hat{\theta}_{\max(i,j)})/2) \tag{4.41}$$

for the special $\hat{\theta}$ under consideration, namely,

$$\hat{\theta} = (\theta_1, \dots, \theta_N, \theta_1 + i\pi/3, \dots, \theta_N + i\pi/3, \theta_1 - i\pi/3, \dots, \theta_N - i\pi/3), \quad \theta_N < \dots < \theta_1. \tag{4.42}$$

We already know from (4.36) that this phase is just the sign s_J . Indeed, we have

$$\sinh((\theta_j - \theta_k - i\psi)/2) \sinh((\theta_j - \theta_k + i\psi)/2) > 0, \quad \psi = \pi/3, \quad 2\pi/3, \tag{4.43}$$

which confirms that Π_I is either positive or negative.

We continue to analyze the contributions to s_J from indices in I (4.30), taking (4.42) into account. First, we consider the contribution of $i_{\sigma^j} \in I$. Due to the ordering of the θ_j in (4.42), any $j \notin I$ with $j \leq N$ yields $s_{i_{\sigma^j}} > 0$. Also, any B_k that is not a subset of I yields a contribution of the form (4.43) with $\psi = \pi/3$. Hence all indices i_1, \dots, i_s in I yield positive signs.

Next, we study the contribution of $B_{j_{\beta}} \subset I$. For $i \in I$ with $i \leq N$ it again follows from (4.43) that we get a positive sign. Now suppose B_k is not a subset of I and consider the pertinent product

$$s_{j_{\beta}+N, k+N} s_{j_{\beta}+N, k+M} s_{j_{\beta}+M, k+N} s_{j_{\beta}+M, k+M}. \tag{4.44}$$

Both for $k < j_{\beta}$ and for $k > j_{\beta}$ this product is easily seen to be negative. Therefore any pair B_j, B_k with B_j included in I and B_k not included in I gives rise to a minus sign in Π_I . Now I contains b breather index sets, so there are $N-b$ breather sets not contained in I . Thus we finally deduce

$$s_J = (-)^{b(N-b)}. \tag{4.45}$$

With this explicit formula in hand, it is routine to verify (4.40). Consequently, our proof of the asserted tau function equality is now complete. \square

V. THE $N=1$ CASE

In this section we consider various aspects of the special case $N=1$, which yields an illuminating illustration of the above constructions and equalities. More specifically, we focus on features of the Tzitzeica one-soliton solution and the space $\Omega_{\mathcal{P}}$ for $c \in (0, \pi/2)$.

For $N=1$ it follows from (3.28) and (3.29) that we have

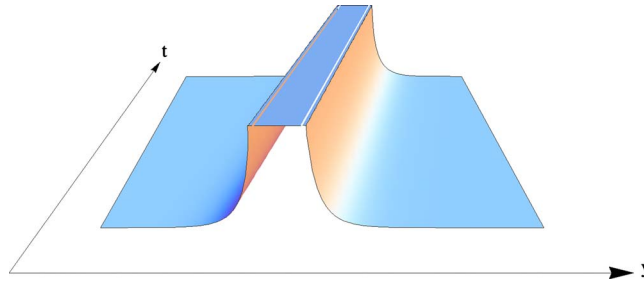


FIG. 1. (Color online) One-soliton solution. $-\Psi(t,y)$ is shown; the y -axis points left to right and the t -axis points into the page. The plateau approximates the region in which $\Psi(t,y)$ is complex valued.

$$\tau_0 = \tau_1 = 1 + 2F + F^2, \quad \tau_2 = 1 - 4F + F^2, \quad F = \exp(q(u,v)/2), \quad (5.1)$$

with

$$q(u,v) = q - 2\sqrt{3}(ue^{-\theta} + ve^\theta) = q + 2\sqrt{3}(t \sinh(\theta) - y \cosh(\theta)), \quad (5.2)$$

see (1.2). Thus the two τ_2 -zeros for $F=2 \pm \sqrt{3}$ yield two parallel space-time lines,

$$y_{\pm}(t) = t \tanh(\theta) + \frac{1}{2\sqrt{3} \cosh(\theta)}(q - 2 \ln(2 \pm \sqrt{3})), \quad (5.3)$$

where Ψ diverges, see Fig. 1. In this figure and in Fig. 2, which appears in Sec. VII B, we plot for clarity $-\Psi$ rather than Ψ and truncate $-\Psi$ at a finite cutoff value. The regions between the singularities, where τ_2 is negative so that $\Psi = \ln(\tau_2/\tau_0)$ takes complex values, are approximated by the “plateaus” in these figures.

Recalling (1.39), we see that the τ_2 -zeros correspond to $x_1^+(t,y)$ or $x_2^+(t,y)$ being zero. The space-time lines may therefore be viewed as the trajectories of the two particles in the one-soliton cluster.

For $N=1$ it might seem an easy matter to locate the two-dimensional space Ω_p within the six-dimensional phase space Ω . In fact, however, it is not even easy to find the one-dimensional submanifold of Ω_p consisting of H -equilibrium points, i.e.,

$$\Omega_E = \{(q, \theta) \in \Omega_p \approx \mathbb{R}^2 | \theta = 0\}, \quad (5.4)$$

in explicit form. (Note that for $c = \pi/3$ this yields the stationary one-soliton tau functions.) Of course, it is immediate from (1.16)–(1.19) that we need

$$p_1^+ = p_2^+ = p_1^- = 0 \quad (5.5)$$

for H to have an equilibrium. Requiring this, one expects from physical considerations to get an equilibrium point for

$$x_1^+ = -x_2^+ = d > 0, \quad x_1^- = 0 \quad (5.6)$$

and a suitable d . We now confirm this for $c \in (0, \pi/2)$. Then we have from (1.16)–(1.19)

$$\frac{H(d, -d, 0, 0, 0, 0)}{M_0} = 2(f^+ f^-)^{1/2} + f^-, \quad f^+ = 1 + \frac{\sin^2 c}{\sinh^2 d}, \quad f^- = 1 - \frac{\sin^2 c}{\cosh^2(d/2)}. \quad (5.7)$$

Thus we have $H \rightarrow 3M_0$ for $d \rightarrow \infty$ and $H \rightarrow \infty$ for $d \rightarrow 0$. Since H is equal to $M_0 \sum_{j=1}^3 \cosh(\theta_j)$, it has an absolute minimum $M_0 p(c)$ for $\hat{\theta} = (0, ic, -ic)$. Now we can still choose $x_1^s, x_1 \in \mathbb{R}$ (see (4.7) and (4.8)), so H has a two-parameter family of stable equilibria. It is straightforward to check that for d equal to

$$d_e = \cosh^{-1}(1 + \cos c), \quad (5.8)$$

we have

$$f^+ = \frac{1 + 2 \cos c}{(2 + \cos c) \cos c}, \quad f^- = \frac{(1 + 2 \cos c) \cos c}{2 + \cos c}. \quad (5.9)$$

Hence the right hand side of (5.7) equals $M_0 p(c)$, so we do get a stable equilibrium for $d=d_e$. This implies that all of the points

$$E(\sigma) = (d_e + \sigma, -d_e + \sigma, \sigma, 0, 0, 0), \quad \sigma \in \mathbb{R}, \quad (5.10)$$

are also equilibria.

Since H equals $M_0 p(c)$ on Ω_E , one might expect that the one-parameter family of equilibria $E(\sigma)$ yields Ω_E . In fact, however, only $E(0)$ belongs to Ω_E . To be specific, we assert that it corresponds to $q=0$. To show this, we use (4.5) with $N=1$ to find the symmetric functions of A on Ω_E . We determined the relevant limits below (4.28). In particular, (4.33) yields $p(c)^2$ for I equal to $\{1\}$ or $\{2,3\}$, since $\theta_1=0$ on Ω_E . Using also (4.25) and (4.28), we readily obtain

$$S_1(A) = p(c) \exp(q/p(c)), \quad (5.11)$$

$$S_2(A) = -p(c) \exp(2 \cos(c) q/p(c)), \quad (5.12)$$

$$S_3(A) = -\exp(q). \quad (5.13)$$

For $q=0$ this implies that the spectrum of A is given by

$$\sigma(A) = \{1 + \cos(c) \pm (\cos^2(c) + 2\cos(c))^{1/2}, -1\}. \quad (5.14)$$

The first two eigenvalues can be written as $\exp(\pm d_e)$, see (5.8). Thus the origin of $\Omega_E \simeq \mathbb{R}$ corresponds to generalized positions $x_1^+ = d_e$, $x_2^+ = -d_e$, $x_1^- = 0$. Since Ω_E consists of H -equilibria, we also have $p_1^+ = p_2^+ = p_1^- = 0$. Hence we deduce $E(0) \in \Omega_E$ and $q=0$, as asserted.

On the other hand, the equilibrium points in Ω_P with $q \neq 0$ are harder to find explicitly. As already announced, they do not include the equilibria $E(\sigma)$ for $\sigma \neq 0$. (Indeed, $E(0)$ does belong to Ω_E , as just shown. Now the σ -shift of the generalized positions corresponds to a shift of x_1^s and x_1 by σ , so that the Ω_P -condition $x_1 = \cos(c)x_1^s$ no longer holds true for $\sigma \neq 0$.) Rather, they are obtained by acting with the commuting flow $\exp(yP/M_0)$, $y \in \mathbb{R}$, on the equilibrium $E(0)$, yielding translated equilibria

$$T(y) = (x_1^+(y), x_2^+(y), x_1^-(y), 0, 0, 0), \quad y \in \mathbb{R}. \quad (5.15)$$

In view of (5.13) and (4.22) this yields the sum rule

$$x_1^+(y) + x_2^+(y) + x_1^-(y) = p(c)y. \quad (5.16)$$

Also, from Hamilton's equations for P/M_0 we have

$$x_j^+(y)' > \cos(c), \quad j=1,2, \quad x_1^-(y) \geq \cos^2(c), \quad (5.17)$$

so that $x_1^+(y)$, $x_2^+(y)$, and $x_1^-(y)$ are strictly increasing functions of y .

Next, we deduce from (5.11)–(5.13) that we have the reflection symmetry

$$x_1^+(-y) = -x_2^+(y), \quad x_1^-(-y) = -x_1^-(y), \quad (5.18)$$

so it remains to determine the functions for $y > 0$. This is presumably possible, but we have not pursued this. However, the large- y asymptotics can be established from the above. Specifically, setting

$$\epsilon = \exp(\cos c - 1)y, \quad (5.19)$$

we obtain for $\epsilon \rightarrow 0$

$$x_1^+(y) = y + \ln p(c) + O(\epsilon^2), \quad (5.20)$$

$$x_2^+(y) = \cos(c)y - \frac{1}{2} \ln p(c) - \frac{1}{2} \sqrt{p(c)}(1 - p(c)^{-2})\epsilon + O(\epsilon^2), \quad (5.21)$$

$$x_1^-(y) = \cos(c)y - \frac{1}{2} \ln p(c) + \frac{1}{2} \sqrt{p(c)}(1 - p(c)^{-2})\epsilon + O(\epsilon^2). \quad (5.22)$$

A final observation of interest concerns the kinetic and potential energy densities of the stationary one-soliton solution, i.e., the functions

$$E_K(y) = (\partial_y \Psi)^2/2, \quad E_P(y) = \exp(\Psi) + \exp(-2\Psi)/2 - 3/2, \quad \Psi = \ln(\tau_2/\tau_0), \quad (5.23)$$

obtained from (5.1) for $\theta=0$. It is far from obvious but true that these functions are equal. This equality can be verified directly by a straightforward but quite tedious calculation.

A more conceptual derivation of this virial-type identity will now be given. First, we note that any t -independent solution to (1.1) satisfies the ordinary differential equation

$$f_{yy} = e^f - e^{-2f}. \quad (5.24)$$

Now from (5.1) we obtain a solution to (5.24) of the form

$$f(y) = g(\exp((q - 2\sqrt{3}y)/2)), \quad g(z) = \ln\left(\frac{1 - 4z + z^2}{1 + 2z + z^2}\right). \quad (5.25)$$

Defining

$$z_{\pm} = 2 \pm \sqrt{3}, \quad (5.26)$$

it satisfies

$$e^{g(z)} > 0, \quad z > z_+, \quad z < z_-, \quad (5.27)$$

$$e^{g(z)} < 0, \quad z \in (z_-, z_+), \quad (5.28)$$

and

$$g(z), \quad g'(z) \rightarrow 0, \quad z \rightarrow \pm \infty, \quad (5.29)$$

$$\exp(g(1)) = -1/2, \quad g'(1) = 0. \quad (5.30)$$

Next, consider the Hamiltonians

$$H_{\pm}(x, p) = p^2/2 - V_{\pm}(x), \quad V_{\pm}(x) = \pm e^x + e^{-2x}/2 - 3/2. \quad (5.31)$$

The potential $-V_+(x)$ has a maximum 0 at $x=0$ and yields a Newton equation

$$\ddot{x} = e^x - e^{-2x}. \quad (5.32)$$

Comparing this to (5.24), (5.27), and (5.29), we see that $g(z)$ for $z > z_+$ corresponds to the $E=0$ orbit with $x(t) < 0$ coming from 0 for $t \rightarrow -\infty$ and $g(z)$ for $z < z_-$ to the $E=0$ orbit with $x(t) < 0$ going to 0 for $t \rightarrow \infty$. By energy conservation, we have $\dot{x}^2/2 = V_+(x)$, which implies $E_K(y) = E_P(y)$ for the y -intervals where $e^{f(y)} > 0$.

It remains to show the identity for the y -interval where $e^{f(y)} < 0$. Then we can compare (5.24) to the Newton equation

$$\ddot{x} = -e^x - e^{-2x}, \tag{5.33}$$

corresponding to H_- . The $E=0$ orbit stays to the left of the origin and has its turning point at $x = \ln(1/2)$, see (5.31). Comparing this to (5.30), we see that it corresponds to the real part of $g(z)$ for $z \in (z_-, z_+)$. Hence $E_K(y) = E_P(y)$ now follows from $\dot{x}^2/2 = V_-(x)$.

VI. THE DARBOUX AND KAPTSOV–SHAN’KO SOLITONS

In this section we begin by showing that the solitons obtained by the B_∞ -reduction from the Kyoto 2D Toda solitons are equal to those obtained by a similar reduction from a seemingly different class of 2D Toda solitons. The latter are constructed via repeated Darboux transformations. As will transpire, this procedure yields a larger class of solutions, involving an arbitrary constant antisymmetric matrix \mathcal{C} . In order to obtain equality to the B_∞ solitons of Sec. II, this matrix must be suitably specialized.

Our demonstration of equality leads to a new representation of the latter solitons. This representation can be exploited to show that τ_0 equals the square of a simpler tau function. This is because it readily leads to τ_0 being the determinant of an antisymmetric $M \times M$ matrix A . Hence it follows that we have

$$\tau_0 = \tau^2, \quad \tau = \text{Pf}(A). \tag{6.1}$$

The Pfaffian can be explicitly evaluated, and when the Tzitzeica substitutions of Secs. II and III are made in A , then the resulting τ is the one obtained by Kaptsov and Shan’ko in their study of the Tzitzeica equation.¹⁶

We proceed with the details. We start from the tau function τ_n (2.19), with \mathcal{J} , \mathcal{D} , and \mathcal{A} given by (2.17), (2.20), and (2.21) and with the quantities χ_j in \mathcal{D} arbitrary at this stage. We claim that τ_n is equal to the determinant $\tilde{\tau}_n$ of the matrix

$$\mathcal{M}_n = \mathcal{C}_R + \mathcal{B}\tilde{\mathcal{A}}_n\mathcal{B}, \tag{6.2}$$

where

$$\mathcal{C}_R = \begin{pmatrix} 0 & \mathcal{R}_N \\ -\mathcal{R}_N & 0 \end{pmatrix}, \tag{6.3}$$

$$\mathcal{B} = \text{diag}(\beta_1, \dots, \beta_M), \tag{6.4}$$

$$\tilde{\mathcal{A}}_{n,jk} = \frac{(-a_j)^n a_k^{-n+1}}{a_j + a_k}, \quad j, k = 1, \dots, M, \tag{6.5}$$

provided that β_1, \dots, β_M are chosen such that

$$\beta_j \beta_{M-j+1} = \chi_j^2, \quad j = 1, \dots, N. \tag{6.6}$$

(Recall that \mathcal{R}_N denotes the $N \times N$ reversal permutation matrix.)

In order to prove this, we denote the columns of \mathcal{M}_n by c_1, \dots, c_M and use

$$\tilde{\tau}_n = |\mathcal{M}_n| = |\text{Col}(c_1, \dots, c_M)| = |\text{Col}(c_M, \dots, c_{N+1}, -c_N, \dots, -c_1)|. \tag{6.7}$$

Thus, $\tilde{\tau}_n$ equals the determinant of the matrix

$$\mathbf{1}_M + \mathcal{N}_n \mathcal{J}, \quad \mathcal{N}_{n,jk} = \beta_j \beta_{M-k+1} \frac{(-a_j)^n (a_{M-k+1})^{-n+1}}{a_j + a_{M-k+1}}. \tag{6.8}$$

Transforming this matrix with the similarity matrix

$$S = \text{diag}(\gamma_1, \dots, \gamma_M), \quad \gamma_j = (\beta_{M-j+1}/\beta_j)^{1/2}, \quad j = 1, \dots, M, \quad (6.9)$$

we obtain

$$\tilde{\tau}_n = |\mathbf{1}_M + \tilde{\mathcal{D}}\mathcal{A}_n\tilde{\mathcal{D}}\mathcal{J}|, \quad (6.10)$$

with

$$\tilde{\mathcal{D}} = \text{diag}((\beta_1\beta_M)^{1/2}, (\beta_2\beta_{M-1})^{1/2}, \dots, (\beta_M\beta_1)^{1/2}). \quad (6.11)$$

Hence the β -constraint (6.6) entails equality of $\tilde{\tau}_n$ and τ_n , as advertised.

We can now compare the new representation

$$\tau_n = |\mathcal{C}_{\mathcal{R}} + \mathcal{D}\tilde{\mathcal{A}}_n\mathcal{D}|, \quad (6.12)$$

obtained by choosing

$$\beta_j = \beta_{M-j+1} = \chi_j, \quad j = 1, \dots, N \quad (6.13)$$

(so that (6.6) is obeyed, and \mathcal{B} and $\tilde{\mathcal{D}}$ both equal \mathcal{D}), with the solitons obtained by a B_∞ symmetry reduction from the Darboux-type 2D Toda solitons. (See Appendix C for a sketch of their construction.) The reduced solitons are of the form

$$\tau_n^{\mathcal{D}} = |\mathcal{C} + \tilde{\mathcal{B}}\tilde{\mathcal{A}}_n\tilde{\mathcal{B}}|, \quad (6.14)$$

where \mathcal{C} is an arbitrary antisymmetric $M \times M$ matrix and $\tilde{\mathcal{B}}$ is a diagonal matrix,

$$\tilde{\mathcal{B}} = \text{diag}(\tilde{\beta}_1, \dots, \tilde{\beta}_M), \quad (6.15)$$

with diagonal elements of the form

$$\tilde{\beta}_j = \alpha_j \exp(-iva_j + iua_j^{-1}), \quad j = 1, \dots, M. \quad (6.16)$$

Recalling the definition (2.13)–(2.15) of χ_j , we see that (6.6) is obeyed when we choose $\alpha_1, \dots, \alpha_M$ such that

$$\alpha_j \alpha_{M-j+1} = \exp(\varphi_j), \quad j = 1, \dots, N. \quad (6.17)$$

Provided we also specialize the arbitrary antisymmetric matrix \mathcal{C} to $\mathcal{C}_{\mathcal{R}}$ (see (6.3)), we therefore conclude equality of the reduced Darboux-type solitons to the reduced Kyoto solitons.

We continue by using the new representation (6.12) to show the square property of τ_0 , see (6.1). As it stands, the matrix $\mathcal{C}_{\mathcal{R}} + \mathcal{D}\tilde{\mathcal{A}}_0\mathcal{D}$ is not antisymmetric. But we have

$$\tilde{\mathcal{A}}_{0,jk} = \frac{a_k}{a_j + a_k} = \frac{1}{2} \left(1 - \frac{a_j - a_k}{a_j + a_k} \right), \quad (6.18)$$

so that we can write

$$\tilde{\mathcal{A}}_0 = \frac{1}{2}\zeta \otimes \zeta - \frac{1}{2}\mathcal{A}, \quad \zeta = (1, \dots, 1), \quad (6.19)$$

where \mathcal{A} is the antisymmetric matrix with elements

$$\mathcal{A}_{jk} = \frac{a_j - a_k}{a_j + a_k}. \quad (6.20)$$

As shown in Appendix B, the determinant

$$\tau_0 = \left| C_{\mathcal{R}} + \frac{1}{2}(\mathcal{D}\zeta \otimes \mathcal{D}\zeta - \mathcal{D}A\mathcal{D}) \right| \tag{6.21}$$

equals the square of

$$\tau = \text{Pf}(C_{\mathcal{R}} - \mathcal{D}A\mathcal{D}/2), \tag{6.22}$$

and the Pfaffian can be explicitly evaluated. The resulting formula is

$$\tau = \sum_{l=0}^N \sum_{1 \leq j_1 < \dots < j_l \leq N} \prod_{m=1}^l d_{j_m} \cdot \prod_{1 \leq m < n \leq l} c_{j_m j_n}, \tag{6.23}$$

with

$$d_j = -\frac{1}{2}\chi_j^2 \mathcal{A}_{j, M-j+1}, \quad j = 1, \dots, N, \tag{6.24}$$

$$c_{ij} = \mathcal{A}_{ij} \mathcal{A}_{i, M-j+1} \mathcal{A}_{j, M-i+1} \mathcal{A}_{M-j+1, M-i+1}, \quad i, j = 1, \dots, N, \tag{6.25}$$

see (B21).

Next, we specialize a_1, \dots, a_M as in (2.26) and (2.27). Using (2.31), this yields

$$d_j = \frac{\cos(c)}{2i \sin(c)} \exp(\varphi_j - 2 \sin(c)[v \kappa_j + u \kappa_j^{-1}]), \tag{6.26}$$

$$c_{ij} = \left(\frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right)^2 \frac{\kappa_i^2 + \kappa_j^2 + 2 \cos(2c) \kappa_i \kappa_j}{\kappa_i^2 + \kappa_j^2 - 2 \cos(2c) \kappa_i \kappa_j}, \tag{6.27}$$

where

$$\kappa_j = \exp(\theta_j), \quad j = 1, \dots, N. \tag{6.28}$$

We now substitute (3.11) and then choose $c = \pi/3$ and introduce new parameters

$$k_j = \sqrt{3} \kappa_j, \quad \exp(s_j) = \exp(q_j/2) F_j(\theta)/2, \quad j = 1, \dots, N, \tag{6.29}$$

$$x = -v, \quad y = -u. \tag{6.30}$$

Then we finally obtain

$$d_j = \exp(s_j + x k_j + 3 y k_j^{-1}), \tag{6.31}$$

$$c_{ij} = \left(\frac{k_i - k_j}{k_i + k_j} \right)^2 \frac{k_i^2 + k_j^2 - k_i k_j}{k_i^2 + k_j^2 + k_i k_j}. \tag{6.32}$$

With (q, θ) varying over \mathcal{P} (1.15), we have $k_j > 0$ and $\exp(s_j) > 0$, so that $d_j > 0$ and $c_{ij} > 0$. The function τ is therefore positive, implying τ_0 is positive. Note that τ can be rewritten in the form (2.1), with M , $\xi_{j,n}$, and f_{jk} replaced by N , d_j , and c_{jk} , respectively.

Last but not least, the tau function (6.23) with the substitutions (6.31) and (6.32) coincides with the tau function obtained in Sec. 2 of Ref. 16. Moreover, combining the relations

$$\exp(\Psi) = \tau_2/\tau_0, \quad \tau_0 = \tau_1, \quad \tau_0 = \tau^2 \tag{6.33}$$

(see (1.13), (1.11), and (6.1)) and the 2D Toda equation of motion (1.10) with $n=1$, we deduce

$$2 \partial_u \partial_v \ln \tau = 1 - \exp(\Psi). \tag{6.34}$$

Therefore, the function $\exp(\Psi)$ coincides with the function v employed in Ref. 16.

VII. CONCLUDING REMARKS

A. Integrability on Ω_p

It is not hard to see that the flows generated by the Hamiltonians

$$H_r = \text{Tr}(L^r), \quad r \in \mathbb{R}^*, \quad (7.1)$$

leave Ω_p invariant provided

$$\cos(rc) = \cos(c). \quad (7.2)$$

Indeed, this readily follows from (4.11) and (4.17), noting that (7.1) corresponds to (4.10) with $h(z) = e^{rz}$. Thus the restricted phase space $\langle \Omega_p, \sum_{j=1}^N dq_j \wedge d\theta_j \rangle$ and the Hamiltonians

$$H_r, \quad \pm r = 1 + 2\pi k/c, \quad k \in \mathbb{Z}, \quad (7.3)$$

give rise to an integrable system on Ω_p . For the Tzitzeica case $c = \pi/3$, one can choose as the N independent Hamiltonians the power traces

$$\text{Tr}(L^{1+6k}), \quad k = 0, 1, \dots, N-1. \quad (7.4)$$

B. Space-time trajectories

As we have shown in Theorem 4.1, on Ω_p the tau function (1.39) equals the Tzitzeica tau function, and hence is real, see Proposition 3.1. This reality property is far from obvious for $n = 0, 1$, but for $n=2$ reality on all of Ω is in fact clear from real-valuedness of A , see (1.32). Specializing to Ω_p , we have

$$\tau_2(u, v) = \prod_{i=1}^{2N} [1 - \exp(x_i^+(u, v))] \cdot \prod_{j=1}^N [1 + \exp(x_j^-(u, v))]. \quad (7.5)$$

Now it follows from Sec. VI that $\tau_0(u, v)$ is positive. Therefore, the Tzitzeica solution

$$\Psi(u, v) = \ln(\tau_2(u, v)/\tau_0(u, v)) \quad (7.6)$$

has logarithmic singularities at the zeros of τ_2 , as we have already seen for $N=1$ in Sec. VI. We proceed to analyze these in terms of the space-time coordinates t and y , see (1.2).

Fixing t , the function τ_2 is positive for $y \rightarrow \pm\infty$ and has $2N$ sign changes for finite y . The locations of these zeros on the y axis are distinct for all t (since $x_{2N}^+ < \dots < x_1^+$). Thus one obtains $2N$ space-time trajectories, which may be viewed as the locations of the particles in the N -soliton solution. For $t \rightarrow \pm\infty$ these trajectories exhibit soliton scattering, with a factorized phase shift in terms of the function $\ln(c_{ij})$, see (6.32). A more systematic analysis would be feasible by following the path laid out in Chap. 7 of Ref. 31, but this is beyond our present scope. See, however, Fig. 2 for a plot of the two-soliton collision.

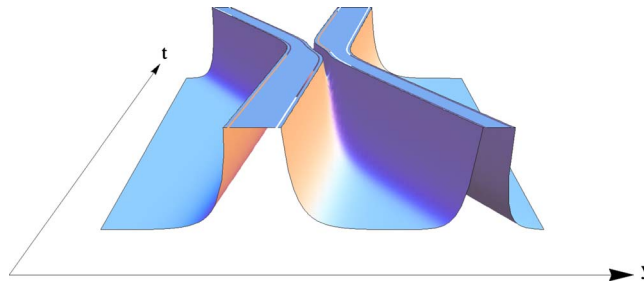


FIG. 2. (Color online) Two-soliton interaction. $-\Psi(t, y)$ is shown; the y -axis points left to right and the t -axis points into the page. The plateau approximates the region in which $\Psi(t, y)$ is complex valued.

C. Comparison with Ref. 33

The 2D Toda soliton tau functions studied in Ref. 33 do not include the above real valued Tzitzeica tau functions. This is because in Ref. 33 the condition $\tau_{-n} = \bar{\tau}_n$ is imposed (see (2.10) in Ref. 33), whereas for the Tzitzeica case we have $\tau_{-1} = \tau_2 \neq \tau_1 = \bar{\tau}_1$. To accommodate this different starting point, the fusion procedure in Sec. 2 of Ref. 33 starts from tau functions that in terms of the relativistic Calogero–Moser systems amount to

$$\tau_n = |\mathbf{1}_M + \exp(-2inc)A(x(u,v))|, \quad A(x) = \text{diag}(\exp(x_1^+), \dots, \exp(x_M^+)). \quad (7.7)$$

Even so, we could specialize the fusion identities of Ref. 33, since they only pertain to the action variables, and the dependence on the latter is governed by the same function ((2.15) in Ref. 33) for particles and antiparticles.

D. Quantum analogs

We have chosen $N_+ = 2N$ and $N_- = N$ throughout the paper (see (1.34)), but we could just as well have started from N particles and $2N$ antiparticles. Indeed, this amounts to working with the “charge conjugate” dual Lax matrix

$$A^C(x) = \text{diag}(\exp(x_1^+), \dots, \exp(x_N^+), -\exp(x_1^-), \dots, -\exp(x_{2N}^-)) \quad (7.8)$$

and tau function

$$\tau_n^C(u,v) = \det(\mathbf{1}_{3N} - \exp(i\pi(1-2n)/3)A^C(x(u,v))) \quad (7.9)$$

instead of (1.32) and (1.39). Clearly, one can then proceed in the same way as before.

On the other hand, one cannot enlarge the three-body soliton correspondence without venturing into complexified phase spaces, losing control of the action-angle map in the process. More precisely, there exist real valued Tzitzeica soliton tau functions that correspond to two different charges, hence giving rise to breatherlike bound states. (Indeed, this is not hard to see from the explicit form of the solitons at the end of Sec. III; for example, one needs only perform a suitable analytic continuation in the two-soliton solution to obtain the one-breather solution.) However, it can be shown that these do not correspond to subspaces of the real Calogero–Moser phase spaces with arbitrary N_+ and N_- .

It may be expected that this picture persists on the quantum level. To be specific, a suitable reduction of the unitary joint eigenfunction transform for the commuting quantum Hamiltonians with $c = \pi/3$ and $3N$ variables (which, to be sure, is not even known to exist to date) should give rise to a unitary transform with N variables, which can be interpreted as a transform for N quantum solitons with the same charge. However, no such reductions are likely to exist when different charges are involved. In particular, if quantum breathers for the Tzitzeica quantum field theory do exist (a feature that is taken for granted in most of the work within the form factor program), then they are not likely to have analogs in the Calogero–Moser particle picture. (By contrast, for the sine-Gordon case a complete correspondence is expected.⁴¹)

E. Demoulin solitons

From Propositions 2.2 and 2.3 it is clear that for

$$c = \pi/6 \quad (7.10)$$

the tau function satisfies

$$\tau_{-n+1} = \tau_n, \quad \tau_{n+6} = \tau_n. \quad (7.11)$$

As a consequence, we have

$$\tau_1 = \tau_0, \quad \tau_2 = \tau_5, \quad \tau_3 = \tau_4. \quad (7.12)$$

Setting

$$h_n = \tau_{n+1}\tau_{n-1}/\tau_n^2 = \exp(\phi_n - \phi_{n-1}) \quad (7.13)$$

(where we used (1.9)), we deduce

$$h_1 = h_0, \quad h_2 = h_5, \quad h_3 = h_4 \quad (7.14)$$

and

$$h_1 h_2 h_3 = 1. \quad (7.15)$$

Now from (1.3) we have

$$(\ln h_n)_{uv} = 2h_n - h_{n+1} - h_{n-1}. \quad (7.16)$$

Hence, setting

$$h = h_1 = \tau_2/\tau_1, \quad k = h_3 = \tau_2/\tau_3 \quad (7.17)$$

and using (7.15), we obtain

$$(\ln h)_{uv} = h - \frac{1}{hk}, \quad (\ln k)_{uv} = k - \frac{1}{kh}. \quad (7.18)$$

This system of relativistic wave equations is the Demoulin system, see Ref. 40, p. 343, Eq. (9.58). Therefore the $c=\pi/6$ tau functions with parameters in \mathcal{P} yield real valued Demoulin solitons in tau function form. In particular, from (3.15)–(3.27) we see that the one-soliton case is given by (7.12) and

$$\tau_1 = 1 + 2G + G^2, \quad \tau_2 = 1 + G^2, \quad \tau_3 = 1 - 2G + G^2, \quad G = \exp(q/2 - ve^\theta - ue^{-\theta}). \quad (7.19)$$

Note added in proof. The identity (B14) is indeed known. It is called Schur's Pfaffian identity (I. Schur, J. Reine Angew. Math. **139**, 155 (1911).)

ACKNOWLEDGMENTS

We would like to thank E. Ferapontov, E. Corrigan, G. Delius, C. Korff, W. Schief, M. Niedermaier, and C. Rogers (in chronological order) for illuminating discussions and for making us aware of relevant literature. Also, thanks to suggestions of the referee, the exposition of this paper has been improved. The paper was completed while the authors were both visiting the Isaac Newton Institute, Cambridge, U.K., as part of the program on Discrete Integrable Systems (January–June 2009). They would like to thank the program organizers for the invitations and the INI for financial support and hospitality.

APPENDIX A: FUSION

In this appendix we detail how formula (4.6) for $p_I(\hat{\theta})^2$ leads to (4.33) in the collapsing torus limit. We begin by recalling that in this limit $p_I(\hat{\theta})$ vanishes, unless the subset I of $\{1, \dots, 3N\}$ has the form

$$I = \bigcup_{j \in J} I_j, \quad (A1)$$

where

$$J = I \cap \{1, \dots, 2N\}, \quad I_j = \{j\}, \quad I_{N+j} = \{N+j, 2N+j\}, \quad j = 1, \dots, N, \tag{A2}$$

see the paragraph preceding (4.30). For a subset I of this form, (4.6) entails

$$p_I(\hat{\theta})^2 = \prod_{j \in J, k \notin J} \left(\prod_{m \in I_j, n \in I_k} \frac{\sinh^2((\hat{\theta}_m - \hat{\theta}_n)/2) + \sin^2 c}{\sinh^2((\hat{\theta}_m - \hat{\theta}_n)/2)} \right). \tag{A3}$$

Next, we substitute (4.26) and (4.27) in this formula and cancel terms in the interior product using fusion identities. A general fusion identity, in which the sets I_k are of arbitrary cardinality, can be found in Ref. 33, but here we only detail the special cases we need.

Using the trigonometric/hyperbolic identity

$$\sinh^2 x + \sin^2 y = \sinh(x + iy)\sinh(x - iy), \tag{A4}$$

the general pair factor in (A3) can be written as

$$\frac{\sinh((\hat{\theta}_m - \hat{\theta}_n)/2 + ic)\sinh((\hat{\theta}_m - \hat{\theta}_n)/2 - ic)}{\sinh^2((\hat{\theta}_m - \hat{\theta}_n)/2)}. \tag{A5}$$

For a soliton-soliton interaction included in J , there is only one pair factor in the product. Specifically, we have $I_j = \{j\}$, $I_k = \{k\}$ and (from (4.26) and (4.32)) $\hat{\theta}_j = \eta_j$, $\hat{\theta}_k = \eta_k$, so the contribution to the product for a soliton-soliton pair is

$$\frac{\sinh^2((\eta_j - \eta_k)/2) + \sin^2 c}{\sinh^2((\eta_j - \eta_k)/2)} = \frac{\sinh((\eta_j - \eta_k)/2 + ic)\sinh((\eta_j - \eta_k)/2 - ic)}{\sinh^2((\eta_j - \eta_k)/2)} = \frac{s(2)s(-2)}{s^2(0)}, \tag{A6}$$

where we have set $s(l) = \sinh((\eta_j - \eta_k + lic)/2)$. For a soliton-breather interaction we have $I_j = \{j\}$, $I_{N+k} = \{N+k, 2N+k\}$ and $\hat{\theta}_j = \eta_j$, $\hat{\theta}_{N+k} = \eta_k + ic$, $\hat{\theta}_{2N+k} = \eta_k - ic$, yielding two pair factors

$$\frac{s(1)s(-3)}{s^2(-1)} \frac{s(3)s(-1)}{s^2(1)} = \frac{s(3)s(-3)}{s(1)s(-1)} = \frac{\sinh^2((\eta_j - \eta_k)/2) + \sin^2(3c/2)}{\sinh^2((\eta_j - \eta_k)/2) + \sin^2(c/2)}. \tag{A7}$$

Finally, for a breather-breather interaction in J , we get $I_{N+j} = \{N+j, 2N+j\}$, $I_{N+k} = \{N+k, 2N+k\}$, and hence four pairs whose contribution to the product is

$$\frac{s(2)s(-2)}{s^2(0)} \frac{s(4)s(0)}{s^2(2)} \frac{s(2)s(-2)}{s^2(0)} \frac{s(0)s(-4)}{s^2(-2)} = \frac{s(4)s(-4)}{s^2(0)} = \frac{\sinh^2((\eta_j - \eta_k)/2) + \sin^2(2c)}{\sinh^2((\eta_j - \eta_k)/2)}. \tag{A8}$$

Using $\eta_{l+N} = \eta_l$ and $c_l = c$, $c_{N+l} = 2c$ for $l = 1, \dots, N$ (see (4.32)), these three cases may be encoded in the single formula

$$\frac{\sinh^2((\eta_j - \eta_k)/2) + \sin^2((c_j + c_k)/2)}{\sinh^2((\eta_j - \eta_k)/2) + \sin^2((c_j - c_k)/2)}, \quad j, k = 1, \dots, 2N. \tag{A9}$$

Hence (4.33) results.

APPENDIX B: PFAFFIAN IDENTITIES

In this appendix we show that the determinant on the right hand side of (6.21) equals the square of the Pfaffian on the right hand side of (6.22), and that the latter has the explicit form (6.23)–(6.25). We proceed in a slightly more general way, since this eases the notation and adds insight.

First we note that the determinant is of the form $|v \otimes v + A|$, where A is a $2N \times 2N$ antisymmetric matrix. Now we invoke the following lemma.

Lemma B1: Assume A is an antisymmetric $2N \times 2N$ matrix and $v \in \mathbb{C}^{2N}$. Then one has

$$|v \otimes v + A| = |A|. \tag{B1}$$

Proof: By continuity it suffices to prove this for an invertible A . Then we have

$$|v \otimes v + A| = |A| |\mathbf{1}_{2N} + A^{-1}v \otimes v| = |A|(1 + (\bar{v}, A^{-1}v)). \tag{B2}$$

Now since $(A^{-1})^t = (A^t)^{-1} = -A^{-1}$, it follows that A^{-1} is also antisymmetric. Hence the inner product $(\bar{v}, A^{-1}v) = v^t A^{-1}v$ vanishes, and (B1) results. \square

As a consequence, we have

$$\tau_0 = |\mathcal{C}_{\mathcal{R}} - \mathcal{D}AD/2| = \tau^2, \tag{B3}$$

with τ given by (6.22). We continue to show (6.23). Again, we first study a slightly more general situation. We begin by recalling the definition

$$\text{Pf}(A) = \sum (-)^{\text{sgn}(\sigma)} A_{i_1 j_1} \cdots A_{i_N j_N}, \quad A \in M_{2N}(\mathbb{C}), \quad A^t = -A, \tag{B4}$$

where the sum is over all index choices with

$$1 = i_1 < \cdots < i_N \leq 2N - 1, \quad i_1 < j_1, \dots, i_N < j_N, \tag{B5}$$

and σ is the permutation

$$\sigma: (i_1, j_1, \dots, i_N, j_N) \mapsto (1, 2, \dots, 2N). \tag{B6}$$

Denoting $\text{Pf}(A)$ by $[1, 2, \dots, 2N]$, this implies an expansion formula

$$[1, 2, \dots, 2N] = \sum_{n=2}^{2N} (-)^n [1, n][2, \dots, \hat{n}, \dots, 2N], \tag{B7}$$

where the notation on the right hand side will be clear from context. Consider now the Pfaffian of the antisymmetric matrix

$$B = \begin{pmatrix} 0 & \lambda \mathcal{R}_N \\ -\lambda \mathcal{R}_N & 0 \end{pmatrix} + A, \quad \lambda \in \mathbb{C}. \tag{B8}$$

The relevant (upper triangular) part of B is

$$\begin{matrix} A_{12} & \cdots & \cdots & \cdots & \cdots & \cdots & \lambda + A_{1,2N} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & A_{N-1,N} & A_{N-1,N+1} & \lambda + A_{N-1,N+2} & \cdots & \cdots \\ & & & \lambda + A_{N,N+1} & A_{N,N+2} & \cdots & \cdots \\ & & & & A_{N+1,N+2} & \cdots & \cdots \\ & & & & & \cdots & A_{2N-2,2N} \\ & & & & & & A_{2N-1,2N} \end{matrix}. \tag{B9}$$

Therefore we have

$$\text{Pf}(B) = \sum_{l=0}^N \lambda^{N-l} S_l, \tag{B10}$$

where S_l is a sum over certain “minors.” Specifically, taking (B7) into account to check signs, we readily get

$$S_k = \sum_{1 \leq j_1 < \dots < j_k \leq N} [j_1, \dots, j_k, 2N - j_k + 1, \dots, 2N - j_1 + 1]. \tag{B11}$$

(Hence, in particular, $S_0 = 1, S_{2N} = [1, \dots, 2N]$.)

Applying this expansion to τ (6.22), we obtain

$$\tau = \sum_{l=0}^N \sum_{1 \leq j_1 < \dots < j_l \leq N} \prod_{m=1}^l \left(-\frac{1}{2} \chi_{j_m}^2 \right) \cdot [j_1, \dots, j_l, 2N - j_l + 1, \dots, 2N - j_1 + 1]_{\mathcal{A}}, \tag{B12}$$

where $[\dots]_{\mathcal{A}}$ denotes the reduced Pfaffians for \mathcal{A} . We now obtain an explicit formula for the latter.

Lemma B2: Suppose $\mathcal{A} \in M_{2l}(\mathbb{C})$ is given by

$$\mathcal{A}_{jk} = \frac{a_j - a_k}{a_j + a_k}, \quad j, k = 1, \dots, 2l. \tag{B13}$$

Then we have

$$\text{Pf}(\mathcal{A}) = \prod_{1 \leq j < k \leq 2l} \mathcal{A}_{jk}. \tag{B14}$$

This explicit evaluation is proved in Ref. 42. It is presumably known, but we do not know a reference. Indeed, up to its sign this lemma can be viewed as a consequence of Cauchy’s identity. To see this, note that the lemma entails

$$|\mathcal{A}| = \prod_{1 \leq j < k \leq 2l} \mathcal{A}_{jk}^2. \tag{B15}$$

If we now substitute $a_j \rightarrow \exp(2q_j)$, then (B15) becomes

$$|(\tanh(q_j - q_k))|_{2l \times 2l} = \prod_{1 \leq j < k \leq 2l} \tanh^2(q_j - q_k), \tag{B16}$$

an identity that readily follows from Cauchy’s identity (see (B42) in Ref. 43). Thus, Cauchy’s identity implies (B15), so (B14) follows save for its sign.

Returning to (B12), we substitute (B14) in the \mathcal{A} -Pfaffians, yielding

$$\prod_{j < k, j, k \in \{j_1, \dots, j_l, 2N - j_l + 1, \dots, 2N - j_1 + 1\}} \mathcal{A}_{jk}. \tag{B17}$$

Introducing

$$c_{jk} = \mathcal{A}_{jk} \mathcal{A}_{j, 2N - k + 1} \mathcal{A}_{k, 2N - j + 1} \mathcal{A}_{2N - k + 1, 2N - j + 1}, \quad 1 \leq j < k \leq N, \tag{B18}$$

we can write (B17) as

$$\prod_{m=1}^l \mathcal{A}_{j_m, 2N - j_m + 1} \cdot \prod_{j < k, j, k \in \{j_1, \dots, j_l\}} c_{jk}. \tag{B19}$$

Substituting this in (B12) and setting

$$d_j = -\frac{1}{2} \chi_j^2 \mathcal{A}_{j,2N-j+1}, \quad j = 1, \dots, N, \quad (\text{B20})$$

we finally obtain

$$\tau = \sum_{l=0}^N \sum_{1 \leq j_1 < \dots < j_l \leq N} \prod_{m=1}^l d_{j_m} \cdot \prod_{j < k, j, k \in \{j_1, \dots, j_l\}} c_{jk}, \quad (\text{B21})$$

which amounts to (6.23).

APPENDIX C: SOLUTIONS OBTAINED BY BINARY DARBOUX TRANSFORMATION

The A_∞ Toda lattice (1.3) has a family of solutions obtained by an iterated binary Darboux transformation. We follow the derivation given in Ref. 44 but make some small modifications so as to conform with the notation and conventions used in the rest of this paper.

A Lax pair for (1.3) is given by

$$A_{n,u} = i \exp(\phi_{n+1} - \phi_n) A_{n+1}, \quad -A_{n,v} = i A_{n-1} + \phi_{n,v} A_n. \quad (\text{C1})$$

(Thus, equality of cross derivatives entails (1.3).) It is covariant with respect to the Darboux transformation⁴⁵

$$\tilde{A}_n = A_{n,v} - a_{n,v} a_n^{-1} A_n, \quad (\text{C2})$$

$$\tilde{\phi}_n = \phi_n + \ln(a_n/a_{n-1}), \quad (\text{C3})$$

where a_n is an eigenfunction (i.e., any particular solution of (C1)). The adjoint Lax pair is

$$-B_{n,u} = i \exp(\phi_n - \phi_{n-1}) B_{n-1}, \quad B_{n,v} = i B_{n+1} + \phi_{n,v} B_n. \quad (\text{C4})$$

Given an eigenfunction a_n and an adjoint eigenfunction b_n for a seed solution $\phi_n = \phi_n^0 = \ln(\tau_{n+1}^0/\tau_n^0)$ to (1.3), an eigenfunction potential $\Omega(a_n, b_n)$ is defined, consistently and uniquely up to an additive constant, by the three requirements

$$\Omega(a_n, b_n)_{,u} = -i \exp(\phi_{n+1}^0 - \phi_n^0) b_n a_{n+1}, \quad (\text{C5})$$

$$\Omega(a_n, b_n)_{,v} = -i b_{n+1} a_n, \quad (\text{C6})$$

$$\Omega(a_n, b_n) - \Omega(a_n, b_{n-1}) = -b_n a_n. \quad (\text{C7})$$

The Lax pair (C1) is also covariant with respect to the binary Darboux transformation

$$\hat{A}_n = A_n - a_n \Omega(A_n, b_n) / \Omega(a_n, b_n), \quad (\text{C8})$$

$$\hat{\phi}_n = \phi_n^0 + \ln(\Omega(a_n, b_n) / \Omega(a_{n-1}, b_{n-1})), \quad (\text{C9})$$

or, equivalently, $\hat{\tau}_n = \Omega(a_{n-1}, b_{n-1}) \tau_n^0$. (In (C8) A_n denotes the general solution to (C1) with ϕ replaced by ϕ^0 .)

The M th iterated binary Darboux transformation is expressed in terms of an M -vector \mathbf{A}_n , whose components satisfy (C1) with $\phi \rightarrow \phi^0$, and an M -vector \mathbf{B}_n , whose components satisfy (C4) with $\phi \rightarrow \phi^0$. These eigenfunction and adjoint eigenfunction vectors are used to define an $M \times M$ matrix eigenfunction potential $\Omega_n = \Omega(\mathbf{A}_n, \mathbf{B}_n)$ satisfying

$$\Omega_{n,u} = -i \exp(\phi_{n+1}^0 - \phi_n^0) \mathbf{B}_n \otimes \mathbf{A}_{n+1}, \quad \Omega_{n,v} = -i \mathbf{B}_{n+1} \otimes \mathbf{A}_n, \quad \Omega_n - \Omega_{n-1} = -\mathbf{B}_n \otimes \mathbf{A}_n. \quad (\text{C10})$$

The solutions obtained are then expressed as

$$\hat{\tau}_n = \det(\Omega_{n-1}) \tau_n^0. \quad (\text{C11})$$

From now on we omit the $\hat{}$ and write $\hat{\tau}_n$ as τ_n .

The B_∞ Toda lattice has $\phi_{-n} = -\phi_n$ for $n \in \mathbb{Z}$, so that $\phi_0 = 0$ and the system is semi-infinite, expressed in terms of ϕ_1, ϕ_2, \dots . This reduction can be achieved by choosing solutions satisfying $\tau_{-n+1} = \tau_n$. In this reduction, it is consistent to choose adjoint eigenfunctions given by $\mathbf{B}_n = (-1)^n \mathbf{A}_{-n}$. It then follows from (C10) that

$$\Omega_{-n} = -\Omega_{n-1}^t \quad (\text{C12})$$

and so provided M is even, $M=2N$, say, we indeed obtain $\tau_{-n+1} = \tau_n$.

In particular, to obtain soliton solutions for the A_∞ Toda lattice, one chooses as seed solution the vacuum solution $\tau_n^0 = 1$, $\phi_n^0 = 0$, and then vectors of eigenfunctions and adjoint eigenfunctions given by

$$\mathbf{A}_{n,j} = \alpha_j a_j^{-n} e^{-i v a_j + i u a_j^{-1}}, \quad \mathbf{B}_{n,j} = \beta_j b_j^n e^{i v b_j - i u b_j^{-1}}, \quad j = 1, \dots, M, \quad (\text{C13})$$

where α_j, β_j, a_j , and b_j are arbitrary constants. Then, one obtains

$$\tau_n = \det \left(C_{jk} + \beta_j \alpha_k \frac{b_j^n a_k^{-n+1}}{a_k - b_j} e^{-i v (a_k - b_j) + i u (a_k^{-1} - b_j^{-1})} \right), \quad (\text{C14})$$

where C_{jk} are arbitrary constants. In the B_∞ -reduction, one must take $b_i = -a_i$ and $\beta_i = \alpha_i$ giving

$$\tau_n = \det \left(C_{jk} + \alpha_j \alpha_k \frac{(-a_j)^n a_k^{-n+1}}{a_j + a_k} e^{-i v (a_j + a_k) + i u (a_j^{-1} + a_k^{-1})} \right). \quad (\text{C15})$$

Here, C_{jk} are arbitrary constants satisfying, in accordance with the symmetry condition (C12), $C_{kj} = -C_{jk}$.

Thus we have arrived at expression (6.14) that is the starting point of comparison with the Kyoto solitons in Sec. VI.

¹G. Tzitzeica, Acad. Sci., Paris, C. R. **150**, 955 (1910).

²G. Tzitzeica, Acad. Sci., Paris, C. R. **150**, 1227 (1910).

³H. Jonas, Ann. Mat. Pura Appl. **30**, 223 (1921).

⁴H. Jonas, Math. Nachr. **10**, 331 (1953).

⁵R. K. Dodd and R. K. Bullough, Proc. R. Soc. London, Ser. A **352**, 481 (1977).

⁶A. V. Zhiber and A. B. Shabat, Sov. Phys. Dokl. **24**, 607 (1979).

⁷A. V. Mikhailov, Physica D **3**, 73 (1981).

⁸W. K. Schief, Phys. Lett. A **223**, 55 (1996).

⁹J. Weiss, J. Math. Phys. **27**, 1293 (1986).

¹⁰C. Athorne and J. J. C. Nimmo, Inverse Probl. **7**, 809 (1991).

¹¹M. Musette and R. Conte, J. Phys. A **27**, 3895 (1994).

¹²W. K. Schief and C. Rogers, Inverse Probl. **10**, 711 (1994).

¹³W. K. Schief, J. Phys. A **29**, 5153 (1996).

¹⁴Y. V. Brezhnev, Phys. Lett. A **211**, 94 (1996).

¹⁵I. Y. Cherdantsev and R. A. Sharipov, Int. J. Mod. Phys. A **5**, 3021 (1990).

¹⁶O. V. Kaptsov and Y. V. Shan'ko, Diff. Eq. **35**, 1683 (1999).

¹⁷A. B. Borisov, S. A. Zykov, and M. V. Pavlov, Theor. Math. Phys. **131**, 550 (2002).

¹⁸R. Hirota and D. Takahashi, Glasg. Math. J. **47**, 77 (2005).

¹⁹A.-M. Wazwaz, Chaos, Solitons Fractals **25**, 55 (2005).

²⁰A. E. Mironov, Math. Notes **82**, 569 (2007).

²¹F. A. Smirnov, Int. J. Mod. Phys. A **6**, 1407 (1991).

²²A. Fring, G. Mussardo, and P. Simonetti, Phys. Lett. B **307**, 83 (1993).

²³C. Acerbi, Nucl. Phys. B **497**, 589 (1997).

- ²⁴V. Fateev, S. Lukyanov, A. Zamolodchikov, and A. Zamolodchikov, *Nucl. Phys. B* **516**, 652 (1998).
- ²⁵R. K. Bullough, *Glasg. Math. J.* **47**, 45 (2005).
- ²⁶P. E. G. Assis and L. A. Ferreira, *Nucl. Phys. B* **800**, 409 (2008).
- ²⁷B. Gaffet, *Physica D* **26**, 123 (1987).
- ²⁸C. Rogers, W. K. Schief, K. W. Chow, and C. C. Mak, *Stud. Appl. Math.* **114**, 271 (2005).
- ²⁹S. N. M. Ruijsenaars, in *Particles and Fields*, CRM Series in Mathematical Physics (Springer, New York, 1999), pp. 251–352.
- ³⁰S. N. M. Ruijsenaars and H. Schneider, *Ann. Phys.* **170**, 370 (1986).
- ³¹S. N. M. Ruijsenaars, *Publ. Res. Inst. Math. Sci.* **30**, 865 (1994).
- ³²S. N. M. Ruijsenaars, in *Integrable and Superintegrable Systems* (World Scientific, Teaneck, NJ, 1990), pp. 165–206.
- ³³S. N. M. Ruijsenaars, *Ann. Phys.* **256**, 226 (1997).
- ³⁴S. N. M. Ruijsenaars, *J. Nonlinear Math. Phys.* **8**, 256 (2001).
- ³⁵A. V. Mikhailov, *JETP Lett.* **30**, 414 (1979).
- ³⁶E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, in *Nonlinear Integrable Systems—Classical Theory and Quantum Theory* (World Scientific, Singapore, 1983), pp. 39–119.
- ³⁷M. Jimbo and T. Miwa, *Publ. Res. Inst. Math. Sci.* **19**, 943 (1983).
- ³⁸E. Date, M. Jimbo, M. Kashiwara, and T. Miwa, *J. Phys. A* **16**, 221 (1983).
- ³⁹A. Demoulin, *Bull. de l'Acad. Belgique* **19**, 479 (1933); **19**, 579 (1933); **19**, 1352 (1933).
- ⁴⁰C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, Cambridge Texts in Applied Mathematics (Cambridge University Press, Cambridge, 2002).
- ⁴¹S. N. M. Ruijsenaars, *Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory*, NATO Science Series II: Mathematics, Physics and Chemistry Vol. 35 (Kluwer Academic, Dordrecht, 2001), pp. 273–292.
- ⁴²J. J. C. Nimmo, *J. Phys. A* **23**, 751 (1990).
- ⁴³S. N. M. Ruijsenaars, *Commun. Math. Phys.* **110**, 191 (1987).
- ⁴⁴J. J. C. Nimmo and R. Willox, *Proc. R. Soc. London, Ser. A* **453**, 2497 (1997).
- ⁴⁵V. B. Matveev, *Lett. Math. Phys.* **3**, 217 (1979).