Generalized Long-Moody functors

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Abstract

In this paper, we generalize the principle of the Long-Moody construction for representations of braid groups to other groups, such as mapping class groups of surfaces. Namely, we introduce endofunctors over a functor category that encodes representations of a family of groups. They are called Long-Moody functors and provide new representations. In this context, notions of polynomial functors are defined and play an important role in the study of homological stability. We prove that, under additional assumptions, a Long-Moody functor increases the *very strong* and *weak* polynomial degrees of functors by one.

Introduction

In 1994, as a result of a collaboration with Moody, Long [28] gave a method to construct a new linear representation of the braid group on n strands $\mathbf{B}_n$ from a representation of $\mathbf{B}_{n+1}$. The underlying framework of this construction, called the *Long-Moody construction*, naturally arises in many situations in connection with topology: the first aim of this paper is to extend this construction to these settings.

Namely, for a family of groups $\{G_n\}_{n\in\mathbb{N}}$ equipped with injections $\gamma_n : G_n \hookrightarrow G_{n+1}$, we give a method to construct a representation of G_n from a representation of G_{n+1}, which generalizes the underlying idea of the original Long-Moody construction. This machinery is based on the following ingredients: a family of groups $\{H_n\}_{n\in\mathbb{N}}$, an action $\mathcal{A}_n : G_n \to \mathrm{Aut}(H_n)$ and a group morphism $\chi_n : H_n \to G_{n+1}$ for all natural numbers n. For instance, we can consider the following situation:

- Let $\Sigma_{g,1}$ be a smooth connected compact surface with genus g and one boundary component. We consider the surfaces $\{\Sigma_{g,1}^n\}_{n\in\mathbb{N}}$ obtained from $\Sigma_{g,1}$ by removing n points from its interior. We take the family of groups $\{G_n\}_{n\in\mathbb{N}}$ to be the family of *mapping class groups* $\{\mathbf{\Gamma}_{g,1}^n\}_{n\in\mathbb{N}}$, that is the group of isotopy classes of homeomorphisms fixing the punctures setwise and restricting to the identity on the boundary component.

- We define the injection $\gamma_n : \mathbf{\Gamma}_{g,1}^n \hookrightarrow \mathbf{\Gamma}_{g,1}^{n+1}$ by extending an element $\varphi \in \mathbf{\Gamma}_{g,1}^n$ to a mapping class of $\Sigma_{g,1}^{n+1}$ by the identity on the complement $\Sigma_{0,1}^1$ of $\Sigma_{g,1}^n \hookrightarrow \Sigma_{g,1}^{n+1}$.

- We take the family of groups $\{H_n\}_{n\in\mathbb{N}}$ to be the fundamental groups of the surfaces $\{\pi_1(\Sigma_{g,1}^n, p)\}_{n\in\mathbb{N}}$, where p is a point in the boundary component which lies on the part of the boundary of each $\Sigma_{g,1}^n$ that is common to all of the surfaces $\{\Sigma_{g,1}^n\}_{n\in\mathbb{N}}$ (and thus p is independent of n); see Figure 2.

- We consider the natural action $\mathcal{A}_n$ of $\mathbf{\Gamma}_{g,1}^n$ on $\pi_1(\Sigma_{g,1}^n, p)$.

- We consider a morphism $\chi_{n,1} : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{\Gamma}_{g,1}^{n+1}$ induced by the *point pushing map* as it called in Farb and Margalit [18, Section 4.2.1]; it is explicitly defined by the formula (15).

The input of the Long-Moody construction is a representation $\rho_{n+1} : G_{n+1} \to GL_R(M_{n+1})$, for R a commutative ring and M_{n+1} an R-module. Then we note that M_{n+1} is endowed with an H_n-module structure via

the composition $\rho_{n+1} \circ \chi_n : H_n \to G_{n+1} \to GL_R(M_{n+1})$. We denote by $\mathcal{I}_{R[H_n]}$ the augmentation ideal of the group H_n – which has a canonical $R[H_n]$-module structure. The key idea of the Long-Moody construction is to give the tensor product

$$\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} M_{n+1}$$

a G_n-module structure as follows. The action $\mathcal{A}_n : G_n \to \mathrm{Aut}(H_n)$ linearly extends to an action on $\mathcal{I}_{R[H_n]}$ and M_{n+1} is a G_n-module structure via the restriction of the morphism ρ_{n+1} along $\gamma_n : G_n \hookrightarrow G_{n+1}$. Then, for all $g \in G_n$, $x \in \mathcal{I}_{R[H_n]}$ and $m \in M_{n+1}$, we define a diagonal action of G_n on the above tensor product by

$$g \cdot (x \underset{R[H_n]}{\otimes} m) = \mathcal{A}_n(g)(x) \underset{R[H_n]}{\otimes} \rho_{n+1}(\gamma_n(g))(m).$$

For this action to be well-defined, it has to be compatible with the H_n-module structures on both sides of the tensor product: this requires technical compatibilities between the morphisms $\mathcal{A}_n$, χ_n and γ_n. This is the delicate point of the construction. However these compatibilities can easily be described by using categorical tools, in particular the *Grothendieck construction*, as follows.

We consider the groupoid $\mathcal{G}$ with objects indexed by the natural numbers (and denoted by $\underline{n}$) and with the groups $\{G_n\}_{n \in \mathbb{N}}$ as automorphism groups. We denote by $\mathfrak{Gr}$ the category of groups and $\int^{\mathcal{G}} F$ is the Grothendieck construction on a functor $F : \mathcal{G} \to \mathfrak{Gr}$; see Section 1. The cornerstone to define the Long-Moody construction is to assume that the families of morphisms $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ and $\{\chi_n\}_{n \in \mathbb{N}}$ assemble to define functors $\mathcal{A} : \mathcal{G} \to \mathfrak{Gr}$ such that $\mathcal{A}(\underline{n}) = H_n$ and $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$. In addition, we require that the following diagram be commutative:

$$
\begin{array}{ccc}
\mathcal{G} & \overset{i}{\hookrightarrow} & \int^{\mathcal{G}} \mathcal{A} \\
& {\scriptstyle \gamma} \searrow & \downarrow {\scriptstyle \chi} \\
& & \mathcal{G},
\end{array}
$$

where i is the evident section of the projection functor induced by the Grothendieck construction and γ is the functor defined by $\underline{n} \mapsto \underline{n+1}$ and induced by the canonical injections $\{\gamma_n : G_n \hookrightarrow G_{n+1}\}_{n \in \mathbb{N}}$. The Grothendieck construction encodes semidirect product structures, so this condition actually reflects the factorization of the injections γ_n through $H_n \rtimes_{\mathcal{A}_n} G_n$; see Lemma 2.12.

The necessary coherence conditions on the group morphisms $\{\chi_n\}_{n \in \mathbb{N}}$ are restrictive. However, the trivial morphisms $\{\chi_{n,tr} : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$ always satisfy the necessary technical condition, and there are many situations where non-trivial $\{\chi_n\}_{n \in \mathbb{N}}$ arise naturally, in particular for some families of groups in connection with topology. For example, the above non-trivial morphisms $\{\chi_{n,1} : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{\Gamma}_{g,1}^{n+1}\}_{n \in \mathbb{N}}$ satisfy the appropriate conditions; see Lemma 3.8.

Furthermore, instead of considering these constructions for only one group G_n, a natural question is how to extend these constructions to *families of representations* of $\{G_n\}_{n \in \mathbb{N}}$. Denoting by $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ the category of functors from a small category $\mathfrak{C}$ to the category of R-modules $R\text{-}\mathfrak{Mod}$, an object M of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$ is a collection of linear representations $\{\rho_n : G_n \to GL_R(M_n)\}_{n \in \mathbb{N}}$. For instance, the groupoid $\mathcal{G}$ for the family of mapping class groups $\{\mathbf{\Gamma}_{g,1}^n\}_{n \in \mathbb{N}}$ is introduced in Section 3.1 and we denote it by $\mathfrak{M}_2^{+,g}$. The condition for the functor M to encode a family of representations consists of requiring that there exist maps $m_n : GL_R(M_n) \to GL_R(M_{n+1})$ for all natural numbers n such that:

$$m_n \circ \rho_n(g) = \rho_{n+1}(\gamma_n(g)) \tag{1}$$

for all $g \in G_n$. Then, we say that the representations $\{M_n\}_{n \in \mathbb{N}}$ form a *family* of linear representations of the groups $\{G_n\}_{n \in \mathbb{N}}$. This notion has been described as a *consistent sequence of representations* in Church, Ellenberg and Farb [13]. However, the extra information (1) on M is not encoded by the fact that M is an object of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$. Quillen's bracket construction (see Grayson [22, p.219]) defines a new category $\mathfrak{U}\mathcal{G}$ with enough additional morphisms to resolve this failure. In particular, the groupoid $\mathcal{G}$ is its maximal subgroupoid, $\mathfrak{U}\mathcal{G}$ contains extra morphisms from each object $\underline{n}$ to the object $\underline{n+1}$ allowing to encode the compatibility condition (1) and γ canonically extends to $\mathfrak{U}\mathcal{G}$. Actually the category $\mathfrak{U}\mathcal{G}$ forces some additional relations on the morphisms which are not used to describe the above compatibilities. For example, the family of symplectic representations of the mapping class groups defines an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{M}_2^{+,g}, R\text{-}\mathfrak{Mod})$. Also the

category $\mathfrak{UG}$ is fundamental since it provides a natural setting to study coefficient systems for homological stability; see Wahl and Randal-Williams [36, Sections 1 and 4].

Therefore, our goal is to define the Long-Moody construction as an endofunctor of the functor category $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$. We thus deal with an extension problem: we require the functors $\mathcal{A}$ and χ to respectively extend along the inclusions $\mathcal{G} \hookrightarrow \mathfrak{UG}$ and $\int^{\mathcal{G}} \mathcal{A} \to \int^{\mathfrak{UG}} \mathcal{A}$ so that the following diagram is commutative

$$
\begin{array}{ccc}
\mathfrak{UG} & \overset{i}{\lhook\joinrel\longrightarrow} & \int^{\mathfrak{UG}} \mathcal{A} \\
& {\scriptstyle \gamma} \searrow & \downarrow {\scriptstyle \chi} \\
& & \mathfrak{UG}.
\end{array}
$$

Under these assumptions, we prove:

Theorem A (Definition 2.9). *There is a right-exact functor* $\mathbf{LM}_{\{\mathcal{G},\chi\}} : \mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$, *called the Long-Moody functor associated to* $\{\mathcal{G}, \chi\}$, *that assigns to all objects F of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and objects $\underline{n}$ of $\mathfrak{UG}$ the R-module*

$$
\mathbf{LM}_{\{\mathcal{G},\chi\}}(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(\underline{n+1}) \tag{2}
$$

and to all elements g of G_n the morphism $\mathbf{LM}_{\{\mathcal{G},\chi\}}(F)(g) = \mathcal{A}_n(g) \otimes_{R[H_n]} F(\gamma_n(g))$.

In particular, if we take the groups $\{G_n\}_{n\in\mathbb{N}}$ to be the family of braid groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$ and the groups $\{H_n\}_{n\in\mathbb{N}}$ to be the family of free groups $\{\mathbf{F}_n\}_{n\in\mathbb{N}}$, Theorem A recovers a previous result of the author [37, Theorem A]. However, the present framework improves the previous one since it allows us to define and recover more representations for the braid groups; see Example 3.15 for instance. Additionally, the families of mapping class groups of surfaces also fit into this framework and are studied in Section 3. Further families of groups also fit into the present framework such as mapping class groups of compact connected oriented 3-manifolds with boundary, loop braid groups or automorphism groups of free products of groups, although they are not presented in this paper. Also, the new abstract approach to the Long-Moody functors presented in this paper indicates that these are particular examples of more general *tensorial constructions* introduced in Section 2.1: further generalizations and wider applications should emerge from these last constructions. They are however not addressed in the present paper.

In addition, we can relate a Long-Moody functor $\mathbf{LM}_{\{\mathcal{G},\chi\}}$ to the first homology groups of the family of groups $\{H_n\}_{n\in\mathbb{N}}$. We recall that $\mathcal{A}$ denotes a functor $\mathfrak{UG} \to \mathfrak{Gr}$ such that $\mathcal{A}(\underline{n}) = H_n$ for all n. We denote by $H_1(\mathcal{A}; R)$ the composition $H_1(-; R) \circ \mathcal{A}$, where $H_1(-; R)$ denotes the first homology group functor. Then, denoting by $R : \mathfrak{UG} \to R\text{-}\mathfrak{Mod}$ the constant functor at R, the functor $\mathbf{LM}_{\{\mathcal{G},\chi\}}(R)$ is equivalent to the composition $H_1(\mathcal{A}; R)$; see Proposition 2.16. For instance, for any functor χ defining a Long-Moody functor for the category $\mathfrak{M}_2^{+,g}$ associated to the mapping class groups $\{\mathbf{\Gamma}_{g,1}^n\}_{n\in\mathbb{N}}$, the functor $\mathbf{LM}_{\{\mathfrak{M}_2^{+,g},\chi\}}(R)$ thus encodes the family of symplectic representations. Moreover, considering the family of trivial morphisms $\{\chi_{n,tr} : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$, we also prove that for all objects F of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$, there is a natural equivalence for the associated Long-Moody functor (see Proposition 2.16):

$$
\mathbf{LM}_{\{\mathcal{G},\chi_{tr}\}}(F) \cong H_1(H_n; R) \underset{R}{\otimes} F(1 + -). \tag{3}
$$

Hence $\mathbf{LM}_{\{\mathcal{G},\chi_{tr}\}}$ is defined as the tensor product of F with the functor $H_1(\mathcal{A}; R)$. Nevertheless, the equivalence (3) does not hold generally speaking for an endofunctor $\mathbf{LM}_{\{\mathcal{G},\chi\}}$ where χ is not a functor induced by the morphisms $\{\chi_{n,tr}\}_{n\in\mathbb{N}}$. Indeed, for each $n \in \mathbb{N}$, the G_n-module $\mathbf{LM}_{\{\mathcal{G},\chi\}}(F)(\underline{n})$ is defined by the twisted tensor product (2), which is not isomorphic in general to $H_1(\mathcal{A}; R) \otimes_R F(\underline{n+1})$ if $F(\underline{n+1})$ is not a trivial H_n-module. Also, for each $n \in \mathbb{N}$, the twisted first homology group $H_1(H_n; F(\underline{n+1}))$ is canonically a submodule $\mathbf{LM}_{\{\mathcal{G},\chi\}}(F)(\underline{n})$; see the 4-term short exact sequence (14). For instance, let $\mathbf{LM}_{\{\mathfrak{M}_2^{+,g},\chi_1\}}$ be the Long-Moody functor defined by the above non-trivial morphisms $\{\chi_{n,1} : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{\Gamma}_{g,1}^{n+1}\}_{n\in\mathbb{N}}$. The application of this functor provides linear representations of the family of groups $\{\mathbf{\Gamma}_{g,1}^n\}_{n\in\mathbb{N}}$ which, as far as the author knows, are unknown in the literature; see Section 3.3.3. In the same vein, new interesting linear representations for the braid groups on orientable surfaces $\{\mathbf{B}_n(\Sigma_{g,1})\}_{n\in\mathbb{N}}$ and for the mapping class groups $\{\mathbf{\Gamma}_{n,1}\}_{n\in\mathbb{N}}$ are defined by the applications of some appropriate Long-Moody functors; see Sections 3.3.3 and 3.4.

Furthermore, among the objects in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ of particular importance are the *strong* and *very strong polynomial functors*. The first notions of polynomial functors date back to Eilenberg and Mac Lane [17] for functors on module categories. Polynomial functors have been the object of an intensive study because of their applications in algebraic topology (see Henn, Lannes and Schwartz [26]), representation theory (see Kuhn [27]) and group cohomology (see Franjou, Friedlander, Scorichenko and Suslin [19]). Also this notion has progressively been extended to deal with a more general framework than module categories; see Pirashvili [33] for instance. Djament and Vespa recently introduced in [16] strong polynomial functors for symmetric monoidal categories in which the monoidal unit is initial. This new definition encompasses all the previous ones. This definition is extended to pre-braided monoidal categories in which the monoidal unit is initial in [37, Section 3]. The notion of a very strong polynomial functor in this context is introduced in [37, Section 3]; it is equivalent to that of coefficient systems of finite degree of Wahl and Randal-Williams [36, Section 4.4]. These notions of strong and very strong polynomial functors extend to the more general context of the present paper; see Section 4.1.

One reason for our interest in very strong polynomial functors is their homological stability properties: in [36], Randal-Williams and Wahl prove homological stability results for certain families of groups $\{G_n\}_{n\in\mathbb{N}}$ with twisted coefficients given by very strong polynomial objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{Z}\text{-}\mathfrak{Mod})$. Their results hold for surface braid groups and mapping class groups of orientable and non-orientable surfaces. The representation theory of these groups is complicated and an active research topic; see Birman and Brendle's survey [6, Section 4.6] or Margalit's expository paper [31]. A fortiori, the very strong polynomial functors associated with these groups are not well-understood.

In addition, we are interested in *weak* polynomial functors, a notion introduced by Djament and Vespa [16, Section 3.1] for symmetric monoidal categories. We prove that this concept extends to the present framework in Section 4.2. A first matter of interest in this last notion is that it is more appropriate for understanding the stable behaviour of a given functor: it reflects more accurately than the strong polynomiality the behaviour of functors for large values. Also weak polynomial functors of degree less or equal to some $d \in \mathbb{N}$ form a category $\mathcal{P}ol_d(\mathfrak{U}\mathcal{G})$ that is *localizing*, which allows one to define quotient categories $\mathcal{P}ol_{d+1}(\mathfrak{U}\mathcal{G})/\mathcal{P}ol_d(\mathfrak{U}\mathcal{G})$. These quotients provide an organizing tool for families of representations of the groups $\{G_n\}_{n\in\mathbb{N}}$; see Section 6.2.

We then investigate the effects of Long-Moody functors on polynomial functors. This analysis requires to make further assumptions on the parameters $\mathcal{A}$ and χ, which are detailed in Assumption 5.1. Namely we assume that there exist two groups H_0 and H such that $\mathcal{A}(\underline{n}) = H^{*n} * H_0$ for all objects $\underline{n}$ of $\mathcal{G}$, that $\mathcal{A}$ satisfies some compatibility conditions with respect to the morphisms and braiding of $\mathfrak{U}\mathcal{G}$ and that a coherence equality between the group morphisms $\{\chi_n : H_n \to G_{n+1}\}_{n\in\mathbb{N}}$ and the braiding of $\mathfrak{U}\mathcal{G}$ is checked. Assumption 5.1 is a technical (but quite natural) hypothesis, which is satisfied in many of the examples of interest, such as mapping class groups of surfaces or surface braid groups; see Section 6. Then we prove:

Theorem B (**Theorems 5.20 and 5.23**). *Under the hypotheses of Theorem A and Assumption 5.1 and assuming that the groups $\{H_n\}_{n\in\mathbb{N}}$ are free, the Long-Moody functor $\mathbf{LM}_{\{\mathcal{G},\chi\}}$ increases by one both the very strong and the weak polynomial degrees.*

The proof of this result relies on keystone relations for the action of the *difference* and *evanescence* functors on Long-Moody functors. For the family of braid groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$, Theorem B recovers in particular the previous result of the author for braid groups [37, Theorem B] and gives the new result for the weak polynomial degree.

Therefore the Long-Moody functors provide new families of (very) strong polynomial and weak polynomial functors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ in any degree. This result allows one to gain a better understanding of polynomial functors for mapping class groups and extends the scope of twisted homological stability to more sophisticated sequences of representations. It is a precise measure of the fact that the Long-Moody functors produce more complicated (hence interesting) representations of families of groups, but not overly complicated since polynomiality is preserved. These methods also introduce new tools to clarify the structures of weak polynomial functors in this context; see Proposition 6.6.

Outline. The paper is organized as follows. In Section 1, we recall Quillen's bracket construction, pre-braided monoidal categories and the Grothendieck construction. In Section 2, after setting up the general framework of the families of groups, we define the generalized Long-Moody functors and give some of their properties. Section 3 is devoted to the application of Long-Moody functors to the mapping class groups

of surfaces and surface braid groups (recovering in particular the case of classical braid groups). Section 4 introduces the notions of strong, very strong and weak polynomial functors in the present framework. In Section 5, we consider the effect of Long-Moody functors on strong and weak polynomial functors. Finally, in Section 6, we explain the applications of the effect of Long-Moody functors on polynomiality, in particular their interest for homological stability results.

General notation. We fix a commutative (associative, unital, non-trivial) ring R throughout this paper. We denote by $R\text{-}\mathfrak{Mod}$ the category of R-modules. We denote by $\mathfrak{Gr}$ the category of groups and by $*$ the coproduct in this category. For G a group, we denote its unit element by e_G.

Let $\mathfrak{Cat}$ denote the category of small categories. Let $\mathfrak{C}$ be an object of $\mathfrak{Cat}$. We use the abbreviation $\text{Obj}(\mathfrak{C})$ to denote the set of objects of $\mathfrak{C}$. If there exists an initial object $\varnothing$ in the category $\mathfrak{C}$, then we denote by $\iota_A : \varnothing \to A$ the unique morphism from $\varnothing$ to A. If t is a terminal object in the category $\mathfrak{C}$, then we denote by $t_A : A \to t$ the unique morphism from A to t. The maximal subgroupoid $\mathscr{Gr}(\mathfrak{C})$ is the subcategory of $\mathfrak{C}$ which has the same objects as $\mathfrak{C}$ and whose morphisms are the isomorphisms of $\mathfrak{C}$. We denote by $\mathscr{Gr} : \mathfrak{Cat} \to \mathfrak{Cat}$ the functor which associates to a category its maximal subgroupoid. For $\mathfrak{D}$ a category and $\mathfrak{C}$ a small category, we denote by $\mathbf{Fct}(\mathfrak{C}, \mathfrak{D})$ the category of functors from $\mathfrak{C}$ to $\mathfrak{D}$. We take the convention that the set of natural numbers $\mathbb{N}$ is the set of nonnegative integers $\{0, 1, 2, \dots\}$.

Contents

1   Preliminaries on some categorical tools

The aim of this section is to introduce the categorical framework necessary for our study. In particular, we recall notions and properties of Quillen's bracket construction introduced in [22, p.219] and pre-braided monoidal categories, based on [36, Section 1] to which we refer the reader for further details. Also, we recall a construction for functors from a small category to the category of small categories, called the *Grothendieck construction*.

Beforehand, we take this opportunity to recall some terminology about monoidal categories, referring to [29] for an introduction to (braided) strict monoidal categories. A strict monoidal category will be denoted by $(\mathfrak{C}, \natural, 0)$, where $\mathfrak{C}$ is a category, $\natural$ is the monoidal product and 0 is the monoidal unit. If it is braided, then we denote its braiding by $b_{A,B}^{\mathfrak{C}} : A \natural B \xrightarrow{\sim} B \natural A$ for all objects A and B of $\mathfrak{C}$. **We fix a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ throughout this section.**

Quillen's bracket construction.   The following definition is a particular case of a more general construction of [22] and we refer the reader to [36, Section 1.1] for further details. *Quillen's bracket construction* on the groupoid $\mathfrak{G}$, denoted by $\mathfrak{U}\mathfrak{G}$, is the category with the same objects as $\mathfrak{G}$ and for morphisms

$$\mathrm{Hom}_{\mathfrak{U}\mathfrak{G}}(A, B) = \mathrm{Colim}_{\mathfrak{G}}[\mathrm{Hom}_{\mathfrak{G}}(-\natural A, B)]$$

for A and B two objects of $\mathfrak{G}$. Thus, a morphism from A to B in the category $\mathfrak{U}\mathfrak{G}$ is an equivalence class of pairs (X, f), where X is an object of $\mathfrak{G}$ and $f : X \natural A \to B$ is a morphism of $\mathfrak{G}$; we denote this by $[X, f] : A \to B$: two morphisms $[X, f]$ and $[X', f']$ of $\mathrm{Hom}_{\mathfrak{U}\mathfrak{G}}(A, B)$ are equivalent if there exists an isomorphism $\psi \in \mathrm{Hom}_{\mathfrak{G}}(X, X')$ such that

$$f' \circ (\psi \natural id_A) = f. \tag{4}$$

For all objects X of $\mathfrak{U}\mathfrak{G}$, the identity morphism in the category $\mathfrak{U}\mathfrak{G}$ is given by $[0, id_X] : X \to X$. Then, the composition in the category $\mathfrak{U}\mathfrak{G}$ is defined by $[Y, g] \circ [X, f] = [Y \natural X, g \circ (id_Y \natural f)]$ for $[X, f] : A \to B$ and $[Y, g] : B \to C$ two morphisms in the category $\mathfrak{U}\mathfrak{G}$. In particular, we note that the unit 0 of the monoidal structure is an initial object in the category $\mathfrak{U}\mathfrak{G}$.

There is a relationship between the automorphisms of the groupoid $\mathfrak{G}$ and those of its associated Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$. We recall that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is said to have *no zero divisors* if, for all objects A and B of $\mathfrak{G}$, $A \natural B \cong 0$ if and only if $A \cong B \cong 0$. In particular, if the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and $\mathrm{Aut}_{\mathfrak{G}}(0) = \{id_0\}$, then [36, Proposition 1.7] proves that the canonical functor $\mathfrak{c}_{\mathfrak{U}\mathfrak{G}} : \mathfrak{G} \hookrightarrow \mathfrak{U}\mathfrak{G}$ defined as the identity on objects and by $\mathfrak{c}_{\mathfrak{U}\mathfrak{G}}(\phi) = [0, \phi]$ for all $\phi \in \mathrm{Aut}_{\mathfrak{G}}(X)$ is fully faithful. The maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{U}\mathfrak{G})$ of $\mathfrak{U}\mathfrak{G}$ then corresponds to $\mathfrak{G}$ in this situation. **Henceforth in this section, we assume that the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ has no zero divisors and that $\mathrm{Aut}_{\mathfrak{G}}(0) = \{id_0\}$.**

Remark 1.1. Let X be an object of $\mathfrak{G}$ and $\phi \in \mathrm{Aut}_{\mathfrak{G}}(X)$. Then, as an element of $\mathrm{Hom}_{\mathfrak{U}\mathfrak{G}}(X, X)$, we abuse the notation and write ϕ for $[0, \phi]$. We also note from the definition of Quillen's bracket construction that if $(\mathfrak{G}, \natural, 0)$ is locally small, then so is $\mathfrak{U}\mathfrak{G}$.

A natural question is to wonder when an object of $\mathbf{Fct}(\mathfrak{G}, \mathscr{C})$ extends to an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{G}, \mathscr{C})$ for a given category $\mathscr{C}$, which is the aim of the following lemma. Analogous statements can be found in [36, Proposition 2.4] and [37, Lemma 1.12] (for the category $\mathfrak{U}\boldsymbol{\beta}$ for this last reference).

Lemma 1.2. *Let $\mathscr{C}$ be a category and F an object of $\mathbf{Fct}(\mathfrak{G}, \mathscr{C})$. Assume that there exist a morphism $\eta_{Q,P} : F(P) \to F(Q \natural P)$ for each pair (P, Q) of objects of $\mathfrak{G}$, such that for all $A, X, Y \in Obj(\mathfrak{G})$:*

$$\eta_{Y, X \natural A} \circ \eta_{X, A} = \eta_{Y \natural X, A} \tag{5}$$

and $\eta_{0,B} = id_{F(B)}$ for all $B \in Obj(\mathfrak{G})$. Then, the assignments $F([X, \gamma]) = F(\gamma) \circ \eta_{X,A}$ to all morphisms $[X, \gamma] : A \to X \natural A$ of $\mathfrak{U}\mathfrak{G}$ extend the functor $F : \mathfrak{G} \to \mathscr{C}$ to a functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ if and only if for all $A, X \in Obj(\mathfrak{G})$, for all $\gamma'' \in Aut_{\mathfrak{G}}(A)$ and all $\gamma' \in Aut_{\mathfrak{G}}(X)$:

$$F([X, id_{X \natural A}]) \circ F(\gamma'') = F(\gamma' \natural \gamma'') \circ F([X, id_{X \natural A}]). \tag{6}$$

Proof. Assume that relation (6) is satisfied. Our assignments imply that $F([0, id_A]) = id_{F(A)}$ for all objects A. First, let us prove that our assignment conforms with the defining equivalence relation (4) of $\mathfrak{U}\mathfrak{G}$. Let $A, X \in$ Obj$(\mathfrak{G})$. Let $\gamma, \gamma' \in \text{Aut}_{\mathfrak{G}}(X \natural A)$ such that there exists $\psi \in \text{Aut}_{\mathfrak{G}}(X)$ such that $\gamma' \circ (\psi \natural id_A) = \gamma$. According to the relation (6) and since F is a functor over $\mathfrak{G}$, we deduce that $F([X, \gamma]) = F(\gamma') \circ F([X, id_{X \natural A}]) \circ F(id_A) = F([X, \gamma'])$. Now, let us check the composition axiom. Let $A, X, Y \in$ Obj$(\mathfrak{G})$, let $[X, \gamma] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(A, X \natural A)$ and $[Y, \gamma'] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(X \natural A, Y \natural X \natural A)$. We deduce from relation (6) that $F([Y, \gamma']) \circ F([X, \gamma]) = F(\gamma') \circ (F(id_Y \natural \gamma) \circ F([Y, id_{Y \natural X \natural A}])) \circ F([X, id_{X \natural A}])$. So, it follows from relation (5) that

$$F([Y, \gamma']) \circ F([X, \gamma]) = F(\gamma' \circ (id_Y \natural \gamma)) \circ F([Y \natural X, id_{Y \natural X \natural A}]) = F([Y, \gamma'] \circ [X, \gamma]).$$

Conversely, assume that the functor $F : \mathfrak{U}\mathfrak{G} \to \mathscr{C}$ is well-defined. In particular, the composition axiom in $\mathfrak{U}\mathfrak{G}$ is satisfied and implies that for all $A, X \in$ Obj$(\mathfrak{G})$, for all $\gamma \in \text{Aut}_{\mathfrak{G}}(A)$, $F([X, id_{X \natural A}]) \circ F(\gamma) = F([X, id_X \natural \gamma])$. So it follows from the defining equivalence relation (4) of $\mathfrak{U}\mathfrak{G}$ that relation (6) is satisfied. $\square$

Pre-braided monoidal categories. If the strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ is braided, Quillen's bracket construction $\mathfrak{U}\mathfrak{G}$ inherits a strict monoidal structure; see Proposition 1.4. However, the braiding $b^{\mathfrak{G}}_{-,-}$ does not extend in general to $\mathfrak{U}\mathfrak{G}$.

First, we recall the notion of a pre-braided monoidal category introduced by Randal-Williams and Wahl in [36, Definition 1.5], generalizing that of a strict braided monoidal category. For $(\mathfrak{C}, \natural, 0)$ a strict monoidal category such that the unit 0 is initial, we say that $(\mathfrak{C}, \natural, 0)$ is *pre-braided* if the maximal subgroupoid $\mathscr{G}\mathfrak{r}(\mathfrak{C})$ is a braided monoidal category – where the monoidal structure is induced by that of $(\mathfrak{C}, \natural, 0)$ – and the braiding associated with the maximal subgroupoid $b^{\mathscr{G}\mathfrak{r}(\mathfrak{C})}_{A,B} : A \natural B \longrightarrow B \natural A$ satisfies

$$b^{\mathscr{G}\mathfrak{r}(\mathfrak{C})}_{A,B} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \longrightarrow B \natural A \tag{7}$$

for all objects A and B of $\mathfrak{C}$ (recall that $\iota_B : 0 \to B$ denotes the unique morphism from 0 to B).

Remark 1.3. A braided monoidal category is automatically pre-braided. However, a pre-braided monoidal category is not necessarily braided; see [36, Remark 5.24] or [37, Remark 1.15]. Namely, for a pre-braided monoidal category, the opposite of condition (7), that is $b^{\mathfrak{C}}_{A,B} \circ (\iota_B \natural id_A) = id_A \natural \iota_B$, does not hold generally speaking, whereas this is a necessary property for a braided monoidal category.

Finally, let us give the following key property when Quillen's bracket construction is applied on a strict braided monoidal groupoid $(\mathfrak{G}, \natural, 0, b^{\mathfrak{G}}_{-,-})$.

Proposition 1.4. *[36, Proposition 1.8] If the groupoid $(\mathfrak{G}, \natural, 0)$ is braided, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is pre-braided monoidal. Namely, the monoidal structure on the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is defined on objects by that of $(\mathfrak{G}, \natural, 0)$ and on morphisms by letting*

$$[X, f] \natural [Y, g] = [X \natural Y, (f \natural g) \circ (id_X \natural (b^{\mathfrak{G}}_{A,Y})^{-1} \natural id_C)] \tag{8}$$

for $[X, f] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, B)$ and $[Y, g] \in Hom_{\mathfrak{U}\mathfrak{G}}(C, D)$. In particular, the canonical functor $\mathfrak{c}_{\mathfrak{U}\mathfrak{G}} : \mathfrak{G} \hookrightarrow \mathfrak{U}\mathfrak{G}$ is monoidal.

The Grothendieck construction. We present here the Grothendieck construction for a functor from a small category to the category of small categories. We refer the reader to [30, Chapter 1, Section 5] for further details.

Let $\mathfrak{C}$ be a *small* category and $\mathcal{A} : \mathfrak{C} \to \mathfrak{Cat}$ a functor. The *Grothendieck construction* for $\mathcal{A}$, also known as the category of elements of $\mathcal{A}$ and denoted by $\int^{\mathfrak{C}} \mathcal{A}$, is defined as follows. The objects are pairs (x, c) where $c \in$ Obj$(\mathfrak{C})$ and $x \in$ Obj$(\mathcal{A}(c))$, and a morphism from (x, c) to (x', c') in $\int^{\mathfrak{C}} \mathcal{A}$ is a pair (α, f) where $f \in \text{Hom}_{\mathfrak{C}}(c, c')$ and $\alpha \in \text{Hom}_{\mathcal{A}(c')}(\mathcal{A}(f)(x), x')$. For $(\alpha, f) \in \text{Hom}_{\int^{\mathfrak{C}} \mathcal{A}}((x_1, c_1), (x_2, c_2))$ and $(\beta, g) \in$ $\text{Hom}_{\int^{\mathfrak{C}} \mathcal{A}}((x_2, c_2), (x_3, c_3))$, the composition in the category $\int^{\mathfrak{C}} \mathcal{A}$ is defined by

$$(\beta, g) \circ (\alpha, f) = (\beta \circ \mathcal{A}(g)(\alpha), g \circ f).$$

There is a canonical projection functor $\int^{\mathfrak{C}} \mathcal{A} \to \mathfrak{C}$, given by sending an object (x, c) to c.

Recall that $\mathfrak{Gr}$ denotes the category of groups. For a functor $\mathcal{A} : \mathfrak{C} \to \mathfrak{Gr}$, the associated Grothendieck construction $\int^{\mathfrak{C}} \mathcal{A}$ is then defined by considering a group G as a category with one object denoted by $\cdot_G$. Denoting by 0 the functor $\mathfrak{C} \to \mathfrak{Gr}$ sending all $c \in \mathrm{Obj}(\mathfrak{C})$ to the trivial group $0_{\mathfrak{Gr}}$, there exists a unique natural transformation $0 \to \mathcal{A}$. Applying the Grothendieck construction, this induces a section $\mathfrak{C} = \int^{\mathfrak{C}} 0 \hookrightarrow \int^{\mathfrak{C}} \mathcal{A}$ to the projection functor $\int^{\mathfrak{C}} \mathcal{A} \to \mathfrak{C}$. We denote this section functor by $\mathfrak{s}_{\mathcal{A}}$.

With the category of groups as the target category of $\mathcal{A}$, the associated Grothendieck construction actually encodes a semidirect product structure:

Lemma 1.5. *For a functor $\mathcal{A} : \mathfrak{C} \to \mathfrak{Gr}$ where $\mathfrak{C}$ is a small groupoid, then $\int^{\mathfrak{C}} \mathcal{A}$ is a groupoid where the automorphism group of any object c is the semidirect product $\mathcal{A}(c) \rtimes Aut_{\mathfrak{C}}(c)$ induced by $\mathcal{A}$.*

Proof. Considering such a functor $\mathcal{A}$ is equivalent to defining a group homomorphism $\mathcal{A}_c : \mathrm{Aut}_{\mathfrak{C}}(c) \to \mathrm{Aut}_{\mathfrak{Gr}}(\mathcal{A}(c))$ for each object c. It follows from the definition of the Grothendieck construction that all the morphisms of $\int^{\mathfrak{C}} \mathcal{A}$ are invertible, and that there is a clear group isomorphism $\mathrm{Aut}_{\int^{\mathfrak{C}} \mathcal{A}}((\cdot_{\mathcal{A}(c)}, c)) \cong \mathcal{A}(c) \rtimes_{\mathcal{A}_c} \mathrm{Aut}_{\mathfrak{C}}(c)$, the composition in the category $\int^{\mathfrak{C}} \mathcal{A}$ following exactly the same rule as the group operation of a semidirect product. $\square$

2 The generalized Long-Moody functors

In this section, we introduce the notion of Long-Moody functors for an abstract family of groups, inspired by the Long-Moody construction for braid groups of [28, Theorem 2.1]. First, we introduce a general construction in Section 2.1, called *the tensorial construction*, using tensor product of functors and the required tools for our study. We then define *the generalized Long-Moody functors* and establish some of their first properties in Section 2.2. In addition to recovering all the results of [37, Section 2], we give a new approach to the tools and conditions previously considered in [37], allowing a deeper understanding of these constructions and a wider application.

2.1 A general construction

In this first subsection, we present a general construction based on a tensor product for functor categories. The generalized Long-Moody functors introduced in Section 2.2 are particular cases of this construction. We refer the reader to [29, Section VII.3] for the notions of monoid objects and modules in a monoidal category, which will be used in this section.

Tensor product over a monoid functors. First, let us introduce the notion of tensor product over a monoid functor. **We fix a small category $\mathfrak{C}$ throughout Section 2.1.**

Let $\otimes_R$ be the pointwise tensor product in the functor category $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ and let R denote the constant functor at R. These endow $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ with a strict monoidal structure $(\mathbf{Fct}\,(\mathfrak{C}, R\text{-}\mathfrak{Mod}), \otimes_R, R)$. Let $\mathcal{M}$ be a monoid object in $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$. We denote by $\mathcal{M}\text{-}\mathfrak{Mod}$ and $\mathfrak{Mod}\text{-}\mathcal{M}$ the categories of left and right modules in $\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ over $\mathcal{M}$ respectively. The *tensor product over $\mathcal{M}$ functor* $- \otimes_{\mathcal{M}} - : \mathfrak{Mod}\text{-}\mathcal{M} \times \mathcal{M}\text{-}\mathfrak{Mod} \to \mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$ is defined by:

• Objects: for $F \in \mathrm{Obj}(\mathfrak{Mod}\text{-}\mathcal{M})$ and $G \in \mathrm{Obj}(\mathcal{M}\text{-}\mathfrak{Mod})$, denoting ρ_F and λ_G the natural transformation actions of $\mathcal{M}$ on F and G respectively, $F \otimes_{\mathcal{M}} G : \mathfrak{C} \to R\text{-}\mathfrak{Mod}$ is the coequalizer of the natural transformations $\rho_F \otimes_R id_G$ and $id_F \otimes_R \lambda_G$.

• Morphisms: let F_1 and F_2 be two objects of $\mathfrak{Mod}\text{-}\mathcal{M}$ and $f : F_1 \to F_2$ be a natural transformation in $\mathfrak{Mod}\text{-}\mathcal{M}$; let G_1 and G_2 be two objects of $\mathcal{M}\text{-}\mathfrak{Mod}$ and $g : G_1 \to G_2$ be a natural transformation in $\mathcal{M}\text{-}\mathfrak{Mod}$. We define $f \otimes_{\mathcal{M}} g : F_1 \otimes_{\mathcal{M}} G_1 \to F_2 \otimes_{\mathcal{M}} G_2$ to be the unique morphism induced from $f \otimes_R g : F_1 \otimes_R G_1 \to F_2 \otimes_R G_2$ by the universal property of the coequalizer $F_1 \otimes_{\mathcal{M}} G_1$.

The functor $- \otimes_{\mathcal{M}} -$ is called *the tensor product functor over $\mathcal{M}$*. In particular, fixing an object F of $\mathfrak{Mod}\text{-}\mathcal{M}$ defines a functor $F \otimes_{\mathcal{M}} - : \mathcal{M}\text{-}\mathfrak{Mod} \to \mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod})$.

Group algebra and augmentation ideal functors. We recall that $\mathfrak{Gr}$ denotes the category of groups. **From now on, we fix a functor $\mathcal{A} : \mathfrak{C} \to \mathfrak{Gr}$ for the remainder of Section 2.1.**

Let R-$\mathfrak{Alg}$ be the category of unital R-algebras. For all objects G of $\mathfrak{Gr}$, the group rings $R[G]$ and augmentation ideals $\mathcal{I}_{R[G]}$ respectively assemble to define the group algebra functor $R[-] : \mathfrak{Gr} \to R$-$\mathfrak{Alg}$ and the augmentation ideal functor $\mathcal{I}_{R[-]} : \mathfrak{Gr} \to R$-$\mathfrak{Mod}$. Let $R[\mathcal{A}]$ be the composition functor $R[-] \circ \mathcal{A} : \mathfrak{C} \to R$-$\mathfrak{Alg}$, called the group algebra functor induced by $\mathcal{A}$. Similarly, let $\mathcal{I}_{R[\mathcal{A}]}$ be the composition functor $\mathcal{I}_{R[-]} \circ \mathcal{A} : \mathfrak{C} \to R$-$\mathfrak{Mod}$, called the augmentation ideal functor induced by $\mathcal{A}$. The unital R-algebra structures of $R[\mathcal{A}(c)]$ for all $c \in \mathrm{Obj}(\mathfrak{C})$ induce an associative unital monoid object structure on $R[\mathcal{A}]$ with respect to the monoidal structure $(\mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod}), \otimes_R, R)$. Therefore:

Lemma 2.1. *The augmentation ideal functor $\mathcal{I}_{R[\mathcal{A}]}$ is a right $R[\mathcal{A}]$-module.*

Proof. The natural transformation $\mathcal{I}_{R[\mathcal{A}]} \otimes_R R[\mathcal{A}] \to \mathcal{I}_{R[\mathcal{A}]}$ is induced by the right $R[\mathcal{A}(c)]$-module structure of the augmentation ideal $\mathcal{I}_{R[\mathcal{A}(c)]}$ for each object c of $\mathfrak{C}$, the associativity and unit axioms of a module over a monoid object being straightforward to check. $\square$

The tensorial construction. We present now a general construction for functor categories, using a tensor product functor. First of all, we have the following key property:

Proposition 2.2. *The precomposition by the section $\mathfrak{s}_\mathcal{A} : \mathfrak{C} \hookrightarrow \int^{\mathfrak{C}} \mathcal{A}$ induces an equivalence of categories $\mathfrak{s}_\mathcal{A}^* : \mathbf{Fct}(\int^{\mathfrak{C}} \mathcal{A}, R\text{-}\mathfrak{Mod}) \cong R[\mathcal{A}]\text{-}\mathfrak{Mod}$.*

Proof. Let F be an object of $\mathbf{Fct}(\int^{\mathfrak{C}} \mathcal{A}, R\text{-}\mathfrak{Mod})$. We consider the functor $\mathfrak{s}_\mathcal{A}^*(F) : \mathfrak{C} \to R\text{-}\mathfrak{Mod}$. We recall that e_G denotes the unit element of a group G. For c and c' two objects of $\mathfrak{C}$, a morphism from $(\cdot_{\mathcal{A}(c)}, c)$ to $(\cdot_{\mathcal{A}(c')}, c')$ in $\int^{\mathfrak{C}} \mathcal{A}$ is of the form $(x, \varphi) = (x, id_{c'}) \circ (e_{\mathcal{A}(c')}, \varphi)$ where $x \in \mathcal{A}(c')$ and $\varphi \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$. We define a left $R[\mathcal{A}]$-module structure natural transformation $\lambda_{\mathfrak{s}_\mathcal{A}^*(F)} : R[\mathcal{A}] \otimes_R \mathfrak{s}_\mathcal{A}^*(F) \to \mathfrak{s}_\mathcal{A}^*(F)$ as follows. For each $c \in \mathrm{Obj}(\mathfrak{C})$, we define the morphism $\lambda_{\mathfrak{s}_\mathcal{A}^*(F)(c)} : R[\mathcal{A}(c)] \otimes_R \mathfrak{s}_\mathcal{A}^*(F)(c) \to \mathfrak{s}_\mathcal{A}^*(F)(c)$ by

$$\lambda_{\mathfrak{s}_\mathcal{A}^*(F)}(y \otimes_R v) = F(y, id_c)(v)$$

for all $y \in \mathcal{A}(c)$ and $v \in F(\mathfrak{s}_\mathcal{A}(c))$. We recall from the composition rule in $\int^{\mathfrak{C}} \mathcal{A}$ that $(e_{\mathcal{A}(c')}, \varphi) \circ (y, id_c) = (\mathcal{A}(\varphi)(y), \varphi)$. Then, since F is a functor over the category $\int^{\mathfrak{C}} \mathcal{A}$, we deduce that for each $\varphi \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$:

$$(\lambda_{\mathfrak{s}_\mathcal{A}^*(F)(c')} \circ (R[\mathcal{A}(\varphi)] \otimes_R \mathfrak{s}_\mathcal{A}^*(F)(\varphi)))(y \underset{R}{\otimes} v) = F((\mathcal{A}(\varphi)(y), \varphi))(v) = (\mathfrak{s}_\mathcal{A}^*(F)(\varphi) \circ \lambda_{\mathfrak{s}_\mathcal{A}^*(F)(c)})(y \underset{R}{\otimes} v).$$

Hence, the morphisms $\lambda_{\mathfrak{s}_\mathcal{A}^*(F)(c)}$ define a natural transformation and the axioms of an action are easily checked. The naturality with respect to F follows straightforwardly from these assignments.

Conversely, let G be a left $R[\mathcal{A}]$-module and we denote by $\lambda_G : R[\mathcal{A}] \otimes_R G \to G$ its associated left module natural transformation. We extend G to a functor $\widehat{G}$ with $\int^{\mathfrak{C}} \mathcal{A}$ as source category as follows. For each $c \in \mathrm{Obj}(\mathfrak{C})$, $\widehat{G}$ sends the object $(\cdot_{\mathcal{A}(c)}, c)$ to $G(c)$. Then, for all (x, φ) where $x \in \mathcal{A}(c')$ and $\varphi \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$, we define $\widehat{G}((x, \varphi))$ to be the composition $G(c) \to R[A(c')] \otimes_R G(c') \to G(c')$ defined by $\lambda_{G(c')} \circ (x \otimes_R G(\varphi))$. Considering another (x', φ') where $x' \in \mathcal{A}(c'')$ and $\varphi' \in \mathrm{Hom}_{\mathfrak{C}}(c', c'')$, since λ_G is a natural transformation, we compute that $G(\varphi') \circ \lambda_{G(c')} \circ (x \otimes_R G(\varphi)) = \lambda_{G(c'')} \circ (\mathcal{A}(\varphi)(x) \otimes_R G(\varphi' \circ \varphi))$ and deduce that

$$\widehat{G}((x', \varphi')) \circ \widehat{G}((x, \varphi)) = \lambda_{G(c'')} \circ (x'\mathcal{A}(\varphi)(x) \otimes_R G(\varphi' \circ \varphi)) = \widehat{G}((x', \varphi') \circ (x, \varphi)).$$

Therefore the assignment for $\widehat{G}$ satisfies the composition axiom on $\int^{\mathfrak{C}} \mathcal{A}$ and the identity axiom is easily checked. Also, the naturality with respect to G follows from the naturality with respect to λ_G of a functor between two $R[\mathcal{A}]$-modules. Then, it follows from these definitions that $\widehat{-}$ defines a functor from $R[\mathcal{A}]$-$\mathfrak{Mod}$ to $\mathbf{Fct}(\int^{\mathfrak{C}} \mathcal{A}, R\text{-}\mathfrak{Mod})$ and that is the inverse of $\mathfrak{s}_\mathcal{A}^*$. $\square$

Now, we can introduce the construction:

Definition 2.3. Let $\chi : \int^{\mathfrak{C}} \mathcal{A} \to \mathfrak{D}$ be a functor where $\mathfrak{D}$ is another small category and we denote by χ^* the precomposition functor induced by χ. We define $\mathfrak{T}_{\mathcal{A},\chi}$ to be the composition:

$$\mathbf{Fct}(\mathfrak{D}, R\text{-}\mathfrak{Mod}) \xrightarrow{\mathfrak{s}_\mathcal{A}^* \circ \chi^*} R[\mathcal{A}]\text{-}\mathfrak{Mod} \xrightarrow{\mathcal{I}_{R[\mathcal{A}]} \otimes_{R[\mathcal{A}]} -} \mathbf{Fct}(\mathfrak{C}, R\text{-}\mathfrak{Mod}).$$

The functor $\mathfrak{T}_{\mathcal{A},\chi}$ is called *the tensorial construction* by the functor $\mathcal{A}$ along χ.

We take this opportunity to spell out how a morphism in $\mathfrak{C}$ acts on $\mathfrak{T}_{\mathcal{A},\chi}$. The following result is a straightforward consequence of the definition of $\mathfrak{T}_{\mathcal{A},\chi}$:

Lemma 2.4. *We take up the notations of Definition 2.3. Let F be an object of $\mathbf{Fct}(\mathfrak{D}, R\text{-}\mathfrak{Mod})$ and $\varphi \in \mathrm{Hom}_{\mathfrak{C}}(c, c')$. Then $\mathfrak{T}_{\mathcal{A},\chi}(F)(\varphi)$ is the morphism from $\mathcal{I}_{R[\mathcal{A}(c)]} \otimes_{R[\mathcal{A}(c)]} F((\chi \circ \mathfrak{s}_{\mathcal{A}})(c))$ to $\mathcal{I}_{R[\mathcal{A}(c')]} \otimes_{R[\mathcal{A}(c')]} F((\chi \circ \mathfrak{s}_{\mathcal{A}})(c'))$ defined by*

$$\mathfrak{T}_{\mathcal{A},\chi}(F)(\varphi)(i \underset{R[\mathcal{A}(c)]}{\otimes} v) = \mathcal{I}_{R[\mathcal{A}]}(\varphi)(i) \underset{R[\mathcal{A}(c')]}{\otimes} F((\chi \circ \mathfrak{s}_{\mathcal{A}})(\varphi))(v)$$

for all $i \in \mathcal{I}_{R[\mathcal{A}(c)]}$ and $v \in F((\chi \circ \mathfrak{s}_{\mathcal{A}})(c))$.

Also, let G be another objects of $\mathbf{Fct}(\mathfrak{D}, R\text{-}\mathfrak{Mod})$ and $\eta : F \to G$ be a natural transformation. Then, the natural transformation $\mathfrak{T}_{\mathcal{A},\chi}(\eta) : \mathfrak{T}_{\mathcal{A},\chi}(F) \to \mathfrak{T}_{\mathcal{A},\chi}(G)$ is given for all $c \in Obj(\mathfrak{C})$ by assigning the morphism $\mathfrak{T}_{\mathcal{A},\chi}(\eta_d)_c$ to be the tensor product morphism $id_{\mathcal{I}_{R[\mathcal{A}(c)]}} \otimes_{R[\mathcal{A}(c)]} \eta_{(\chi \circ \mathfrak{s}_{\mathcal{A}})(c)}$.

Let us give some immediate properties of a tensorial construction.

Proposition 2.5. *The tensorial construction $\mathfrak{T}_{\mathcal{A},\chi}$ is additive, right-exact and commutes with all colimits.*

Proof. Let $0 : \mathfrak{E} \to R\text{-}\mathfrak{Mod}$ denote the null functor for a small category $\mathfrak{E}$. It follows from the definition that $\mathfrak{T}_{\mathcal{A},\chi}(0) = 0$, and so additivity follows from the commutation with all colimits.

Since the precomposition functors χ^* and $\mathfrak{s}_{\mathcal{A}}^*$ are exact, it is enough to prove the results for the functor $\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} -$. As a consequence of the properties of the tensor product of modules, the functor $\mathcal{I}_{R[\mathcal{A}(c)]} \otimes_{R[\mathcal{A}(c)]} - : R[\mathcal{A}(c)]\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ is additive, right-exact and commutes with all colimits for all $c \in \mathrm{Obj}(\mathfrak{C})$. Then the naturality of each of these properties with respect to the morphisms of $\mathfrak{C}$ is a formal consequence of the definition of $\mathcal{I}_{\mathcal{A}} \otimes_{R[\mathcal{A}]} -$. $\qquad\square$

2.2 The Long-Moody functors

Using the tensorial construction of Section 2.1, we introduce here the generalized Long-Moody functors, inspired from the Long-Moody construction [28]. While the original construction was associated with braid groups, the following framework encompasses a much broader class of groups; see Section 3.

Categorical framework. First, we require the following categorical framework to define generalized Long-Moody functors. Let $(\mathcal{G}', \natural, 0_{\mathcal{G}'}, b_{-,-}^{\mathcal{G}'})$ be a locally small strict braided monoidal groupoid with no zero divisors and such that $\mathrm{Aut}_{\mathcal{G}'}(0_{\mathcal{G}'}) = \{id_{0_{\mathcal{G}'}}\}$. We recall from Proposition 1.4 that Quillen's bracket construction $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'})$ is a locally small pre-braided monoidal category such that the unit $0_{\mathcal{G}'}$ is an initial object. Let $\underline{0}$ and 1 be two objects of $\mathcal{G}'$, such that 1 is not isomorphic to $0_{\mathcal{G}'}$.

Notation 2.6. For all natural numbers n, we denote the object $1^{\natural n} \natural \underline{0}$ of $\mathcal{G}'$ by $\underline{n}$ and the object $1^{\natural n}$ of $\mathcal{G}'$ by n. Note that $m \natural \underline{n} = \underline{m + n}$ for all natural numbers m and n.

Definition 2.7. Let $\mathcal{G}$ be the small full subgroupoid of $\mathcal{G}'$ on the objects $\{\underline{n}\}_{n \in \mathbb{N}}$. We denote by G_n the automorphism group $\mathrm{Aut}_{\mathcal{G}}(\underline{n})$ for all n. For convenience, we denote the small full subcategory of Quillen's bracket construction $\mathfrak{U}\mathcal{G}'$ on the objects $\{\underline{n}\}_{n \in \mathbb{N}}$ by $\mathfrak{U}\mathcal{G}$. Let $1 \natural - : \mathfrak{U}\mathcal{G} \to \mathfrak{U}\mathcal{G}$ be the functor defined by $(1 \natural -)(\underline{n}) = 1 \natural \underline{n}$ for all $\underline{n} \in \mathrm{Obj}(\mathcal{G})$ and $(1 \natural -)([n' - n, g]) = id_1 \natural [n' - n, g]$ for all morphism $[n' - n, g]$ of $\mathfrak{U}\mathcal{G}$.

Warning: the category $\mathfrak{U}\mathcal{G}$ is not in general Quillen's bracket construction introduced in Section 1 applied to $\mathcal{G}$, but Quillen's bracket construction applied to the ambient groupoid $\mathcal{G}'$. We choose this convention to avoid too heavy notations. Note that $\mathfrak{U}\mathcal{G}$ is Quillen's bracket construction on $\mathcal{G}$ if and only if $0_{\mathcal{G}'} = \underline{0}$. The present framework allows one to handle families of groups such as mapping class groups of surfaces; see Sections 3.3 and 3.4.

We fix a braided monoidal groupoid $\mathcal{G}'$, the small full subgroupoid $\mathcal{G}$ of Definition 2.7 and a functor $\mathcal{A} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ for the remainder of Section 2.2.

2.2.1 Definition of the Long-Moody functors

The idea to define a Long-Moody functor is to use the tensorial construction by the functor $\mathcal{A}$ along some χ, such that the composition the section $\mathfrak{s}_{\mathcal{A}}$ is the functor $1 \natural -$. We have the choice to consider $\mathcal{A}$ over $\mathcal{G}$ or $\mathfrak{U}\mathcal{G}$.

Actually, this choice leads to an extension problem: the restriction along the canonical functor $\mathfrak{c}_{\mathfrak{U}\mathcal{G}} : \mathcal{G} \hookrightarrow \mathfrak{U}\mathcal{G}$ defines the Grothendieck construction $\int^{\mathcal{G}} \mathcal{A}$ together with an inclusion functor $\int^{\mathcal{G}} \mathcal{A} \hookrightarrow \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$, so that the following diagram is commutative

$$
\begin{array}{ccc}
\mathcal{G} & \hookrightarrow & \mathfrak{U}\mathcal{G} \\
\downarrow & & \downarrow \\
\int^{\mathcal{G}} \mathcal{A} & \hookrightarrow & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}.
\end{array}
$$

This is not a pushout of categories: given $\mathcal{G} \hookrightarrow \int^{\mathcal{G}} \mathcal{A} \to \mathfrak{E}$ a functor for $\mathfrak{E}$ a small category, we deal with the extension problem

$$
\begin{array}{ccc}
\mathcal{G} & \hookrightarrow & \mathfrak{U}\mathcal{G} \\
\downarrow & & \downarrow \searrow \\
\int^{\mathcal{G}} \mathcal{A} & \hookrightarrow & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \\
& & \searrow \quad \vdots \quad \downarrow \\
& & \mathfrak{E}.
\end{array}
$$

In the present situation, $\mathfrak{E}$ is taken to be $\mathfrak{U}\mathcal{G}$ and the composites $\mathcal{G} \to \int^{\mathcal{G}} \mathcal{A} \to \mathfrak{E}$ and $\mathfrak{U}\mathcal{G} \to \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathfrak{E}$ to be $1\natural-$. This motivates the following:

Definition 2.8. The setting $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is a *Long-Moody system* if $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ is a functor such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathcal{G} & \overset{\mathfrak{s}_{\mathcal{A}}}{\hookrightarrow} & \int^{\mathcal{G}} \mathcal{A} \\
& {}_{1\natural-}\searrow & \downarrow \chi \\
& & \mathcal{G}.
\end{array}
\tag{9}
$$

The functor $\mathcal{A}$ equipped with such functor χ is said to define a Long-Moody system, denoted by $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$. If χ extends along the inclusion $\int^{\mathcal{G}} \mathcal{A} \hookrightarrow \int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ to define a functor $\chi : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathfrak{U}\mathcal{G}$, such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{U}\mathcal{G} & \overset{\mathfrak{s}_{\mathcal{A}}}{\hookrightarrow} & \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \\
& {}_{1\natural-}\searrow & \downarrow \chi \\
& & \mathfrak{U}\mathcal{G},
\end{array}
\tag{10}
$$

then the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is then said to be *coherent*.

Then, we can introduce the main concept of Section 2:

Definition 2.9. The Long-Moody functor associated with the Long-Moody system (respectively coherent Long-Moody system) $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$, denoted by $\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}}$ (respectively $\mathbf{LM}^{\mathfrak{U}}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}}$), is the tensorial construction $\mathfrak{T}_{\mathcal{A}, \chi}$ by the functors $\mathcal{A} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ along $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ (respectively along $\chi : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathfrak{U}\mathcal{G}$) of Definition 2.3.

Notation 2.10. When there is no ambiguity, once the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is fixed, we omit it from the notation. If the Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is coherent, we omit it the $\mathfrak{U}$ from the notation if there is no risk of confusion. In this case, we denote both the endofunctor of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$ and the one of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ by $\mathbf{LM}$ for simplicity. Indeed, since the clear diagram relating them – induced by the precomposition functor $\mathfrak{c}^{*}_{\mathfrak{U}\mathfrak{G}} : \mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$ – is commutative.

For sake of clarity, let us spell out the explicit definition of the Long-Moody functor associated to a coherent system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$. For an object F if $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, for all objects $\underline{n}$ of $\mathcal{G}$ we have

$$
\mathbf{LM}_{\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}}(F)(\underline{n}) = \mathcal{I}_{R[\mathcal{A}(\underline{n})]} \underset{R[\mathcal{A}(\underline{n})]}{\otimes} F(id_1 \natural \underline{n}).
$$

11

Also, the actions of $\mathbf{LM}_{\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}}(F)$ on morphisms of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and of $\mathbf{LM}_{\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}}$ on morphisms of $\mathfrak{U}\mathcal{G}$ are induced by Lemma 2.4.

Non-trivial coherent Long-Moody systems arise naturally in many situations in connection with topology; see Section 3. We give a first example here:

Example 2.11. Let us fix $(\mathcal{G}',\natural,0_{\mathcal{G}'}) = (\mathcal{G},\natural,0_{\mathcal{G}}) = (\boldsymbol{\beta},\natural,0)$, where $\boldsymbol{\beta}$ is the braid groupoid. It has the natural numbers as its objects the natural numbers and its automorphisms are the braid groups $\{\mathbf{B}_n\}_{n\in\mathbb{N}}$. The strict monoidal structure $\natural$ is defined by the usual addition for the objects and laying two braids side by side for the morphisms; see [29, Chapter XI, Section 4] for more details.

Let $\mathbf{F}_n$ be the free group of rank n. In this case, the Artin representations $\{\mathcal{A}_{1,n}^{\boldsymbol{\beta}} : \mathbf{B}_n \to \mathrm{Aut}(\mathbf{F}_n)\}_{n\in\mathbb{N}}$, defined by the action $\mathbf{B}_n$ on the fundamental group of a n-punctured disc, assemble to define a functor $\mathcal{A}_1^{\boldsymbol{\beta}} : \mathfrak{U}\boldsymbol{\beta} \to \mathfrak{Gr}$; see Section 3.2. Moreover, there exists a family of non-trivial morphisms $\{\chi_{n,1} : \mathbf{F}_n \to \mathbf{B}_{n+1}\}_{n\in\mathbb{N}}$ (see Definition 3.6) such that the morphism given by the coproduct $\chi_{n,1} * (id_1\natural-) : \mathbf{F}_n * \mathbf{B}_n \to \mathbf{B}_{n+1}$ factors through the canonical surjection to the semidirect product $\mathbf{F}_n \rtimes_{\mathcal{A}_{1,n}^{\boldsymbol{\beta}}} \mathbf{B}_n$ and such that the corresponding diagram (12) is commutative; see Section 3.3.1. We thus define a non-trivial functor $\chi_1 : \int^{\mathfrak{U}\boldsymbol{\beta}} \mathcal{A}_1^{\boldsymbol{\beta}} \to \mathfrak{U}\boldsymbol{\beta}$ such that we have a coherent Long-Moody system $\{\mathcal{A}_1^{\boldsymbol{\beta}}, \boldsymbol{\beta}, \boldsymbol{\beta}, \chi_1\}$. We refer to Section 3.3.2 for more details. Hence Definitions 2.8 and 2.9 recover their analogues [37, Definition 2.14 and Theorem 2.19].

2.2.2 Properties of Long-Moody systems

Equivalent characterization of Long-Moody systems. We give now an equivalent description of the functor χ introduced in Definition 2.8 to define a Long-Moody system. For all natural numbers n, we denote by $\mathcal{A}_n : G_n \to \mathrm{Aut}_{\mathfrak{Gr}}(\mathcal{A}(\underline{n}))$ the group morphisms induced by the functor $\mathcal{A}$. We note that the coproduct $\mathcal{A}(\underline{n}) * G_n$ canonically surjects onto the semidirect product $\mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n$ and we denote this canonical surjection by $s_\rtimes$. Also, considering the functor $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ is equivalent to considering a family of group morphisms $\{\chi_n^\rtimes : \mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n \to G_{n+1}\}_{n\in\mathbb{N}}$ by Lemma 1.5. Hence, we deduce:

Lemma 2.12. *Given a family of group morphisms $\{\chi_n : \mathcal{A}(\underline{n}) \to G_{n+1}\}_{n\in\mathbb{N}}$ such that the coproduct morphism $\chi_n * (id_1\natural-) : \mathcal{A}(\underline{n}) * G_n \to G_{n+1}$ factors across the canonical surjection $s_\rtimes$, ie there exist group morphisms $\{\chi_n^\rtimes : \mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n \to G_{n+1}\}_{n\in\mathbb{N}}$ such that the following diagram is commutative*

$$\mathcal{A}(\underline{n}) \hookrightarrow \mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n \hookleftarrow G_n$$
$$\chi_n \searrow \quad \downarrow \chi_n^\rtimes \quad \swarrow id_1\natural-$$
$$G_{1+n},$$

then there is a functor $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ such that the diagram (9) is commutative. In particular, these conditions hold taking the composition $\mathcal{A}(\underline{n}) \hookrightarrow \mathcal{A}(\underline{n}) \rtimes G_n \xrightarrow{\chi_n^\rtimes} G_{n+1}$ for χ_n.

Proof. The commutation of the diagram (9) is equivalent to the equality in G_{n+1}

$$(id_1\natural g) \circ \chi_n(h) = \chi_n(\mathcal{A}_n(g)(h)) \circ (id_1\natural g) \tag{11}$$

for all $g \in G_n$ and $h \in \mathcal{A}(\underline{n})$. This is exactly the definition of the fact that $\chi_n * (id_1\natural-)$ factors across the semidirect product $\mathcal{A}(\underline{n}) \rtimes_{\mathcal{A}_n} G_n$. $\qquad\square$

Furthermore, the following result highlights the underlying subtleties when extending a Long-Moody system to a coherent one.

Proposition 2.13. *We consider a functor $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$. If the functor χ extends to the Grothendieck construction $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ such that the diagram (10) is commutative, then the extension is unique.*

Furthermore, given a family of group morphisms $\{\chi_n : \mathcal{A}(\underline{n}) \to G_{n+1}\}_{n \in \mathbb{N}}$ *satisfying the conditions of Lemma 2.12 and such that the following diagram is commutative in the category* $\mathfrak{U}\mathcal{G}$

$$
\begin{array}{ccc}
1 \natural \underline{n} & \xrightarrow{\ \chi_n(h)\ } & 1 \natural \underline{n} \\
{\scriptstyle id_1 \natural [n'-n, id_{\underline{n'}}]} \big\downarrow & & \big\downarrow {\scriptstyle id_1 \natural [n'-n, id_{\underline{n'}}]} \\
1 \natural \underline{n'} & \xrightarrow[\chi_{n'}(\mathcal{A}([n'-n, id_{\underline{n'}}])(h))]{} & 1 \natural \underline{n'},
\end{array}
\tag{12}
$$

for all elements $h \in \mathcal{A}(\underline{n})$, *for all natural numbers* n *and* n' *such that* $n' \geq n$, *then there is a functor* $\chi : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathcal{G}$ *such that the diagram (10) is commutative. In particular, if the Long-Moody system* $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ *is* **coherent**, *these conditions hold taking the composition* $\mathcal{A}(\underline{n}) \hookrightarrow \mathcal{A}(\underline{n}) \rtimes G_n \xrightarrow{\chi_n^{\rtimes}} G_{n+1}$ *for* χ_n.

Proof. We recall that $e_{\mathcal{A}(\underline{n'})}$ denotes the unit elements of the groups $\mathcal{A}(\underline{n'})$. It follows from the definition of the Grothendieck construction $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ that an extension of $\chi : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ is defined by $\chi((\cdot_{\underline{n}}, \underline{n})) = 1 \natural \underline{n}$ for all $\underline{n} \in \mathrm{Obj}(\mathcal{G})$ for objects, and for morphisms by $\chi((h, id_{\underline{n}})) = \chi_n(h)$ for all $h \in \mathcal{A}(\underline{n})$ and

$$
\chi((e_{\mathcal{A}(\underline{n'})}, [n' - n, \varphi])) = id_1 \natural [n' - n, \varphi]
$$

for all $[n'-n, \varphi] \in \mathrm{Hom}_{\mathfrak{U}\mathcal{G}}(\underline{n}, \underline{n'})$. In particular, the uniqueness of the extension follows from these assignments. Actually, the functor χ extends to define a coherent Long-Moody system if and only if these assignments on morphisms satisfy the composition axiom for a functor. We note that the additional composition axiom which has to be checked for extending χ to $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ is for the morphisms of type $(e_{\mathcal{A}(\underline{n'})}, [n' - n, id_{\underline{n'}}]) \circ (h, id_{\underline{n}})$. Namely, χ extends to $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$ if and only if

$$
\chi((e_{\mathcal{A}(\underline{n'})}, [n' - n, id_{\underline{n'}}])) \circ \chi((h, id_{\underline{n}})) = \chi((e_{\mathcal{A}(\underline{n'})}, [n' - n, id_{\underline{n'}}]) \circ (h, id_{\underline{n}}))
$$

for all natural numbers $n' \geq n$ and $h \in \mathcal{A}(\underline{n})$. The second statement is then a direct consequence of the assignments for χ and of the composition rule in $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$. $\qquad \square$

Alternatively, the following proposition gives a *sufficient criterion* for a Long-Moody system to be coherent. This result is in practice more convenient to handle and easier to check that the technical equivalent condition of Proposition 2.13; see Section 3. We denote by σ_1 the braiding $b_{1,1}^{\mathcal{G}'} \natural id_{\underline{n}}$ of $\mathcal{G}'$ for simplicity; this is consistent with the usual notation for the Artin generators of braid groups.

Proposition 2.14. *Given a family of group morphisms* $\{\chi_n : \mathcal{A}(\underline{n}) \to G_{n+1}\}_{n \in \mathbb{N}}$ *satisfying the conditions of Lemma 2.12 and such that following equality holds in* G_{n+2}

$$
id_1 \natural \chi_n(h) \ = \ \sigma_1 \circ \chi_{n+1}(\mathcal{A}([1, id_{\underline{n+1}}])(h)) \circ \sigma_1^{-1},
\tag{13}
$$

for all elements $h \in \mathcal{A}(\underline{n})$ *and for all* n, *then there is a functor* $\chi : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathcal{G}$ *defining a* **coherent** *Long-Moody system* $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$.

Conversely, if $\mathrm{Aut}_{\mathcal{G}'}(1) = \{id_1\}$ *and considering a* **coherent** *Long-Moody system* $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$, *then the equality (13) holds for each* n *taking the composition* $\mathcal{A}(\underline{n}) \hookrightarrow \mathcal{A}(\underline{n}) \rtimes G_n \xrightarrow{\chi_n^{\rtimes}} G_{n+1}$ *for* χ_n.

Proof. First, note that by definition of the braiding $b_{-,-}^{\mathcal{G}'}$, we have:

$$
(b_{1,1}^{\mathcal{G}'})^{-1} \natural id_n = ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{n'-1}) \circ (id_1 \natural (b_{1,(n'-n)-1}^{\mathcal{G}'})^{-1} \natural id_n).
$$

Hence, a straightforward recursion proves that the commutation of the diagram (12) is equivalent to assuming that the morphisms $\{\chi_n\}_{n \in \mathbb{N}}$ satisfy the following equality, as morphisms in the category $\mathfrak{U}\mathcal{G}$, for all elements $h \in \mathcal{A}(\underline{n})$:

$$
[1, ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_n) \circ (id_1 \natural \chi_n(h))] = [1, \chi_{n+1}(\mathcal{A}([1, id_{\underline{n+1}}])(h)) \circ ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_n)].
$$

Hence, by Proposition 2.13, the statements follow from the equivalence relation (4) for morphisms in $\mathfrak{U}\mathcal{G}$. $\quad \square$

Remark 2.15. In Section 5, we will have to assume that the stronger equality (13) holds; see Assumption 5.1.

Connection with (twisted) first homology. Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ be a (possibly coherent) Long-Moody system and let F be an object of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$. We fix a natural number n. We recall that the augmentation ideal defines the short exact sequence of $R[\mathcal{A}(\underline{n})]$-modules $0 \to \mathcal{I}_{R[\mathcal{A}(\underline{n})]} \to R[\mathcal{A}(\underline{n})] \to R \to 0$. We consider the long exact sequence obtained by applying the functor $\mathrm{Tor}_*^{R[\mathcal{A}(\underline{n})]}(-, F(\underline{n+1}))$ to that short exact sequence. We note that $\mathrm{Tor}_i^{R[\mathcal{A}(\underline{n})]}(R[\mathcal{A}(\underline{n})], F(\underline{n+1})) = 0$ for all $i > 0$ since $R[\mathcal{A}(\underline{n})]$ is a free $R[\mathcal{A}(\underline{n})]$-module and recall that the twisted homology group $H_i(\mathcal{A}(\underline{n}); F(\underline{n+1}))$ is equal to $\mathrm{Tor}_i^{R[\mathcal{A}(\underline{n})]}(R, F(\underline{n+1}))$. Therefore, we obtain the following 4-term exact sequence of R-modules:

$$0 \longrightarrow H_1(\mathcal{A}(\underline{n}); F(\underline{n+1})) \longrightarrow \mathbf{LM}(F)(\underline{n}) \longrightarrow F(\underline{n+1}) \longrightarrow H_0(\mathcal{A}(\underline{n}); F(\underline{n+1})) \longrightarrow 0. \quad (14)$$

If $F(\underline{n+1})$ is not a trivial G_n-module, the first homology group of the group $\mathcal{A}(\underline{n})$ with twisted coefficients thus defines a *proper* subrepresentation of the one encoded by the Long-Moody functor.

If the group $\mathcal{A}(\underline{n})$ is free, the augmentation ideal $\mathcal{I}_{R[\mathcal{A}(\underline{n})]}$ has the significant advantage to automatically be a free module equipped with a natural basis (see [41, Proposition 6.2.6] for instance), whereas the first twisted homology of a free group can be less convenient to handle. Moreover, the action of the group G_n on the Long-Moody representation has an explicit formula from the definition of the Long-Moody construction and can therefore be directly fully computed. On the contrary, understanding the G_n-action on the first twisted homology requires much more work generally speaking.

Iteration and triviality. Considering the categories $\mathcal{G}'$, $\mathcal{G}$ and a functor $\mathcal{A} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$, there always exists at least one functor $\chi : \int^{\mathfrak{U}\mathcal{G}} \mathcal{A} \to \mathfrak{U}\mathcal{G}$ such that the diagram (10) is commutative: the functor $\chi_{tr} : \int^{\mathcal{G}} \mathcal{A} \to \mathcal{G}$ induced by the family of morphisms $\{\chi_{n,tr} : \mathcal{A}(\underline{n}) \to 0_{\mathfrak{Gr}} \to G_{n+1}\}_{n \in \mathbb{N}}$ factoring across the trivial group $0_{\mathfrak{Gr}}$ (considered as a category with one object) always works. This functor χ_{tr} trivially extends to $\int^{\mathfrak{U}\mathcal{G}} \mathcal{A}$, a fortiori defining a coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi_{tr}\}$.

We denote by $R : \mathfrak{U}\mathcal{G} \to R\text{-}\mathfrak{Mod}$ the constant functor at R. We also recall that the first homology group $H_1(-; R)$ defines a functor from the category of groups $\mathfrak{Gr}$ to the category $R\text{-}\mathfrak{Mod}$ (see [9, Section 8] for example), and we denote the composition $H_1(-; R) \circ \mathcal{A}$ by $H_1(\mathcal{A}; R)$. The following result provides a general description of a Long-Moody functor when the input has a trivial $\mathcal{A}(\underline{n})$-module structure for all n. In particular, this description fully determines the Long-Moody functor associated to the coherent system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi_{tr}\}$.

Proposition 2.16. *Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ be a (possibly coherent) Long-Moody system and F be an object of $\mathbf{Fct}(\mathcal{G}, R\text{-}\mathfrak{Mod})$ (or possibly $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ if the Long-Moody system is coherent).*

If $F(\underline{n+1})$ is a trivial $R[\mathcal{A}(\underline{n})]$-module for all natural numbers n, then $\mathbf{LM}(F) \cong H_1(\mathcal{A}; R) \otimes_R F(1\natural-)$.

Proof. Let n be a natural number. Since $F(1\natural\underline{n})$ a trivial $R[\mathcal{A}(\underline{n})]$-module, we obtain that $H_0(\mathcal{A}(\underline{n}); F(1\natural\underline{n})) \cong F(1\natural\underline{n})$. Then, we deduce from the exact sequence (14) and from the universal coefficient theorem for group homology that there is an R-module isomorphism $\mathcal{I}_{R[\mathcal{A}(\underline{n})]} \otimes_{R[\mathcal{A}(\underline{n})]} F(1\natural\underline{n}) \cong H_1(\mathcal{A}; R)(\underline{n}) \otimes_R F(1\natural\underline{n})$. It is straightforward to check that this isomorphism is natural with respect to the morphisms of $\mathfrak{U}\mathcal{G}$. $\qquad\square$

Using free products of groups for $\mathcal{A}$. We obtain more properties for the associated Long-Moody functors assuming that there exist two *free* groups H_0 and H such that $\mathcal{A}(\underline{n}) = H^{*n} * H_0$ for all natural numbers n. First, we recall the following classical result:

Lemma 2.17. *Let G be a torsion-free group. The augmentation ideal $\mathcal{I}_{R[G]}$ is a projective $R[G]$-module if and only if G is a free group.*

Proof. Let us assume that $\mathcal{I}_{R[G]}$ is a projective $R[G]$-module. The short exact sequence $0 \to \mathcal{I}_{R[G]} \to R[G] \to R \to 0$ is thus a projective resolution of R as an $R[G]$-module. Hence the cohomological dimension of G is one. Then it follows from [38, Theorem A] that G is a free group. The converse is a classical result of homological algebra; see [41, Corollary 6.2.7]. $\qquad\square$

Corollary 2.18. *Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ be a (possibly coherent) Long-Moody system such that $\mathcal{A}(\underline{n}) = H^{*n} * H_0$ for all natural numbers n, where the groups H and H_0 are free. Then the associated functor $\mathbf{LM}$ is exact and commutes with all finite limits.*

Proof. Let n be a natural number. Since the augmentation ideal $\mathcal{I}_{R[\mathcal{A}(\underline{n})]}$ is a projective $R[\mathcal{A}(\underline{n})]$-module by Lemma 2.17, it is a flat $R[\mathcal{A}(\underline{n})]$-module: hence the functor $\mathcal{I}_{R[\mathcal{A}(\underline{n})]} \otimes_{R[\mathcal{A}(\underline{n})]} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ is exact. Then the naturality of this exactness property with respect to the morphisms of $\mathfrak{UG}$ is a formal consequence of the definition of the Long-Moody functor; see Theorem 2.9. The commutation result for finite limits is a general property of exact functors; see [29, Chapter 8, section 3] for example. $\qquad\square$

3 Examples

The groups $\{G_n\}_{n \in \mathbb{N}}$ for which it is natural to define some generalized Long-Moody functors are mapping class groups of surfaces. In this section, we present various coherent Long-Moody systems which are defined for several families of mapping class groups and surface braid groups. In particular, we recover the Long-Moody functors for braid groups of [37] in Section 3.3.2, and we introduce new ones thanks to the more general framework of Section 2.

3.1 The monoidal groupoid associated with surfaces

Let us first introduce a suitable category for our work, inspired by [36, Section 5.6]. Namely, the decorated surfaces groupoid $\mathcal{M}_2$ is the locally small groupoid defined by:

⋆ **Objects:** decorated surfaces $(S, I, \mathscr{P})$, where:

- S is a smooth connected compact surface with one boundary component denoted by $\partial_0 S$;

- $I : [-1, 1] \hookrightarrow \partial_0 S$ is a parametrized interval in the boundary and $p = 0 \in I$ a basepoint; when there is no ambiguity, we omit the parametrized interval I from the notation;

- $\mathscr{P}$ is a finite set of points removed from the interior of S (in other words with *punctures*); we usually omit $\mathscr{P}$ from the notation for convenience or if it is empty.

⋆ **Morphisms:** the isotopy classes of homeomorphisms of S restricting to the identity on a neighbourhood of the parametrized interval I and fixing the punctures setwise, denoted by $\pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$.

Remark 3.1. We recall that $\pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$ is isomorphic to the isotopy classes of homeomorphisms of S restricting to the identity on the boundary $\partial_0 S$ and fixing the punctures setwise, which is the traditional way to define the mapping class group of S. Also, if the surface S is orientable, the orientation on S is induced by the orientation of I: then the elements of $\pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$ automatically preserve that orientation.

Alternatively, the morphisms of the groupoid $\mathcal{M}_2$ may be described as follows. We denote by $\widehat{S}$ the surface obtained from $S \in \mathrm{Obj}(\mathcal{M}_2)$ removing an open disc on a neighbourhood of each punctures, so that we obtain boundary components instead of punctures; see [36, Section 5.6.1]. Let $\mathrm{Diff}^{\partial_0}(\widehat{S})$ be the group of diffeomorphisms of $\hat{S}$ fixing pointwise the boundary component $\partial_0 S$ and moving freely the other boundary components. We recall that $\pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$ is isomorphic to the group of isotopy classes $\pi_0 \mathrm{Diff}^{\partial_0}(\widehat{S})$; see [18, Section 1.4.2] for instance.

The groupoid $\mathcal{M}_2$ has a monoidal structure induced by gluing; for completeness, the definition is outlined below; we refer to [36, Section 5.6.1] for technical details. Let (S_1, I_1) and (S_2, I_2) be for two decorated surfaces. For a parametrized interval I, the left half-interval $[-1, 0] \hookrightarrow \partial_0 S$ of I is denoted by I^- and the right half-interval $[0, 1] \hookrightarrow \partial_0 S$ of I is denoted by I^+, defining $I = I^- \cup I^+$. We define the boundary connected sum $(S_1, I_1) \natural (S_2, I_2)$ as $(S_1 \natural S_2, I_1 \natural I_2)$, where $S_1 \natural S_2$ is the surface obtained from gluing S_1 and S_2 along the half-interval I_1^+ and the half-interval I_2^-, and $I_1 \natural I_2 = I_1^- \cup I_2^+$. The homeomorphisms being the identity on a neighbourhood of the parametrized intervals I_1 and I_2, we canonically extend the homeomorphisms of S_1 and S_2 to $S_1 \natural S_2$. The braiding of the monoidal structure $b_{(S_1, I_1), (S_2, I_2)}^{\mathcal{M}_2}$ is the morphism $(S_1, I_1) \natural (S_2, I_2) \to (S_2, I_2) \natural (S_1, I_1)$ given by doing half a Dehn twist in a pair of pants neighbourhood of $\partial_0 S_1$ and $\partial_0 S_2$; see Figure 1.

Notation 3.2. We consider the unit 2-disc $\mathbb{D}^2$. We denote by D the disc $\mathbb{D}^2$ with one puncture, we fix a torus with one boundary component denoted by T and a Möbius band denoted by M. By the classification of surfaces, for S an object of the groupoid $\mathcal{M}_2$, there exist $g, s, h \in \mathbb{N}$ such that there is an homeomorphism $S \simeq (\natural_s D) \natural (\natural_g T) \natural (\natural_h M)$. Moreover, if $h = 0$, then g and s are unique.
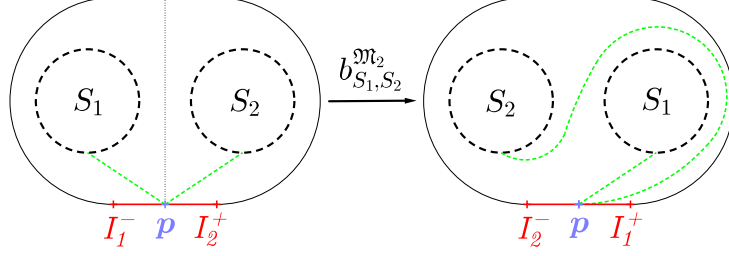
Figure 1: Braiding

By [36, Proposition 5.18], the boundary connected sum $\natural$ induces a braided monoidal structure on $\mathcal{M}_2$. There are no zero divisors in the category $\mathcal{M}_2$ and $\mathrm{Aut}_{\mathcal{M}_2}(\mathbb{D}^2) = \{id_{\mathbb{D}^2}\}$ by Alexander's trick. For consistency with respect to the framework of Section 2, we consider the following subgroupoid of $\mathcal{M}_2$ for the monoidal structure $\natural$ to become strict: let $\mathfrak{M}_2$ be the full subgroupoid of $\mathcal{M}_2$ on the same objects modulo the identification that $\mathbb{D}^2 \natural S = S \natural \mathbb{D}^2 = S$ for any surface S. Hence the groupoid $(\mathfrak{M}_2, \natural, \mathbb{D}^2, b_{-,-}^{\mathfrak{M}_2})$ is locally small, strict braided monoidal with no zero divisors and such that $\mathrm{Aut}_{\mathfrak{M}_2}(\mathbb{D}^2) = \{id_{\mathbb{D}^2}\}$. We denote by $\mathfrak{UM}_2$ Quillen's bracket construction on the groupoid $(\mathfrak{M}_2, \natural, \mathbb{D}^2)$; by Proposition 1.4, we obtain a strict pre-braided monoidal category $(\mathfrak{UM}_2, \natural, \mathbb{D}^2)$ where the unit $\mathbb{D}^2$ is an initial object. Following the framework of Section 2.2.1, the groupoid $\mathfrak{M}_2$ plays the role of the category $\mathcal{G}'$.

3.2 Fundamental group functor

As detailed in Definition 2.8, we have to fix a functor $\mathcal{A} : \mathfrak{UG} \to \mathfrak{Gr}$ in order to define a Long-Moody functor. In this section, we choose the functor which will play the role of that parameter introduced in the next section.

We define the functor $\pi_1(-, p) : (\mathfrak{M}_2, \natural, \mathbb{D}^2) \to \mathfrak{Gr}$ by assigning the fundamental group $\pi_1(S, p)$ to each object S of $\mathfrak{M}_2$ and the corresponding natural action, denoted by $\mathcal{A}_S$, of the mapping class group on $\pi_1(S, p)$ for the morphisms of $\mathfrak{M}_2$.

Let $\mathfrak{gr}$ be a skeleton of the full subcategory of $\mathfrak{Gr}$ of finitely-generated free groups. The free product $* : \mathfrak{gr} \times \mathfrak{gr} \to \mathfrak{gr}$ defines a monoidal structure on $\mathfrak{gr}$, with 0 the unit, denoted by $(\mathfrak{gr}, *, 0)$. We recall that the group $\pi_1(S, p)$ is a finitely generated free group for any object S of $\mathfrak{M}_2$. We use the category $\mathfrak{gr}$ for the functor $\pi_1(-, p)$ to be strict monoidal:

Lemma 3.3. *The functor* $\pi_1(-, p) : (\mathfrak{M}_2, \natural, \mathbb{D}^2) \to (\mathfrak{gr}, *, 0_{\mathfrak{Gr}})$ *is strict monoidal.*

Proof. Since the category $\mathfrak{gr}$ is skeletal, it follows from Van Kampen's theorem that $\pi_1(S' \natural S, p) = \pi_1(S', p) * \pi_1(S, p)$ for any $S, S' \in \mathrm{Obj}(\mathfrak{M}_2)$. Let ϕ and ϕ' be elements of $\pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$ and $\pi_0 \mathrm{Homeo}^I(S', \mathscr{P})$ respectively. The isomorphism $id_{S'} \natural \phi$ (respectively $\phi' \natural id_S$) corresponds to extending ϕ (resp. ϕ') to $S' \natural S$ by the identity on S' (resp. S). It is clear that $id_{S'} \natural \phi$ acts trivially on $\pi_1(S', p)$ in $\pi_1(S', p) * \pi_1(S, p)$ and has the same action as ϕ on $\pi_1(S, p)$ in $\pi_1(S', p) * \pi_1(S, p)$. Similarly, $\phi' \natural id_S$ acts trivially on $\pi_1(S, p)$ in $\pi_1(S', p) * \pi_1(S, p)$ and has the same action as ϕ' on $\pi_1(S', p)$ in $\pi_1(S', p) * \pi_1(S, p)$. Hence the action of $\phi' \natural \phi$ on $\pi_1(S' \natural S, p) = \pi_1(S', p) * \pi_1(S, p)$ is the same as $\mathcal{A}_{S'}(\phi') * \mathcal{A}_S(\phi)$, which ends the proof. □

The object $0_{\mathfrak{Gr}}$ is null in the category of groups $\mathfrak{Gr}$, and we let $\iota_G : 0_{\mathfrak{Gr}} \to G$ denote the unique morphism from $0_{\mathfrak{Gr}}$ to the group G. The next proposition tells us that the functor $\pi_1(-, p)$ naturally extends to the category $\mathfrak{UM}_2$.

Proposition 3.4. *The functor* $\pi_1(-, p)$ *of Lemma 3.3 extends to a functor* $\pi_1(-, p) : \mathfrak{UM}_2 \to \mathfrak{gr}$ *by assigning* $\pi_1(-, p)([S', id_{S' \natural S}]) = \iota_{\pi_1(S', p)} * id_{\pi_1(S, p)}$ *to all* $S, S' \in \mathrm{Obj}(\mathfrak{M}_2)$.

Proof. Let S and S' be objects of $\mathfrak{M}_2$. It follows from the definitions that relation (5) of Lemma 1.2 is automatically satisfied. Let $\varphi \in \pi_0 \mathrm{Homeo}^I(S, \mathscr{P})$ and $\varphi' \in \pi_0 \mathrm{Homeo}^I(S', \mathscr{P}')$. By Lemma 3.3, we have $\pi_1(-, p)(\varphi' \natural \varphi) \circ \pi_1(-, p)([S', id_{S' \natural S}]) = (\mathcal{A}_{S'}(\varphi') * \mathcal{A}_S(\varphi)) \circ \pi_1(-, p)([S', id_{S' \natural S}])$. Hence, by definition of the morphism $\iota_{\pi_1(S', p)}$, we obtain that $\pi_1(-, p)(\varphi' \natural \varphi) \circ \pi_1(-, p)([S', id_{S' \natural S}])$ is equal to $\pi_1(-, p)([S', id_{S' \natural S}]) \circ \mathcal{A}_S(\varphi)$. Relation (6) of Lemma 1.2 is thus satisfied, which implies the desired result. □
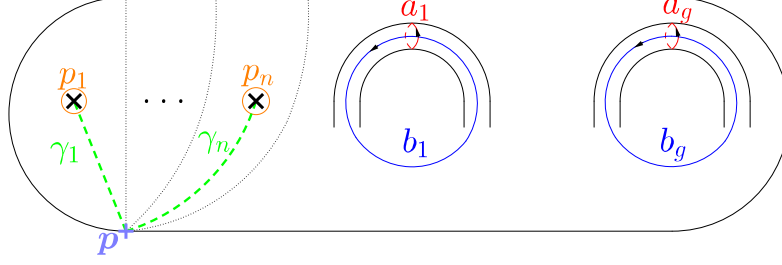
Figure 2: Generators and paths

In the constructions of the following sections, we take restrictions of the functor $\pi_1(-,p)$ to some small subcategories of $\mathfrak{UM}_2$ for the parameter $\mathcal{A}$ to define the Long-Moody systems. Note that we could make other choices for such parameter: for instance, in the case of braid groups, Wada introduces a families of morphisms $\mathbf{B}_n \to \mathrm{Aut}(\mathbf{F}_n)$ which are not Artin's representation [40]. We may use such other choices to construct alternative Long-Moody functors.

3.3 Modifying the number of punctures

This section is devoted to the construction of Long-Moody functors for the mapping class groups of surfaces when the number of punctures varies (see Section 3.3.1) and surface braid groups (see Section 3.3.2). We will detail the case of orientable surfaces for simplicity, and then we will see by the end of this section that the non-orientable case follows mutatis mutandis. Finally, we study the applications of these Long-Moody functors in Section 3.3.3.

3.3.1 Mapping class groups

We fix a natural number $g \geq 0$. For all natural numbers n, we denote by $\Sigma_{g,1}^n$ the surface $D^{\natural n} \natural T^{\natural g}$ (and $\Sigma_{0,1}^0 := \mathbb{D}^2$ for consistency), and we denote the corresponding mapping class group by $\mathbf{\Gamma}_{g,1}^n$. Let $\mathfrak{M}_2^{+,g}$ be the small full subgroupoid of $\mathfrak{M}_2$ on the objects $\{\Sigma_{g,1}^n\}_{n\in\mathbb{N}}$. In particular, in the notations of Section 2.2.1, the groupoid $\mathfrak{M}_2$ plays the role of $\mathcal{G}'$ and $\mathfrak{M}_2^{+,g}$ corresponds to $\mathcal{G}$ where $\underline{0} := \Sigma_{g,1}^0$ and $1 := \Sigma_{0,1}^1$.

Let H be the free group $\pi_1(\Sigma_{0,1}^1, p) \cong \mathbb{Z}$, let H_0 be the free group $\pi_1(\Sigma_{g,1}^0, p) \cong \mathbf{F}_{2g}$ and thus $H_n = H^{*n} * H_0 = \pi_1(\Sigma_{g,1}^n, p)$ for all natural numbers n. We denote by $\pi_1(\Sigma_{g,1}^-, p) : \mathfrak{UM}_2^{+,g} \to \mathfrak{Gr}$ the restriction of the functor of Proposition 3.4 along the inclusion functor $\mathfrak{UM}_2^{+,g} \hookrightarrow \mathfrak{UM}_2$.

Let $\{a_i, b_i \mid i \in \{1, \ldots, g\}\}$ be a direct system of meridians and longitudes of the surface $\Sigma_{g,1}^0$ with orientations as pictured in Figure 2. For n a natural number and $j \in \{1, \ldots, n\}$, let p_j be an oriented closed curve encircling the puncture in the j-th surface $\Sigma_{0,1}^1$ in $\Sigma_{0,1}^1 \natural \cdots \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^0$ as in Figure 2; we abuse the notation and also denote by p_j the puncture encircled by the curve p_j.

The generator of the j-th copy of H in H_n is the homotopy class of a simple closed curve α_{p_j} of $\Sigma_{0,1}^1$ in $\Sigma_{g,1}^n$ based at p and encircling the curve p_j. A generator h' of H_0 in H_n is the homotopy class of a simple closed curve $\alpha_{h'}$ of $\Sigma_{g,1}^0$ in $\Sigma_{g,1}^n$ based at p and encircling on the curves $\{a_i, b_i \mid i \in \{1, \ldots, g\}\}$. From now on, we fix a choice of such simple closed curves α_h for each standard generator h of H_n.

To introduce the group morphisms $\{\chi_n : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{\Gamma}_{g,1}^{1+n}\}_{n\in\mathbb{N}}$ in order to define a Long-Moody functor in this section, we first need to introduce additional tools and recall some classical facts about mapping class groups of surfaces.

For the unit disc with one puncture $\Sigma_{0,1}^1$, we consider x_1 a marked point filling in the puncture p_1 and denote by $\Sigma_{0,1}^{[x_1]}$ the obtained surface. Let γ_1 be a path in $\Sigma_{0,1}^{[x_1]}$ connecting the point $p \in I$ to x_1. For n a natural number and $j \in \{1, \ldots, n\}$, we denote by x_j the corresponding filling point of the j-th copy of $\Sigma_{0,1}^1$ in $\Sigma_{0,1}^1 \natural \cdots \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^0$. Also, we denote by γ_j the corresponding path γ_1 of the j-th copy of $\Sigma_{0,1}^1$ in $\Sigma_{0,1}^1 \natural \cdots \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^0$; see Figure 2. For all natural numbers n, we denote the surface $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{g,1}^n$ by $\Sigma_{g,1}^{[x_1],n}$.
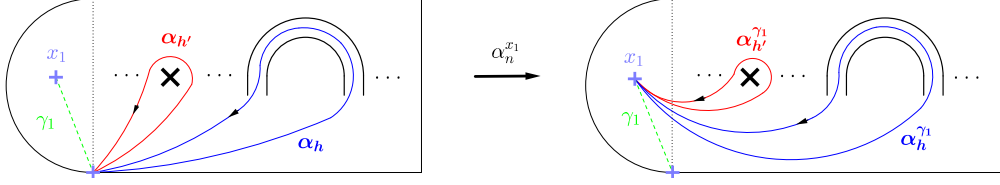
Figure 3: The simple closed curve α^{γ_1}

Now, we fix a canonical isomorphism between the group H_n and the fundamental group of $\Sigma_{g,1}^{[x_1],n}$ with x_1 as basepoint.

Definition 3.5. For a generator h of H_n and its representative simple closed curve α_h, we consider the path $\gamma_1 \cdot \alpha_h \cdot \gamma_1^{-1}$ obtained by changing the basepoint of the curve α_h from p to x_1 via the path γ_1. This procedure for each generator h of H_n induces an isomorphism $\alpha_n^{x_1} : H_n \xrightarrow{\sim} \pi_1(\Sigma_{g,1}^{[x_1],n}, x_1)$, which is independent of the choice of path γ_1 up to homotopy of $\Sigma_{0,1}^{[x_1]}$. For a generator h of H_n, we denote by $\alpha_h^{\gamma_1}$ a simple closed curve of $\Sigma_{g,1}^{[x_1],n}$ based at x_1, homotopic to $\gamma_1 \cdot \alpha_h \cdot \gamma_1^{-1}$ and thus representing the element $\alpha_n^{x_1}(h)$; see Figure 3 for an illustration.

For all natural numbers n, let $\mathbf{\Gamma}_{g,1}^{[1],n}$ be the subgroup of the mapping class group $\mathbf{\Gamma}_{g,1}^{1+n}$ where the first puncture p_1 is sent to itself. We denote by $\mathscr{E}_n : \mathbf{\Gamma}_{g,1}^{[1],n} \hookrightarrow \mathbf{\Gamma}_{g,1}^{1+n}$ the associated canonical embedding. Note that the group $\mathbf{\Gamma}_{g,1}^{[1],n}$ is also clearly isomorphic to the isotopy classes of homeomorphisms of the surface $\Sigma_{g,1}^{[x_1],n}$ restricting to the identity on the boundary component, fixing the punctures setwise and sending the marked point x_1 to itself; see [18, Section 1.1.1].

The central ingredient to define the morphism $\chi_n : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{\Gamma}_{g,1}^{1+n}$ is the use of the well-known *point pushing map* $\mathrm{Push}_{p_1} : \pi_1(\Sigma_{g,1}^{[x_1],n}, x_1) \hookrightarrow \mathbf{\Gamma}_{g,1}^{[1],n}$: it is basically a reinterpretation of the morphism Push_{p_1} in our context, namely replacing the source with $\pi_1(\Sigma_{g,1}^n, p)$ via the isomorphism $\alpha_n^{x_1}$ of Definition 3.5. The formula (15) below describes the morphism Push_{p_1} and we refer the reader to [18, Section 4.2.1] for further details. For a simple loop ε, we denote by τ_ε the Dehn twist along it. We consider a generator h of H_n and the associated simple closed curve $\alpha_h^{\gamma_1}$ of Definition 3.5. Let $(\alpha_h^{\gamma_1})^-$ and $(\alpha_h^{\gamma_1})^+$ be the isotopy classes of the simple closed curves in $\Sigma_{g,1}^{[x_1],n}$ obtained by pushing the simple closed curve $\alpha_h^{\gamma_1}$ off itself to the left and right respectively. It follows from [18, Fact 4.7] that:

$$\mathrm{Push}_{p_1}(\alpha_n^{x_1}(h)) = \tau_{(\alpha_h^{\gamma_1})^-} \circ \tau_{(\alpha_h^{\gamma_1})^+}^{-1}. \tag{15}$$

Figure 4 allows us to visualize the effect of the point-pushing map; see also [18, Section 4.2.1]. In this figure, f_1 denotes the generator of $\pi_1(\Sigma_{0,1}^1, p)$ corresponding to the first copy of $\Sigma_{0,1}^1$ in H_n; note that this $\Sigma_{0,1}^1$ is the second copy of $\Sigma_{0,1}^1$ in $\Sigma_{g,1}^{1+n} = \Sigma_{0,1}^1 \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^{n-1}$. We now can define each morphism χ_n from the morphisms $\alpha_n^{x_1}$, Push_{p_1} and $\mathscr{E}_n$:

Definition 3.6. Let n be a natural number. We define the morphism $\chi_{n,1} : H_n \to \mathbf{\Gamma}_{g,1}^{1+n}$ to be the composition:

$$\mathscr{E}_n \circ \mathrm{Push}_{p_1} \circ \alpha_n^{x_1} : H_n \xrightarrow{\sim} \pi_1(\Sigma_{g,1}^{[x_1],n}, x_1) \hookrightarrow \mathbf{\Gamma}_{g,1}^{[1],n} \hookrightarrow \mathbf{\Gamma}_{g,1}^{1+n}.$$

In particular, when $g = n = 0$, the morphism $\chi_{0,1} : \pi_1(\Sigma_{0,1}^0, p) \cong 0_{\mathfrak{Gr}} \to \mathbf{\Gamma}_{0,1}^1 \cong 0_{\mathfrak{Gr}}$ is trivial. Also, using the notations of Definition 3.5 and following (15), if $2g + n \geq 1$ then the morphisms $\chi_{n,1}$ sends h to $\tau_{(\alpha_h^{\gamma_1})^-} \circ \tau_{(\alpha_h^{\gamma_1})^+}^{-1}$ for each n and each generator h either of H_0 or of one of the copies of H in H_n. Each element of H_n is thus identified with a 1-strand braid on the surface $\Sigma_{g,1}^n$. For instance, we denote by $\mathbf{PB_2}$ the pure braid group $\mathbf{\Gamma}_{0,1}^{[1],1}$ and by σ_1 the Artin generator of the braid group on two strands $\mathbf{B_2}$ which is identified with the braiding $b_{\Sigma_{0,1}^1, \Sigma_{0,1}^1}^{\mathcal{M}_2}$ as pictured in Figure 1. Then, with the orientation convention of Figure 4, the morphism $\chi_{1,1} : \pi_1(\Sigma_{0,1}^1, p) \to \mathbf{PB_2}$ sends the generator f_1 of $\pi_1(\Sigma_{0,1}^1, p)$ to σ_1^2.
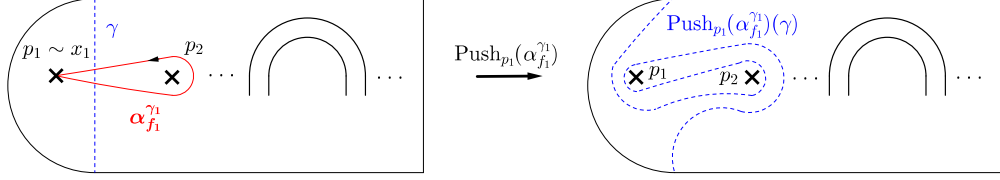
Figure 4: Illustration of the point-pushing map Push_{p_1}

In addition, we recall from [18, Theorem 4.6] that the point pushing map Push_{p_1} is part of the following *Birman short exact sequence* (16): for all natural numbers n, there is a short exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,1}^{[x_1],n}, x_1) \xrightarrow{\text{Push}_{p_1}} \Gamma_{g,1}^{[1],n} \xrightarrow{\text{Forget}} \Gamma_{g,1}^n \longrightarrow 1. \tag{16}$$

Namely, viewing the surface $\Sigma_{g,1}^{[x_1],n}$ as the surface obtained from $\Sigma_{g,1}^n = \Sigma_{0,1}^0 \natural \Sigma_{g,1}^n$ by marking a point x_1 in the interior of $\Sigma_{0,1}^0$ and denoting by $\mathscr{P}$ the set $\{p_2, \ldots, p_{n+1}\}$ of punctures of $\Sigma_{g,1}^n$, we recall that (16) is the long exact sequence in homotopy of the locally trivial fibration $\text{Homeo}^I(\Sigma_{g,1}^n, \mathscr{P}) \twoheadrightarrow \Sigma_{g,1}^n$ defined by evaluating homeomorphisms at x_1 in $\Sigma_{g,1}^n$; see [18, Section 4.2.3] for further details. In particular, the morphism Forget is the map induced in homotopy by the fiber inclusion defined by forgetting that the point x_1 is marked. This short exact sequence has the following stronger property, which is probably known to the experts.

Lemma 3.7. *Let n be a natural number. The Birman short exact sequence (16) splits, inducing an isomorphism $\mathscr{B}_n : H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n \xrightarrow{\sim} \Gamma_{g,1}^{[1],n}$.*

Proof. Identifying $\Sigma_{g,1}^{[x_1],n}$ with $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{g,1}^n$, the surface $\Sigma_{g,1}^n$ can be viewed as a subsurface of $\Sigma_{g,1}^{[x_1],n}$ as the complement of the disc $\Sigma_{0,1}^{[x_1]}$ with the marked point x_1. Extending the elements of $\text{Homeo}^I(\Sigma_{g,1}^n, \mathscr{P})$ by the identity on $\Sigma_{0,1}^{[x_1]}$ induces in homotopy the morphism $s_n : \Gamma_{g,1}^n \to \Gamma_{g,1}^{[1],n}$. This is known as the stabilization map for the mapping class groups with respect to the punctures; see [7] for instance. It is a routine to check from the definitions that s_n is a section of the map Forget (which implies in particular that s_n is injective), thus providing a splitting of the exact sequence (16). Hence we have an isomorphism $\pi_1(\Sigma_{g,1}^{[x_1],n}, x_1) \rtimes_{\mathcal{A}_{x_1}} \Gamma_{g,1}^n \cong \Gamma_{g,1}^{[1],n}$, where $\mathcal{A}_{x_1}$ denotes the natural action of $\Gamma_{g,1}^n$ on $\pi_1(\Sigma_{g,1}^{[x_1],n}, x_1)$. Then, since the splitting is induced by considering $\Sigma_{g,1}^n$ as a subsurface of $\Sigma_{g,1}^{[x_1],n}$, the precomposition by the canonical isomorphism $\alpha_n^{x_1}$ of Definition 3.5 on $\pi_1(\Sigma_{g,1}^{[x_1],n}, x_1)$ induces the required isomorphism. $\square$

Therefore all the above ingredients define an appropriate Long-Moody functor:

Lemma 3.8. *The setting $\{\pi_1(\Sigma_{g,1}^-, p), \mathfrak{M}_2^{+,g}, \mathfrak{M}_2, \chi_1\}$ is a Long-Moody system.*

Proof. It is clear from the definitions that the composition $\mathscr{E}_n \circ s_n : \Gamma_{g,1}^n \hookrightarrow \Gamma_{g,1}^{[1],n} \hookrightarrow \Gamma_{g,1}^{1+n}$ is equal to the injection $id_{\Sigma_{0,1}^1} \natural -$. Also, it follows from Lemma 3.7 that the restriction of the isomorphism $\mathscr{B}_n : H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n \xrightarrow{\sim} \Gamma_{g,1}^{[1],n}$ to H_n is equal to the composition $\text{Push}_{p_1} \circ \alpha_n^{x_1}$. Hence, the following diagram is commutative:

$$
\begin{array}{ccccc}
H_n & \hookrightarrow & H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}} \Gamma_{g,1}^n & \hookleftarrow & \Gamma_{g,1}^n \\
& \searrow {\scriptstyle \chi_{n,1}} & \downarrow {\scriptstyle \mathscr{E}_n \circ \mathscr{B}_n} & \swarrow {\scriptstyle id_{\Sigma_{0,1}^1} \natural -} & \\
& & \Gamma_{g,1}^{1+n}. & &
\end{array}
$$

Hence it follows Lemma 2.12 that the diagram (9) of Definition 2.8 is commutative, which ends the proof. $\square$

Moreover, we have the stronger property:

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Proposition 3.9. *The setting* $\{\pi_1(\Sigma_{g,1}^-, p), \mathfrak{M}_2^{+,g}, \mathfrak{M}_2, \chi_1\}$ *is a coherent Long-Moody system.*

Proof. By Proposition 2.14 and Lemma 3.8, it is enough to prove that the morphism $\chi_{n,1}$ satisfies the equality (13) in the present context and for any natural number n. Let f be a generator either of H_0 or of one of the copies of H in H_n. We recall that the braiding of $\mathfrak{M}_2$ is defined by doing half a Dehn twist in a pair of pants neighbourhood of the boundaries of two copies of $\Sigma_{0,1}^1$ in $\Sigma_{g,1}^{2+n}$. Hence the morphism $b_{\Sigma_{0,1}^1, \Sigma_{0,1}^1}^{\mathfrak{M}_2} \natural id_n$ is the element $\sigma_1 \in \mathbf{B}_2 \leq \boldsymbol{\Gamma}_{g,1}^{2+n}$ which exchanges the punctures p_1 and p_2 in $\Sigma_{g,1}^{2+n}$; see Figure 1. Note that $\pi_1(-, p)([\Sigma_{0,1}^1, id_{\Sigma_{0,1}^{1+n}}])(f) = id_{\pi_1(\Sigma_{0,1}^1, p)} * f$ in H_{n+1}, and we denote this element by f' for simplicity. Hence, following the equality (13), it is enough to prove that as elements of $\boldsymbol{\Gamma}_{g,1}^{2+n}$:

$$\sigma_1 \circ \chi_{n+1,1}(f') \circ \sigma_1^{-1} \quad = \quad id_{\Sigma_{0,1}^1} \natural \chi_{n,1}(f). \tag{17}$$

If $g = n = 0$, then the equality (17) is trivially checked since f and f' are trivial. We thus assume that $2g + n \geq 1$. Let x_1 and x_2 be marked points filling in the punctures p_1 and p_2 respectively of the two first copies of $\Sigma_{0,1}^1$ in $\Sigma_{g,1}^{2+n} = \Sigma_{0,1}^1 \natural \Sigma_{0,1}^1 \natural \Sigma_{g,1}^n$, and we denote by $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{0,1}^{[x_2]} \natural \Sigma_{g,1}^n$ the obtained surface. Let γ_1 and γ_2 respectively be paths in $\Sigma_{0,1}^{[x_1]}$ and $\Sigma_{0,1}^{[x_2]}$ connecting the point $p \in I$ to x_1 and x_2; see Figure 2. Similarly to Definition 3.5, for each generator h of H_n and its representative curve α_h, we denote by $\alpha_h^{\gamma_1}$ (respectively $\alpha_h^{\gamma_2}$) a simple closed curve of $\Sigma_{0,1}^{[x_1]} \natural \Sigma_{0,1}^{[x_2]} \natural \Sigma_{g,1}^n$ based at x_1 (respectively x_2) obtained after changing the basepoint of the curve α_h from p to x_1 (respectively x_2) via the path γ_1 (respectively γ_2). Then it follows from (15), Definition 3.6 and [18, Fact 3.7] that

$$\sigma_1 \circ \chi_{n+1,1}(f') \circ \sigma_1^{-1} = (\sigma_1 \circ \tau_{(\alpha_{f'}^{\gamma_1})^-} \circ \sigma_1^{-1}) \circ (\sigma_1 \circ \tau_{(\alpha_{f'}^{\gamma_1})^+}^{-1} \circ \sigma_1^{-1}) = \tau_{\sigma_1((\alpha_{f'}^{\gamma_1})^-)} \circ \tau_{\sigma_1((\alpha_{f'}^{\gamma_1})^+)}^{-1},$$

and using the shift in the enumeration given by adding a surface $\Sigma_{0,1}^1$ on the left that

$$id_{\Sigma_{0,1}^1} \natural \chi_{n,1}(f) = id_{\Sigma_{0,1}^1} \natural (\tau_{(\alpha_f^{\gamma_1})^-} \circ \tau_{(\alpha_f^{\gamma_1})^+}^{-1}) = \tau_{(\alpha_f^{\gamma_2})^-} \circ \tau_{(\alpha_f^{\gamma_2})^+}^{-1}.$$

The braiding σ_1 exchanges the two first punctures p_1 and p_2 of $\Sigma_{g,1}^{2+n}$ as in Figure 1. In particular, the image of the curve $\alpha_{f'}^{\gamma_1}$ by σ_1 is isotopic to $\alpha_f^{\gamma_2}$. Therefore $(\alpha_f^{\gamma_2})^- = \sigma_1((\alpha_{f'}^{\gamma_1})^-)$ and $(\alpha_f^{\gamma_2})^+ = \sigma_1((\alpha_{f'}^{\gamma_1})^+)$ as isotopy classes of simple closed curves. Since $\tau_\gamma = \tau_\theta$ for γ and θ two isotopy classes of simple closed curves which are equal (see [18, Fact 3.6] for instance), we deduce that the equality (17) is satisfied, which ends the proof. $\square$

Non-orientable surfaces. Instead of considering orientable surfaces, we can fix a natural number $h \geq 2$ and consider the non-orientable surfaces $D^{\natural n} \natural M^{\natural h}$ for all natural numbers n, that we denote by $\mathscr{N}\Sigma_{h,1}^n$. We denote the corresponding mapping class group by $\boldsymbol{\mathcal{N}}_{h,1}^n$. Let $\mathfrak{M}_2^{-,h}$ be the small full subgroupoid of $\mathfrak{M}_2$ on these objects. In this case, H is the group $\pi_1(\mathscr{N}\Sigma_{0,1}^1, p) \cong \mathbb{Z}$, H_0 is the free group $\pi_1(\mathscr{N}\Sigma_{h,1}^0, p) \cong \mathbf{F}_h$ and $H_n = H^{*n} * H_0 = \pi_1(\mathscr{N}\Sigma_{h,1}^n, p)$ for all natural numbers n.

Then the above work adapts mutatis mutandis to this situation. Let $\boldsymbol{\mathcal{N}}_{h,1}^{[1],n}$ be the subgroup of $\boldsymbol{\mathcal{N}}_{h,1}^{1+n}$ of isotopy classes of homeomorphisms of the surface $\mathscr{N}\Sigma_{h,1}^{1+n}$ restricting to the identity on the boundary component, fixing the first puncture and fixing the other ones setwise. The non-orientable version of the Birman short exact sequence (16) is obtained by the long exact sequence on homotopy associated with the locally trivial fibration $\mathrm{Homeo}^I(\mathscr{N}\Sigma_{h,1}^n, \mathscr{P}) \to \mathscr{N}\Sigma_{h,1}^n$, defined by evaluating homeomorphisms at x_1 in $\mathscr{N}\Sigma_{h,1}^n$. The associated fiber inclusion induces the map $\boldsymbol{\mathcal{N}}_{h,1}^{[1],n} \twoheadrightarrow \boldsymbol{\mathcal{N}}_{h,1}^n$ which forgets that the point x_1 is marked. Moreover, identifying $\mathscr{N}\Sigma_{h,1}^{[x_1],n}$ with $\Sigma_{0,1}^{[x_1]} \natural \mathscr{N}\Sigma_{h,1}^n$, extending the homeomorphisms of $\mathscr{N}\Sigma_{h,1}^n$ to $\mathscr{N}\Sigma_{h,1}^{[x_1],n}$ by the identity on $\Sigma_{0,1}^{[x_1]}$ defines a section $\boldsymbol{\mathcal{N}}_{h,1}^n \to \boldsymbol{\mathcal{N}}_{h,1}^{[1],n}$. We thus define $\chi_{n,2} : H_n \hookrightarrow H_n \rtimes \boldsymbol{\mathcal{N}}_{h,1}^n \cong \boldsymbol{\mathcal{N}}_{h,1}^{[1],n} \hookrightarrow \boldsymbol{\mathcal{N}}_{h,1}^{1+n}$ for all n. The proofs of Lemma 3.8 and Proposition 3.9 repeat mutatis mutandis and we deduce that:

Proposition 3.10. *The setting* $\{\pi_1(\mathscr{N}\Sigma_{h,1}^-, p), \mathfrak{M}_2^{-,h}, \mathfrak{M}_2, \chi_2\}$ *is a coherent Long-Moody system.*

3.3.2 Surface braid groups

Drawing on the work of Section 3.3.1, we can construct Long-Moody functors for the surface braid groups. We review some basic results about these groups and refer the reader to [5] or [23] for further details.

Let (S, I) be an object of $\mathfrak{M}_2$ where S is a surface without punctures and n be a natural number. We denote by $\mathfrak{S}_n$ the symmetric group on a set of n elements. For a partition (λ_1, λ_2) of n we denote by $\mathfrak{S}_{\lambda_1, \lambda_2}$ the product $\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2}$; for the partitions $(n, 0)$ or $(0, n)$ we simply denote it by $\mathfrak{S}_n$. Let $F_n(S) = \{(x_1, \ldots, x_n) \in \mathring{S}^{\times n} \mid x_i \neq x_j \text{ if } i \neq j\}$ be the *ordered* configuration spaces of n distinct points in the interior of S. For the partition (λ_1, λ_2) of n, we denote by $C_{(\lambda_1, \lambda_2)}(S)$ the quotient $F_n(S)/\mathfrak{S}_{\lambda_1, \lambda_2}$ and its fundamental group by $\mathbf{B}_{\lambda_1, \lambda_2}(S)$. For the partitions $(n, 0)$ or $(0, n)$, the set $C_n(S)$ corresponds to the classical *unordered* configuration space of n distinct points in the interior of S. The surface braid group $\mathbf{B}_n(S)$ is the fundamental group of $C_n(S)$.

Let $\mathscr{P}$ be a set of n points removed from the interior of S and (λ_1, λ_2) be a partition of n. Identifying S with the boundary connected sum $\mathbb{D}^2 \natural S$, the surface $S \setminus \mathscr{P}$ can be viewed as a subsurface of S obtained by removing $\mathscr{P}$ in the interior of $\mathbb{D}^2$. We denote by $\pi_0 \mathrm{Homeo}^I(S, (\lambda_1, \lambda_2))$ the isotopy classes of homeomorphisms of S restricting to the identity on a neighborhood of the parametrized interval I and fixing the punctures of λ_1 and λ_2 setwise.

Proposition 3.11. *For the partition (λ_1, λ_2) of a natural number n, we have a* short *exact sequence:*

$$1 \longrightarrow \mathbf{B}_{\lambda_1, \lambda_2}(S) \xrightarrow{i_\lambda^\flat} \pi_0 \mathrm{Homeo}^I(S, (\lambda_1, \lambda_2)) \xrightarrow{p_\lambda^\flat} \pi_0 \mathrm{Homeo}^I(S) \longrightarrow 1 \qquad (18)$$

Proof. We consider the map $\epsilon : \mathrm{Homeo}^I(S) \to F_n(S)/\mathfrak{S}_{\lambda_1, \lambda_2}$ defined by evaluating homeomorphisms at a base configuration in $S \setminus \mathscr{P}$. Let $\epsilon_F : \mathrm{Homeo}^I(S) \to F_n(S)$ be the map defined by evaluating homeomorphisms at a base configuration in $S \setminus \mathscr{P}$. We recall from [4, Lemma 1.2] that ϵ_F is a locally trivial fibration, whose fiber identifies with the group of homeomorphisms of S restricting to the identity on a neighborhood of the parametrized interval I and fixing each puncture pointwise denoted by $\mathrm{Homeo}^I(S, [n])$. The natural action of $\mathfrak{S}_{\lambda_1, \lambda_2}$ on $F_n(S)$ defined by permutation of coordinates is free. Therefore the associated canonical projection $p : F_n(S) \to F_n(S)/\mathfrak{S}_{\lambda_1, \lambda_2}$ is a regular covering space map with deck transformation group $\mathfrak{S}_{\lambda_1, \lambda_2}$; see [25, Proposition 1.40] for instance. In particular, the map p defines a locally trivial fibration with discrete fiber $\mathfrak{S}_{\lambda_1, \lambda_2}$. Hence the evaluation map ϵ is equal to the composition $p \circ \epsilon_F$ and it is therefore a locally trivial fibration. Its fiber identifies with $\mathrm{Homeo}^I(S, (\lambda_1, \lambda_2))$. We denote by $\widehat{\omega}_\lambda : \mathrm{Homeo}^I(S, (\lambda_1, \lambda_2)) \to \mathrm{Homeo}^I(S)$ the associated inclusion. We obtain the exact sequence (18) from the long exact sequence of homotopy groups of this locally trivial fibration, using the fact that $\pi_1(\mathrm{Homeo}^I(S)) = 0$ by [24]. $\square$

In particular, we consider the surface braid group $\mathbf{B}_n(S)$ as a normal subgroup of the mapping class group $\pi_0 \mathrm{Homeo}^I(S, n)$ via (18) for the partition $(0, n)$. We define $\mathfrak{B}_2$ to be the subgroupoid of $\mathfrak{M}_2$ with the same objects and with morphisms for each surface S given by the surface braid group $\mathbf{B}_n(S)$. The monoidal structure $(\mathfrak{M}_2, \natural, 0)$ clearly restricts to a braided monoidal structure on the subgroupoid $\mathfrak{B}_2$, denoted in the same way $(\mathfrak{B}_2, \natural, 0)$. We thus have a canonical inclusion $\mathfrak{UB}_2 \hookrightarrow \mathfrak{UM}_2$. We denote by $\pi_1(-, p)^\flat$ the restriction of the fundamental group functor $\pi_1(-, p) : \mathfrak{UM}_2 \to \mathfrak{gr}$ introduced in Section 3.2 along this inclusion. In particular, it follows from Lemma 3.3 that the functor $\pi_1(-, p)^\flat : (\mathfrak{B}_2, \natural, \mathbb{D}^2) \to (\mathfrak{gr}, *, 0_{\mathfrak{gr}})$ is strict monoidal.

We fix a natural number $g \geq 0$ throughout Section 3.3.2 and consider the surface $\Sigma_{g,1}$. Let $\mathfrak{B}_2^g$ be the small full subgroupoid of $\mathfrak{B}_2$ on the objects $\underline{n} = 1^{\natural n} \natural \underline{0}$ where $\underline{0} := \Sigma_{g,1}^0$ and $1 := \Sigma_{0,1}^1$. In the notations of Section 2.2.1, the groupoid $\mathfrak{B}_2$ plays the role of $\mathcal{G}'$ and $\mathfrak{B}_2^g$ corresponds to $\mathcal{G}$.

As in Section 3.3.1, let H be the free group $\pi_1(\Sigma_{0,1}^1, p) \cong \mathbf{F}_1$, H_0 be the free group $\pi_1(\Sigma_{g,1}^0, p) \cong \mathbf{F}_{2g}$ and thus $H_n = H^{*n} * H_0 = \pi_1(\Sigma_{g,1}^n, p)$ for all natural numbers n. Precomposing by $\mathfrak{UB}_2^g \to \mathfrak{UB}_2$, the restriction of the functor $\pi_1(-, p)^\flat$ to $\mathfrak{UB}_2^g$ sends $\Sigma_{g,1}^n$ to H_n. We denote by $\mathcal{A}_{\Sigma_{g,1}^n}^\flat : \mathbf{B}_n(\Sigma_{g,1}) \to \mathrm{Aut}(\pi_1(\Sigma_{g,1}^n, p))$ the associated natural action for natural number n: it corresponds to the restriction of the natural action $\mathcal{A}_{\Sigma_{g,1}^n} : \mathbf{\Gamma}_{g,1}^n \to \mathrm{Aut}(\pi_1(\Sigma_{g,1}^n, p))$ to the normal subgroup $\mathbf{B}_n(\Sigma_{g,1}) \hookrightarrow \mathbf{\Gamma}_{g,1}^n$.

For sake of coherence with the notation of Section 3.3.1, we henceforth denote the group $\mathbf{B}_{1,n}(\Sigma_{g,1})$ by $\mathbf{B}_{[1],n}(\Sigma_{g,1})$ since it is the kernel of the morphism $\mathbf{\Gamma}_{g,1}^{[1],n} \twoheadrightarrow \mathbf{\Gamma}_{g,1}$ by (18). The following lemma is the key to define a Long-Moody system in the present situation.

Lemma 3.12. *For all $n \in \mathbb{N}$, there is an isomorphism $\mathscr{B}_n^\flat : H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}^\flat} \mathbf{B}_n(\Sigma_{g,1}) \xrightarrow{\sim} \mathbf{B}_{[1],n}(\Sigma_{g,1})$. In particular, there is a canonical injection $Push_{p_1}^\flat : H_n \hookrightarrow \mathbf{B}_{[1],n}(\Sigma_{g,1})$ such that $i_{(1,n)}^\flat \circ Push_{p_1}^\flat = Push_{p_1} \circ \alpha_n^{x_1}$.*

Proof. It follows from the definitions of the morphism Forget of the exact sequence (16), of its splitting $s_n : \mathbf{\Gamma}_{g,1}^n \hookrightarrow \mathbf{\Gamma}_{g,1}^{[1],n}$ introduced in Lemma 3.7 and of the morphisms $p_{(1,n)}^\flat$ and $p_n^\flat$ of (18) that the following diagrams are commutative:

$$
\begin{array}{ccc}
\mathbf{\Gamma}_{g,1}^{[1],n} \xrightarrow{\ \text{Forget}\ } \mathbf{\Gamma}_{g,1}^n & \qquad & \mathbf{\Gamma}_{g,1}^{[1],n} \xleftarrow{\ s_n\ } \mathbf{\Gamma}_{g,1}^n \\
\searrow_{p_{(1,n)}^\flat} \quad \swarrow_{p_n^\flat} & & \searrow_{p_{(1,n)}^\flat} \quad \swarrow_{p_n^\flat} \\
\mathbf{\Gamma}_{g,1}, & & \mathbf{\Gamma}_{g,1}.
\end{array}
$$

We deduce from the universal property of the kernel that there exist morphisms $\text{Forget}^\flat : \mathbf{B}_{[1],n}(\Sigma_{g,1}) \to \mathbf{B}_n(\Sigma_{g,1})$ and $s_n^\flat : \mathbf{B}_n(\Sigma_{g,1}) \to \mathbf{B}_{[1],n}(\Sigma_{g,1})$ such that the following diagrams are commutative:

$$
\begin{array}{ccc}
\mathbf{B}_{[1],n}(\Sigma_{g,1}) \xrightarrow{\ \text{Forget}^\flat\ } \mathbf{B}_n(\Sigma_{g,1}) & \qquad & \mathbf{B}_{[1],n}(\Sigma_{g,1}) \xleftarrow{\ s_n^\flat\ } \mathbf{B}_n(\Sigma_{g,1}) \\
\downarrow_{i_{(1,n)}^\flat} \qquad\qquad \downarrow_{i_n^\flat} & & \downarrow_{i_{(1,n)}^\flat} \qquad\qquad \downarrow_{i_n^\flat} \\
\mathbf{\Gamma}_{g,1}^{[1],n} \xrightarrow{\ \text{Forget}\ } \mathbf{\Gamma}_{g,1}^n, & & \mathbf{\Gamma}_{g,1}^{[1],n} \xleftarrow{\ s_n\ } \mathbf{\Gamma}_{g,1}^n.
\end{array}
$$

The commutativity of the second square implies that the morphism $s_n^\flat$ is a monomorphism. Also it is a formal fact that $s_n^\flat$ defines a section of $\text{Forget}^\flat$, and the latter is therefore an epimorphism. Denoting by K the kernel of $\text{Forget}^\flat$, we thus obtain that $\mathbf{B}_{[1],n}(\Sigma_{g,1}) \cong K \rtimes \mathbf{B}_n(\Sigma_{g,1})$.

We deduce from the universal property of $\pi_1(\Sigma_{g,1}^n, p)$ as the kernel of Forget that there exists a unique monomorphism $K \hookrightarrow \pi_1(\Sigma_{g,1}^n, p)$. Since $p_{(1,n)}^\flat \circ (\text{Push}_{p_1} \circ \alpha_n^{x_1}) = (p_n^\flat \circ \text{Forget}) \circ (\text{Push}_{p_1} \circ \alpha_n^{x_1}) = 0$, the universal property of $\mathbf{B}_{[1],n}(\Sigma_{g,1})$ as the kernel of $p_{(1,n)}^\flat$ provides a unique monomorphism $\theta : \pi_1(\Sigma_{g,1}^n, p) \hookrightarrow \mathbf{B}_{[1],n}(\Sigma_{g,1})$ such that $\text{Push}_{p_1} \circ \alpha_n^{x_1} = i_{(1,n)}^\flat \circ \theta$. Then, the commutativity of the above first square and the universal property of K provide a unique monomorphism $\pi_1(\Sigma_{g,1}^n, p) \hookrightarrow K$ which is a section of $K \hookrightarrow \pi_1(\Sigma_{g,1}^n, p)$: hence K is isomorphic to $\pi_1(\Sigma_{g,1}^n, p)$. The action of $\mathbf{B}_n(\Sigma_{g,1})$ on $\pi_1(\Sigma_{g,1}^n, p)$ is formally induced by the restriction $\mathcal{A}_{\Sigma_{g,1}^n}^\flat$ of $\mathcal{A}_{\Sigma_{g,1}^n}$ to the surface braid group, which ends the proof. $\qquad\square$

Furthermore, we consider the short exact sequences of type (18) associated with the morphisms $p_{(1,n)}^\flat$: $\mathbf{\Gamma}_{g,1}^{[1],n} \twoheadrightarrow \mathbf{\Gamma}_{g,1}$ and $p_{n+1}^\flat : \mathbf{\Gamma}_{g,1}^{1+n} \twoheadrightarrow \mathbf{\Gamma}_{g,1}$. We recall that we have a canonical injection $\mathscr{E}_n : \mathbf{\Gamma}_{g,1}^{[1],n} \hookrightarrow \mathbf{\Gamma}_{g,1}^{1+n}$ induced by viewing $\mathbf{\Gamma}_{g,1}^{[1],n}$ as a subgroup of $\mathbf{\Gamma}_{g,1}^{1+n}$ where the first puncture is fixed. It is clear from the definitions that $p_{n+1}^\flat \circ \mathscr{E}_n = p_{(1,n)}^\flat$. Therefore the universal property of $\mathbf{B}_{[1],n}(\Sigma_{g,1})$ as the kernel of $p_{(1,n)}^\flat$ provides a canonical embedding $\mathscr{E}_n^\flat : \mathbf{B}_{[1],n}(\Sigma_{g,1}) \hookrightarrow \mathbf{B}_{1+n}(\Sigma_{g,1})$. Hence we now can define the appropriate morphisms $\{\chi_n : \pi_1(\Sigma_{g,1}^n, p) \to \mathbf{B}_{1+n}(\Sigma_{g,1})\}_{n \in \mathbb{N}}$ to define a Long-Moody functor for surface braid groups:

Definition 3.13. Let n be a natural number. We define the morphism $\chi_{n,1}^\flat : H_n \to \mathbf{B}_{1+n}(\Sigma_{g,1})$ to be the composition $\mathscr{E}_n^\flat \circ \mathscr{B}_n^\flat \circ \text{Push}_{p_1}^\flat : H_n \hookrightarrow \mathbf{B}_{[1],n}(\Sigma_{g,1}) \hookrightarrow \mathbf{B}_{1+n}(\Sigma_{g,1})$.

If $g = 0$, we note that $\chi_{0,1}^\flat : \pi_1(\Sigma_{0,1}^0, p) \to 0_{\mathfrak{Gr}}$ is the trivial morphism and $\chi_{1,1}^\flat : \pi_1(\Sigma_{0,1}^1, p) \to \mathbf{PB}_2$ is the morphism sending the generator f_1 of $\pi_1(\Sigma_{0,1}^1, p)$ to σ_1^2 (where σ_1 denotes the Artin generator of the braid group on two strands $\mathbf{B}_2$).

We now have all the required ingredients to define an appropriate Long-Moody functor:

Proposition 3.14. *The setting* $\{\pi_1(-, p)^\flat, \mathfrak{B}_2^g, \mathfrak{B}_2, \chi_1^\flat\}$ *is a coherent Long-Moody system.*

Proof. First, we note from the definitions that the composition of the canonical injection of $\mathbf{B}_n(\Sigma_{g,1})$ into $H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}^\flat} \mathbf{B}_n(\Sigma_{g,1})$ with the composition $\mathscr{E}_n^\flat \circ \mathscr{B}_n^\flat : H_n \rtimes_{\mathcal{A}_{\Sigma_{g,1}^n}^\flat} \mathbf{B}_n(\Sigma_{g,1}) \hookrightarrow \mathbf{B}_{1+n}(\Sigma_{g,1})$ is the canonical morphism $id_{\Sigma_{0,1}^1} \natural - : \mathbf{B}_n(\Sigma_{g,1}) \hookrightarrow \mathbf{B}_{1+n}(\Sigma_{g,1})$ induced by the monoidal structure. Then, following mutatis mutandis the proof of Lemma 3.8, the setting $\{\pi_1(\Sigma_{g,1}^-, p)^\flat, \mathfrak{B}_2^g, \mathfrak{B}_2, \chi_1^\flat\}$ is a Long-Moody system.

By Proposition 2.14, to prove that this system is coherent, it is enough to prove that the morphisms $\chi_{n,1}^\flat$ satisfy the equality (13) for all natural numbers n. It follows from Definitions 3.6 and 3.13 that the composition $i_n^\flat \circ \chi_{n,1}^\flat : H_n \hookrightarrow \mathbf{B}_{1+n}(\Sigma_{g,1}) \hookrightarrow \mathbf{\Gamma}_{g,1}^{1+n}$ is actually equal to $\chi_{n,1}$: therefore the morphisms $\chi_{n,1}^\flat$ satisfy the equality (13) because the morphisms $\chi_{n,1}$ do; see the proof of Proposition 3.9. $\qquad\square$

Connection with previous work. Assuming that $g = 0$, we recover the results of an earlier paper of the author [37]. Indeed, in this case we consider the category $\mathfrak{UB}_2^0 = \mathfrak{U}\beta$, which is Quillen's bracket construction on the braid groupoid β. The choice $\chi_{n,1} = \chi_{n,1}^{\flat} : \mathbf{F}_n \to \mathbf{B}_{n+1}$ of Definitions 3.6 and 3.13 corresponds to the morphism introduced in [37, Example 2.7]. The actions $\mathcal{A}_{\Sigma_{0,1}^n} : \mathbf{B}_n \to \mathrm{Aut}_{\mathfrak{Gr}}(\pi_1(\Sigma_{0,1}^n, p))$, which correspond to Artin's representations for all natural numbers n. We deduce that the Long-Moody functor associated with the coherent system $\{\pi_1(-, p)^{\flat}, \mathfrak{B}_2^0, \mathfrak{B}_2, \chi_1^{\flat}\}$ is isomorphic the Long-Moody functor of [37, Section 2.3.1] denoted by $\mathbf{LM}_1$. We could also have chosen other actions $\mathcal{A}_n : \mathbf{B}_n \to \mathrm{Aut}(\mathbf{F}_n)$ and morphisms $\chi_n : \mathbf{F}_n \to \mathbf{B}_{n+1}$ so that the framework of Section 2 is satisfied. Hence, we recover all the Long-Moody functors introduced in [37].

In addition, the following example shows that the new framework developed in the present paper recovers even more families of representations of braid groups that the work of [37] could not obtain.

Example 3.15. For each natural number n, there is a classical geometric embedding $\mathscr{W}_n : \mathbf{B}_{2n+1} \hookrightarrow \mathbf{\Gamma}_{n,1}$ that sends the standard generators of the braid group to Dehn twists around a fixed system of meridians and longitudes on the surface $\Sigma_{n,1}$; we refer the reader to [8, Section 1] for more details about this embedding. Let $\mathcal{W}$ be the subgroupoid of $\mathfrak{M}_2^{+,0}$ defined by the embeddings $\{\mathscr{W}_n\}_{n\in\mathbb{N}}$. We assign H to be the group $\pi_1(\Sigma_{n,1}, p)$ and H_0 to be the trivial group. Hence, the restriction of the functor $\pi_1(\Sigma_{-,1}, p)$ provides $\pi_1(\Sigma_{-,1}, p)^{\flat,2} : \mathfrak{U}\mathcal{W} \to \mathfrak{U}\mathfrak{M}_2^{+,0} \to \mathfrak{Gr}$. Using the trivial functor χ_{tr}, we have a coherent Long-Moody system $\{\pi_1(-, p)^{\flat,2}, \mathcal{W}, \mathfrak{M}_2^{+,0}, \chi_{tr}\}$. By Proposition 2.16 the associated Long-Moody functor applied to the constant functor R is isomorphic to the restriction of the functor $H_1(\Sigma_{-,1}, R)$ to the category $\mathfrak{U}\mathcal{W}$, denoted by $H_1(\Sigma_{-,1}, \mathbb{Z})_{\mathfrak{U}\mathcal{W}}$. The encoded representations are considered by Callegaro and Salvetti in [10].

In particular, considering a coherent Long-Moody system $\{\pi_1(-, p)^{\flat}, \mathfrak{B}_2, \mathfrak{B}_2, \chi\}$ fitting in the previous framework of [37], the group H_n is the free group on n generators $\mathbf{F}_n$. Then $\mathbf{LM}(M)(n) \cong M^{\oplus n}$ for all objects M of $\mathbf{Fct}(\mathfrak{U}\beta, R\text{-}\mathfrak{Mod})$ and all natural numbers n. Since $H_1(\Sigma_{-,1}, \mathbb{Z})_{\mathfrak{U}\mathcal{W}}(n) \cong \mathbb{Z}^{\oplus 2n}$, it is impossible to directly recover the functor $H_1(\Sigma_{-,1}, \mathbb{Z})_{\mathfrak{U}\mathcal{W}}$ applying a Long-Moody functor with this setting.

For non-orientable surfaces. As in Section 3.3.1, the above work adapts verbatim to the case of the braid group on the non-orientable surface $\mathcal{N}\Sigma_{h,1}^n$, with a natural number $h \geq 2$. Let $\mathfrak{B}_2^{-,h}$ be the small full subgroupoid of $\mathfrak{B}_2$ on the objects $\underline{n} = 1^{\natural n}\natural\underline{0}$ where $\underline{0} := \mathcal{N}\Sigma_{h,1}$ and $1 := D$. The morphisms $\{\chi_{n,2}\}_{n\in\mathbb{N}}$ induce in the same way a family of morphisms $\{\chi_{n,2}^{\flat} : \pi_1(\mathcal{N}\Sigma_{h,1}^n, p) \to \mathbf{B}_{1+n}(\mathcal{N}\Sigma_{h,1})\}_{n\in\mathbb{N}}$. Then we can analogously prove that:

Proposition 3.16. *The setting* $\{\pi_1(\mathcal{N}\Sigma_{h,1}^n, p)^{\flat}, \mathfrak{B}_2^{-,h}, \mathfrak{B}_2, \chi_2^{\flat}\}$ *is a coherent Long-Moody system.*

3.3.3 Applications

Finally, we briefly state some applications of the Long-Moody functors defined by Propositions 3.9 and 3.14 to construct new families of representations for mapping class groups and surface braid groups respectively. We do not detail the proofs showing that the constructed representations are not certain known ones: these follow from some standard (but quite lengthy) computations, which are left to the reader.

Mapping class groups. We fix $g \geq 1$ and take the commutative ring R to be $\mathbb{Q}$ for this paragraph. We consider the coherent Long-Moody system of Proposition 3.9 and denote its associated functor by $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}$ for simplicity. The surface $\Sigma_{g,1}^m$ being a classifying space of $\pi_1(\Sigma_{g,1}^m, p)$, we denote by $H_1(\Sigma_{g,1}^-, R)$ the composition functor $H_1(-, R) \circ \pi_1(\Sigma_{g,1}^-, p)$. It follows from Proposition 2.16 that $H_1(\Sigma_{g,1}^-, \mathbb{Q})$ is isomorphic to $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(\mathbb{Q})$. However, some straightforward (but lengthy) matrix computations show that the functor $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(H_1(\Sigma_{g,1}^-, \mathbb{Q}))$ is not isomorphic to $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(\mathbb{Q}) \otimes_{\mathbb{Q}} H_1(\Sigma_{g,1}^-, \mathbb{Q})(1\natural -)$. The computation of the Long-Moody functor associated with $\{\pi_1(\Sigma_{g,1}^-, p), \mathfrak{M}_2^{+,g}, \mathfrak{M}_2, \chi_1\}$ on an object F of $\mathbf{Fct}(\mathfrak{U}\mathfrak{M}_2^{+,g}, R\text{-}\mathfrak{Mod})$ is therefore not generally speaking given by Proposition 2.16. By applying the associated Long-Moody functor, we thus provide new families of linear representations for the mapping class groups $\{\mathbf{\Gamma}_{g,1}^n\}_{n\in\mathbb{N}}$, which have very few examples of linear representations in the literature.

Surface braid groups. We fix $g \geq 0$ and consider the coherent Long-Moody system of Proposition 3.9 with $R = \mathbb{Z}$ as commutative ring. We denote its associated functor by $\mathbf{LM}_{\mathfrak{B}_2^g}$ for simplicity.

Instead of applying the Long-Moody functors on the first homology of the surface as above for the mapping class groups, there is a more natural and interesting alternative for surface braid groups. For a group G, we denote by $\{\Gamma_l(G), l \geq 0\}$ its *lower central series*. When there is no ambiguity, we omit G from the notations. For $g = 0$, we recall that the abelianization $\mathbf{B}_n/\Gamma_2\mathbf{B}_n$ is isomorphic to $\mathbb{Z}$ for $n \geq 2$. Let $\mathbb{Z}[\mathbb{Z}] : \boldsymbol{\beta} \to \mathbb{Z}\text{-}\mathfrak{Mod}$ be the functor defined by sending a natural number n to the group ring $\mathbb{Z}[\mathbb{Z}]$ if $n \geq 2$, on which the action of $\mathbf{B}_n$ is induced by the left multiplication on the quotient $\mathbf{B}_n/\Gamma_2\mathbf{B}_n$, or to the trivial group if $n \leq 1$. Then, the iterates of $\mathbf{LM}_{\boldsymbol{\beta}}$ on $\mathbb{Z}[\mathbb{Z}]$ provide new families of representations; see [37, Section 2.3].

Now we fix $g \geq 1$. A direct computation from [1, Theorem 1.1] shows that $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_2(\mathbf{B}_n(\Sigma_{g,1}))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{2g}$, where the braid generators are sent to $\mathbb{Z}/2\mathbb{Z}$. A fortiori the generators σ_i^2 act trivially on that quotient: we thus go one step further in the lower central series to get a module which has a non-trivial action via $\chi_{n-1,1}^{\flat} : H_{n-1} \to \mathbf{B}_n(\Sigma_{g,1})$. The third lower central quotient $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_n(\Sigma_{g,1}))$ is isomorphic to the semidirect product $(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g$ for $n \geq 3$ by [3, Corollary 3.14]. Let $\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3] : \mathfrak{B}_2^g \to \mathbb{Z}\text{-}\mathfrak{Mod}$ be the functor defined by assigning the group ring $\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]$ to each natural number n, on which the action of $\mathbf{B}_n(\Sigma_{g,1})$ is induced by the left multiplication on the quotient $\mathbf{B}_n(\Sigma_{g,1})/\Gamma_3(\mathbf{B}_n(\Sigma_{g,1}))$ if $n \geq 3$ and the trivial action if $n \leq 2$. Hence we deduce from some direct elementary (but lengthy) computations on the obtained matrices that the functor $\mathbf{LM}_{\mathfrak{B}_2^g}^{\circ d+1}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is not isomorphic to the tensor product functor $\mathbf{LM}_{\mathfrak{B}_2^g}(\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbf{LM}_{\mathfrak{B}_2^g}^{\circ d}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]))(1\flat-)$ for $d \in \{0,1\}$. Hence, the iterates of $\mathbf{LM}_{\mathfrak{B}_2^g}$ are not generally speaking determined by Proposition 2.16 and thus define new representations for surface braid groups. As far as the author knows, there are very few explicit examples of linear representations of surface braid groups for $g \geq 1$.

3.4 Modifying the genus

In this section, we construct Long-Moody functors for the mapping class groups of surfaces when the orientable genus varies. For all natural numbers n, we denote by $\Sigma_{n,1}$ the surface $T^{\natural n}\natural\mathbb{D}^2$. For consistency if $n = 0$, we assign $\Sigma_{0,1} = \mathbb{D}^2$. Let $\mathfrak{M}_2^+$ be the small full subgroupoid of $\mathfrak{M}_2$ on the objects $\{\Sigma_{n,1}\}_{n\in\mathbb{N}}$. The monoidal structure $(\mathfrak{M}_2, \natural, 0)$ clearly restricts to a braided monoidal structure on the subgroupoid $\mathfrak{M}_2^+$, denoted in the same way $(\mathfrak{M}_2^+, \natural, 0)$. In particular, in the notations of Section 2.2.1, the groupoid $\mathfrak{M}_2^+$ plays the role of both $\mathcal{G}'$ and $\mathcal{G}$ where $\underline{0} := \Sigma_{0,1}$ and $\underline{1} := \Sigma_{1,1}$. We denote the mapping class group $\pi_0\mathrm{Homeo}^I(\Sigma_{n,1})$ by $\boldsymbol{\Gamma}_{n,1}$, for all $n \in \mathbb{N}$.

The key point to define a non-trivial Long-Moody system in the present situation is the use of *pure surface framed braids*: for completeness, the definition is recalled below and we refer the reader to [2] for further details. We fix a natural number $n \geq 0$ and choose a marked point x in the interior of $\Sigma_{n,1}$. There is no loss of generality in endowing the surface $\Sigma_{n,1}$ with an arbitrary smooth structure and a Riemannian metric. We denote by $U\Sigma_{n,1}$ the total space of the unit tangent bundle and fix a unit tangent vector v of $\Sigma_{n,1}$ at a point x in the interior of $\Sigma_{n,1}$. Using the long exact sequence in homotopy for the fibration $\mathbb{S}^1 \to U\Sigma_{n,1} \to \Sigma_{n,1}$ (where $\mathbb{S}^1$ is the 1-dimensional unit sphere) and the contractibility results of [21, Théorème 5], we deduce that the fundamental group $\pi_1(U\Sigma_{n,1}, (x, v))$ is isomorphic to $\mathbb{Z} \times \pi_1(\Sigma_{n,1}, x)$ as a group if $n \geq 1$. As $\Sigma_{n,1}$ is a path-connected space, the group $\pi_1(U\Sigma_{n,1}, (x, v))$ is independent of the base (x, v) up to an isomorphism (in particular, if x is fixed then the isomorphism is unique). The *pure framed braid group* $FP_1(\Sigma_{n,1})$ is defined as the fundamental group of $U\Sigma_{n,1}$ (see [2, Definition 1]) and is isomorphic to $\mathbb{Z} \times \pi_1(\Sigma_{n,1}, x)$ (see [2, Theorem 5, 1)]) if $n \geq 1$. The natural action of the diffeomorphisms of $\Sigma_{n,1}$ on the unit tangent bundle induce a canonical action of $\boldsymbol{\Gamma}_{n,1}$ on $FP_1(\Sigma_{n,1})$, denoted by $\mathcal{A}_{U\Sigma_{n,1}}$. Then we define a functor

$$FP_1 : (\mathfrak{M}_2^+, \natural, \mathbb{D}^2) \to \mathfrak{Gr}$$

assigning the pure framed braid group $FP_1(\Sigma_{n,1})$ to each object $\Sigma_{n,1}$ and the corresponding natural action of the mapping class group on morphisms. The functor FP_1 plays the role of the parameter $\mathcal{A} : \mathcal{G} \to \mathfrak{Gr}$ for the Long-Moody functor of this section.

The remaining ingredient to be introduced to define an appropriate Long-Moody system is the set of suitable morphisms $\{\chi_n : FP_1(\Sigma_{n,1}) \to \boldsymbol{\Gamma}_{1+n,1}\}_{n\in\mathbb{N}}$. Let $\Sigma_{n,2}$ be a smooth connected compact surface with two boundary components: one of the two is marked by the parametrized interval I and the other one is denoted by $\partial'\Sigma_{n,2}$. Denoting by $\mathrm{Diff}^{+,\partial}(\Sigma_{n,2})$ the group of orientation preserving diffeomorphisms of $\Sigma_{n,2}$ which fix the boundary pointwise, we denote by $\boldsymbol{\Gamma}_{n,2}$ the mapping class group $\pi_0\mathrm{Diff}^{+,\partial}(\Sigma_{n,2})$. We recall from [2, Corollary 3] that we have a short exact sequence for each $n \geq 1$:

$$1 \longrightarrow FP_1(\Sigma_{n,1}) \xrightarrow{\overrightarrow{\mathrm{Push}_n}} \boldsymbol{\Gamma}_{n,2} \xrightarrow{\mathrm{Cap}} \boldsymbol{\Gamma}_{n,1} \longrightarrow 1. \tag{19}$$

Namely, the short exact sequence (19) is the long exact sequence in homotopy of a locally trivial fibration $\text{Diff}^{\partial_0}(\Sigma_{n,1}) \to U\Sigma_{n,1}$ which fiber identifies with the group $\text{Diff}^{+,\partial}(\Sigma_{n,2})$; see [2, Proposition 2] for further details. In particular, the morphism Cap is the map induced in homotopy by the associated fiber inclusion ϖ_n, corresponding to the capping of the boundary component $\partial'\Sigma_{n,2}$ with a closed disc. Moreover, we prove that:

Lemma 3.17. *The sequence (19) splits, inducing an isomorphism $\mathscr{C}_n : \Gamma_{n,2} \cong FP_1(\Sigma_{n,1}) \rtimes_{\mathcal{A}_{U\Sigma_{n,1}}} \Gamma_{n,1}$.*

Proof. Identifying $\Sigma_{n,2}$ with the boundary connected sum $\Sigma_{0,2}\natural\Sigma_{n,1}$ along the marked intervals, the surface $\Sigma_{n,1}$ can be viewed as a subsurface of $\Sigma_{n,2}$ as the complement of the cylinder $\Sigma_{0,2}$ with the extra boundary component $\partial'\Sigma_{n,2}$. Let $\nu_n : \text{Diff}^{\partial_0}(\Sigma_{n,1}) \to \text{Diff}^{+,\partial}(\Sigma_{n,2})$ be the inclusion map defined by sending each $\varphi \in \text{Diff}^{+,\partial}(\Sigma_{n,1})$ to $id_{\Sigma_{0,2}}\natural\varphi$, thus inducing a morphism $\pi_0(\nu_n) : \Gamma_{n,1} \to \Gamma_{n,2}$ in homotopy. The composition $\varpi_n \circ \nu_n(\varphi)$ is clearly isotopic to φ and a fortiori $\pi_0(\nu_n)$ is a section of Cap. Hence the short exact sequence (19) splits, thus providing the required isomorphism. $\square$

On another note, we define a morphism $id_{\mathring{\mathbb{T}}}\natural- : \Gamma_{n,2} \to \Gamma_{1+n,1}$ by gluing a torus with a boundary component $\mathring{\mathbb{T}}$ along the boundary component $\partial'\Sigma_{n,2}$, and by extending the diffeomorphisms of $\Sigma_{n,2}$ by the identity on $\mathring{\mathbb{T}}$. We now are able to define the appropriate morphisms $\{\chi_n : FP_1(\Sigma_{n,1}) \to \Gamma_{1+n,1}\}_{n\in\mathbb{N}}$ in the present context:

Definition 3.18. We assign $\chi_{0,fr} : FP_1(\Sigma_{0,1}) \to \Gamma_{1,1}$ to be the trivial morphism. For $n \geq 1$ a natural number, we define the morphism $\chi_{n,fr} : FP_1(\Sigma_{n,1}) \to \Gamma_{1+n,1}$ to be the composite:

$$FP_1(\Sigma_{n,1})\hookrightarrow FP_1(\Sigma_{n,1}) \rtimes_{\mathcal{A}_{U\Sigma_{n,1}}} \Gamma_{n,1} \overset{\mathscr{C}_n}{\cong} \Gamma_{n,2} \xrightarrow{id_{\mathring{\mathbb{T}}}\natural-} \Gamma_{1+n,1}.$$

We now have all the required ingredients to define a Long-Moody functor:

Proposition 3.19. *The setting $\{FP_1, \mathfrak{M}_2^+, \mathfrak{M}_2, \chi_{fr}\}$ is a Long-Moody system.*

Proof. If $n \geq 1$, note that the injection $\pi_0(\nu_n) : \Gamma_{n,1} \hookrightarrow FP_1(\Sigma_{n,1}) \rtimes \Gamma_{n,1} \cong \Gamma_{n,2}$ is equal to $id_{\Sigma_{0,2}}\natural-$. Let $id_{\Sigma_{1,1}}\natural- : \Gamma_{n,1} \to \Gamma_{1+n,1}$ be the morphism induced by extending the diffeomorphisms of $\Sigma_{n,1}$ by the identity on $\Sigma_{1+n,1}$. It is straightforward from the above definitions that the composition $(id_{\mathring{\mathbb{T}}}\natural-) \circ (id_{\Sigma_{0,2}}\natural-)$ is equal to $id_{\Sigma_{1,1}}\natural-$. Hence, the following diagram is commutative:

For $n = 0$, the commutativity of the analogous diagram is trivially checked. Hence it follows Lemma 2.12 that the diagram (9) of Definition 2.8 is commutative, which ends the proof. $\square$

For simplicity, we denote by $\mathbf{LM}_{\mathfrak{M}_2^+}$ the functor associated to the Long-Moody system of Proposition 3.19 and take the commutative ring R to be $\mathbb{Z}$. We recall that the framed pure braid group $FP_1(\Sigma_{n,1})$ corresponds to the fundamental group of the unit tangent bundle $U\Sigma_{n,1}$. Recalling that the framed pure braid groups define a functor $FP_1 : (\mathfrak{M}_2^+, \natural, \mathbb{D}^2) \to \mathfrak{Gr}$, we denote by $H_1(U\Sigma_{-,1}; \mathbb{Z})$ the composition functor $H_1(-, \mathbb{Z}) \circ FP_1$. It follows from Proposition 2.16 that $\mathbf{LM}_{\mathfrak{M}_2^+}(\mathbb{Z})(\underline{n})$ is isomorphic to $H_1(U\Sigma_{n,1}; \mathbb{Z})$ as an $\mathbb{Z}$-module for each $n \in \mathbb{N}$. The $\Gamma_{n,1}$-module structure of $H_1(U\Sigma_{n,1}; \mathbb{Z})$ has been described in details by Trapp in [39]. The following result shows that the evaluation of the functor $\mathbf{LM}_{\mathfrak{M}_2^+}$ is not determined by Proposition 2.16 generally speaking. Identifying $\Sigma_{1,1}\natural\Sigma_{n,1}$ with $\Sigma_{n+1,1}$, we fix a point x in the interior of $\Sigma_{1,1}$ in $\Sigma_{n+1,1}$.

Proposition 3.20. *Let M be an object of $\mathbf{Fct}(\mathfrak{M}_2^+, \mathbb{Z}\text{-}\mathfrak{Mod})$ such that $M(1\natural\underline{n})$ is a free abelian group for some $n \geq 1$. Using the group isomorphism $FP_1(\Sigma_{n,1}) \cong \mathbb{Z} \times \pi_1(\Sigma_{n,1}, x)$, we consider $\pi_1(\Sigma_{n,1}, x)$ as a subgroup of $FP_1(\Sigma_{n,1})$. We assume that the $\pi_1(\Sigma_{n,1}, x)$-action induced by $\overrightarrow{Push_n} : FP_1(\Sigma_{n,1}) \hookrightarrow \Gamma_{n+1,1}$ on $M(1\natural\underline{n})$ is not trivial. Then the functor $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(M)$ is not isomorphic to $\mathbf{LM}_{\mathfrak{M}_2^+}(\mathbb{Z}) \otimes_{\mathbb{Z}} M(1\natural-)$.*

Proof. Since $H_1(U\Sigma_{n,1};\mathbb{Z})$ is a free abelian group of rank $2n+1$, the module $\mathbf{LM}_{\mathfrak{M}_2^+}(\mathbb{Z})(\underline{n}) \otimes_{\mathbb{Z}} M(1\natural\underline{n})$ is isomorphic to $M(1\natural\underline{n})^{\oplus(2n+1)}$. Also, we have a surjection $\mathbb{Z}[FP_1(\Sigma_{n,1})] \twoheadrightarrow \mathbb{Z}[\pi_1(\Sigma_{n,1},x)]$ (induced from $\pi_1(U\Sigma_{n,1},(x,v)) \twoheadrightarrow \pi_1(\Sigma_{n,1},x)$) which defines a right $\mathbb{Z}[FP_1(\Sigma_{n,1})]$-module surjection

$$p_{FP_1(\Sigma_{n,1})} : \mathcal{I}_{\mathbb{Z}[FP_1(\Sigma_{n,1})]} \twoheadrightarrow \mathcal{I}_{\mathbb{Z}[\pi_1(\Sigma_{n,1},x)]} \otimes_{\mathbb{Z}[\pi_1(\Sigma_{n,1},x)]} \mathbb{Z}[FP_1(\Sigma_{n,1})].$$

We recall from Lemma 2.17 that $\mathcal{I}_{\mathbb{Z}[\pi_1(\Sigma_{n,1},x)]}$ is a free $\mathbb{Z}[\pi_1(\Sigma_{n,1},x)]$-module of rank $2n$. Therefore, the tensor product morphism $p_{FP_1(\Sigma_{n,1})} \otimes_{\mathbb{Z}[FP_1(\Sigma_{n,1})]} id_{M(1\natural\underline{n})}$ provides a surjection of abelian groups from $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(M)(\underline{n})$ to $M(1\natural\underline{n})^{\oplus 2n}$. We denote by K the kernel of this surjection. Also, since the target of this surjection is a projective abelian group, this surjection has a section which associated quotient is isomorphic to K.

Now suppose that $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(M)(\underline{n})$ is isomorphic to $H_1(U\Sigma_{n,1},\mathbb{Z}) \otimes_{\mathbb{Z}} M(1\natural\underline{n})$ as an abelian group. Then the kernel K is isomorphic to $M(1\natural\underline{n})$. Using the decomposition $\mathbb{Z} \times \pi_1(\Sigma_{n,1},x)$ of $FP_1(\Sigma_{n,1})$, an element of $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(H_1(U\Sigma_{-,1},\mathbb{Z}))(\underline{n})$ is of the form $(\sigma\alpha - 1) \otimes v$ where $\sigma \in \mathbb{Z}$, $\alpha \in \pi_1(\Sigma_{n,1},x)$ and $v \in M(1\natural\underline{n})$. Since $\sigma\alpha = \alpha\sigma$ in $FP_1(\Sigma_{n,1})$, we compute that $\sigma\alpha - 1 = (\alpha - 1) + (\sigma - 1)\alpha = (\alpha - 1)\sigma + (\sigma - 1)$. Hence we have $(\sigma - 1)\alpha \otimes v = (\sigma - 1) \otimes v$ in K (seen as a quotient of $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(M)(\underline{n})$). However, since the abelian group $M(1\natural\underline{n})$ is a non-trivial $\pi_1(\Sigma_{n,1},x)$-module, we deduce that there cannot exist an injective abelian group morphism from $M(1\natural\underline{n})$ to K: this contradicts the fact that K is isomorphic to $M(1\natural\underline{n})$ and ends the proof. $\qquad\square$

Example 3.21. The kernel of the $\Gamma_{n,1}$-module $H_1(U\Sigma_{n,1};\mathbb{Z})$ is called the *Chillingworth subgroup* and denoted by $\mathfrak{C}_{n,1}$; see [39, Definition 2.6]. We recall that equipping the surface $\Sigma_{n,1}$ with a continuous non-vanishing vector field X, the winding number $\omega_X(c)$ with respect to X of an oriented regular curve c is the number of times its tangent rotates with respect to the framing induced by X; see [11, 12]. Then $\mathfrak{C}_{n,1}$ is the group of all elements f in the *Torelli subgroup* such that $\omega_X(f(c)) = \omega_X(c)$ for all oriented regular curve c.

We identify $\Sigma_{1+n,1}$ with $\Sigma_{1,1}\natural\Sigma_{n,1}$ and we consider the standard basis $\{\alpha_i, \beta_i \mid i \in \{1,\ldots,n\}\}$ of generators of $\pi_1(\Sigma_{n,1},p)$ associated with the system of meridians and longitudes $\{a_1, b_1, \ldots, a_n, b_n\}$ fixed in Section 3.3.1; see Figure 2. The action of $\overrightarrow{\mathrm{Push}}_n(\beta_1)$ on a_1 is analogous to the one pictured in Figure 4: the difference is that the puncture p_1 is replaced by a glued handle $\Sigma_{1,1}$ and that we act on the meridian curve. In particular, the winding number of $\overrightarrow{\mathrm{Push}}_n(\beta_1)(a_1)$ is clearly different from the one of a_1, and so $\overrightarrow{\mathrm{Push}}_n(\beta_1)$ does not belong to the Chillingworth subgroup $\mathfrak{C}_{n+1,1}$. Hence $H_1(U\Sigma_{n+1,1};\mathbb{Z})$ is not a trivial $\pi_1(\Sigma_{n,1},x)$-module and thus $H_1(U\Sigma_{-,1};\mathbb{Z})$ satisfies the assumptions of Proposition 3.20. The functor $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}(H_1(U\Sigma_{-,1};\mathbb{Z}))$ thus provides a family of representations of the mapping class groups which (as far as the author knows) does not appear in the literature.

4 Strong and weak polynomial functors

This section introduces the notions of (very) strong and weak polynomial functors with respect to the framework of this paper. Namely, the first subsection presents strong and very strong polynomial functors and their basic properties. In the second subsection, we introduce weak polynomial functors for some subcategories of pre-braided monoidal categories with an initial object, generalizing the previous notion of [16, Section 1]. We also detail some first properties of these functors and present their use to organize families of representations.

4.1 Strong and very strong polynomial functors

For the remainder of Section 4.1, $(\mathfrak{M}',\natural,0)$ is a pre-braided strict monoidal category where the unit 0 is an initial object. We consider $\mathfrak{M}$ a small full subcategory of $(\mathfrak{M}',\natural,0)$. Finally, we fix $\mathscr{A}$ an abelian category.

In this section, we introduce the notions of strong and very strong polynomiality for objects in the functor category $\mathbf{Fct}(\mathfrak{M},\mathscr{A})$. In [37, Section 3], a framework is given for defining these notions in the category $\mathbf{Fct}(M,\mathscr{A})$, where M is a small pre-braided monoidal category where the unit is an initial object. It generalizes the previous work of Djament and Vespa in [16, Section 1]. We also refer to [32] for a comparison of the various instances of the notions of twisted coefficient system and polynomial functor. This section thus extends the definitions and properties of [37, Section 3] to the present larger framework, the various proofs being direct generalizations of this previous work.

Notation 4.1. We denote by $\text{Obj}(\mathfrak{M})_{\natural}$ the set of objects m of $\mathfrak{M}'$ such that $m\natural n$ is an object of $\mathfrak{M}$ for all objects n of $\mathfrak{M}$.

Let m be an element of $\text{Obj}(\mathfrak{M})_{\natural}$. We denote by $\tau_m : \mathbf{Fct}(\mathfrak{M}, \mathscr{A}) \to \mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ the *translation* functor defined by $\tau_m(F) = F(m\natural-)$, $i_m : Id \to \tau_m$ the natural transformation of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ induced by the unique morphism $\iota_m : 0 \to m$. We define $\delta_m = \text{coker}(i_m)$ the *difference* functor and $\kappa_m = \ker(i_m)$ the *evanescence* functor. The following basic properties are direct generalizations of [37, Propositions 3.2 and 3.5]:

Proposition 4.2. *Let $m, m' \in Obj(\mathfrak{M})_{\natural}$. Then the translation functor τ_m is exact and we have the following exact sequence of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$:*

$$0 \longrightarrow \kappa_m \xrightarrow{\Omega_m} Id \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \tag{20}$$

Moreover, for a short exact sequence $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$ in the category $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$, there is a natural exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$:

$$0 \longrightarrow \kappa_m(F) \longrightarrow \kappa_m(G) \longrightarrow \kappa_m(H) \longrightarrow \delta_m(F) \longrightarrow \delta_m(G) \longrightarrow \delta_m(H) \longrightarrow 0. \tag{21}$$

In addition, the functors τ_m and $\tau_{m'}$ commute up to natural isomorphism and they commute with limits and colimits; the difference functors δ_m and $\delta_{m'}$ commute up to natural isomorphism and they commute with colimits; the functors κ_m and $\kappa_{m'}$ commute up to natural isomorphism and they commute with limits; the functor τ_m commute with the functors $\delta_{m'}$ and $\kappa_{m'}$ up to natural isomorphism.

We can define the notions of strong and very strong polynomial functors using Proposition 4.2. Namely:

Definition 4.3. We recursively define on $d \in \mathbb{N}$ the categories $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ and $\mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A})$ of *strong* and *very strong polynomial* functors of degree less than or equal to d to be the full subcategories of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ as follows:

1. If $d < 0$, $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A}) = \mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A}) = \{0\}$;

2. if $d \geq 0$, the objects of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ are the functors F such that for all $m \in \text{Obj}(\mathfrak{M})_{\natural}$, the functor $\delta_m(F)$ is an object of $\mathcal{P}ol_{d-1}^{strong}(\mathfrak{M}, \mathscr{A})$; the objects of $\mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A})$ are the objects F of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ such that $\kappa_m(F) = 0$ and the functor $\delta_m(F)$ is an object of $\mathcal{VP}ol_{d-1}(\mathfrak{M}, \mathscr{A})$ for all $m \in \text{Obj}(\mathfrak{M})_{\natural}$.

For an object F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ which is strong polynomial of degree less than or equal to $n \in \mathbb{N}$, the *smallest* natural number $d \leq n$ for which F is an object of $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ is called the *strong degree* of F. If in addition F is very strong polynomial, its strong degree is also the smallest natural number $d \leq n$ for which F is an object of $\mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A})$ and is then also called the *very strong degree* of F.

Finally, we recall useful properties of the categories associated with strong and very strong polynomial functors. Beforehand, for a strict monoidal category $(\mathfrak{C}, \natural, 0)$, we say that a full subcategory $\mathfrak{D}$ of $\mathfrak{C}$ is *finitely generated by the monoidal structure* if there exists a finite set E of objects of the category $\mathfrak{C}$ such that all objects d of $\mathfrak{D}$ is isomorphic to a finite monoidal product of objects of E. The following properties are direct generalizations of [37, Propositions 3.9 and 3.19], the proofs carry over mutatis mutandis to the present framework.

Proposition 4.4. *We assume that the category $\mathfrak{M}$ is finitely generated by the monoidal structure in $(\mathfrak{M}', \natural, 0)$. We denote by E a finite generating set of $\mathfrak{M}$. Let d be a natural number. Then:*

1. *The category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ is closed under the translation functor, under quotient, under extension and under colimits. The category $\mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A})$ is closed under the translation functor, under normal subobjects and under extensions.*

2. *An object F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ or to $\mathcal{VP}ol_d(\mathfrak{M}, \mathscr{A})$ if and only if the conditions on the evanescence and difference functors of Definition 4.3 are satisfied (only) by the objects of $E \cap Obj(\mathfrak{M})_{\natural}$.*

3. *An object F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ belongs to $\mathcal{P}ol_0^{strong}(\mathfrak{M}, \mathscr{A})$ if and only if it is the quotient of a constant object of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$.*

4.2 Weak polynomial functors

We deal here with the concept of *weak* polynomial functor. It is introduced by Djament and Vespa in [16, Section 2] in the category $\mathbf{Fct}(S, A)$ where S is a symmetric monoidal category where the unit is an initial object, and A is a Grothendieck category. Weak polynomial functors form a *localizing* subcategory of $\mathbf{Fct}(S, A)$; see Proposition 4.10. In particular, this notion happens to be more appropriate to study the stable behaviour for objects of the category $\mathbf{Fct}(S, A)$ (see for example [16, Section 6], [15] and Section 6.2) and provide a new organizing tool for polynomial functors and a fortiori representations of families of groups; see Section 6.2.

We introduce the definition and properties of weak polynomial functors to the present larger setting. In particular, considering a reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$, the notion of weak polynomial functor is well-defined for the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ where $\mathfrak{U}\mathcal{G}$ is the full subcategory of Quillen's bracket construction $\mathfrak{U}\mathcal{G}'$ on the objects of $\mathcal{G}$; see Definition 2.7. We refer the reader to [20, Chapitres II et III] for general notions on abelian categories and quotient abelian category which are required for this section. We recall that a *Grothendieck category* is a cocomplete abelian category, which admits a generator, and in which filtered colimits of exact sequences are exact.

For the remainder of Section 4.2, we assume that the abelian category $\mathscr{A}$ is a Grothendieck category. We recall that we consider $(\mathfrak{M}', \natural, 0)$ a strict pre-braided monoidal small category where the unit 0 is an initial object and $\mathfrak{M}$ a small full subcategory of $(\mathfrak{M}', \natural, 0)$ finitely generated by the monoidal structure. Therefore, the functor category $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ is a Grothendieck category; see [20]. We recall that we defined a particular set of objects of $\mathfrak{M}'$ denoted $\mathrm{Obj}(\mathfrak{M})_\natural$ in Notation 4.1.

Let F be an object of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$. We denote the subfunctor $\sum_{m \in \mathrm{Obj}(\mathfrak{M})_\natural} \kappa_m F$ of F by $\kappa(F)$. Let $K(\mathfrak{M}, \mathscr{A})$ be the full subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ of the objects F such that $\kappa(F) = F$. We have the following basic properties:

Lemma 4.5. *The functor κ is left exact. Moreover, the functor $\kappa(F)$ is an object of $K(\mathfrak{M}, \mathscr{A})$ for all objects F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$.*

Proof. A filtration on the evanescence functors $\{\kappa_m\}_{m \in \mathrm{Obj}(\mathfrak{M})_\natural}$ is given by the canonical inclusions $\kappa_{n'} \hookrightarrow \kappa_{n'\natural n}$ and $\kappa_n \hookrightarrow \kappa_{n'\natural n}$ induced by the morphisms $n \to n'\natural n$ and $n' \to n'\natural n$. Hence, since filtered colimits are exact in $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ as it is a Grothendieck category, the functor κ is left exact as the filtered colimit of the left exact functors $\{\kappa_m\}_{m \in \mathrm{Obj}(\mathfrak{M})_\natural}$. Also, we note that the functor κ commutes with filtered colimits since it is a filtered colimit of finite limits; see [29, Chapter IX, section 2].

Let x be an object of $\mathrm{Obj}(\mathfrak{M})_\natural$. By definition, there is a canonical inclusion $\kappa_m(\kappa_x F) \hookrightarrow \kappa_x F$ for each $m \in \mathrm{Obj}(\mathfrak{M})_\natural$. Since κ_x is the kernel of a natural transformation between the identity functor and a left exact functor, we note that the inclusion $\kappa_x(\kappa_x F) \hookrightarrow \kappa_x F$ is an isomorphism. Hence, as the functor $\kappa(\kappa_x F)$ is the filtered colimit of the functors $\{\kappa_m(\kappa_x F)\}_{m \in \mathrm{Obj}(\mathfrak{M})_\natural}$, we deduce from the universal property of a colimit that $\kappa(\kappa_x F) = \kappa_x F$. Therefore, viewing the functor $\kappa(F)$ as the filtered colimit of the functors $\{\kappa_x F\}_{x \in \mathrm{Obj}(\mathfrak{M})_\natural}$, the second result follows from the commutation of κ with filtered colimits. $\square$

The following proposition is the key property to define weak polynomial functors. It extends the result [16, Corollary 2.15], although its proof is quite different. We recall that we follow the terminology of [20]: a subcategory is *thick* if it closed under subobjects, quotients and extensions; this notion is also known as a *Serre subcategory*.

Proposition 4.6. *The category $K(\mathfrak{M}, \mathscr{A})$ is a thick subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ and it is closed under colimits.*

Proof. Let us prove that $K(\mathfrak{M}, \mathscr{A})$ is a thick subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$. First, let B be a subfunctor of an object F of $K(\mathfrak{M}, \mathscr{A})$. As $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ is a Grothendieck category, we denote by F/B the quotient. Since κ is left exact, the following diagram, where the rows are exact and the vertical arrows are the inclusions, is commutative:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \kappa(B) & \longrightarrow & \kappa(F) & \longrightarrow & \kappa(F/B) \\
 & & \cup\uparrow & & \cup\uparrow & & \cup\uparrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & B & \longrightarrow & F & \longrightarrow & F/B & \longrightarrow & 0.
\end{array}
$$

Since $\kappa(F) = F$, it follows from the 4-lemma that the inclusion $\kappa(B) \hookrightarrow B$ is an equality: $K(\mathfrak{M}, \mathscr{A})$ is thus closed under subobject.

Now, let $f : F \to Q \to 0$ be an epimorphism of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ such that F is an object of $K(\mathfrak{M}, \mathscr{A})$. We consider the following commutative diagram where the vertical arrows are the inclusions:

$$
\begin{array}{ccc}
\kappa(F) & \xrightarrow{\kappa(f)} & \kappa(Q) \\
\downarrow & & \downarrow \\
F & \xrightarrow{\quad f \quad} & Q \longrightarrow 0.
\end{array}
$$

Thus, since $\kappa(F) = F$, then the arrow $\kappa(Q) \hookrightarrow Q$ is also an epimorphism and a fortiori an equality. Hence, $K(\mathfrak{M}, \mathscr{A})$ is closed under quotient.

Finally, let $0 \to B \to F \to Q \to 0$ be a short exact sequence of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ such that B and Q are objects $K(\mathfrak{M}, \mathscr{A})$. Let m be an object of $\mathrm{Obj}(\mathfrak{M})_\natural$. Let F_m be the pullback of the morphisms $F \twoheadrightarrow Q$ and $\kappa_m(Q) \hookrightarrow Q$: the functor F is thus the filtered colimit (with respect to the inclusions) of the pullbacks $\{F_m\}_{m \in \mathrm{Obj}(\mathfrak{M})_\natural}$. We recall that the functor κ is the filtered colimit with respect to the inclusions of the evanescence functors $\{\kappa_x\}_{x \in \mathrm{Obj}(\mathfrak{M})_\natural}$ and that filtered colimits in $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ are exact since it is a Grothendieck category. Hence, the functor κ commutes with filtered colimits. Therefore, it is enough to prove that each F_m is in $K(\mathfrak{M}, \mathscr{A})$ for all $m \in \mathrm{Obj}(\mathfrak{M})_\natural$ to show that $K(\mathfrak{M}, \mathscr{A})$ is closed under extension.

Let B_m be the kernel of $F_m \twoheadrightarrow \kappa_m(Q)$. It follows from the universal property of a kernel and the four lemma that the canonical inclusions $\kappa_m(Q) \hookrightarrow Q$ and $F_m \hookrightarrow F$ induce an inclusion $B_m \hookrightarrow B$. We deduce from the closure $K(\mathfrak{M}, \mathscr{A})$ under subobjects (proved above) that B_m is an object of $K(\mathfrak{M}, \mathscr{A})$. Furthermore, we recall that, since κ_m is the kernel of a natural transformation between the identity functor and a left exact functor, the composition $\kappa_m \circ \kappa_m$ is isomorphic to κ_m and therefore $i_m(\kappa_m(Q)) = 0$. By the universal property of the kernel, there exists a unique morphism φ_m such that the following diagram is commutative:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_m & \xrightarrow{\ \alpha\ } & F_m & \longrightarrow & \kappa_m(Q) & \longrightarrow & 0 \\
& & {\scriptstyle i_m(B_m)}\downarrow & {\scriptstyle \varphi_m}\swarrow & \downarrow{\scriptstyle i_m(F_m)} & & \downarrow{\scriptstyle i_m(\kappa_m(Q))=0} & & \\
0 & \longrightarrow & \tau_m(B_m) & \longrightarrow & \tau_m(F_m) & \longrightarrow & \tau_m(\kappa_m(Q)) & \longrightarrow & 0.
\end{array}
$$

For all $n \in \mathrm{Obj}(\mathfrak{M})_\natural$, let $\varphi_m^{-1}(\tau_m(\kappa_n(B_m)))$ be the pullback of the morphisms $\varphi_m : F_m \to \tau_m(B_m)$ and $\tau_m(\kappa_n(B_m)) \hookrightarrow \tau_m(B_m)$. As a pullback commutes with a filtered colimit in an abelian category and since τ_m commutes with filtered colimits and since $\kappa(B_m) = B_m$, we deduce that

$$
\underset{n \in \mathrm{Obj}(\mathfrak{M})_\natural}{\mathrm{Colim}} \left(\varphi_m^{-1}(\tau_m(\kappa_n(B_m))) \right) = F_m.
$$

In addition, since $\mathfrak{M}'$ is pre-braided monoidal, we deduce from the relation (7) that $(b_{m,n}^{\mathfrak{M}'})^{-1} \circ (\iota_n \natural id_m) = id_m \natural \iota_n$. Hence the precomposition by $(b_{m,n}^{\mathfrak{M}'})^{-1} \natural id_{\mathfrak{M}}$ defines a natural isomorphism $((b_{m,n}^{\mathfrak{M}'})^{-1} \natural id_{\mathfrak{M}})^* : \tau_m \circ \tau_n \xrightarrow{\sim} \tau_n \circ \tau_m$ for all $n \in \mathrm{Obj}(\mathfrak{M})_\natural$, such that

$$
((b_{m,n}^{\mathfrak{M}'})^{-1} \natural id_{\mathfrak{M}})^* \circ (\tau_m(i_n F)) = ((b_{m,n}^{\mathfrak{M}'})^{-1} \natural id_{\mathfrak{M}})^* \circ F(\iota_n \natural id_m \natural -) = F(id_m \natural \iota_n \natural -) = i_n(\tau_m F)
$$

for each object F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$. Hence, the following diagram is commutative for all $n \in \mathrm{Obj}(\mathfrak{M})_\natural$:

$$
\begin{array}{c}
F_m \\
{\scriptstyle \varphi_m}\swarrow \quad \downarrow{\scriptstyle i_m(F_m)} \quad \searrow{\scriptstyle i_{n\natural m}(F_m)} \\
\tau_m(B_m) \xrightarrow{\ \tau_m(\alpha)\ } \tau_m(F_m) \xrightarrow{\ i_n(\tau_m(F_m))\ } \tau_n\tau_m(F_m) \cong \tau_{n\natural m}(F_m). \\
{\scriptstyle \tau_m(i_n(B_m))}\swarrow \quad \downarrow{\scriptstyle i_n(\tau_m(B_m))} \quad \nearrow{\scriptstyle \tau_n\tau_m(\alpha)} \\
\tau_m\tau_n(B_m) \xrightarrow[\ ((b_{m,n}^{\mathfrak{M}'})^{-1}\natural id_{\mathfrak{M}})^*\ ]{\cong} \tau_n\tau_m(B_m)
\end{array}
$$

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We deduce from the previous commutative diagram and the universal property of the kernel that there exists an inclusion morphism $\varphi_m^{-1}(\tau_m(\kappa_n(B_m))) \hookrightarrow \kappa_{n\natural m}(F_m)$ for all $n \in \mathrm{Obj}(\mathfrak{M})_\natural$. Using the definition of κ as a filtered colimit, we deduce that $\mathrm{Colim}_{n\in\mathrm{Obj}(\mathfrak{M})_\natural}(\varphi_m^{-1}(\tau_m(\kappa_n(B_m))))$ is a subobject of $\kappa(F_m)$. Hence, we have $\kappa(F_m) = F_m$ and thus $K(\mathfrak{M}, \mathscr{A})$ is closed under extension.

Let us now prove that $K(\mathfrak{M}, \mathscr{A})$ is closed under colimits. We recall that coproducts are exact in $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ since it is a Grothendieck category; see [34, Corollary 2.8.9] for instance. Hence, for each object m in $\mathrm{Obj}(\mathfrak{M})_\natural$, the evanescence functor κ_m commutes with the coproducts of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$. Since the functor κ is the filtered colimit of the functors $\{\kappa_m\}_{m\in\mathrm{Obj}(\mathfrak{M})_\natural}$, we deduce that κ commutes with the coproducts of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$: therefore, the category $K(\mathfrak{M}, \mathscr{A})$ is closed under coproducts. As $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ is a Grothendieck category, any colimit of $K(\mathfrak{M}, \mathscr{A})$ may be expressed as a quotient of a coproduct. Since $K(\mathfrak{M}, \mathscr{A})$ is closed under quotient (see above), this category is thus closed under colimits. □

Remark 4.7. We see in the proof of Proposition 4.6 why we require the category $\mathscr{A}$ to have more properties than just being an abelian category: it is necessary to assume that the filtered colimits in the category $\mathscr{A}$ are exact, which is the case for a Grothendieck category. Actually, we could only assume that the category $\mathscr{A}$ is an $AB5$-category, ie a cocomplete abelian category in which filtered colimits of exact sequences are exact, and all the work of Section 4.2 extends mutatis mutandis. However, this more general notion is less standard than the one of Grothendieck category that we chose to use for sake of simplicity.

The thickness property of Proposition 4.6 ensures that we can consider the quotient category of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ by $K(\mathfrak{M}, \mathscr{A})$; see [20, Chapitre III]. Let $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ denote the quotient category $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})/K(\mathfrak{M}, \mathscr{A})$. We consider the canonical functor associated with this quotient by $\pi_\mathfrak{M} : \mathbf{Fct}(\mathfrak{M}, \mathscr{A}) \to \mathbf{St}(\mathfrak{M}, \mathscr{A})$. It is exact, essentially surjective and commutes with all colimits. Since the functor category $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ is a Grothendieck category, it follows from Proposition 4.6 that the subcategory $K(\mathfrak{M}, \mathscr{A})$ is *localizing* (see [20, Proposition III.4.8] or [35, 4.7.14, p.315]), in the sense that the functor $\pi_\mathfrak{M}$ admits a right adjoint functor denoted by $s_\mathfrak{M} : \mathbf{Fct}(\mathfrak{M}, \mathscr{A})/K(\mathfrak{M}, \mathscr{A}) \to \mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ and called the *section functor*.

The following proposition introduces the induced translation and difference functors on the category $\mathbf{St}(\mathfrak{M}, \mathscr{A})$. Its proof is verbatim that of [16, Proposition 2.19], using Proposition 4.2.

Proposition 4.8. *Let $m \in Obj(\mathfrak{M})_\natural$. The translation functor τ_m and the difference functor δ_m of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ respectively induce an exact endofunctor of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ which commute with colimits, again called the translation functor τ_m and the difference functor δ_m respectively. In addition:*

1. *The following relations hold: $\delta_m \circ \pi_\mathfrak{M} = \pi_\mathfrak{M} \circ \delta_m$ and $\tau_m \circ \pi_\mathfrak{M} = \pi_\mathfrak{M} \circ \tau_m$.*

2. *The exact sequence (20) induces a short exact sequence of endofunctors of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$:*

$$0 \longrightarrow Id \xrightarrow{i_m} \tau_m \xrightarrow{\Delta_m} \delta_m \longrightarrow 0. \tag{22}$$

3. *For another object m' of $\mathfrak{M}$, the endofunctors δ_m, $\delta_{m'}$, τ_m and $\tau_{m'}$ of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ pairwise commute up to natural isomorphism.*

We can now introduce the notion of a weak polynomial functor, extending that of [16, Definition 2.22].

Definition 4.9. We recursively define on $d \in \mathbb{N}$ the category $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ of polynomial functors of degree less than or equal to d to be the full subcategory of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ as follows:

1. If $d < 0$, the objects of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ are those of $K(\mathfrak{M}, \mathscr{A})$;

2. if $d \geq 0$, the objects of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ are the functors F such that the functor $\delta_x(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M}, \mathscr{A})$ for all $x \in \mathrm{Obj}(\mathfrak{M})_\natural$.

For an object F of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ which is polynomial of degree less than or equal to $d \in \mathbb{N}$, the smallest natural number $n \leq d$ for which F is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ is called the degree of F. An object F of $\mathbf{Fct}(\mathfrak{M}, \mathscr{A})$ is *weak polynomial* of degree at most d if its image $\pi_\mathfrak{M}(F)$ is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$. The degree of polynomiality of $\pi_\mathfrak{M}(F)$ is called the (weak) degree of F.

Let us give some important properties of the categories of weak polynomial functors used in Sections 5 and 6. Their proofs follow verbatim their analogues in [16, Section 2].

Proposition 4.10. *[16, Propositions 2.24-2.26] For d a natural number, the subcategory $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$ is thick and closed under limits and colimits. Furthermore, there is an equivalence of categories $\mathscr{A} \simeq \mathcal{P}ol_0(\mathfrak{M}, \mathscr{A})$.*

We assume that the category $\mathfrak{M}$ is finitely generated by the monoidal structure in $(\mathfrak{M}', \natural, 0)$ and we denote by E a finite generating set of $\mathfrak{M}$. Let F be an object of $\mathbf{St}(\mathfrak{M}, \mathscr{A})$. Then, the functor F is an object of $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ if and only if the functor $\delta_e(F)$ is an object of $\mathcal{P}ol_{d-1}(\mathfrak{M}, \mathscr{A})$ for all objects e of $E \cap Obj(\mathfrak{M})_\natural$.

5 Behaviour of the Long-Moody functors on polynomial functors

In this section, we study the effect of some generalized Long-Moody functors on (very) strong and weak polynomial functors. Indeed, under some additional assumptions, they have the property to increase by one both the very strong and the weak polynomial degrees; see Theorems 5.20 and 5.23.

For all the work of this section, we fix a *coherent* Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$; see Section 2.2. We recall that $\mathfrak{Gr}$ denotes the category of groups and that the free product is denoted by $*$. Let $\mathcal{G}'_{(0,1)}$ be the small full subgroupoid of $(\mathcal{G}', \natural, 0_{\mathcal{G}'})$ of the finite monoidal products on the objects $0_{\mathcal{G}'}$, 0 and 1 of $\mathcal{G}'$. Note that the monoidal structure $\natural$ restricts to give $\mathcal{G}'_{(0,1)}$ a braided monoidal structure. We assume that the functors $\mathcal{A}$ and χ satisfy the following additional properties:

Assumption 5.1. *The functor $\mathcal{A} : \mathfrak{U}\mathcal{G} \to \mathfrak{Gr}$ is the restriction of a functor $\mathfrak{U}\mathcal{G}' \to \mathfrak{Gr}$ (that we also denote by $\mathcal{A}$) along the canonical inclusion $\mathfrak{U}\mathcal{G} \hookrightarrow \mathfrak{U}\mathcal{G}'$, and there exist two groups H_0 and H (with H non-trivial), such that:*

- *for all objects $\underline{n}$ of $\mathcal{G}$, $\mathcal{A}(\underline{n}) = H^{*n} * H_0$ that we denote by H_n, and $\mathcal{A}(1^{\natural m}) = H^{*m}$ for all $m \in \mathbb{N}$.*

- *for all objects X and Y of $\mathcal{G}'_{(0,1)}$, $\mathcal{A}(\iota_X \natural id_Y) = \iota_{\mathcal{A}(X)} * id_{\mathcal{A}(Y)}$ and $\mathcal{A}(id_Y \natural \iota_X) = id_{\mathcal{A}(Y)} * \iota_{\mathcal{A}(X)}$, where $\iota_G : 0_{\mathfrak{Gr}} \to G$ denotes the unique morphism from $0_{\mathfrak{Gr}}$ to a group G.*

Moreover, the family of group morphisms $\{\chi_n : H_n \to G_{n+1}\}_{n \in \mathbb{N}}$ induced by χ satisfies the equality (13) of Proposition 2.14. Namely we assume that in G_{n+2}, for all elements $h \in H_n$:

$$((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}) \circ (id_1 \natural \chi_n(h)) \quad = \quad \chi_{n+1}((\iota_H * id_{H_n})(h)) \circ ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}). \tag{23}$$

Definition 5.2. *A coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is said to be reliable if Assumption 5.1 is satisfied.*

Consequences of Assumption 5.1 are heavily used in our study; see the proofs of Lemmas 5.4 and 5.13 and of Proposition 5.11. Some of the results presented in Section 5.1 still hold without the hypotheses of Assumption 5.1. However, these additional properties are necessary to prove some of the further results. In particular, we point out some relations which are used in the proof of Proposition 5.11.

Corollary 5.3. *Let m be a natural number and X be an object of $\mathcal{G}'_{(0,1)}$. We have:*

$$\mathcal{A}((b_{m,X}^{\mathcal{G}'})^{-1}) \circ (\iota_{\mathcal{A}(X)} * id_{H^{*m}}) = id_{H^{*m}} * \iota_{\mathcal{A}(X)}. \tag{24}$$

Also, for n another natural number and for all $g \in G_n$, we have

$$\mathcal{A}(id_m \natural g) \circ (id_{H^{*m}} * \iota_{H_n}) = id_{H^{*m}} * \iota_{H_n}. \tag{25}$$

Proof. We first recall from Assumption 5.1 that $\mathcal{A}(\iota_X \natural id_m) = \iota_{\mathcal{A}(X)} * id_{H^{*m}}$. Since $\mathfrak{U}\mathcal{G}'$ is a pre-braided monoidal category, we deduce from the relation (7) that $(b_{m,X}^{\mathcal{G}'})^{-1} \circ (\iota_X \natural id_m) = id_m \natural \iota_X$. Therefore, the equality (24) follows from the functoriality of $\mathcal{A}$ over the category $\mathfrak{U}\mathcal{G}'$. Furthermore, we recall that $0_{\mathcal{G}'}$ is a initial object in the category $\mathfrak{U}\mathcal{G}'$ and that $\iota_{\underline{n}}$ is the unique morphism from $0_{\mathcal{G}'}$ to $\underline{n}$ in $\mathfrak{U}\mathcal{G}'$. We deduce that $(id_m \natural g) \circ (id_m \natural \iota_{\underline{n}}) = id_m \natural (g \circ \iota_{\underline{n}}) = id_m \natural \iota_{\underline{n}}$. Hence, since $id_{H^{*m}} * \iota_{H_n} = \mathcal{A}(id_m \natural \iota_{\underline{n}})$, the functoriality of $\mathcal{A}$ over the category $\mathfrak{U}\mathcal{G}'$ gives the equality (25). $\square$

We assume that the fixed coherent Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ is reliable. Note that such functors $\mathcal{A}$ and χ always exist: we can at least consider the functor $\mathcal{A}_{id}$ defined assigning $\mathcal{A}(g) = id_{\mathcal{A}(\underline{n})}$ to

all $g \in G_n$ and all natural numbers n, and the trivial functor χ_{tr}. Also, we show in Section 6 that most of the coherent Long-Moody systems introduced in Section 3 are reliable.

We consider the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}}$ associated with the reliable Long-Moody system $\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}$, which is fixed throughout this section. In particular, we omit $\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}$ from the notation most of the time. Since the category $\mathfrak{UG}$ is generated by the objects $\underline{0}$ and 1 using the monoidal product $\natural$, it is enough for our work to only consider the translation functor τ_1 by Propositions 4.4 and 4.10.

5.1 Relation with evanescence and difference functors

In this section, we describe the decomposition of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}}$ with respect to the translation functor τ_1; see Corollary 5.16. We then establish the crucial results stated in Theorem 5.17, describing the behaviour of the Long-Moody functor $\mathbf{LM}_{\{\mathcal{A},\mathcal{G},\mathcal{G}',\chi\}}$ with respect to the evanescence and difference functors.

5.1.1 Factorization of the natural transformation $i_1\mathbf{LM}$ by $\mathbf{LM}(i_1)$

Recall from Proposition 4.2 the exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$, which defines the natural transformation i_1:

$$0 \longrightarrow \kappa_1 \xrightarrow{\ \Omega_1\ } Id \xrightarrow{\ i_1\ } \tau_1 \xrightarrow{\ \Delta_1\ } \delta_1 \longrightarrow 0 \ . \tag{26}$$

As we are interested in the effect of the considered Long-Moody functor $\mathbf{LM}$ on (very) strong and weak polynomial functors, our objective is to study the cokernel of the natural transformation $i_1\mathbf{LM} : \mathbf{LM} \to \tau_1 \circ \mathbf{LM}$. We recall from the relation (8) that $\iota_1\natural id_{\underline{n}} = [1, id_{\underline{n+1}}]$. Then, for F an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and for all natural numbers n, $i_1\mathbf{LM}$ is defined by the morphisms:

$$(i_1\mathbf{LM})(F)_{\underline{n}} = \mathbf{LM}(F)(\iota_1\natural id_{\underline{n}}) = \mathbf{LM}(F)([1, id_{\underline{n+1}}]) : \mathbf{LM}(F)(\underline{n}) \to \tau_1\mathbf{LM}(F)(\underline{n}).$$

Observe that, since the associated Long-Moody functor is right-exact by Proposition 2.5, we have the following exact sequence:

$$\mathbf{LM} \xrightarrow{\ \mathbf{LM}(i_1)\ } \mathbf{LM} \circ \tau_1 \xrightarrow{\ \mathbf{LM}(\Delta_1)\ } \mathbf{LM} \circ \delta_1 \longrightarrow 0 \ . \tag{27}$$

Moreover, if the groups H_0 and H are free, as the associated Long-Moody functor is then exact by Corollary 2.18, we note that the following sequence is exact:

$$0 \longrightarrow \mathbf{LM} \circ \kappa_1 \xrightarrow{\ \mathbf{LM}(\Omega_1)\ } \mathbf{LM} \xrightarrow{\ \mathbf{LM}(i_1)\ } \mathbf{LM} \circ \tau_1 \xrightarrow{\ \mathbf{LM}(\Delta_1)\ } \mathbf{LM} \circ \delta_1 \longrightarrow 0 \ . \tag{28}$$

First, we prove that the functor $i_1\mathbf{LM}$ factors through $\mathbf{LM}(i_1)$ via a natural transformation ξ defined as follows. Let F be an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and n be a natural number. We recall from the definition of a Long-Moody functor and of the translation endofunctor τ_1 that

$$(\mathbf{LM} \circ \tau_1)(F)(\underline{n}) = \mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} F(2\natural\underline{n}) \text{ and } (\tau_1 \circ \mathbf{LM})(F)(\underline{n}) = \mathcal{I}_{R[H_{n+1}]} \underset{R[H_{n+1}]}{\otimes} F(2\natural\underline{n}).$$

In particular, for $(\tau_1 \circ \mathbf{LM})(F)(\underline{n})$, the $R[G_{n+2}]$-module $F(2\natural\underline{n})$ is an $R[H_{n+1}]$-module via the composition $F \circ \chi_{n+1} : H_{n+1} \to G_{n+2} \to \mathrm{Aut}_R(F(2\natural\underline{n}))$. We also note that $F(2\natural\underline{n})$ is an $R[H_n]$-module using $F \circ (id_1\natural\chi_n(-)) : H_n \to G_{n+2} \to \mathrm{Aut}_R(F(2\natural\underline{n}))$ for $(\mathbf{LM} \circ \tau_1)(F)(\underline{n})$.

By Assumption 5.1, the augmentation ideal functor $\mathcal{I}_{R[\mathcal{A}]}$ defines the monomorphism $\mathcal{I}_{R[\mathcal{A}]}(\iota_1\natural id_{\underline{n}}) : \mathcal{I}_{R[H_n]} \hookrightarrow \mathcal{I}_{R[H_{n+1}]}$. We also consider the automorphism $F((b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}})$ of $F(2\natural\underline{n})$. Let $\otimes_{R[H_{n+1}]}$ denote the canonical $R[H_{n+1}]$-balanced projection from $\mathcal{I}_{R[H_{n+1}]} \times F(2\natural\underline{n})$ to $\mathcal{I}_{R[H_{n+1}]} \otimes_{R[H_{n+1}]} F(2\natural\underline{n})$. We denote by $\hat{\xi}(F)_{\underline{n}}$ the composition $\otimes_{R[H_{n+1}]} \circ (\mathcal{I}_{R[\mathcal{A}]}(\iota_1\natural id_{\underline{n}}) \times F((b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}}))$.

Lemma 5.4. *The morphism $\hat{\xi}(F)_{\underline{n}}$ is $R[H_n]$-balanced.*

Proof. We first note that it is enough to check that $\hat{\xi}(F)_{\underline{n}}$ is H_n-balanced, the result for $R[H_n]$ following by linearity. We fix $h \in H_n$, $i \in \mathcal{I}_{R[H_n]}$ and $v \in F(2\natural\underline{n})$. We deduce from the condition on $\mathcal{A}(\iota_1\natural id_{\underline{n}})$ of Assumption 5.1 that

$$\mathcal{I}_{R[\mathcal{A}]}(\iota_1\natural id_{\underline{n}})(i \cdot h) = \mathcal{I}_{R[\mathcal{A}]}(\iota_1\natural id_{\underline{n}})(i) \cdot (\iota_H * id_{H_n})(h).$$

Then, it follows from the functoriality of F and from the relation (23) of Assumption 5.1 that:

$$
\begin{aligned}
\hat{\xi}(F)_{\underline{n}}(i \cdot h \underset{R[H_n]}{\otimes} v) &= \mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n}})(i) \underset{R[H_{n+1}]}{\otimes} F(\chi_{n+1}((\iota_H * id_{H_n})(h))) \circ F((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})(v) \\
&= \mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n}})(i) \underset{R[H_{n+1}]}{\otimes} F((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}) \circ F(id_1 \natural \chi_n(h))(v) \\
&= \hat{\xi}(F)_{\underline{n}}(i \underset{R[H_n]}{\otimes} \tau_1 F(\chi_n(h))(v)),
\end{aligned}
$$

which ends the proof. $\qquad\square$

Remark 5.5. We stress that the condition (23) of Assumption 5.1 is required for Lemma 5.4.

It follows from Lemma 5.4 that the universal property of the tensor product over $R[H_n]$ defines a unique morphism $\xi(F)_{\underline{n}}$ from $(\mathbf{LM} \circ \tau_1)(F)(\underline{n})$ to $\tau_1 \mathbf{LM}(F)(\underline{n})$ for each $n \in \mathbb{N}$. Actually, there is a more explicit description of the morphism $\xi(F)_{\underline{n}}$: it is a routine to check that the conjugation of $\xi(F)_{\underline{n}}$ by the canonical isomorphism from $\mathcal{I}_{R[H_n]} \otimes_{R[H_n]} F(2\natural\underline{n})$ to $(\mathcal{I}_{R[H_n]} \otimes_{R[H_n]} R[H * H_n]) \otimes_{R[H_{n+1}]} F(2\natural\underline{n})$ is equal to the morphism

$$
(\mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n}}) \underset{R[H_n]}{\otimes} id_{R[H * H_n]}) \underset{R[H_{n+1}]}{\otimes} F((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}). \tag{29}
$$

This explicit definition allows in particular to prove the following property:

Lemma 5.6. *The morphism $\xi(F)_{\underline{n}}$ is a monomorphism for each n.*

Proof. We recall that $t_G : G \to 0_{\mathfrak{Gr}}$ denotes the unique morphism from the group G to $0_{\mathfrak{Gr}}$. We denote by $\mathcal{I}_{R[\mathcal{A}]}^{-1}(\iota_1 \natural id_{\underline{n}}) : \mathcal{I}_{R[H_{n+1}]} \twoheadrightarrow \mathcal{I}_{R[H_n]}$ the R-module surjection induced by the group surjection $t_H * id_{H_n} : H * H_n \twoheadrightarrow H_n$. Since $\mathcal{I}_{R[\mathcal{A}]}^{-1}(\iota_1 \natural id_{\underline{n}})$ is a left inverse for $\mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n}})$, we deduce that the morphism $(\mathcal{I}_{R[\mathcal{A}]}^{-1}(\iota_1 \natural id_{\underline{n}}) \otimes_{R[H_n]} id_{R[H * H_n]}) \otimes_{R[H_{n+1}]} F(b_{1,1}^{\mathcal{G}'} \natural id_{\underline{n}})$ defines a left inverse for $\xi(F)_{\underline{n}}$. $\qquad\square$

The morphisms of the form $\xi(F)_{\underline{n}}$ are the key to define the natural transformation ξ:

Proposition 5.7. *The morphisms $\{\xi(F)_{\underline{n}}\}$ define a natural transformation $\xi(F) : (\mathbf{LM} \circ \tau_1)(F) \to (\tau_1 \circ \mathbf{LM})(F)$. Then this yields a natural transformation $\xi : \mathbf{LM} \circ \tau_1 \to \tau_1 \circ \mathbf{LM}$.*

Proof. Let n and n' be natural numbers such that $n' \geq n$, and $[n'-n, g] \in \mathrm{Hom}_{\mathfrak{UG}}(\underline{n}, \underline{n'})$. Since $\mathfrak{s}_{\mathcal{A}}^* \circ \chi^* = 1\natural-$ as endofunctors of $\mathfrak{UG}$, we recall from Lemma 2.4 that for all $i \in \mathcal{I}_{R[H_n]}$ and $v \in F(2\natural\underline{n})$:

$$
(\mathbf{LM} \circ \tau_1)(F)([n'-n, g])(i \underset{R[H_n]}{\otimes} v) = \mathcal{I}_{R[\mathcal{A}]}([n'-n, g])(i) \underset{R[H_{n'}]}{\otimes} F(id_1 \natural id_1 \natural [n'-n, g])(v),
$$

$$
(\tau_1 \circ \mathbf{LM})(F)([n'-n, g])(i \underset{R[H_n]}{\otimes} v) = \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural [n'-n, g])(i) \underset{R[H_{n'+1}]}{\otimes} F(id_1 \natural id_1 \natural [n'-n, g])(v).
$$

It follows from the defining equivalence relation (4) of $\mathfrak{UG}'$ that $[1+n'-n, (b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}] = [1+n'-n, id_{n'+1}]$. We then compute that $(id_1 \natural [n'-n, g]) \circ (\iota_1 \natural id_{\underline{n}}) = [1 + n' - n, id_1 \natural g]$. Since $\mathcal{I}_{R[\mathcal{A}]}$ is a functor over $\mathfrak{UG}'$, we deduce that:

$$
\mathcal{I}_{R[\mathcal{A}]}(id_1 \natural [n'-n, g]) \circ (\iota_1 \natural id_{\underline{n}}) = \mathcal{I}_{R[\mathcal{A}]}([1 + n' - n, id_1 \natural g]) = \mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n'}}) \circ \mathcal{I}_{R[\mathcal{A}]}([n'-n, g]).
$$

Also, as a consequence of the properties of a monoidal structure, we compute that:

$$
(id_2 \natural [n'-n, g]) \circ ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}) = (b_{1,1}^{\mathcal{G}'})^{-1} \natural [n'-n, g] = ((b_{1,1}^{\mathcal{G}'})^{-1} \natural id_{\underline{n'}}) \circ (id_2 \natural [n'-n, g]).
$$

Hence, we deduce from the definitions that for all $i \in \mathcal{I}_{R[H_n]}$ and $v \in F(2\natural\underline{n})$

$$
\begin{aligned}
((\tau_1 \circ \mathbf{LM})(F)([n'-n, g])) \circ \xi(F)_{\underline{n}}(i \underset{R[H_n]}{\otimes} v) &= \mathcal{I}_{R[\mathcal{A}]}([1 + n' - n, id_1 \natural g])(i) \underset{R[H_{n'+1}]}{\otimes} F((b_{1,1}^{\mathcal{G}'})^{-1} \natural [n'-n, g])(v) \\
&= \xi(F)_{\underline{n'}} \circ ((\mathbf{LM} \circ \tau_1)(F)([n'-n, g]))(i \underset{R[H_n]}{\otimes} v).
\end{aligned}
$$

Therefore, $\xi(F)$ is a natural transformation from $(\tau_1 \circ \mathbf{LM})(F)$ to $(\mathbf{LM} \circ \tau_1)(F)$. Let us now check that assembling all these natural transformations for all the objects of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ defines a natural transformation from $\tau_1 \circ \mathbf{LM}$ to $\mathbf{LM} \circ \tau_1$. Let M and N be two objects of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and $\eta : M \to N$ be a natural transformation. It follows from Lemma 2.4 that the natural transformations $(\mathbf{LM} \circ \tau_1)(\eta)$ and $(\tau_1 \circ \mathbf{LM})(\eta)$ are respectively given by:

$$(\mathbf{LM} \circ \tau_1)(\eta)_{\underline{n}} = id_{\mathcal{I}_{R[H_n]}} \underset{R[H_n]}{\otimes} \eta_{2\natural\underline{n}} \text{ and } (\tau_1 \circ \mathbf{LM})(\eta)_{\underline{n}} = id_{\mathcal{I}_{R[H_{1+n}]}} \underset{R[H_{1+n}]}{\otimes} \eta_{2\natural\underline{n}}$$

for all natural numbers n. Since η is a natural transformation, we have $N((b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}}) \circ \eta_{2\natural\underline{n}} = \eta_{2\natural\underline{n}} \circ M((b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}})$ and we deduce that $\xi(N)_{\underline{n}} \circ (\mathbf{LM} \circ \tau_1)(\eta)_{\underline{n}} = (\tau_1 \circ \mathbf{LM})(\eta)_{\underline{n}} \circ \xi(M)_{\underline{n}}$. Hence $\xi(M) \circ (\mathbf{LM} \circ \tau_1)(\eta)$ and $(\tau_1 \circ \mathbf{LM})(\eta) \circ \xi(N)$ are equal as natural transformations from $(\mathbf{LM} \circ \tau_1)(M)$ to $(\mathbf{LM} \circ \tau_1)(N)$, which ends the proof. $\qquad\square$

Finally, we prove that the natural transformation $i_1\mathbf{LM}$ factors across $\mathbf{LM}(i_1)$. As a result, we obtain that the endofunctor $\mathbf{LM} \circ \delta_1$ is a subfunctor of $\delta_1 \circ \mathbf{LM}$; see diagram (30).

Proposition 5.8. *As natural transformations from* $\mathbf{LM}$ *to* $\tau_1 \circ \mathbf{LM}$*, the following equality holds:*

$$\xi \circ (\mathbf{LM}(i_1)) = i_1\mathbf{LM}.$$

Moreover, there exists a unique natural transformation $\mathbf{LM} \circ \delta_1 \to \delta_1 \circ \mathbf{LM}$ *such that the following diagram is commutative and the rows are exact sequences in the category of endofunctors of* $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{LM} \circ \tau_1 & \xrightarrow{\xi} & \tau_1 \circ \mathbf{LM} & \xrightarrow{\Upsilon} & Coker(\xi) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbf{LM} \circ \delta_1 & \longrightarrow & \delta_1 \circ \mathbf{LM} & \longrightarrow & Coker(\xi) & \longrightarrow & 0.
\end{array}
\tag{30}
$$

Proof. Let F be an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and n be a natural number. It follows from the definition of a Long-Moody functor on morphisms and natural transformations (see Lemma 2.4) and the definition of the natural transformation i_1 (see Section 4.1) that

$$\mathbf{LM}(i_1F)_{\underline{n}} = id_{\mathcal{I}_{R[H_n]}} \underset{R[H_{1+n}]}{\otimes} F(\iota_1\natural id_1\natural id_{\underline{n}}),$$

$$(i_1\mathbf{LM})(F)_{\underline{n}} = \mathbf{LM}(F)(\iota_1\natural id_{\underline{n}}) = \mathcal{I}_{\mathcal{A}}(\iota_1\natural id_{\underline{n}}) \underset{R[H_{1+n}]}{\otimes} F(id_1\natural\iota_1\natural id_{\underline{n}}).$$

Since $\mathfrak{UG}'$ is pre-braided, we have $(b_{1,1}^{\mathcal{G}'})^{-1} \circ (\iota_1\natural id_1) = id_1\natural\iota_1$ by the relation (7) and we deduce from the definition of ξ that $\xi(F)_{\underline{n}} \circ \mathbf{LM}(i_1F)_{\underline{n}} = (i_1\mathbf{LM})(F)_{\underline{n}}$. Hence the following diagram is commutative for any object F of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$

$$
\begin{array}{ccc}
& & \mathbf{LM}(F) \\
{\scriptstyle(i_1\mathbf{LM})(F)}\Big\downarrow & \overset{\mathbf{LM}(i_1F)}{\searrow} & \\
(\mathbf{LM} \circ \tau_1)(F) & \xrightarrow[\xi(F)]{} & (\tau_1 \circ \mathbf{LM})(F),
\end{array}
$$

which proves the first statement. Then, we recall that for two morphisms f and g of an abelian category such that the target of f is the source of g and g is a monomorphism, the snake lemma provides a short exact sequence for the cokernels $0 \to \mathrm{Coker}(f) \to \mathrm{Coker}(g \circ f) \to \mathrm{Coker}(g) \to 0$. Since $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ is an abelian category, the bottom short exact sequence of (30) is an instance of that result assigning $f = \mathbf{LM}(i_1)$ and $g = \xi$ by using the equality $\xi \circ (\mathbf{LM}(i_1)) = i_1\mathbf{LM}$ and Lemma (5.6). Finally, in the diagram (30), the top short exact sequence is obtained from the definition of ξ and Lemma (5.6), and the commutativities for the two squares formally follow from the definitions of the arrows of the bottom short exact sequence. $\qquad\square$

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5.1.2   Study of Coker(ξ)

By Proposition 5.8, we know that the endofunctor $\tau_1 \circ \mathbf{LM}$ is an extension of Coker(ξ) by $\mathbf{LM} \circ \tau_1$; see diagram (30). In this section, we prove that this extension is actually trivial; see Corollary 5.16. For this purpose, we first make the following observations.

Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and n be a natural number. We recall a classical decomposition for the augmentation ideal of a free group, which may be found in [41, Proposition 6.2.9] (stated for $R = \mathbb{Z}$, but which can be straightforwardly generalized to any commutative ring R) or in [14, Section 4, Lemma 4.3 and Theorem 4.7] (we note that these statements are done for groups which are not necessarily of cohomological dimension at most 1). For two groups A and B, there is a right $R[A * B]$-module isomorphism:

$$(\mathcal{I}_{R[A]} \underset{R[A]}{\otimes} R[A * B]) \oplus (\mathcal{I}_{R[B]} \underset{R[B]}{\otimes} R[A * B]) \cong \mathcal{I}_{R[A*B]}. \tag{31}$$

where $R[A * B]$ is an $R[A]$-module via the canonical inclusion $id_A * \iota_B : A \to A * B$ and an $R[B]$-module via $\iota_A * id_B : B \to A * B$. We recall that $t_G : G \to 0_{\mathfrak{Gr}}$ denotes the unique morphism from the group G to $0_{\mathfrak{Gr}}$. We denote by $\mathcal{I}(id_A * \iota_B) : \mathcal{I}_{R[A]} \hookrightarrow \mathcal{I}_{R[A*B]}$ and $\mathcal{I}(\iota_A * id_B) : \mathcal{I}_{R[B]} \hookrightarrow \mathcal{I}_{R[A*B]}$ the R-module injections induced by the group injections $id_A * \iota_B : A \hookrightarrow A * B$ and $\iota_A * id_B : B \hookrightarrow A * B$. The isomorphism (31) is explicitly defined by the direct sum of the injections $\mathcal{I}(id_A * \iota_B) \otimes_{R[A]} id_{R[A*B]}$ and $\mathcal{I}(\iota_A * id_B) \otimes_{R[B]} id_{R[A*B]}$; we refer the reader to [41, Proposition 6.2.9] for further details.

Applying the isomorphism (31) provides the following Lemma 5.10. Before this, we recall and introduce some notations:

Notation 5.9. Let n and n' be natural numbers such that $n' \geq n$. We recall from Assumption 5.1 that the functor $\mathcal{A}$ is defined over the category $\mathfrak{U}\mathcal{G}'$ and that $\mathcal{A}(id_n \natural \iota_{n'-n})$ is equal to $id_{H^{*n}} * \iota_{H_{n'-n}}$. Also, by definition (see Section 2.1), the augmentation ideal functor $\mathcal{I}_{R[\mathcal{A}]}$ is defined over the category $\mathfrak{U}\mathcal{G}'$ and in particular defines the R-module injection $\mathcal{I}_{R[\mathcal{A}]}(id_n \natural \iota_{n'-n}) : \mathcal{I}_{R[H^{*n}]} \hookrightarrow \mathcal{I}_{R[H_{n'}]}$ induced by the group injection $id_{H^{*n}} * \iota_{H_{n'-n}}$. We denote by $\mathcal{I}_{R[\mathcal{A}]}^{-1}(id_n \natural \iota_{n'-n}) : \mathcal{I}_{R[H_{n'}]} \twoheadrightarrow \mathcal{I}_{R[H^{*n}]}$ the R-module surjection induced by the group surjection $id_{H^{*n}} * t_{H_{n'-n}} : H_{n'} \twoheadrightarrow H^{*n}$.

Lemma 5.10. *The direct sum of* $(\mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}) \otimes_{R[H]} id_{R[H*H_n]}) \otimes_{R[H_{1+n}]} id_{F(2\natural \underline{n})}$ *and* $\xi(F)_{\underline{n}}$ *provides an R-module isomorphism:*

$$((\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} R[H * H_n]) \underset{R[H_{n+1}]}{\otimes} F(2\natural \underline{n})) \oplus (\mathbf{LM} \circ \tau_1)(F)(\underline{n}) \xrightarrow{\cong} \tau_1 \mathbf{LM}(F)(\underline{n}). \tag{32}$$

Proof. We deduce from the isomorphism (31), from the definition of $\tau_1 \mathbf{LM}(F)(\underline{n})$ and from the distributivity of the tensor product that the direct sum of the morphisms $(\mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}) \otimes_{R[H]} id_{R[H*H_n]}) \otimes_{R[H_{n+1}]} id_{F(2\natural \underline{n})}$ and $(\mathcal{I}_{R[\mathcal{A}]}(\iota_1 \natural id_{\underline{n}}) \otimes_{R[H_n]} id_{R[H*H_n]}) \otimes_{R[H_{n+1}]} id_{F(2\natural \underline{n})}$ defines an R-module isomorphism:

$$\left((\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} R[H * H_n]) \underset{R[H_{n+1}]}{\otimes} F(2\natural \underline{n})\right) \oplus \left((\mathcal{I}_{R[H_n]} \underset{R[H_n]}{\otimes} R[H * H_n]) \underset{R[H_{n+1}]}{\otimes} F(2\natural \underline{n})\right) \xrightarrow{\cong} \tau_1 \mathbf{LM}(F)(\underline{n}).$$

Also, precomposing the right-hand summand with the automorphism $(id_{R[H_n]} \otimes_{R[H_n]} id_{R[H*H_n]}) \otimes_{R[H_{n+1}]} F((b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}})$, the injection associated with the right hand summand is equal to the morphism (29). Hence, using the canonical isomorphism between $\mathcal{I}_{R[H_n]} \otimes_{R[H_n]} F(2\natural \underline{n})$ and $(\mathcal{I}_{R[H_n]} \otimes_{R[H_n]} R[H*H_n]) \otimes_{R[H_{n+1}]} F(2\natural \underline{n})$, we identify the right-hand summand with the R-module $(\mathbf{LM} \circ \tau_1)(F)(\underline{n})$ and associated injection $\xi(F)_{\underline{n}}$. $\square$

It follows from Lemma 5.10 and from the universal property of a cokernel that the R-module Coker$(\xi)(F(\underline{n}))$ is isomorphic to the left hand summand of (32). We consider the natural transformation $\Upsilon : \tau_1 \circ \mathbf{LM} \to$ Coker(ξ) of Proposition 5.8. We deduce from the isomorphism (32) that the morphism $\Upsilon(F)_{\underline{n}}$ is equal to $(\mathcal{I}_{R[\mathcal{A}]}^{-1}(id_1 \natural \iota_{\underline{n}}) \otimes_{R[H]} id_{R[H*H_n]}) \otimes_{R[H_{1+n}]} id_{F(2\natural \underline{n})}$. This leads ineluctably to wonder if the left-hand side of the isomorphism (32) is a direct sum of two endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.

Identification with a translation functor. We aim at identifying the left-hand summand the isomorphism (32) as the defining input of a certain endofunctor of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ for each object F of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and each natural number n. Beforehand, we highlight the following technical relation, which is used in the proof of Proposition 5.12. We recall that e_G denotes the unit element of a group G.

Proposition 5.11. *Let n and n' be natural numbers such that $n' \geq n$, let $[n' - n, g] \in Hom_{\mathfrak{UG}}(\underline{n}, \underline{n'})$ and let $h \in H$. Then:*

$$\chi_{n'+1}(h * e_{H_{n'}}) \circ (id_2 \natural [n' - n, g]) = (id_2 \natural [n' - n, g]) \circ \chi_{n+1}(h * e_{H_n}). \tag{33}$$

Proof. We deduce from the relation (23) that the following square of the groupoid $\mathcal{G}$ is commutative:

$$
\begin{array}{ccc}
2 \natural \underline{n'} & \xrightarrow{id_{n'-n-k} \natural \chi_{n+k}(e_{H*k-1}*h*e_{H_n})} & 2 \natural \underline{n'} \\
{\scriptstyle id_{n'-n-k}\natural(b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{k+n}} \downarrow & & \downarrow {\scriptstyle id_{n'-n-k}\natural(b_{1,1}^{\mathcal{G}'})^{-1}\natural id_{k+n}} \\
2 \natural \underline{n'} & \xrightarrow[id_{n'-n-k} \natural \chi_{n+k+1}(e_{H*k}*h*e_{H_n})]{} & 2 \natural \underline{n'}
\end{array}
$$

for each $1 \leq k \leq n' - n$. Composing the above squares for all $1 \leq k \leq n' - n$ and recalling that we have

$$b_{1,n'-n}^{\mathcal{G}'} = (id_{n'-n-1} \natural b_{1,1}^{\mathcal{G}'}) \circ (id_{n'-n-2} \natural b_{1,1}^{\mathcal{G}'} \natural id_1) \circ \cdots \circ (id_1 \natural b_{1,1}^{\mathcal{G}'} \natural id_{n'-n-2}) \circ (b_{1,1}^{\mathcal{G}'} \natural id_{n'-n-1})$$

by the definition of a braiding, we deduce the following equality:

$$((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_1 \natural id_{\underline{n}}) \circ (id_{n'-n} \natural \chi_{n+1}(h * e_{H_n})) = \chi_{n'+1}(e_{H*n'-n} * h * e_{H_n}) \circ ((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_1 \natural id_{\underline{n}}) \tag{34}$$

Furthermore, it follows from the relation (11) induced by Lemma 2.12 that

$$
\begin{aligned}
& (id_1 \natural ((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})) \circ \chi_{n'+1}(e_{H*n'-n} * h * e_{H_n}) \\
= \ & \chi_{n'+1}(\mathcal{A}((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})(e_{H*n'-n} * h * e_{H_n})) \circ (id_1 \natural ((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})).
\end{aligned}
$$

We deduce from the relation (24) that $\mathcal{A}((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})(e_{H*n'-n} * h * e_{H_n}) = h * e_{H_{n'}}$ and thus note that

$$(id_1 \natural ((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})) \circ \chi_{n'+1}(e_{H*n'-n} * h * e_{H_n}) = \chi_{n'+1}(h * e_{H_{n'}}) \circ (id_1 \natural ((b_{1,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}})). \tag{35}$$

We recall from the definition of the monoidal structure in $\mathfrak{UG}$ (see the relation (8)) that:

$$(id_2 \natural [n' - n, id_{\underline{n'}}]) \circ \chi_{n+1}(h * e_{H_n}) = [n' - n, ((b_{2,n'-n}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}) \circ (id_{n'-n} \natural \chi_{n+1}(h * e_{H_n})].$$

By the definition of a braiding, we have $b_{2,n'-n}^{\mathcal{G}'} = (b_{1,n'-n}^{\mathcal{G}'} \natural id_1) \circ (id_1 \natural b_{1,n'-n}^{\mathcal{G}'})$. Therefore, using the relations (34) and (35), we prove the following key equality as morphisms in $\mathfrak{UG}$:

$$(id_2 \natural [n' - n, id_{\underline{n'}}]) \circ \chi_{n+1}(h * e_{H_n}) = \chi_{n'+1}(h * e_{H_{n'}}) \circ (id_2 \natural [n' - n, id_{\underline{n'}}]). \tag{36}$$

On another note, since $(id_2 \natural g) \circ \chi_{n'+1}(h * e_{H_{n'}}) = \chi_{n'+1}(\mathcal{A}(id_1 \natural g)(h * e_{H_{n'}})) \circ (id_2 \natural g)$ by the relation (11) induced from Lemma 2.12 , we deduce from the relation (25) that

$$(id_2 \natural g) \circ \chi_{n'+1}(h * e_{H_{n'}}) = \chi_{n'+1}(h * e_{H_{n'}}) \circ (id_2 \natural g). \tag{37}$$

Finally, the equality (33) follows from the combination of the relations (36) and (37) since $id_2 \natural [n' - n, g] = (id_2 \natural g) \circ [n' - n, id_{\underline{n'}}]$. $\qquad \square$

We now introduce the endofunctor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ to which we will identify $\mathrm{Coker}(\xi)$. For each object F of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$, we assign $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)(\underline{n})$ to be $\mathcal{I}_{R[H]} \otimes_{R[H]} (\tau_2 F)(\underline{n})$ for each $n \in \mathbb{N}$. In particular, $(\tau_2 F)(\underline{n}) = F(2 \natural \underline{n})$ is an $R[H]$-module via the composition $H \to G_{n+2} \to \mathrm{Aut}_R(F(2 \natural \underline{n}))$ defined by $F \circ \chi_{n+1} \circ (id_H * \iota_{H_n})$. For each morphism $[n' - n, g]$ of $\mathfrak{UG}$, we assign $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)([n' - n, g])$ to be the tensor product morphism $id_{\mathcal{I}_{R[H]}} \otimes_{R[H]} F(id_2 \natural [n' - n, g])$. Finally, for each natural transformation $\eta : F \to G$ of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$, we assign $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(\eta)$ to be the natural transformation defined by $id_{\mathcal{I}_{R[H]}} \otimes_{R[H]} \eta_{2\natural\underline{n}}$ for each $n \in \mathbb{N}$.

Proposition 5.12. *The above assignments define an endofunctor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$.*

Proof. Let us prove that $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 F$ is an object of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ for any functor $F : \mathfrak{UG} \to R\text{-}\mathfrak{Mod}$. First, we check that the assignment $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)([n'-n,g])$ is well-defined with respect to the tensor product over $R[H]$ for any morphism $[n'-n,g]$ of $\mathfrak{UG}$ as follows. We fix $h \in H$, $i \in \mathcal{I}_{R[H_n]}$ and $v \in F(2\natural\underline{n})$. Then, it follows from the relation (33) of Proposition 5.11 and from the functoriality of F that:

$$
\begin{aligned}
(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 F)([n'-n,g])(i \cdot h \underset{R[H]}{\otimes} v) &= i \underset{R[H]}{\otimes} F(\chi_{n'+1}(h * e_{H_{n'}})) \circ F(id_2\natural[n'-n,g])(v) \\
&= i \underset{R[H]}{\otimes} F(id_2\natural[n'-n,g]) \circ F(\chi_{n+1}(h * e_{H_n}))(v) \\
&= (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2 F)([n'-n,g])(i \underset{R[H]}{\otimes} F(\chi_{n+1}(h * e_{H_n}))(v)),
\end{aligned}
$$

which proves the consistency of the assignment on morphisms with respect to the tensor product structure. Then, it is clear from the functoriality of F that the identity and composition axioms for the morphisms are satisfied by the assignments defining $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)$, which is therefore a functor $\mathfrak{UG} \to R\text{-}\mathfrak{Mod}$.

Furthermore, the fact that $\eta : F \to G$ is a natural transformation directly implies that the assignment for $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(\eta)$ defines a natural transformation from $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)$ to $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(G)$. Also, the identity and composition axioms for the natural transformations (ie the morphisms of the category $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$) are straightforwardly checked from the definitions. $\square$

Let us now introduce a natural transformation from $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ to $\tau_1 \circ \mathbf{LM}$. We fix an object F of $\mathbf{Fct}(\mathfrak{UG}, R\text{-}\mathfrak{Mod})$ and a natural number n. We consider the R-module injection $\mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}}) : \mathcal{I}_{R[H]} \hookrightarrow \mathcal{I}_{R[H_{n+1}]}$ recalled in Notation 5.9. We recall that $\otimes_{R[H_{n+1}]}$ denotes the canonical $R[H_{n+1}]$-balanced projection from $\mathcal{I}_{R[H_{n+1}]} \times F(2\natural\underline{n})$ to $\mathcal{I}_{R[H_{n+1}]} \otimes_{R[H_{n+1}]} F(2\natural\underline{n})$. In particular, we recall that the $R[H_{n+1}]$-module structure of $F(2\natural\underline{n})$ is defined by the composition $F \circ \chi_{n+1} : H_{n+1} \to G_{n+2} \to \mathrm{Aut}_R(F(2\natural\underline{n}))$. We denote by $\hat{v}(F)_{\underline{n}}$ the composition $\otimes_{R[H_{n+1}]} \circ (\mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}}) \times id_{F(2\natural\underline{n})})$.

Lemma 5.13. *The morphism $\hat{v}(F)_{\underline{n}}$ is $R[H]$-balanced.*

Proof. Again, it is enough to check that $\hat{v}(F)_{\underline{n}}$ is H-balanced, the result for $R[H]$ following by linearity. We fix $h \in H$, $i \in \mathcal{I}_{R[H]}$ and $v \in F(2\natural\underline{n})$. We deduce from the condition on $\mathcal{A}(id_1\natural\iota_{\underline{n}})$ of Assumption 5.1 that $\mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}})(i \cdot h) = \mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}})(i) \cdot (h * e_{H_n})$. It then follows from the definitions that:

$$
\hat{v}(F)_{\underline{n}}(i \cdot h \underset{R[H]}{\otimes} v) = \mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}})(i) \underset{R[H_n]}{\otimes} F(\chi_{n+1}(h * e_{H_n}))(v) = \hat{v}(F)_{\underline{n}}(i \underset{R[H]}{\otimes} F(\chi_{n+1}(h * e_{H_n}))(v))
$$

which ends the proof. $\square$

It follows from Lemma 5.13 that the universal property of the tensor product over $R[H]$ defines a unique morphism $v(F)_{\underline{n}} : \mathcal{I}_{R[H]} \otimes_{R[H]} F(2\natural\underline{n}) \to \tau_1 \mathbf{LM}(F)(\underline{n})$ for each $n \in \mathbb{N}$. It is a routine to check that the conjugation of $v(F)_{\underline{n}}$ by the canonical isomorphism from $(\mathcal{I}_{R[H]} \otimes_{R[H]} R[H * H_n]) \otimes_{R[H_{n+1}]} F(2\natural\underline{n})$ to $\mathcal{I}_{R[H]} \otimes_{R[H]} F(2\natural\underline{n})$ is equal to the morphism $(\mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}}) \otimes_{R[H]} id_{R[H * H_n]}) \otimes_{R[H_{1+n}]} id_{F(2\natural\underline{n})}$ (used in Lemma 5.10). We can now introduce the natural transformation v:

Proposition 5.14. *We define a natural transformation $v(F) : \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 F \to (\tau_1 \circ \mathbf{LM})(F)$ from the monomorphisms $\{v(F)_{\underline{n}}\}_{n \in \mathbb{N}}$. This yields a natural transformation $v : \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \to \tau_1 \circ \mathbf{LM}$.*

Proof. Let n and n' be natural numbers such that $n' \geq n$, and $[n'-n,g] \in \mathrm{Hom}_{\mathfrak{UG}}(\underline{n}, \underline{n'})$. Since $\mathfrak{s}_{\mathcal{A}}^* \circ \chi^* = 1\natural-$ as endofunctors of $\mathfrak{UG}$, we recall from the definitions that for all $i \in \mathcal{I}_{R[H]}$ and $v \in F(2\natural\underline{n})$:

$$
(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(F)([n'-n,g])(i \underset{R[H]}{\otimes} v) = i \underset{R[H]}{\otimes} F(id_1\natural id_1\natural[n'-n,g])(v),
$$

$$
(\tau_1 \circ \mathbf{LM})(F)([n'-n,g])(i \underset{R[H]}{\otimes} v) = \mathcal{I}_{R[\mathcal{A}]}(id_1\natural[n'-n,g])(i) \underset{R[H_{n'+1}]}{\otimes} F(id_1\natural id_1\natural[n'-n,g])(v).
$$

We recall from the definition of the monoidal structure in $\mathfrak{UG}$ (see the relation (8)) that $id_1\natural[n'-n,g] = (id_1\natural g) \circ ((b_{1,n'-n}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}}) \circ (\iota_{n'-n}\natural id_{n+1})$. First, we deduce from the relation (7) that $((b_{1,n'-n}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}}) \circ (\iota_{n'-n}\natural id_1\natural\iota_{\underline{n}}) = id_1\natural\iota_{\underline{n'}}$. Then, it follows from the fact that $\mathcal{I}_{R[\mathcal{A}]}$ is a functor over $\mathfrak{UG}'$ that:

$$
\mathcal{I}_{R[\mathcal{A}]}((b_{1,n'-n}^{\mathcal{G}'})^{-1}\natural id_{\underline{n}}) \circ \mathcal{I}_{R[\mathcal{A}]}(\iota_{n'-n}\natural id_{n+1}) \circ \mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n}}) = \mathcal{I}_{R[\mathcal{A}]}(id_1\natural\iota_{\underline{n'}}). \tag{38}
$$

Furthermore, we recall that $(id_1 \natural g) \circ (id_1 \natural \iota_{\underline{n}}) = id_m \natural (g \circ \iota_{\underline{n}}) = id_1 \natural \iota_{\underline{n}}$ since $0_{\mathcal{G}'}$ is a initial object in the category $\mathfrak{U}\mathcal{G}'$ and that $\iota_{\underline{n}}$ is the unique morphism from $0_{\mathcal{G}'}$ to $\underline{n}$ in $\mathfrak{U}\mathcal{G}'$. We deduce from the functoriality of $\mathcal{I}_{R[\mathcal{A}]}$ over $\mathfrak{U}\mathcal{G}'$ that:

$$\mathcal{I}_{R[\mathcal{A}]}(id_1 \natural g) \circ \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}) = \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}). \tag{39}$$

Then, combining the relations (38) and (39), we prove the following key equality as morphisms in $\mathfrak{U}\mathcal{G}'$:

$$\mathcal{I}_{R[\mathcal{A}]}(id_1 \natural [n'-n, g]) \circ \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}) = \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}). \tag{40}$$

Therefore, we deduce from the definitions and from the equality (40) that for all $i \in \mathcal{I}_{R[H_n]}$ and $v \in F(2 \natural \underline{n})$:

$$
\begin{aligned}
((\tau_1 \circ \mathbf{LM})(F)([n'-n, g])) \circ \upsilon(F)_{\underline{n}}(i \underset{R[H_n]}{\otimes} v) &= \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}})(i) \underset{R[H_{n'+1}]}{\otimes} F(id_1 \natural id_1 \natural [n'-n, g])(v) \\
&= \upsilon(F)_{\underline{n'}} \circ ((\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)([n'-n, g]))(i \underset{R[H_n]}{\otimes} v).
\end{aligned}
$$

Hence, $\upsilon(F)$ is a natural transformation from $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 F$ to $(\mathbf{LM} \circ \tau_1)(F)$.

Let us now check that assembling all these natural transformations for all the objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ defines a natural transformation from $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ to $\mathbf{LM} \circ \tau_1$. Let M and N be two objects of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ and $\eta : M \to N$ be a natural transformation. We recall that the natural transformations $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(\eta)$ and $(\tau_1 \circ \mathbf{LM})(\eta)$ are respectively given by:

$$(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(\eta)_{\underline{n}} = id_{\mathcal{I}_{R[H]}} \underset{R[H]}{\otimes} \eta_{2 \natural \underline{n}} \text{ and } (\tau_1 \circ \mathbf{LM})(\eta)_{\underline{n}} = id_{\mathcal{I}_{R[H_{1+n}]}} \underset{R[H_{1+n}]}{\otimes} \eta_{2 \natural \underline{n}}$$

for all natural numbers n. We straightforwardly deduce from the definitions that $\upsilon(N)_{\underline{n}} \circ (\mathbf{LM} \circ \tau_1)(\eta)_{\underline{n}} = (\tau_1 \circ \mathbf{LM})(\eta)_{\underline{n}} \circ \upsilon(M)_{\underline{n}}$. Hence $\upsilon(M) \circ (\mathbf{LM} \circ \tau_1)(\eta)$ and $(\tau_1 \circ \mathbf{LM})(\eta) \circ \upsilon(N)$ are equal as natural transformations from $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(M)$ to $(\mathbf{LM} \circ \tau_1)(N)$, which ends the proof. $\qquad \square$

Remark 5.15. Let us assume that H is a free group which rank is denoted by $r(H)$. Let M be an $R[H]$-module. Since H is free, $\mathcal{I}_{R[H]}$ is a free $R[H]$-module of rank $r(H)$ (see [41, Proposition 6.2.6] for instance), hence there are canonical isomorphisms of R-modules $\mathcal{I}_{R[H]} \otimes_{R[H]} M \cong (R[H])^{\oplus r(H)} \otimes_{R[H]} M \cong M^{\oplus r(H)}$ by the distributive property of the tensor product.

Corollary 5.16. *For $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ a reliable Long-Moody system, as endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, there is an isomorphism $\tau_1 \circ \mathbf{LM} \cong (\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2) \oplus (\mathbf{LM} \circ \tau_1)$. Furthermore, if the group H is free of rank $r(H)$, there is a natural equivalence $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \cong \tau_2^{\oplus r(H)}$.*

Proof. Since $\mathcal{I}_{R[\mathcal{A}]}^{-1}(id_1 \natural \iota_{\underline{n}}) \circ \mathcal{I}_{R[\mathcal{A}]}(id_1 \natural \iota_{\underline{n}}) = id_{\mathcal{I}_{R[H]}}$ for all natural numbers n, the natural transformation $\upsilon : \mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \to \tau_1 \circ \mathbf{LM}$ is a right inverse of the natural transformation $\Upsilon : \tau_1 \circ \mathbf{LM} \to \mathrm{Coker}(\xi)$.

We now assume that the group H is free and consider an object F of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. The isomorphisms of Remark 5.15 provide the R-module isomorphism $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 F(\underline{n}) \cong \tau_2^{\oplus r(H)} F(\underline{n})$ for all natural numbers n. The naturality with respect to the morphisms of $\mathfrak{U}\mathcal{G}$ and that we define a natural transformation in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ both directly follow from the definitions of $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ and $\tau_2^{\oplus r(H)}$. $\qquad \square$

5.1.3 Key relations with the difference and evanescence functors

This section presents the key commutation relations of the generalized Long-Moody functors with the evanescence and difference functors. Considering the endofunctor $\delta_1 \circ \mathbf{LM}$ as an extension of $\mathrm{Coker}(\xi)$ by $\mathbf{LM} \circ \delta_1$ (see diagram (30)), we prove that this extension is trivial by using Corollary 5.16.

Theorem 5.17. *Let $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$ be a reliable Long-Moody system. There is a natural equivalence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$*

$$\delta_1 \circ \mathbf{LM} \cong (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2) \oplus (\mathbf{LM} \circ \delta_1). \tag{41}$$

Moreover, if we assume that the groups H_0 and H are free, then the evanescence endofunctor κ_1 commutes with the endofunctor $\mathbf{LM}$ and the isomorphisms of Remark 5.15 provide a natural equivalence:

$$\delta_1 \circ \mathbf{LM} \cong \tau_2^{\oplus r(H)} \oplus (\mathbf{LM} \circ \delta_1). \tag{42}$$

Proof. To prove the first statement, we consider the diagram (30). Then the isomorphism $\tau_1 \circ \mathbf{LM} \cong (\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2) \oplus (\mathbf{LM} \circ \tau_1)$ of Corollary 5.16 provides a splitting of the top short exact sequence, which automatically induces a splitting for the bottom short exact sequence and thus proves the relation (41).

Furthermore, assuming that the groups H_0 and H are free, we have the exact sequence (28). We recall that for two morphisms f and g of an abelian category such that the target of f is the source of g and g is a monomorphism, the snake lemma provides an isomorphism $\ker(f) \cong \ker(g \circ f)$. Assigning $f = \mathbf{LM}(i_1)$ and $g = \xi$ and using the equality $\xi \circ (\mathbf{LM}(i_1)) = i_1\mathbf{LM}$, we thus conclude that $\mathbf{LM} \circ \kappa_1 \cong \kappa_1 \circ \mathbf{LM}$. Also, the relation (42) is obtained from (41) using the second statement of Corollary 5.16. $\qquad\square$

Finally, the above work on the evanescence endofunctor κ_1 straightforwardly generalizes to any other evanescence endofunctor. This property is used to prove the further Lemma 5.21.

Proposition 5.18. *Let $m \geq 1$ be a natural number and we assume that the groups H_0 and H are free. Then the evanescence endofunctor κ_m commutes with the Long-Moody functor.*

Proof. The result follows repeating verbatim the work on the evanescence endofunctor of Section 5.1 simply by modifying the index 1 into any $m \geq 1$. Indeed, we define a natural transformation $\xi(F) : (\mathbf{LM} \circ \tau_m)(F) \to (\tau_m \circ \mathbf{LM})(F)$ analogously Proposition 5.7 by assigning to each $n \in \mathbb{N}$ the morphism $\xi(F)_{\underline{n}}$ to be $(\mathcal{I}_{\mathcal{A}}(\iota_m \natural id_{\underline{n}}) \otimes_{R[H_{m+n}]} F((b_{1,m}^{\mathcal{G}'})^{-1} \natural id_{\underline{n}}))$. Then the analogue of Proposition 5.8 is satisfied, the key point being that $(b_{1,m}^{\mathcal{G}'})^{-1} \circ (\iota_m \natural id_1) = id_1 \natural \iota_m$ by Relation (7) since $\mathfrak{U}\mathcal{G}'$ is pre-braided. Then the hypotheses of Assumption 5.1 are sufficient to prove the analogue results of that of Section 5.1.2: in other words we define a natural equivalence $\tau_m \circ \mathbf{LM} \cong \tau_{m+1}^{\oplus m \cdot r(H)} \oplus (\mathbf{LM} \circ \tau_m)$ in the category $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. The result thus follows repeating mutatis mutandis the proof of Theorem 5.17. $\qquad\square$

5.2 Effect on strong polynomial functors

In this section, we focus on the behaviour of the generalized Long-Moody functor on (very) strong polynomial functors. We recover in particular the results of [37, Section 4] when $(\mathfrak{U}\mathcal{G}', \natural, 0_{\mathcal{G}'}) = (\mathfrak{U}\mathcal{G}, \natural, 0_{\mathcal{G}}) = (\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$. First, we have the following property:

Lemma 5.19. *The functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ commutes with the difference functor δ_1. Moreover, if H is free, then $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ commutes with the evanescence functor κ_m for all natural numbers $m \geq 1$.*

Proof. The commutation result with the difference functor δ_1 is a consequence of the right-exactness of the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ and of the exactness and the commutation property of the translation functor τ_2 of Proposition 4.2. Assuming that the group H is free, the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} - : R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ is exact as a consequence of Lemma 2.17. Hence, the claim follows from the commutation of the evanescence functor κ_m with the translation functor τ_2 of Proposition 4.2. $\qquad\square$

Theorem 5.20. *Let d be a natural number and F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. We recall that we consider a reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$. If the functor F is strong polynomial of degree d, then:*

- *the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ belongs to $\mathcal{P}ol_d^{strong}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$;*

- *the functor $\mathbf{LM}(F)$ belongs to $\mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.*

Moreover, if the groups H_0 and H are free and F is very strong polynomial of degree d, then the functor $\mathbf{LM}(F)$ is a very strong polynomial functor of degree equal to $d + 1$.

Proof. The result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ follows from Lemma 5.19. By induction on the polynomial degree, we deduce the first result on $\mathbf{LM}(F)$ using the relation (41).

Assume now that the groups H_0 and H are free. We recall that H is non-trivial. For a very strong polynomial functor F of degree d, we deduce from Lemma 5.19 that $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 F \cong \tau_2^{\oplus r(H)} F$ is also very strong polynomial of degree d. A fortiori, the result follows from a clear induction using the relation (42) and the commutation property of the evanescence endofunctor κ_1 with the endofunctor $\mathbf{LM}$ of Theorem 5.17. $\qquad\square$

5.3 Effect on weak polynomial functors

We investigate the effect on weak polynomial functors of the Long-Moody functor associated with the *reliable* Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$. The first step of this study consists in defining the Long-Moody functor on the quotient category $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. First, we note the following property.

Lemma 5.21. *Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. Assume that the groups H_0 and H are free. If the functor F is in $K(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, then the functors $\mathbf{LM}(F)$ and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ are in $K(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.*

Proof. We recall that the functor κ is the colimit $\sum_{m \in \mathrm{Obj}(\mathfrak{U}\mathcal{G})_{\natural}} \kappa_m$. By Proposition 5.18 and Lemma 5.19, the endofunctors $\mathbf{LM}$ and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ commute with the evanescence functor κ_m for all natural numbers $m \geq 1$. We recall from Proposition 2.5 that the endofunctor $\mathbf{LM}$ commutes with all colimits, and thus commutes with κ. Furthermore, we deduce the commutation with all colimits of the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(-)$ from the exactness of the translation functor τ_2 (see Proposition 4.2) and the exactness of the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} -\ :\ R\text{-}\mathfrak{Mod} \to R\text{-}\mathfrak{Mod}$ (since the augmentation ideal $\mathcal{I}_{R[H]}$ is a projective $R[H]$-module by Lemma 2.17). Therefore, the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(-)$ also commutes with κ. $\square$

From now until the end of Section 5.3, we assume that the groups H_0 and H are free. By Lemma 5.21, the endofunctors $\mathbf{LM}$ and $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$ induce two functors on the quotient category $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, denoted by

$$\mathbf{LM_{St}} : \mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \text{ and } (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2)_{\mathbf{St}} : \mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}).$$

Proposition 5.22. *The induced functor $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)_{\mathbf{St}}$ is equivalent to the functor $\tau_2^{\oplus r(H)}$, where the last τ_2 is the translation endofunctor of $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$.*

Furthermore, for F an object of $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$, there are natural equivalences as objects of $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$:

$$\delta_1(\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2(F))_{\mathbf{St}} \cong (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2)_{\mathbf{St}}(\delta_1 F), \tag{43}$$

$$\delta_1 \mathbf{LM_{St}}(F) \cong (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2(F))_{\mathbf{St}} \oplus \mathbf{LM_{St}}(\delta_1 F). \tag{44}$$

Proof. By Corollary 5.16, we have a natural equivalence $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2 \cong \tau_2^{\oplus r(H)}$ and that $K(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ is closed under colimits by Proposition 4.6. Let G be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. We recall that κ is left-exact and that $\kappa(\kappa_2 G) = \kappa_2 G$ (see Lemma 4.5 and its proof). We deduce that G is in $K(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ if the functor $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(G)$ is in $K(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$. This fact and Lemma 5.21 prove the first statement.

As a consequence of the definitions of the induced difference functor of Proposition 4.8 and of the induced functors $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)_{\mathbf{St}}$ and $\mathbf{LM_{St}}$, we obtain the following two natural equivalences $\delta_1(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)_{\mathbf{St}} \cong (\delta_1(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2))_{\mathbf{St}}$ and $\delta_1 \mathbf{LM_{St}} \cong (\delta_1 \circ \mathbf{LM})_{\mathbf{St}}$. The result then follows from Lemma 5.19 and Theorem 5.17. $\square$

Then we can now prove:

Theorem 5.23. *Let d be a natural number and recall that the groups H_0 and H are assumed to be free. Let F be an object of $\mathbf{Fct}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ which is weak polynomial of degree d. Then the functor $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2(F)$ is a weak polynomial functor of degree d and the functor $\mathbf{LM}(F)$ is a weak polynomial functor of degree $d+1$.*

Proof. We proceed by induction on the degree of polynomiality of F. If F is weak polynomial of degree 0, then there exists a constant functor C of $\mathbf{St}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ such that $\pi_{\mathfrak{U}\mathcal{G}}(F) \cong C$ by Proposition 4.10. Hence, we deduce from Proposition 5.22 that $(\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)_{\mathbf{St}}(C) \cong C^{\oplus r(H)}$, which is a degree 0 weak polynomial functor. Now, assume that F is weak polynomial functor of degree $d \geq 0$. Then, the result follows from the relation (43) and the inductive hypothesis.

For the endofunctor $\mathbf{LM}$, we also proceed by induction as follows. Assume that F is a weak polynomial functor of degree 0. By the equivalence (44), we obtain that $\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(\mathbf{LM}(F))) \cong (\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2)(\pi_{\mathfrak{U}\mathcal{G}}(F))$. Following the result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$, the functor $\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(\mathbf{LM}(F)))$ is polynomial of degree 0. Therefore, the functor $\mathbf{LM}(F)$ is therefore weak polynomial of degree 1. Now, assume that F is a weak polynomial functor of degree $d \geq 1$. By the equivalence (44):

$$\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(\mathbf{LM}(F))) \cong (\mathcal{I}_{R[H]} \underset{R[H]}{\otimes} \tau_2)(\pi_{\mathfrak{U}\mathcal{G}}(F)) \oplus \mathbf{LM_{St}}(\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(F))).$$

The result follows from the inductive hypothesis and the result on $\mathcal{I}_{R[H]} \otimes_{R[H]} \tau_2$. $\square$

6 Examples and applications

This last section presents applications of the results of Section 5. Namely, the generalized Long-Moody functors provide very strong and weak polynomial functors in any degree for the families of groups of Section 3. In particular, they give twisted coefficients for which homological stability is satisfied (see Section 6.1) and introduce a tool for classifying weak polynomial functors with $\mathfrak{U}\mathcal{G}$ as source category.

6.1 Strong polynomial functors

Proposition 6.1. *The coherent Long-Moody systems of Sections 3.3.1 and 3.3.2 are reliable.*

Proof. We recall from Lemma 3.3 that the functor $\pi_1(-, p)$ is strict monoidal. Hence, in particular, the assignments conditions of Assumption 5.1 on the functor $\mathcal{A}$ are satisfied. Also, for the families of morphisms $\{\chi_{n,1}\}_{n \in \mathbb{N}}$, $\{\chi_{n,2}\}_{n \in \mathbb{N}}$, $\{\chi_{n,1}^{\flat}\}_{n \in \mathbb{N}}$ and $\{\chi_{n,2}^{\flat}\}_{n \in \mathbb{N}}$, the equality (17) of Proposition 3.9 implies that the relation (23) of Assumption 5.1 is satisfied. A fortiori, it follows from the definition of the functor $\pi_1(-, p)$ that Assumption 5.1 is satisfied. $\square$

Hence, applying a Long-Moody functor on the constant functor $\mathbb{Z}$, we prove:

Corollary 6.2. *For each $d \geq 0$, the functors $\mathbf{LM}_{\mathfrak{M}_2^{+,g}}^{\circ d}(H_1(\Sigma_{g,1}^-, \mathbb{Z}))$, $\mathbf{LM}_{\mathfrak{B}_2^g}^{\circ d}(H_1(\Sigma_{g,1}^-, \mathbb{Z}))$, $\mathbf{LM}_{\mathfrak{M}_2^{-,h}}^{\circ d}(H_1(\mathcal{N}\Sigma_{h,1}^-, \mathbb{Z}))$ and $\mathbf{LM}_{\mathfrak{B}_2^{-,h}}^{\circ d}(H_1(\mathcal{N}\Sigma_{h,1}^-, \mathbb{Z}))$ are very strong polynomial functors of degree $d + 1$.*

Moreover, we have the following application for the Long-Moody functor associated with surface braid group of Section 3.3.3. Although $\mathbf{LM}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is not an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{B}_2^g, \mathbb{Z}\text{-}\mathfrak{Mod})$, the following slight modification of that functor defines a new functor with $\mathfrak{U}\mathfrak{B}_2^g$ as source category. An analogous manipulation was actually made in [37, Section 2.3.1]. Let $\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]^* : \mathfrak{B}_2^g \to \mathbb{Z}\text{-}\mathfrak{Mod}$ be the functor induced by the dual representations of that defined by $\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]$. Let $\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ denote the object $\mathbf{LM}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]) \otimes_{\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]} \mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]^*$ of the category $\mathbf{Fct}(\mathfrak{B}_2^g, \mathbb{Z}\text{-}\mathfrak{Mod})$, where $\otimes_{\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]}$ is the pointwise tensor product for the functor category. It is a routine to check that, by assigning to all natural numbers n and n' such that $n' \geq n$ the morphism $\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])([\Sigma_{g,1}^{n'-n}, id_{\Sigma_{g,1}^{n'}}])$ to be the embedding

$$\mathcal{I}_{\pi_1(\Sigma_{g,1}^-, p)^{\flat}}([\Sigma_{g,1}^{n'-n}, id_{\Sigma_{g,1}^{n'}}]) \underset{\pi_1(\Sigma_{g,1}^{n'}, p)}{\otimes} id_{\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3](\Sigma_{g,1}^{n'+1})},$$

the relations (5) and (6) are then satisfied. It follows from Lemma 1.2 that the functor $\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]) : \mathfrak{B}_2^g \to \mathbb{Z}\text{-}\mathfrak{Mod}$ thus defines an object of $\mathbf{Fct}(\mathfrak{U}\mathfrak{B}_2^g, \mathbb{Z}\text{-}\mathfrak{Mod})$. Hence, we deduce from Proposition 6.1:

Proposition 6.3. *For each $d \geq 0$, the functor $\mathbf{LM}_{\mathfrak{B}_2^g}^{\circ d}(\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3]))$ is very strong polynomial of degree $d + 1$.*

Proof. Let n be a natural number. We note that the application $i_1\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])([0, id_{\underline{n}}])$ is a monomorphism which cokernel is $\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]$. Hence $\kappa_1\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is the null functor. For a natural number $n' \geq n$ and $[\Sigma_{g,1}^{n'-n}, \sigma] \in \mathrm{Hom}_{\mathfrak{U}\mathfrak{B}_2^g}(\Sigma_{g,1}^n, \Sigma_{g,1}^{n'})$. It follows from the universal property of the cokernel and naturality that $\delta_1\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])([\Sigma_{g,1}^{n'-n}, \sigma]) = id_{\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]}$. Hence, $\delta_1\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is the constant functor equal to $\mathbb{Z}[(\mathbb{Z} \times \mathbb{Z}^g) \rtimes \mathbb{Z}^g]$, which is very strong polynomial of degree 0. Hence the functor $\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is very strong polynomial of degree one. The result is thus a consequence of Theorem 5.20 and Proposition 6.1. $\square$

In [36, Section 5], Randal-Williams and Wahl prove homological stability for the families of mapping class groups of surfaces and surface braid groups considered in Section 3, with twisted coefficients given by very strong polynomial functors. Namely, for all the groupoids $\mathcal{G}$ and $\mathcal{G}'$ introduced in the examples of Section 3, they show that if $F : \mathfrak{U}\mathcal{G} \to \mathbb{Z}\text{-}\mathfrak{Mod}$ is a very strong polynomial functor of degree d, then the canonical maps

$$H_*(G_n, F(\underline{n})) \to H_*(G_{n+1}, F(\underline{n+1}))$$

are isomorphisms for $N(*, d) \leq n$ with $N(*, d) \in \mathbb{N}$ depending on $*$ and d; see [36, Theorem A]. The representation theory of mapping class groups of surfaces is wild and an active research topic; see [6, Section

4.6] or [31] for example. Hence there are very few known examples of functors appropriate for homological stability. Using Long-Moody functors, we thus construct very strong polynomial functors in any degree for these families of groups.

6.2 Weak polynomial functors

By Proposition 4.10, the constant functor $\mathbb{Z}$ is weak polynomial of degree 0. Also, it is clear that $\widetilde{\mathbf{LM}}_{\mathfrak{B}_2^g}(\mathbb{Z}[\mathfrak{B}_2^g/\Gamma_3])$ is weak polynomial of degree 1 since all its evanescence functors are trivial. It follows from Theorem 5.23 that:

Proposition 6.4. *For each $d \geq 0$, the functors of Corollary 6.2 and Proposition 6.3 are weak polynomial of degree $d + 1$.*

A strong polynomial functor of degree d is always weak polynomial of degree less than or equal to d by the first property of Proposition 4.8. The converse is false; see [16, Example 5.4] for a counterexample. Also, the weak polynomial degree of a strong polynomial functor can be strictly smaller than its strong polynomial degree as the following example shows. We recall from [37, Section 1.3] the functor $\overline{\mathfrak{Bur}} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod}$ which encodes the family of reduced Burau representations.

Proposition 6.5. *The functor $\overline{\mathfrak{Bur}} : \mathfrak{U}\boldsymbol{\beta} \to \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod}$ is a strong polynomial functor of degree 2 and weak polynomial of degree 1.*

Proof. Let $\mathbb{C}[t^{\pm 1}]_{\geq 1}$ be the subfunctor of the constant functor $\mathbb{C}[t^{\pm 1}]$ which is null at 0 and equal to R elsewhere. The result on the strong polynomial degree is proved in [37, Proposition 3.28], using the short exact sequence $0 \to \overline{\mathfrak{Bur}}_t \to \tau_1 \overline{\mathfrak{Bur}}_t \to \mathbb{C}[t^{\pm 1}]_{\geq 1} \to 0$ of $\mathbf{Fct}(\mathfrak{U}\boldsymbol{\beta}, \mathbb{C}[t^{\pm 1}]\text{-}\mathfrak{Mod})$. Since $\pi_{\mathfrak{U}\mathcal{G}}$ is exact, we deduce that $\delta_1(\pi_{\mathfrak{U}\mathcal{G}}(\overline{\mathfrak{Bur}}_t)) \cong \pi_{\mathfrak{U}\mathcal{G}}(\mathbb{C}[t^{\pm 1}]_{\geq 1})$. The functor $\mathbb{C}[t^{\pm 1}]_{\geq 1}$ is a subfunctor of a weak polynomial functor of degree 0 and $\kappa(\mathbb{C}[t^{\pm 1}]_{\geq 1}) \neq \mathbb{C}[t^{\pm 1}]_{\geq 1}$. So, we deduce from Proposition 4.10 that $\mathbb{C}[t^{\pm 1}]_{\geq 1}$ is weak polynomial of degree 0 and then the functor $\overline{\mathfrak{Bur}}_t$ is weak polynomial of degree 1. $\square$

On the contrary, it is clear that the translation $\tau_1 \overline{\mathfrak{Bur}}$ is both very strong and weak polynomial of degree 1. This exemplifies a benefit of the notion of weak polynomiality compared to the strong one: it reflects more accurately the behaviour of functors, in particular for large values.

Another fundamental reason for the notion of weak polynomial functors to be introduced is that, contrary to the category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$, the category $\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ is *localizing* by Proposition 4.10. This allows one to define the quotient categories $\mathcal{P}ol_{d+1}(\mathfrak{M}, \mathscr{A})/\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$. Generally speaking, a refined description of the category $\mathcal{P}ol_d^{strong}(\mathfrak{M}, \mathscr{A})$ is difficult and seems out of reach even for small d. On the contrary, understanding the quotient categories $\mathcal{P}ol_{d+1}(\mathfrak{M}, \mathscr{A})/\mathcal{P}ol_d(\mathfrak{M}, \mathscr{A})$ is more attainable: for example, when $\mathfrak{M}$ is the category FI of finite sets and bijections, [16, Proposition 5.9] gives a general equivalence of these quotients in terms of module categories.

Also, considering $\mathfrak{M} = \mathfrak{U}\mathcal{G}$, these quotients thus provide a new classifying tool for families of representations of the groups $\{G_n\}_{n \in \mathbb{N}}$. Even if such a description seems generally speaking out of reach for this kind of pre-braided monoidal categories, the Long-Moody functors give a new implement to understand these quotients. Indeed, as a consequence of Theorem 5.23, we obtain:

Proposition 6.6. *For a reliable Long-Moody system $\{\mathcal{A}, \mathcal{G}, \mathcal{G}', \chi\}$, if the groups H_0 and H are free, the associated Long-Moody functor induces a functor:*

$$\mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})/\mathcal{P}ol_{d-1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}) \to \mathcal{P}ol_{d+1}(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})/\mathcal{P}ol_d(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod}),$$

The first quotient category $\mathcal{P}ol_1(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})/\mathcal{P}ol_0(\mathfrak{U}\mathcal{G}, R\text{-}\mathfrak{Mod})$ being the most reasonably computable directly (it is for instance achievable for $\mathcal{G} = \boldsymbol{\beta}$), the Long-Moody functors thus allow to describe subcategories of the further quotients and can therefore be keys to understand them.

References

[1] Paolo Bellingeri. On presentations of surface braid groups. *Journal of Algebra*, 274(2):543–563, 2004.

[2] Paolo Bellingeri and Sylvain Gervais. Surface framed braids. *Geom. Dedicata*, 159:51–69, 2012.

[3] Paolo Bellingeri, Eddy Godelle, and John Guaschi. Abelian and metabelian quotient groups of surface braid groups. *Glasg. Math. J.*, 59(1):119–142, 2017.

[4] Joan S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.

[5] Joan S. Birman. *Braids, links, and mapping class groups.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 82.

[6] Joan S. Birman and Tara E. Brendle. Braids: a survey. *Handbook of knot theory*, pages 19–103, 2005.

[7] Carl-Friedrich Bödigheimer and Ulrike Tillmann. Stripping and splitting decorated mapping class groups. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 47–57. Birkhäuser, Basel, 2001.

[8] Carl-Friedrich Bödigheimer and Ulrike Tillmann. Embeddings of braid groups into mapping class groups and their homology. In *Configuration spaces*, volume 14 of *CRM Series*, pages 173–191. Ed. Norm., Pisa, 2012.

[9] Kenneth S Brown. *Cohomology of groups*, volume 87. Springer Science & Business Media, 2012.

[10] Filippo Callegaro and Mario Salvetti. Homology of the family of hyperelliptic curves. *Israel J. Math.*, 230(2):653–692, 2019.

[11] D. R. J. Chillingworth. Winding numbers on surfaces. I. *Math. Ann.*, 196:218–249, 1972.

[12] D. R. J. Chillingworth. Winding numbers on surfaces. II. *Math. Ann.*, 199:131–153, 1972.

[13] Thomas Church, Jordan S. Ellenberg, and Benson Farb. FI-modules and stability for representations of symmetric groups. *Duke Math. J.*, 164(9):1833–1910, 2015.

[14] Daniel E. Cohen. *Groups of cohomological dimension one.* Lecture Notes in Mathematics, Vol. 245. Springer-Verlag, Berlin-New York, 1972.

[15] Aurélien Djament. On stable homology of congruence groups. *arXiv:1707.07944*, 2017.

[16] Aurélien Djament and Christine Vespa. Foncteurs faiblement polynomiaux. *Int. Math. Res. Not. IMRN*, (2):321–391, 2019.

[17] Samuel Eilenberg and Saunders Mac Lane. On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. (2)*, 60:49–139, 1954.

[18] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton University Press, 2011.

[19] Vincent Franjou, Eric M. Friedlander, Alexander Scorichenko, and Andrei Suslin. General linear and functor cohomology over finite fields. *Ann. of Math. (2)*, 150(2):663–728, 1999.

[20] Pierre Gabriel. Des catégories abéliennes. *Bulletin de la Société Mathématique de France*, 90:323–448, 1962.

[21] André Gramain. Le type d'homotopie du groupe des difféomorphismes d'une surface compacte. In *Annales scientifiques de l'École Normale Supérieure*, volume 6, pages 53–66. Elsevier, 1973.

[22] Daniel Grayson. Higher algebraic K-theory: II (after Daniel Quillen). In *Algebraic K-theory*, pages 217–240. Lectures Notes in Math., Vol.551, Springer, Berlin, 1976.

[23] John Guaschi and Daniel Juan-Pineda. A survey of surface braid groups and the lower algebraic K-theory of their group rings. In *Handbook of group actions. Vol. II*, volume 32 of *Adv. Lect. Math. (ALM)*, pages 23–75. Int. Press, Somerville, MA, 2015.

[24] Mary-Elizabeth Hamstrom. Homotopy groups of the space of homeomorphisms on a 2-manifold. *Illinois J. Math.*, 10:563–573, 1966.

[25] Allen Hatcher. Algebraic topology. 2002. *Cambridge UP, Cambridge*, 606(9), 2002.

[26] Hans-Werner Henn, Jean Lannes, and Lionel Schwartz. The categories of unstable modules and unstable algebras over the Steenrod algebra modulo nilpotent objects. *Amer. J. Math.*, 115(5):1053–1106, 1993.

[27] Nicholas J. Kuhn. A stratification of generic representation theory and generalized Schur algebras. *K-Theory*, 26(1):15–49, 2002.

[28] D. D. Long. Constructing representations of braid groups. *Comm. Anal. Geom.*, 2(2):217–238, 1994.

[29] Saunders Mac Lane. *Categories for the working mathematician*, volume 5. Springer Science & Business Media, 2013.

[30] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[31] Dan Margalit. Problems, questions, and conjectures about mapping class groups. In *Breadth in contemporary topology*, volume 102 of *Proc. Sympos. Pure Math.*, pages 157–186. Amer. Math. Soc., Providence, RI, 2019.

[32] Martin Palmer. A comparison of twisted coefficient systems. *arXiv:1712.06310*, 2017.

[33] Teimuraz Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup. (4)*, 33(2):151–179, 2000.

[34] N. Popescu. *Abelian categories with applications to rings and modules*. Academic Press, London-New York, 1973. London Mathematical Society Monographs, No. 3.

[35] Nicolae Popescu and Liliana Popescu. *Theory of categories*. Martinus Nijhoff Publishers, The Hague; Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn, 1979.

[36] Oscar Randal-Williams and Nathalie Wahl. Homological stability for automorphism groups. *Adv. Math.*, 318:534–626, 2017.

[37] Arthur Soulié. The long–moody construction and polynomial functors. *Annales de l'Institut Fourier*, 69(4):1799–1856, 2019.

[38] Richard G Swan. Groups of cohomological dimension one. *Journal of Algebra*, 12(4):585–610, 1969.

[39] Rolland Trapp. A linear representation of the mapping class group $\mathscr{M}$ and the theory of winding numbers. *Topology Appl.*, 43(1):47–64, 1992.

[40] Masaaki Wada. Group invariants of links. *Topology*, 31(2):399–406, 1992.

[41] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.