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Stochastic Volatility: A Tale of Co-Jumps, Non-Normality, GMM and High Frequency Data

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Abstract

In this article we introduce a linear quadratic volatility model with co-jumps and show how to calibrate this model to a rich dataset. We apply GMM and more specifically match the moments of realized power and multi-power variations, which are obtained from high-frequency stock market data. Our model incorporates two salient features: the setting of simultaneous jumps in both return process and volatility process and the superposition structure of a continuous linear quadratic volatility process and a Lévy-driven Ornstein-Uhlenbeck process. We compare the quality of fit for several models, and show that our model outperforms the conventional jump diffusion or Bates model. Besides that, we find evidence that the jump sizes are not normally distributed and that our model performs best when the distribution of jump-sizes is only specified through certain (co-) moment conditions. Monte Carlo experiments are employed to confirm this.

Keywords: linear quadratic volatility, jump process, general method of moments, power variations, multi-power variations, Monte Carlo

JEL classification: C51, C52, G17

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1 Introduction

The increasing availability of high frequency observations of financial asset prices presents opportunities in modeling asset prices, but it also demands more robust and effective estimation methods. Cheng & Scaillet (2007) introduced a general linear quadratic jump diffusion (LQJD) class and showed that the class of conventional affine jump diffusion (AJD) defined in Duffie et al. (2000) is nested in the LQJD class. In line with the settings in Cheng & Scaillet (2007), suppose an m-dimensional state vector X_t driven by an n-dimensional Brownian motion ($n \leq m$) and a pure jump process N such that

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t + dN_t. \quad (1)$$

For X_t to be a LQJD model, it is required that the drift coefficients matrix $\mu(X_t, t)$, the co-variance coefficients matrix $\Omega(X_t, t) = \sigma(X_t, t)\sigma(X_t, t)^\intercal$, and the jump intensity $\lambda(X_t, t)$ are linear quadratic with respect to X_t, that is, of such form

$$\frac{1}{2}X^\intercal \Lambda(t)X + b(t)^\intercal X + c(t), \tag{2}$$

where $\intercal$ denotes transposition, and $\Lambda(t)$, $b(t)$ and $c(t)$ are time dependent deterministic functions. Conventional continuous stochastic volatility models have been rendered deficient in capturing abrupt changes in asset prices (Bakshi et al., 1997; Eraker et al., 2003), which in turn has triggered numerous studies on the impacts of including jumps in the price process. Further Pan (2002), Eraker et al. (2003) and Aït-Sahalia & Jacod (2009b) indicate that incorporating jumps in the volatility process is as important in accounting for sudden changes in realized volatilities. Using the methodology of power variations and multi-power variations adopted for high frequency returns, we calibrate a linear quadratic volatility jump diffusion model, in which the continuous part of the variance process $\sigma(X_t, t)^2$ belongs to the LQJD class. Our setup so far is semi-parametric. However, we later investigate a fully parametric specification and assess whether the assumption of normally distributed jump sizes is suitable for our model and data.

The variance process in our linear quadratic volatility jump diffusion model is a superposition of the continuous linear quadratic part and the discontinuous part, denoted as

$$\sigma^2(t) = \sigma_c^2(t) + V_d(t). \tag{3}$$

We use the general Gaussian Ornstein-Uhlenbeck (hereafter OU) process as the continuous part of the volatility process, i.e.

$$d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW(t),$$

where θ, μ and ν are constant parameters, and $W(t)$ is a Brownian motion. This causes the continuous part of variance to be linear quadratic. One may doubt the validity of OU process for modelling volatility, since the OU process does not guarantee the volatility process to be positive, which is the key advantage of the CIR process. However, the CIR process is used to model the variance process (the square of the volatility process), and the squared OU process here we used for the variance process is non-negative. We model the discontinuous part $V_d(t)$ as a moving average of the past jumps. The linear quadratic feature is distinct from the Cox-Ingersoll-Ross process (hereafter CIR process), which belongs to the AJD class, in allowing the co variance structure of shocks to state variables to be unrestricted, according to Santa-Clara & Yan (2010) and Christoffersen et al. (2012). The discontinuous part of the variance is incorporating analytically tractable jumps in variance while preserving the mean-reverting feature of variance (Todorov, 2011). The superposition structure in our model is close to Todorov (2009a), but we use a linear quadratic volatility structure instead of a CIR process.

In the empirical part, we assess the quality of fit of our model to high-frequency stock data and compare it with the classical affine jump diffusion model, also called Bates model, and the linear quadratic volatility model with jumps in price. We estimate these by matching moments of daily power variations and bi-power variations within a semi-parametric setting. The setting is semi-parametric, as we do not explicitly specify a parametric form for the distribution of jumps in the price as well as volatility process. Our empirical analysis shows that our model fits the data well. It is capable to capture the abrupt changes in volatility processes, and it performs outstandingly better than those models which only feature jumps in the price processes. This is similar to the result of Todorov (2009a). We further show that our semi parametric setup is superior to any fully parametric setup in which the two jump sizes are specified as being normally distributed, in a sense that the normality assumption is not supported by the empirical evidence. To do this, we repeat our estimation procedure within a fully parametric setting of our model, and find that while the result rejects over-identification, the coefficients of the normal distribution for the jump size are not significant. To investigate it, we conduct a Monte Carlo analysis, which shows that our estimation methodology would indeed return significant parameters for the normal distributions in the two jump sizes, were the jump sizes indeed normally distributed. In conclusion, they are not.

Bates (1996) introduced jumps into the price process within the well known Heston model. Following this, there have been two main trends in the jump diffusion literature, either assuming that the compensator of the price jumps (the jump intensity in the compound Poisson jumps case) is stochastic or allowing jumps in the volatility process. Duffie et al. (2000) analyzed the general AJD class and assumes that the intensity of jumps is a deterministic function of volatility. Time-changed Lévy processes have also been discussed in the literature (for example see Carr et al. (2003) and Carr & Wu (2004)). Santa-Clara & Yan (2010) employed correlated linear quadratic processes in both variance and compensator of jumps and estimated it under both physical and risk-neutral measure. Barndorff-Nielsen & Shephard (2001) proposed Lévy-driven OU processes, i.e. the moving average of realized Lévy processes, to capture sudden changes in volatilities. Later Barndorff-Nielsen & Shephard (2003a) explicitly investigated the distributional properties of those processes. Brockwell (2001) and Brockwell & Lindner (2012) explored different weighting structures of moving averages and generalized this to Lévy-driven continuous autoregressive moving average (CARMA) models. The setting of simultaneous jumps in price process and volatility process was also statistically confirmed by Jacod & Todorov (2010) and Jacod et al. (2017). Further, Jacod et al. (2017) tested for non-correlation between jumps in price and volatility, rejecting the non-correlation hypothesis.

Both, the empirical part as well as the Monte Carlo experiment, are inspired by the idea of matching closed form moment conditions of returns. In the low frequency (i.e. daily frequency) case, Andersen &

Sørensen (1996) adopted a Monte Carlo study to test the feasibility of inference in matching different moments when estimating stochastic auto-regressive volatility (SARV) models using the general method of moments (GMM). Pan (2002) used joint moments inference of both returns and volatility to estimate affine diffusion processes. Turning to high frequency data, early research on financial markets using intraday data dates back to Andersen & Bollerslev (1997) and Bollen & Inder (2002). It was Andersen et al. (2003) who introduced a nonparametric estimator for realized volatility, which is simply the sum of squared returns. Jacod (1994) provided a general description for the asymptotic behavior of that estimator for the case of Brownian semi-martingales. Bollerslev & Zhou (2002) used realized volatility as a proxy of integrated volatility, and estimated jump diffusion models by matching sample moments of integrated volatility. Further, Aït-Sahalia (2004) analyzed the practicality of using moments of realized volatility to estimate stochastic volatility with jumps models. Barndorff-Nielsen & Shephard (2004) introduced a robust method named bipower variation to disentangle jumps from the continuous part of returns. Following this Barndorff-Nielsen et al. (2006) derived asymptotic properties by imposing the Central Limit Theorem (hereafter CLT) for multipower variations. Boudt et al. (2011) propose a robust estimator for small price jumps and find that including periodicity improves the intraday jump detection. Todorov (2009b) implemented Monte Carlo analysis to justify the effectiveness of using joint moments inference of realized power variation and multipower variations when estimating jump diffusion processes. In addition, Mancini (2009) estimated coefficients in jump diffusion models using a threshold truncation method called truncated variation, which had been introduced by Mancini (2001). Salvatierra & Patton (2015) introduce high frequency data into copula models and show that the generalized auto-regressive score (GAS) model with high frequency data is superior to other simpler models. A rough fractional stochastic volatility (RFSV) model, adopted by Gatheral et al. (2018), depicted the logarithmic volatility as a fractional Brownian motion, and was proved to be remarkably consistent with high frequency financial time series.

The remainder of this paper is structured as follows. Sections 2 and 3 provide the specifications of our model and provide parameters and proofs for essential moments which are used in the estimation. Section 4 presents the results of the estimation process based on high frequency data. In this section we also provide several robustness test results that support our methodology. Section 5 provides the details of the Monte Carlo experiment that supports our semi parametric methodology. Finally, section 6 provides some concluding remarks.

2 Model Specification

We consider a linear quadratic volatility jump diffusion model in which the underlying asset price is affected by two types of risk: a continuous diffusive risk, denoted by a Brownian Motion, and discontinuous risk, denoted by a general jump process, both possibly multi-dimensional. The linear quadratic structure is similar to Santa-Clara & Yan (2010), where it is assumed that diffusive volatility and the square root jump intensity follow a Gaussian Ornstein-Uhlenbeck (OU) process. We set our model within a probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geqslant 0},\ \mathbb{P})$ equipped with the natural filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. We denote with $P(t)$ the underlying asset price and let $R(t)$ denote its logarithmic returns. We then specify our model as follows:

$$dR(t) = \alpha(t)dt + \sigma(t-)dW_1(t) + \int_{\mathbb{R}} J(x)\tilde{\mu}(dt, dx), \tag{4}$$

$$\sigma^2(t) = V_c(t) + V_d(t), \tag{5}$$

$$V_c(t) = \sigma_c^2(t), \text{ where } d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW_2(t), \tag{6}$$

$$dV_d(t) = -\kappa V_d(t)dt + \int_{\mathbb{R}} Q(x)\mu(dt, dx), \tag{7}$$

where κ is a constant, and $W_1(t)$ and $W_2(t)$ are two independent standard Brownian motions. The measure $\mu(dt, dx)$ is a time-homogeneous Poisson random measure, and $\tilde{\mu}(dt, dx)$ its compensated version, with compensator $\nu(dt, dx) = dtG(dx)$ with $G : \mathbb{R} \to \mathbb{R}^+$. In the special case of a compound Poisson process, the compensator simplifies to $\lambda f(x)dt$, where λ is the intensity of Poisson jumps and $f(x)$ is the probability density function of the jump size. Here we assume possibly simultaneous jumps in the return process $R(t)$ and the discontinuous part of volatility process $V_d(t)$. $J(x)$ and $Q(x)$ are jump sizes and left-continuous deterministic functions, with $J : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}^+$. The fact that jumps in the returns and volatility are driven by the same random source, the measure $\mu(dt, dx)$, results in possibly simultaneous jumps, i.e. co-jumps, a feature that has recently been studied in the context of COGARCH in Klüppelberg et al. (2004) and in the context of jump driven stochastic volatility in Todorov (2011) for example. As indicated previously, we set our model semi-parametrically, i.e. we do not restrict the distribution of the jump parts. This feature, also used in Todorov (2011), takes on some advantages of GMM in contrast to the Maximum Likelihood Estimator (MLE). Note that because of the use of high-frequency, the drift term $\alpha(t)$ is practically negligible when estimating the quadratic variation and integrated volatility and similar quantities, hence we can ignore it. (see Bollerslev & Zhou (2002)).

By imposing a superposition assumption on the coefficients of the diffusive part in the return process, we form a stochastic volatility process with jumps similar as in Todorov (2009a). The continuous part of the volatility process is assumed to follow an OU process. By employing Itô's formula, it is not difficult to find that the drift part features a linear quadratic structure with respect to the volatility $\sigma_c(t)$ if the variance follows a squared OU process. We find that the variance process satisfies

$$V_c(t) = \sigma_c^2(t), \text{ where } d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW_2(t),$$
$$dV_c(t) = (2\theta\mu\sigma_c(t) + \nu^2 - 2\theta V_c(t))dt + 2\nu\sigma_c(t)dW_2(t). \tag{8}$$

The discontinuous part of the volatility $V_d(t)$ follows a Non-Gaussian OU process, which is a special case of a Lévy-Driven processes (see Barndorff-Nielsen & Shephard (2001) and Brockwell & Lindner (2012)). The discontinuous part of the volatility is a CARMA(1,0) process, and it can be represented as a weighted sum of past jumps (assuming the initial value to be zero),

$$V_d(t) = \int_{-\infty}^{t} \int_{\mathbb{R}} e^{\kappa(s-t)} Q\mu(ds, dx) \tag{9}$$

In order to be consistent with moment conditions used in the estimation part, we then denote the *Quadratic Variation* (QV) process of the underlying return process during period $(t, t+a]$ as,

$$[R, R]_{(t,t+a]} = \int_t^{t+a} \sigma^2(s)ds + \int_t^{t+a} \int_{\mathbb{R}} J^2 \mu(ds, dx). \tag{10}$$

In addition, the *integrated variance* (IV) during the interval $(t, t+a]$ can be separated into continuous part and discontinuous part, denoted as

$$IV_{(t,t+a]} = \int_t^{t+a} \sigma^2(s)ds = \int_t^{t+a} V_c(s)ds + \int_t^{t+a} V_d(s)ds. \tag{11}$$

Let us note at this point that we do not restrict our jump process to be a compound Poisson process. We only estimate the cumulants of jumps, as Aït-Sahalia & Jacod (2009a) claimed evidence of small infinite activities of jumps. That enables us to make assumptions only on the integrability of those cumulants.

The classical leverage effect is implicitly specified in our model. Most often the leverage effect is realized by assuming negative correlation between the two Brownian Motions in return and volatility processes. However, the leverage effect can also be captured by assuming a negative co-variance in the jumps of returns and volatility. In our model this can be realized by appropriately choosing the functions J and Q. The empirical analysis makes this evident through the estimation of the mixed moments of jump sizes in return and volatility processes. It is worth noting that our setting of correlated jumps can be seen as a special case of a sort of discontinuous leverage effect, which has indeed been investigated (jointly with the continuous leverage effect) by Aït-Sahalia et al. (2017). They also obtained the central limit theorems of the estimators for each type of leverage effect. It may be of great interest to explore the joint inference of the power and multi-power variations and the leverage estimator for existing models by using high frequency data.

3 Moment conditions

As the GMM estimator we employed here is mainly composed of moments of the *quadratic variation* and the *integrated variance* that are defined in Section 4.1.2, it is worthwhile to investigate the tractability of such moment conditions. We start by the basic components of those moment conditions. The following Lemma 3.1 provides the moments of the integrand of the continuous part in the *quadratic variation* and the *integrated variance.*

Lemma 3.1 (Moments of the Linear Quadratic Volatility Process)**.** *Given the initial value σ_0, for a Linear Quadratic Volatility process of the type $V(t) = \sigma^2(t)$ with $d\sigma(t) = \theta(\mu - \sigma(t))dt + \nu dW(t)$ ($\theta > 0, \mu > 0, \nu > 0$ and $W(t)$ is a Wiener Process), we have the following moments for $\sigma(t)$*[1]

$$\mathbb{E}(\sigma_T|\mathcal{F}_t) = e^{-\theta(T-t)}\sigma_t + \mu(1 - e^{-\theta(T-t)}),\ \forall t \leqslant T, \tag{12}$$

$$\begin{aligned}\mathbb{E}(\sigma_T^2|\mathcal{F}_t) = {}& e^{-2\theta(T-t)}\sigma_t^2 + (2\mu e^{-\theta(T-t)} - 2\mu e^{-2\theta(T-t)})\sigma_t + 2\mu^2(1 - e^{-\theta(T-t)}) \\ & + \frac{\nu^2 - 2\theta\mu^2}{2\theta}(1 - e^{-2\theta(T-t)}),\end{aligned} \tag{13}$$

$$\lim_{t\to\infty} \mathbb{E}(\sigma_t) = \mu, \tag{14}$$

$$\lim_{t\to\infty} \mathbb{E}(\sigma_t^2) = \frac{\nu^2}{2\theta} + \mu^2, \tag{15}$$

$$\mathbb{E}(\sigma_s^2\sigma_u^2) = \left(\mu^2 + \frac{\nu^2}{2\theta}\right)^2 + \frac{2\mu^2\nu^2}{\theta}e^{-\theta(s-u)} + \frac{\nu^4}{2\theta^2}e^{-2\theta(s-u)},\ \forall u \leqslant s. \tag{16}$$

Proof. To obtain equation (12), the first order (uncentered) conditional moment of the OU process, we multiply σ_t by $e^{\theta t}$, through Itô's lemma we get,

$$e^{\theta T}\sigma_T = e^{\theta t}\sigma_t + \mu(e^{\theta T} - e^{\theta t}) + \int_t^T \nu e^{\theta t} dW_s. \tag{17}$$

Taking the expectation on both sides conditional on $\mathcal{F}_t$, we get

$$\mathbb{E}(\sigma_T|\mathcal{F}_t) = e^{-\theta(T-t)}\sigma_t + \mu(1 - e^{-\theta(T-t)}).$$

For the second order conditional moment, we apply Itô's lemma to $e^{2\theta t}\sigma_t^2 - 2\mu e^{2\theta t}\sigma_t$, jointly with equation (17), we have,

$$\begin{aligned}\sigma_T^2 = {}& 2\mu\Big(e^{-\theta(T-t)}\sigma_t + \mu(1 - e^{-\theta(T-t)})\Big) + e^{-2\theta(T-t)}\sigma_t^2 - 2\mu e^{-2\theta(T-t)}\sigma_t \\ & + \frac{\nu^2 - 2\theta\mu^2}{2\theta}(1 - e^{-2\theta(T-t)}) + \int_t^T \Big(2\nu e^{2\theta(s-T)}(\sigma_s - \mu) + 2\mu\nu e^{\theta(s-T)}\Big)dW_s,\end{aligned} \tag{18}$$

[1] Here we identify $\mathbb{E}(\cdot) = \mathbb{E}(\cdot|\mathcal{F}_0)$

and by taking expectations on both sides conditional on $\mathcal{F}_t$ we get equation (13). Particularly at $t = 0$,

$$\mathbb{E}(\sigma_T) = e^{-\theta T}\mathbb{E}(\sigma_0) + \mu(1 - e^{-\theta T}).$$

The stationary unconditional moments can be obtained by simply taking the limit of time to infinity, so we have equation (14) and (15) (substituting T with t). Additionally, we have unconditional third and fourth moments, $\lim_{t\to\infty} \mathbb{E}(\sigma_t^3) = \mu^3 + \frac{3\mu\nu^2}{2\theta}$ and $\lim_{t\to\infty} \mathbb{E}(\sigma_t^4) = \mu^4 + \frac{3\mu^2\nu^2}{\theta} + \frac{3\nu^4}{4\theta^2}$, since the stationary distribution of σ_t is Gaussian.

The mixed moment (16), provided σ_t is covariance stationary, can be easily obtained by utilizing equation (13), (14), (15) and third and fourth moment through the *law of iterated expectations* $\mathbb{E}\Big(\sigma_u^2\mathbb{E}(\sigma_s^2|\mathcal{F}_u)\Big) = \mathbb{E}(\sigma_s^2\sigma_u^2)$, $\forall u \leqslant s$. □

We then obtain the moment conditions of the *integrated variance* by decomposing it into continuous part and discontinuous part, as the variance process is defined to be the superposition of a linear-quadratic process and a non-Gaussian Lévy driven OU process. Given the previous lemma, we find those moments still tractable, though a little bit tedious. We present results in Theorem 3.1. Moment conditions for the *quadratic variation* are introduced in Theorem 3.2.

Theorem 3.1 (Moments of the Integrated Variance)**.** *For the integrated variance we defined as* $IV_{(t,t+a]} = \int_t^{t+a}\sigma^2(\tau)d\tau = \int_t^{t+a}V_c(\tau)d\tau + \int_t^{t+a}V_d(\tau)d\tau$, $\forall a \in \mathbb{R}^+$, *where* $V_c(t)$ *follows a linear quadratic diffusion process and* $V_d(t)$ *follows a Non-Gaussian Lévy driven OU process, we have the following moments,*

$$\mathbb{E}\left(\int_t^{t+a} V_d(\tau)d\tau|\mathcal{F}_s\right) = \int_{-\infty}^{s}\int_{\mathbb{R}} \frac{e^{\kappa(u-t)} - e^{\kappa(u-t-a)}}{\kappa} Q\tilde{\mu}(du,dx) + a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}, \tag{19}$$

$$\lim_{t\to\infty} \mathbb{E}(IV_{(t,t+a]}) = a\left(\frac{\nu^2}{2\theta} + \mu^2 + \frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right), \tag{20}$$

$$\begin{aligned}\lim_{t\to\infty} Var(IV_{(t,t+a]}) = &\frac{4\mu^2\nu^2(e^{-\theta a} + \theta a - 1)}{\theta^3} + \frac{\nu^4(e^{-2\theta a} + 2\theta a - 1)}{4\theta^3}\\ &+ \frac{e^{2\kappa a} - 2e^{\kappa a} - 2e^{-2\kappa a} + 6e^{-\kappa a} + 2\kappa a - 3}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx),\end{aligned} \tag{21}$$

$$\begin{aligned}\lim_{t\to\infty} Cov(IV_{(t,t+a]}, IV_{(t+h,t+h+a]}) = &\, a\left(\frac{\nu^2}{2\theta} + \mu^2\right)\frac{(\nu^2 - 2\theta\nu^2)(e^{-2\theta h} - e^{-2\theta(h-a)})}{4\theta^2} + \left(a\left(\frac{\nu^2}{2\theta} + \mu^2\right)^2\right.\\ &\left. + \frac{2\mu^2\nu^2(1 - e^{-\theta a})}{\theta^2} + \frac{\nu^4(1 - e^{-2\theta a})}{4\theta^3}\right)\cdot\left(\frac{e^{-2\theta(h-a)} - e^{-\theta h}}{2\theta}\right)\\ &+ \left(\frac{1 - e^{-\theta a}}{\theta}\left(\mu^3 + \frac{3\mu\nu^2}{2\theta}\right) + \mu\left(a - \frac{1 - e^{-\theta a}}{\theta}\right)\left(\frac{\nu^2}{2\theta} + \mu^2\right)\right)\\ &\cdot\left(\frac{e^{-\theta(h-a)} - e^{-\theta h}}{\theta} - \frac{e^{-2\theta(h-a)} - e^{-2\theta h}}{2\theta}\right)2\mu + \frac{(e^{\kappa a} - 1)e^{-\kappa(2a+h)}}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx),\\ &for\ h = ia,\ i \in \mathbb{N}^+.\end{aligned} \tag{22}$$

Proof. We start by proving equation (19). We use *Fubini's Theorem* and obtain[2]

$$\begin{aligned}\int_t^{t+a} V_d(\tau)d\tau &= \int_t^{t+a}\int_{-\infty}^{\tau}\int_{\mathbb{R}} e^{\kappa(u-\tau)}Q\mu(du,dx)d\tau\\ &= \int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx)\\ &+\int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx)+a\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}. \end{aligned} \tag{23}$$

Taking conditional expectations with regards to $\mathcal{F}_s$ on both sides we obtain equation (19).

For equation (20), by the *law of iterated expectations* we know that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}(IV_{(t,t+a]}) &= \lim_{t\to\infty}\int_t^{t+a}\mathbb{E}(V_c(\tau))d\tau+\lim_{t\to\infty}\int_t^{t+a}\mathbb{E}(V_d(\tau))d\tau\\ &= a\left(\frac{\nu^2}{2\theta}+\mu^2+\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}\right).\end{aligned}$$

To prove equation (21), we first calculate $\lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]})$,

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]}) &= \lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau+\int_t^{t+a}V_d(\tau)d\tau\Big)^2\Big)\\ &= \lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^2\Big)+\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_d(\tau)d\tau\Big)^2\Big)\\ &+2a^2\left(\frac{\nu^2}{2\theta}+\mu^2\right)\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}.\end{aligned}$$

To compute these expressions, we first derive the mean of the squared integrated continuous variance with the result of equation (16),

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^2\Big) &= \int_0^a\int_0^a\mathbb{E}\Big(\sigma_c^2(s)\sigma_c^2(u)\Big)dsdu\\ &= a^2\left(\mu^2+\frac{\nu^2}{2\theta}\right)^2+\frac{4\mu^2\nu^2(e^{-\theta a}+\theta a-1)}{\theta^3}+\frac{\nu^4(e^{-2\theta a}+2\theta a-1)}{4\theta^3},\end{aligned} \tag{24}$$

[2] We swap the order of integration of the jump part of the integrated variance, in order to obtain the compensated Poisson measure and further the expectation of it.

then we deal with the squared integrated discontinuous variance of equation (23),

$$
\begin{aligned}
\mathbb{E}\Big(\Big(\int_t^{t+a} V_d(\tau)d\tau\Big)^2\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau \int_{\mathbb{R}} Q\tilde{\mu}(du,dx) \\
&+ \int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau \int_{\mathbb{R}} Q\tilde{\mu}(du,dx) + a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\Big)^2\Big) \\
&= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau \int_{\mathbb{R}} Q\tilde{\mu}(du,dx)\Big)^2\Big) \\
&+ \mathbb{E}\Big(\Big(\int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau \int_{\mathbb{R}} Q\tilde{\mu}(du,dx)\Big)^2\Big) + a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2 \\
&= \mathbb{E}\Big(\int_t^{t+a}(\int_u^{t+a} e^{\kappa(u-\tau)}d\tau)^2 \int_{\mathbb{R}} Q^2\mu(du,dx)\Big) \\
&+ \mathbb{E}\Big(\int_{-\infty}^{t}(\int_t^{t+a} e^{\kappa(u-\tau)}d\tau)^2 \int_{\mathbb{R}} Q^2\mu(du,dx)\Big) + a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2 \\
&= \frac{e^{2\kappa a}-2e^{\kappa a}-2e^{-2\kappa a}+6e^{-\kappa a}+2\kappa a-3}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx) + a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2, \qquad (25)
\end{aligned}
$$

where we use the time-homogeneity of the jump Process and the isometry formula.[3]

Through equations (24) and (25) we calculate $\lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]})$.

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]}) &= a^2\left(\mu^2+\frac{\nu^2}{2\theta}\right)^2 + \frac{4\mu^2\nu^2(e^{-\theta a}+\theta a-1)}{\theta^3} + \frac{\nu^4(e^{-2\theta a}+2\theta a-1)}{4\theta^3} \\
&+ \frac{e^{2\kappa a}-2e^{\kappa a}-2e^{-2\kappa a}+6e^{-\kappa a}+2\kappa a-3}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx) + a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2 \\
&+ 2a^2\left(\frac{\nu^2}{2\theta}+\mu^2\right)\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}. \qquad (26)
\end{aligned}
$$

We know that $\lim_{t\to\infty} Var(IV_{(t,t+a]}) = \lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]}) - \lim_{t\to\infty}\mathbb{E}(IV_{(t,t+a]})^2$, which shows (21).

As for the covariance of IV, we have $\lim_{t\to\infty} Cov(IV_{(t,t+a]}, IV_{(t+h,t+h+a]}) = \mathbb{E}(IV_{(t,t+a]}IV_{(t+h,t+h+a]}) - \mathbb{E}(IV_{(t,t+a]})^2$, for $h = ia$, $i \in \mathbb{N}^+$. We obtain,

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}(IV_{(t,t+a]}IV_{(t+h,t+h+a]}) &= \lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big) \\
&+ \lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_d(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\Big) + \lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\Big) \\
&+ \lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_d(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big),
\end{aligned}
$$

[3] See Proposition 8.7 in Cont & Tankov (2003) for the isometry formula for general martingales.

where the third term $\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\Big)$ can be expressed as

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\left(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\right) &= \lim_{t\to\infty}\mathbb{E}\left(\int_t^{t+a} V_c(\tau)d\tau\left(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}\; d\tau \int_{\mathbb{R}} Q\tilde{\mu}(du,dx)\right.\right.\\
&\quad\left.\left.+\int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) + a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)\right)\\
&= a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\lim_{t\to\infty}\mathbb{E}\left(\int_t^{t+a} V_c(\tau)d\tau\right)\\
&= a^2\left(\frac{\nu^2}{2\theta}+\mu^2\right)\frac{\int_{\mathbb{R}} QG(dx)}{\kappa},
\end{aligned}\tag{27}$$

using the independence of the continuous and discontinuous parts of the integrated variance. The fourth term $\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_d(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big)$ also equals to $a^2\left(\frac{\nu^2}{2\theta}+\mu^2\right)\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}$.

With respect to $\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big)$ and $\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_d(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\Big)$, we have,

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big) &= \lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau\mathbb{E}\Big(\int_{t+h}^{t+h+a} V_c(\tau)d\tau|\mathcal{F}_{t+a}\Big)\Big)\\
&= \left(\left(a-\frac{e^{-2\theta h}-e^{-2\theta(h-a)}}{-2\theta}\right)\frac{\nu^2-2\theta\mu^2}{2\theta}+2\mu^2 a\right)\lim_{t\to\infty}\mathbb{E}\left(\int_t^{t+a}\sigma_c^2(\tau)d\tau\right)\\
&+\frac{e^{-2\theta h}-e^{-2\theta(h-a)}}{-2\theta}\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a}\sigma_c^2(\tau)\sigma_c^2(t+a)d\tau\Big)+2\mu\Big(\frac{e^{-\theta h}-e^{-\theta(h-a)}}{-\theta}\\
&-\frac{e^{-2\theta h}-e^{-2\theta(h-a)}}{-2\theta}\Big)\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a}\sigma_c^2(\tau)\sigma_c(t+a)d\tau\Big).
\end{aligned}$$

Using the *law of iterated expectations* jointly with equation (16) we get,

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a}\sigma_c^2(\tau)\sigma_c^2(t+a)d\tau\Big) &= \lim_{t\to\infty}\int_t^{t+a}\mathbb{E}\Big(\sigma_c^2(\tau)\mathbb{E}\big(\sigma_c^2(t+a)|\mathcal{F}_\tau\big)\Big)d\tau\\
&= a\left(\frac{\nu^2}{2\theta}+\mu^2\right)^2+\frac{2\mu^2\nu^2}{\theta^2}\left(1-e^{-\theta a}\right)+\frac{\nu^4}{4\theta^3}\left(1-e^{-2\theta a}\right),
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\sigma_c^2(\tau)\sigma_c(t+a)d\tau\Big) &= \lim_{t\to\infty}\int_t^{t+a}\mathbb{E}\Big(\sigma_c^2(\tau)\mathbb{E}\big(\sigma_c(t+a)|\mathcal{F}_\tau\big)\Big)d\tau\\
&= \frac{1-e^{-\theta a}}{\theta}\left(\mu^3+\frac{3\mu\nu^2}{2\theta}\right)+\mu\left(a-\frac{1-e^{-\theta a}}{\theta}\right)\left(\frac{\nu^2}{2\theta}+\mu^2\right).
\end{aligned}$$

We already know that the integrated continuous variance satisfies $\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a}\sigma_c^2(\tau)d\tau\Big)=a\left(\frac{\nu^2}{2\theta}+\mu^2\right)$.

Therefore, we have,

$$
\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\Big(\int_t^{t+a} V_c(\tau)d\tau \int_{t+h}^{t+h+a} V_c(\tau)d\tau\Big) = a\left(\frac{\nu^2}{2\theta}+\mu^2\right)\Bigg(\frac{(\nu^2-2\theta\nu^2)(e^{-2\theta h}-e^{-2\theta(h-a)})}{4\theta^2} \\
&+\frac{(\nu^2+2\theta\mu^2)a}{2\theta}\Bigg)+\Bigg(a\left(\frac{\nu^2}{2\theta}+\mu^2\right)^2+\frac{2\mu^2\nu^2(1-e^{-\theta a})}{\theta^2}+\frac{\nu^4(1-e^{-2\theta a})}{4\theta^3}\Bigg) \\
&\cdot\left(\frac{e^{-2\theta(h-a)}-e^{-\theta h}}{2\theta}\right)+\Bigg(\frac{1-e^{-\theta a}}{\theta}\left(\mu^3+\frac{3\mu\nu^2}{2\theta}\right)+\mu\left(a-\frac{1-e^{-\theta a}}{\theta}\right)\Bigg(\frac{\nu^2}{2\theta} \\
&+\mu^2\Bigg)\Bigg)\cdot\left(\frac{e^{-\theta(h-a)}-e^{-\theta h}}{\theta}-\frac{e^{-2\theta(h-a)}-e^{-2\theta h}}{2\theta}\right)2\mu, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\Big(\int_t^{t+a} V_d(\tau)d\tau \int_{t+h}^{t+h+a} V_d(\tau)d\tau\Big) = \mathbb{E}\Bigg(\Bigg(\Big(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) \\
&+\int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx)+a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\Bigg) \\
&\cdot\Bigg(\int_{t+h}^{t+h+a}\int_u^{t+h+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) \\
&+\int_{-\infty}^{t+h}\int_{t+h}^{t+h+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx)+a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\Bigg)\Bigg) \\
&=\mathbb{E}\Big(\int_{-\infty}^{t}\big(\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{t+h}^{t+h+a} e^{\kappa(u-\tau)}d\tau\big)Q^2\mu(du,dx)\Big)+a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2 \\
&=\frac{(e^{\kappa a}-1)e^{-\kappa(2a+h)}}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx)+a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2. \qquad (29)
\end{aligned}
$$

Thus with equation (27), (28) and (29) we have,

$$
\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}(IV_{t,t+a}IV_{t+h,t+h+a}) = a\left(\frac{\nu^2}{2\theta}+\mu^2\right)\left(\frac{(\nu^2-2\theta\nu^2)(e^{-2\theta h}-e^{-2\theta(h-a)})}{4\theta^2}+\frac{(\nu^2+2\theta\mu^2)a}{2\theta}\right) \\
&+\left(a\left(\frac{\nu^2}{2\theta}+\mu^2\right)^2+\frac{2\mu^2\nu^2(1-e^{-\theta a})}{\theta^2}+\frac{\nu^4(1-e^{-2\theta a})}{4\theta^3}\right)\cdot\left(\frac{e^{-2\theta(h-a)}-e^{-\theta h}}{2\theta}\right) \\
&+\left(\frac{1-e^{-\theta a}}{\theta}(\mu^3+\frac{3\mu\nu^2}{2\theta})+\mu\left(a-\frac{1-e^{-\theta a}}{\theta}\right)\left(\frac{\nu^2}{2\theta}+\mu^2\right)\right) \\
&\cdot\left(\frac{e^{-\theta(h-a)}-e^{-\theta h}}{\theta}-\frac{e^{-2\theta(h-a)}-e^{-2\theta h}}{2\theta}\right)2\mu+\frac{(e^{\kappa a}-1)e^{-\kappa(2a+h)}}{2\kappa^3}\int_{\mathbb{R}} Q^2G(dx) \\
&+a^2\left(\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\right)^2+2a^2\left(\frac{\nu^2}{2\theta}+\mu^2\right)\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}. \qquad (30)
\end{aligned}
$$

With equation (30) we then obtain the covariance of IV (22). □

Theorem 3.2 (Moments of the Quadratic Variation)**.** *For the Quadratic Variation defined as* $QV_{(t,t+a]} =$

$\int_t^{t+a}\sigma^2(s)ds+\int_t^{t+a}\int_{\mathbb{R}}J^2\mu(ds,dx)$, *we find the following moments,*

$$\lim_{t\to\infty}\mathbb{E}(QV_{(t,t+a]})=a\left(\frac{\nu^2}{2\theta}+\mu^2+\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}\right)+a\int_{\mathbb{R}}J^2G(dx), \tag{31}$$

$$\begin{aligned}\lim_{t\to\infty}Var(QV_{(t,t+a]})&=\frac{4\mu^2\nu^2(e^{-\theta a}+\theta a-1)}{\theta^3}+\frac{\nu^4(e^{-2\theta a}+2\theta a-1)}{4\theta^3}\\&+\frac{e^{2\kappa a}-2e^{\kappa a}-2e^{-2\kappa a}+6e^{-\kappa a}+2\kappa a-3}{2\kappa^3}\int_{\mathbb{R}}Q^2G(dx)\\&+a\frac{2(\kappa a+e^{-\kappa a}-1)}{\kappa^2}\frac{\int_{\mathbb{R}}QJ^2G(dx)}{\kappa}+a\int_{\mathbb{R}}J^4G(dx).\end{aligned}\tag{32}$$

Proof. We omit the derivation of the mean of QV here as it is straightforward. As for the variance of QV, we start from deriving $\lim_{t\to\infty}\mathbb{E}(QV^2_{(t,t+a]})$:

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}(QV^2_{(t,t+a]})&=\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau+\int_t^{t+a}V_d(\tau)d\tau+\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(du,dx)+a\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}\Big)^2\Big)\\&=\lim_{t\to\infty}\mathbb{E}(IV^2_{(t,t+a]})+\lim_{t\to\infty}2a^2\mathbb{E}(V_c(t))\int_{\mathbb{R}}J^2G(dx)+a\frac{2(\kappa a+e^{-\kappa a}-1)}{\kappa^2}\int_{\mathbb{R}}QJ^2G(dx)\\&+2a^2\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}\int_{\mathbb{R}}J^2G(dx)+a\int_{\mathbb{R}}J^4G(dx)+a^2(\int_{\mathbb{R}}J^2G(dx))^2\\&=a^2(\mu^2+\frac{\nu^2}{2\theta})^2+\frac{4\mu^2\nu^2(e^{-\theta a}+\theta a-1)}{\theta^3}+\frac{\nu^4(e^{-2\theta a}+2\theta a-1)}{4\theta^3}\\&+\frac{e^{2\kappa a}-2e^{\kappa a}-2e^{-2\kappa a}+6e^{-\kappa a}+2\kappa a-3}{2\kappa^3}\int_{\mathbb{R}}Q^2G(dx)+a^2(\frac{\int_{\mathbb{R}}QG(dx)}{\kappa})^2\\&+2a^2(\frac{\nu^2}{2\theta}+\mu^2)\frac{\int_{\mathbb{R}}QG(dx)}{\kappa}+2a^2(\frac{\nu^2}{2\theta}+\mu^2+\frac{\int_{\mathbb{R}}QG(dx)}{\kappa})\int_{\mathbb{R}}J^2G(dx)\\&+a\frac{2(\kappa a+e^{-\kappa a}-1)}{\kappa^2}\int_{\mathbb{R}}QJ^2G(dx)+a\int_{\mathbb{R}}J^4G(dx)+a^2(\int_{\mathbb{R}}J^2G(dx))^2.\end{aligned}$$

Therefore we can obtain equation (32) by $\lim_{t\to\infty}Var(QV_{(t,t+a]})=\lim_{t\to\infty}\mathbb{E}(QV^2_{(t,t+a]})-\lim_{t\to\infty}\mathbb{E}(QV_{(t,t+a]})^2$. □

Note that in Theorem 3.1 and Theorem 3.2 we express the moments of jumps in the form of cumulants, as we do not impose additional assumptions on jumps. The cumulants can be easily transferred into analytic forms when specifying the distribution of jumps.

4 Details of Estimation

4.1 Theoretic Foundations

In this section we introduce some general theory on asymptotic properties of power and multipower variations, as well as techniques for disentangling jumps from observed asset returns. Our methodology is based on using moment conditions for power variations and multipower variations to obtain a general method of moments (GMM) estimator for high frequency data.

The discussion below is valid for a return process which can be represented via an arbitrary semi-martingale X_t of type $X_t = X_0 + \int_0^t \sigma_s dW_s + Y_t$, where the volatility process σ_t ($\sigma > 0$) is adapted to the same filtration $\mathcal{F}_t$, while W_t is a Brownian Motion and Y_t is a pure jump process. We let Δ_n denote the width of the sampling interval, and the process X_t is observed at equally spaced times $i\Delta_n$ for $i = 0, 1, ..., \lfloor T/\Delta_n \rfloor$. We denote the observed return as,

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n},$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x for any real number x. We use $n = \lfloor 1/\Delta_n \rfloor$ to represent the observation frequency during a time interval of unit length. In conclusion there will be nT observations during the period $[0, T]$.

4.1.1 Power Variations

For any $p > 0$, we define the *realized power variation* of order p for X_t as

$$B(p, \Delta_n)_t = \sum_{i=1}^{n} |\Delta_i^n X|^p. \tag{33}$$

This corresponds to daily realized variance (hereafter RV) for $p = 2$ and one observation period is one day. This has been extensively exploited in financial econometrics, compare Andersen et al. (2009). We have

$$RV_t = B(2, \Delta_n) = \sum_{i=1}^{n} |\Delta_i^n X|^2,$$

and for $\Delta_n \to 0$ (the observation frequency n goes to infinity), RV converges to *realized quadratic variation* in probability (see Barndorff-Nielsen & Shephard (2003b)), i.e.

$$RV_t \xrightarrow{\mathbb{P}} [X, X]_t, \tag{34}$$

where $[X, X]_t$ is the *quadratic variation* (hereafter QV) process of X_t (in this case $[X, X]_t = \int_0^t \sigma_s^2 ds + \sum_{0<s\leqslant t} Y_t^2$). This result is crucial in order to match moment conditions when implementing the general method of moments (GMM). One may refer to Jacod & Protter (2012) for the related central limit theorem (CLT) for the convergence rate and the limiting distribution, in order to make inference about the estimation statistics.

Another special case is when $p = 4$, the *realized fourth variation* (hereafter FV) is able to eliminate the continuous part of returns, and it converges to the sum of jumps raised to power four during one single period[4], i.e.

$$FV_t = B(4, \Delta_n) = \sum_{i=1}^{n} |\Delta_i^n X|^4 \xrightarrow{\mathbb{P}} \sum_{0<s\leqslant t} Y_t^4. \tag{35}$$

4.1.2 Bipower Variation

There are several ways to disentangle the various sources from QV, one common and jump-robust estimator for the *integrated variance* (IV) is multipower variations

$$IV_t = \int_0^t \sigma_s^2 ds.$$

Following Barndorff-Nielsen & Shephard (2004), we define the *realized bipower variation* (BV) as

$$BV_t = \frac{\pi}{2} \sum_{i=2}^{n} |\Delta_i^n X||\Delta_{i-1}^n X|. \tag{36}$$

By defining BV, we are able to express RV as the sum of BV and *jump variation* JV

$$JV_t = RV_t - BV_t.$$

4.2 GMM Estimator

We construct a GMM type estimator by using moment conditions of the power variations and the bipower variations. In our Linear quadratic volatility with co-jumps model, the parameter set is given as

$$\xi = \left(\theta, \mu, \nu, \kappa, \int_{\mathbb{R}} J^2 G(dx), \int_{\mathbb{R}} J^4 G(dx), \int_{\mathbb{R}} QG(dx), \int_{\mathbb{R}} Q^2 G(dx), \int_{\mathbb{R}} QJ^2 G(dx)\right)'.$$

We employ the following assumptions in order to guarantee the existence and consistency of the asymptotic distribution of our estimator.

Assumption 4.1. (i) $\alpha(t)$ *is a locally bounded predictable process, and* $\int_{\mathbb{R}} J^2 G(dx) < \infty$.

(ii) $\theta > 0$, $\mu > 0$, *and* $\nu > 0$.

(iii) $\kappa > 0$, $\int_{\mathbb{R}} QG(dx) < \infty$ *and* $\int_{\mathbb{R}} Q^2 G(dx) < \infty$.

(iv) $\int_{\mathbb{R}} QJ^2 G(dx) < \infty$ *and* $\int_{\mathbb{R}} J^4 G(dx) < \infty$.

Parts (i) and (iii) of the assumption guarantee the local existence of the quadratic variation process of the underlying return process in (10), which is obvious. Part (ii) guarantees the stationarity of the OU process $\sigma_c(t)$[5], and Part (iii) implies that the jump part of the volatility $V_d(t)$ is weakly stationary and square-integrable. Part (iv) is used to guarantee the existence of the second moment of the QV (see equation (32)).

[4] Todorov (2009b) and Todorov (2011) also used this as one of moment conditions.
[5] See Doob (1942).

Let $f = \mathbb{E}(f_t)$ be a finite subset of the collection of all moment conditions that are sufficient to identify all the parameters in ξ. We identify the exact moments chosen for estimation later in this section. Then we use the empirical expectation $g(\xi)$ as an approximation for the unconditional expectation, defined as the mean of sample moments,

$$g(\xi) = T^{-1} \sum_{t=1}^{T} (f_t - f),$$

where T is the number of observations. Here our feasible estimator, i.e. the objective optimization function is defined as

$$\hat{\xi} = \arg\min_{\xi} \; g(\xi)' \hat{W}^{-1} g(\xi), \tag{37}$$

where $g(\xi)$ is calculated from moment conditions where QV and IV are replaced by RV and BV correspondingly, and $\hat{W}$ is a consistent estimator of the asymptotic positive semi-definite variance-covariance matrix W of moment conditions f_t,

$$W = \sum_{l=-\infty}^{\infty} \mathbb{E}(g_t g'_{t-l}) = \sum_{l=-\infty}^{\infty} Cov(f_t f'_{t-l}).$$

The matrix W^{-1} is also called the optimal weighting matrix. The asymptotic variance-covariance matrix of $\sqrt{T}(\hat{\xi} - \xi)$ is $(DW^{-1}D')^{-1}$, where $D' = \mathbb{E}(\frac{\partial g}{\partial \xi'})$.[6] We provide explicit expressions for W in the supplementary appendix. However, we do not use these results in this article as it would require further parametrization and leave this as an application for future research. Note that the estimator (37) resembles that in Todorov (2009b), where inference and convergence rate are established. In our article we have a similar central limit theorem result as we merely replace the realized tripower variation by BV. In the robustness test we also use the realized tripower variation (RTV) as a proxy for IV, for which the performance has been tested by Mancini (2001).

As for the exact moment conditions we have used for matching, these are

1. Mean and Variance of QV

2. Mean, Variance and Auto-correlation of IV

3. Mean of FV

With respect to the auto-correlation of IV, we use auto-correlation for lag 1, lag 2, lag 3, lag 5, lag 6, lag 7, lag 9, and the average of auto-correlation for lag 11-20, lag 21-30 and lag 31-40. Overall we have 15 moments, hence we have 6 degrees of freedom for the Linear Quadratic Stochastic Volatility with Co-Jumps model. Using more than 15 moments did not improve the result, while slowing down the computational process.

[6] We refer to Hamilton (1994) for more details.

Regarding the truncation level of the discrete return observations, we use $\alpha = 4.5$ to truncate the return process. The RV, BV and truncated returns of the sample are presented in Figure 1. As for the consistent estimator $\hat{W}^{-1}$ of the optimal weighting matrix W^{-1}, we use the inverse of a Heteroskedasticity and Autocorrelation Consistent (HAC) covariance matrix, specifying a Parzen kernel[7] with a lag length of 80.

4.3 Affine jump diffusion model

We assess our model against the conventional affine jump diffusion model, also referred to as the Bates model:

$$dR(t) = \alpha(t)dt + \sqrt{V(t)}dW_1(t) + \int_{\mathbb{R}} J\tilde{\mu}(dt, dx), \tag{38}$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t). \tag{39}$$

Notice, that this model does not feature jumps in the volatility process. However we make similar assumptions as before on the square integrability of jumps in the return process, in order to guarantee the existence of moment conditions. The moment conditions for the general affine jump diffusion model have been provided in Todorov (2011). Additionally, we apply the reparameterizations $\sigma_v = \sigma\sqrt{\frac{\theta}{2\kappa}}$ to avoid identification problems. The variance follows a CIR process, hence we impose the Feller condition $\sigma_v < \theta$ and $\kappa > 0$ to guarantee stationarity and positivity.

4.4 Data and Estimation Outputs

In our empirical analysis, we use 5-min returns from the S&P 500 index from Jan 1, 2004 to Dec 31, 2016, acquired from Tick Data Database. Excluding weekends, holidays and non-trading days, we have 3250 days of trading data in total. Due to the limitation of the database, we only use high frequency data from 09.00 - 15.00 each day, which consists of 73 5-min observations in total. Estimation results for different models are reported in Table 5.1 and Table 5.2.

Panel A in Table 5.1 shows that our linear quadratic volatility with jumps model works well. Evidence for this is the significance of parameters and the small J-statistics in the over-identification test. Moreover, the two (arguably) most important parameters, those that reflect the two mean-reverting speeds in our model, indicate two different half-life periods, i.e. the discontinuous part of the volatility dies out much more quickly than the continuous one. This interesting result is in line with Todorov (2009a). Another interesting observation is that there is evidence for non-zero correlation between the jump sizes in the return and volatility processes from the estimation results of cumulants in the same panel.

[7] See Chapter 6 of Gallant (2009).

On the other hand, the Bates model (which excludes jumps in the volatility process) gets rejected by the J-test, as shown in Panel A of Table 5.1. This highlights the importance of incorporating jumps in the volatility process. Table 5.2 provides further evidence of the superiority of our model as compared with the traditional Bates model.

In the following discussion we estimate our model by using different proxies for integrated volatility, in order to verify the robustness of our results. We use the Realized Tripower Variation (hereafter RTV) and Truncated Variation (TV) to replace the BV used in the previous part. These were also used in Todorov (2009b) and Mancini (2009) respectively. The estimation results obtained from using these two proxies are presented in panel B of Table 5.1. We do not discuss these further as they are similar to the previous part. The RTV and TV are defined as follows.

$$RTV_t = \mu_{2/3}^{-3} \sum_{i=3}^{n} |\Delta_i^n X|^{2/3} |\Delta_{i-1}^n X|^{2/3} |\Delta_{i-2}^n X|^{2/3},$$

$$TV_t = \sum_{i=1}^{n} |r_{t,i}^c|^2,$$

where $\mu_a = \mathbb{E}(|u|^a)$ and $u \sim \mathcal{N}(0,1)$, and $r_{t,i}^c$ is the continuous part of the discretely observed return. By defining the discretely observed return,

$$r_{t,i} = \Delta_{(t-1)n+i}^n X, \; i = 1, 2, ..., n,$$

and a truncation threshold

$$\text{CUT}_{t,i} = \alpha \Delta_n^{0.49} \sqrt{\tau_i BV_{t-1}},$$

$r_{t,i}^c$ is denoted as

$$r_{t,i}^c = r_{t,i} \mathbb{1}_{|r_{t,i}| \leqslant \text{CUT}_{t,i}},$$

where $\mathbb{1}_E(x)$ is an indicator function, and α is suggested to lie in the interval $[3.5, 4.5]$.(Aït-Sahalia & Jacod, 2009a; Andersen et al., 2011)

Finally, we run our methodology through a fully parametrized version of our model. We follow the conventional assumption that jump sizes in both return and volatility process are normally distributed ($J(x) \sim \mathcal{N}(\mu_J, \sigma_J^2)$ and $Q(x) \sim \mathcal{N}(\mu_Q, \sigma_Q^2)$), and further that these two normal distributions may be correlated with correlation coefficient ρ. We observe that the parameters that relate to these distributions become non-significant, although the J-test does not reject the fully parametrized model, see 5.3. In conclusion, we want to know whether the distribution assumption affects our result. If not, the assumption of normally distributed jump-sizes for both return and volatility may be fundamentally flawed, as the previous results clearly indicate the existence of jumps. To follow up on this analysis, we provide the following Monte Carlo experiment.

4.5 Monte Carlo Experiment

In order to support our conclusion on the non-normality of jump sizes in our model, we conduct a small panel of Monte Carlo experiments. The idea is to run a simulation based on our fully parametrized model, and estimate it again to see whether the coefficients of the jump distribution are significant or not.[8] The simulated data replace the high frequency data from the S&P 500 previously used. Details of the data generating process are stated in Table 5.4. The discretization method used in the Monte Carlo simulation is Euler-Maruyama. Since the randomness of jump processes originates from two sources, i.e. the jump time and the jump size, we follow the method provided by Kloeden & Platen (2013) when simulating compound Poisson processes. We model the high frequency observations by determining that the data are observed in 1-min frequency, and the number of trading days is 1000 with 8 trading hours in each day. Again the RV and BV are summed up in 5-min frequency. The results are presented in Table 5.5, Table 5.6 and Table 5.7.

We summarize the results of our Monte Carlo experiment as follows:

- Our estimation based on the simulated data with distribution assumption shows satisfactory performance, since each calibrated parameter is significant based on the simulated data. Moreover, observing the root mean square error (RMSE), the calibration results on the simulated data are nearly invariant regardless of the jump frequency or whether distribution assumptions are imposed.[9] Therefore, this contradicts the assumption of normally distributed jump sizes for the S&P 500 high frequency dataset.

- The similarity of results with and without distribution specifications, implies that our estimator is indeed consistent with parameters fully specified or with only cumulants estimated.

- The mean-reversion speed of the discontinuous part of volatility is slightly upward biased. That makes the stationary volatility, i.e. the long-run mean, downward biased, which was indicated by Andersen & Sørensen (1996) and Todorov (2011) as well.

[8] One can also conduct a similar simulation approach to test how accurate the asymptotic analysis of the estimator is based on the unobservable QV and IV in practice. We refer to Todorov (2009b) for such analysis. We thank an anonymous referee for pointing this out.

[9] We observe that the bias and RMSE are slightly larger when using the normality assumption than without it; this at first seems counter-intuitive. The reason however is likely that with the normality assumption, we have one more additional parameter. We thank an anonymous referee for suggesting this explanation to us.

5 Conclusion

In this paper, we construct a linear quadratic volatility with jumps model, in which the variance process is a superposition of two separate parts, i.e. the continuous part (a squared Gaussian OU process) and the discontinuous part (a Lévy-driven OU process). We derive the analytic moment conditions for the quadratic variation and the integrated variance of our model and then calibrate our model by matching sample moments of realized power variations and realized bipower variations. Our results indicate that our model fits the market data well and shows superior performance compared with the conventional affine model, i.e. Bates model, which does not incorporate jumps in the volatility process. We also conduct two robustness tests, i.e. use other types of estimator (realized tripower variation and truncated variation) to consolidate our basic results.

While the distribution of both jump sizes of our model in the context are left unrestricted, i.e. we estimate them in cumulants, it is interesting that we get insignificant parameters when we specify the jump size distribution to be normal. That leads us to implement a Monte Carlo experiment in order to investigate the identification problem in the distribution of the jump sizes. The Monte Carlo experiments confirms our view that the traditional normal distribution may not be suitable for S&P 500 high frequency data.

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Table 5.1: Estimation for Linear Quadratic Volatility with Jump Diffusion Models

$$dR(t) = \alpha(t)dt + \sigma(t-)dW_1(t) + \int_{\mathbb{R}} J\tilde{\mu}(dt, dx)$$
$$\sigma^2(t) = V_c(t) + V_d(t)$$
$$V_c(t) = \sigma_c^2(t), \text{ where } d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW_2(t)$$
$$dV_d(t) = -\kappa V_d(t)dt + \int_{\mathbb{R}} Q\mu(dt, dx)$$

Panel A: Linear Quadratic Volatility with Jump Diffusion (By (Multi)Power Variations)		
	With Price and Volatility Jumps	With Only Price Jumps
θ	0.7032 (0.0661)***	0.445 (0.042)***
μ	5.181e-3 (2.998e-3)*	6.573e-3 (1.358e-3)***
ν	6.289e-3 (1.79e-3)***	5.202e-3 (1.103e-3)***
κ	2.257 (1.57e-3)***	
$\int_{\mathbb{R}} J^2G(dx)$	2e-6 (3e-6)	6e-6 (3.5e-5)
$\int_{\mathbb{R}} J^4G(dx)$	6.982e-3 (1e-6)***	0.102 (5.96e-3)
$\int_{\mathbb{R}} QG(dx)$	1.03e-4 (2.9e-5)***	
$\int_{\mathbb{R}} Q^2G(dx)$	2.69e-3 (4e-6)***	
$\int_{\mathbb{R}} QJ^2G(dx)$	0.016 (1e-5)***	
Overidentification Test		
Test Statistics	0.0031	25.4494
Degree of Freedom	6	8
P-Value	1.0000	0.0013

Panel B: Linear Quadratic Volatility with Jump Diffusion (Robustness Tests)		
	By Tripower Variation	By Truncated Variation
θ	0.7986 (0.0518)***	0.5741 (0.1014)***
μ	4.126e-3 (1.247e-3)***	3.251e-3 (1.086e-3)***
ν	7.489e-3 (1.146e-3)***	6.816e-3 (1.493e-3)***
κ	2.3815 (1.85e-4)***	2.3539 (0.0392)***
$\int_{\mathbb{R}} J^2G(dx)$	2e-6 (9e-6)	4.5e-5 (3e-6)***
$\int_{\mathbb{R}} J^4G(dx)$	7.457e-3 (1e-6)***	1.905e-3 (1e-6)***
$\int_{\mathbb{R}} QG(dx)$	1e-4 (2.2e-5)***	1.08e-4 (1.2e-5)***
$\int_{\mathbb{R}} Q^2G(dx)$	4.025e-3 (1e-6)***	2.7e-5 (1e-6)***
$\int_{\mathbb{R}} QJ^2G(dx)$	0.0179 (2e-6)***	1.194e-3 (4.6e-5)***
Overidentification Test		
Test Statistics	0.0061	0.0090
Degree of Freedom	6	6
P-Value	1.0000	1.0000

Note: Standard errors in parentheses. [***] Significant at 1% level; [**] Significant at 5% level; [*] Significant at 10% level. In Sagan's J test, the null hypothesis is that the over-identifying restrictions are valid.

Table 5.2: Estimation for the Affine Jump Diffusion Model

$$dR(t) = \alpha(t)dt + \sqrt{V(t)}dW_1(t) + \int_{\mathbb{R}} J\tilde{\mu}(dt, dx)$$

$$dV(t) = \kappa(\theta - V(t))dt + \sigma\sqrt{V(t)}dW_2(t)$$

Affine Jump Diffusion Model	
	With Only Price Jumps
θ	0.0346 (2.075e-3)***
κ	0.3541 (0.0292)***
σ	0.0346 (2.288e-3)***
$\int_{\mathbb{R}} J^2 G(dx)$	0.0216 (1.5e-3)***
$\int_{\mathbb{R}} J^4 G(dx)$	2.462e-3 (4.32e-4)***
Overidentification Test	
Test Statistics	699.6899
Degree of Freedom	8
P-Value	0.0000

Table 5.3: Estimation for Linear Quadratic Volatility Model with Normal Jumps

$$dR(t) = \alpha(t)dt + \sigma(t-)dW_1(t) + \int_{\mathbb{R}} J\tilde{\mu}(dt, dx)$$
$$\sigma^2(t) = V_c(t) + V_d(t)$$
$$V_c(t) = \sigma_c^2(t), \text{ where } d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW_2(t)$$
$$dV_d(t) = -\kappa V_d(t)dt + \int_{\mathbb{R}} Q\mu(dt, dx)$$

With Normal Jumps in Price and Volatility	
θ	0.6145 (0.2215)***
μ	0.0003 (0.0006)
ν	0.0060 (0.0010)***
κ	0.0216 (1.5e-3)
λ	4e-5 (101.377)
μ_J	0.5619 (96.454)
σ_J	0.5913 (63.715)
μ_Q	0.3920 (90.203)
σ_Q	0.5406 (21.047)
ρ	0.0110 (10.487)
Overidentification Test	
Test Statistics	4.3168
Degree of Freedom	5
P-Value	0.5048

$$dR(t) = \alpha(t)dt + \sigma(t-)dW_1(t) + \int_{\mathbb{R}} J\tilde{\mu}(dt, dx)$$

$$\sigma^2(t) = V_c(t) + V_d(t)$$

$$V_c(t) = \sigma_c^2(t), \text{ where } d\sigma_c(t) = \theta(\mu - \sigma_c(t))dt + \nu dW_2(t)$$

$$dV_d(t) = -\kappa V_d(t)dt + \int_{\mathbb{R}} Q\mu(dt, dx)$$

Table 5.4: Details on Monte Carlo

Case	Parameters									
	θ	μ	ν	κ	λ	μ_J	σ_J	μ_Q	σ_Q	ρ
low frequency	0.6	5e-3	6e-3	2.5	0.1	1e-4	3.2e-3	1e-3	0.01	-0.7
high frequency	0.6	5e-3	6e-3	2.5	0.3	1e-4	3.2e-3	1e-3	0.01	-0.7
large mean and variance	0.6	5e-3	6e-3	2.5	0.1	5e-4	0.01	5e-3	0.02	-0.7

Note: We simulate the jump process as a compound Poisson type here for the sake of parsimony, while other types are also feasible. In addition we simulate the jump sizes to be correlated normal distributions, and the number of iteration is 100 for each simulation.

Table 5.5: Monte Carlo Results with low jump frequency

Parameter	True Value	Mean	RMSE	5th percentile	Median	95th percentile
Panel A: low frequency without distribution assumption						
θ	0.6	0.4995	0.1005	0.4989	0.4995	0.4999
μ	5e-3	4.893e-3	1.184e-4	4.807e-3	4.888e-3	4.981e-3
ν	6e-3	4.825e-3	1.18e-3	4.741e-3	4.82e-3	4.914e-3
κ	2.5	2.469	0.0314	2.465	2.469	2.472
$\int_{\mathbb{R}} J^2 G(dx)$	1e-6	9.9997e-7	3.060e-11	9.9996e-7	9.9997e-7	9.9997e-7
$\int_{\mathbb{R}} J^4 G(dx)$	3e-11	2.5e-8	2.497e-8	2.5e-8	2.5e-8	2.5e-8
$\int_{\mathbb{R}} QG(dx)$	1e-4	7.953e-5	2.045e-5	7.948e-5	7.955e-5	7.962e-5
$\int_{\mathbb{R}} Q^2 G(dx)$	1.01e-5	1.546e-5	5.364e-6	1.504e-5	1.548e-5	1.586e-5
$\int_{\mathbb{R}} QJ^2 G(dx)$	1.44e-9	1.5e-8	1.356e-8	1.5e-8	1.5e-8	1.5e-8
Panel B: low frequency with distribution assumption						
θ	0.6	0.50104	0.099	0.49991	0.50099	0.50206
μ	5e-3	4.933e-3	1.46e-4	4.463e-3	4.969e-3	4.984e-3
ν	6e-3	4.925e-3	1.083e-3	4.454e-3	4.959e-3	4.974e-3
κ	2.5	2.4992	3.084e-3	2.4886	2.4999	2.5004
λ	0.1	0.1239	0.0239	0.12169	0.12410	0.12413
μ_J	1e-4	2.5e-4	1.5e-4	2.49e-4	2.5e-4	2.5e-4
σ_J	3.16e-3	4.97e-3	1.81e-3	4.58e-3	4.997e-3	5.001e-3
μ_Q	1e-3	9.83e-4	3.3e-5	8.82e-4	9.91e-4	9.92e-4
σ_Q	0.01	9.96e-3	6.1e-5	9.91e-3	9.95e-3	1.01e-2
ρ	-0.7	-0.7513	0.0513	-0.7528	-0.7512	-0.7509

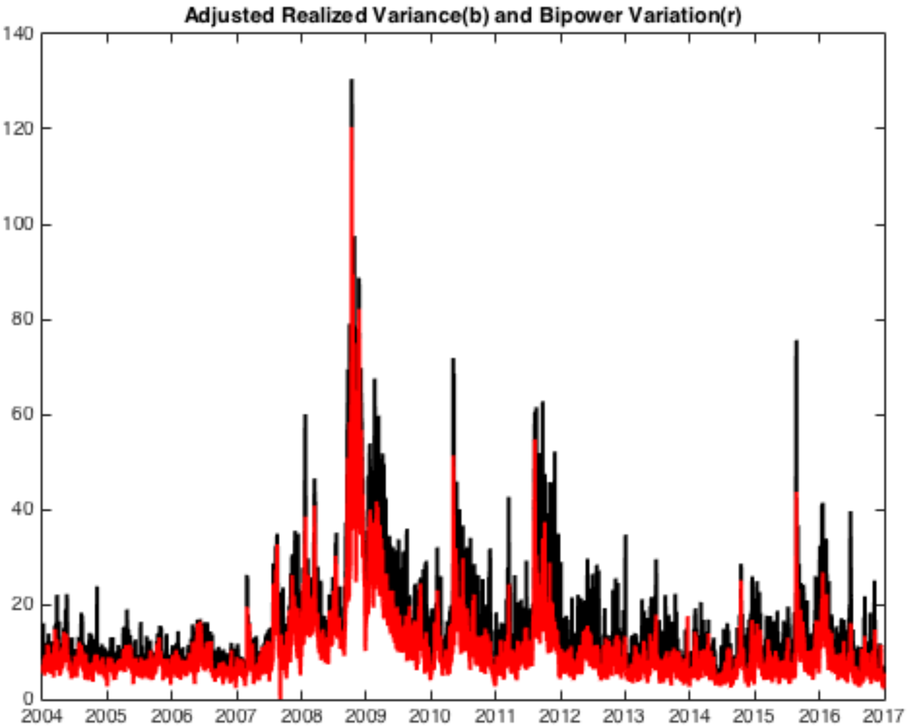
Table 5.6: Monte Carlo Results with high jump frequency

Parameter	True Value	Mean	RMSE	5th percentile	Median	95th percentile
Panel A: high frequency without distribution assumption						
θ	0.6	0.5444	0.056	0.5405	0.5415	0.5493
μ	5e-3	5.55e-3	8.52e-4	4.671e-3	6.023e-3	6.127e-3
ν	6e-3	5.385e-3	8.65e-4	4.652e-3	5.844e-3	5.943e-3
κ	2.5	2.492	0.029	2.458	2.514	2.519
$\int_{\mathbb{R}} J^2 G(dx)$	3e-6	2.0003e-6	1e-6	2e-6	2.0004e-6	2.0004e-6
$\int_{\mathbb{R}} J^4 G(dx)$	9e-11	2.5e-8	2.491e-8	2.5e-8	2.5e-8	2.5e-8
$\int_{\mathbb{R}} QG(dx)$	3e-4	2.6878e-4	3.29e-5	2.5615e-4	2.7656e-4	2.7836e-4
$\int_{\mathbb{R}} Q^2 G(dx)$	3.03e-5	3.2319e-5	5.24e-6	2.6331e-5	3.6203e-5	3.6543e-5
$\int_{\mathbb{R}} QJ^2 G(dx)$	4.32e-9	2.5e-8	2.068e-8	2.5e-8	2.5e-8	2.5e-8
Panel B: high frequency with distribution assumption						
θ	0.6	0.501	0.099	0.4945	0.4949	0.5149
μ	5e-3	5.103e-3	1.04e-4	5.071e-3	5.105e-3	5.129e-3
ν	6e-3	5.088e-3	9.13e-4	5.054e-3	5.093e-3	5.111e-3
κ	2.5	2.5064	6.92e-3	2.5040	2.5051	2.5115
λ	0.3	0.2997	4.54e-4	0.2992	0.2996	0.3003
μ_J	1e-4	2.5e-4	1.5e-4	2.49e-4	2.5e-4	2.5e-4
σ_J	3.16e-3	5.007e-3	1.85e-3	5.001e-3	5.008e-3	5.011e-3
μ_Q	1e-3	1.041e-3	4.1e-5	1.035e-3	1.04e-3	1.048e-3
σ_Q	0.01	1.086e-2	6.3e-5	1.066e-2	1.089e-2	1.099e-2
ρ	-0.7	-0.7505	0.0505	-0.7509	-0.7506	-0.7501

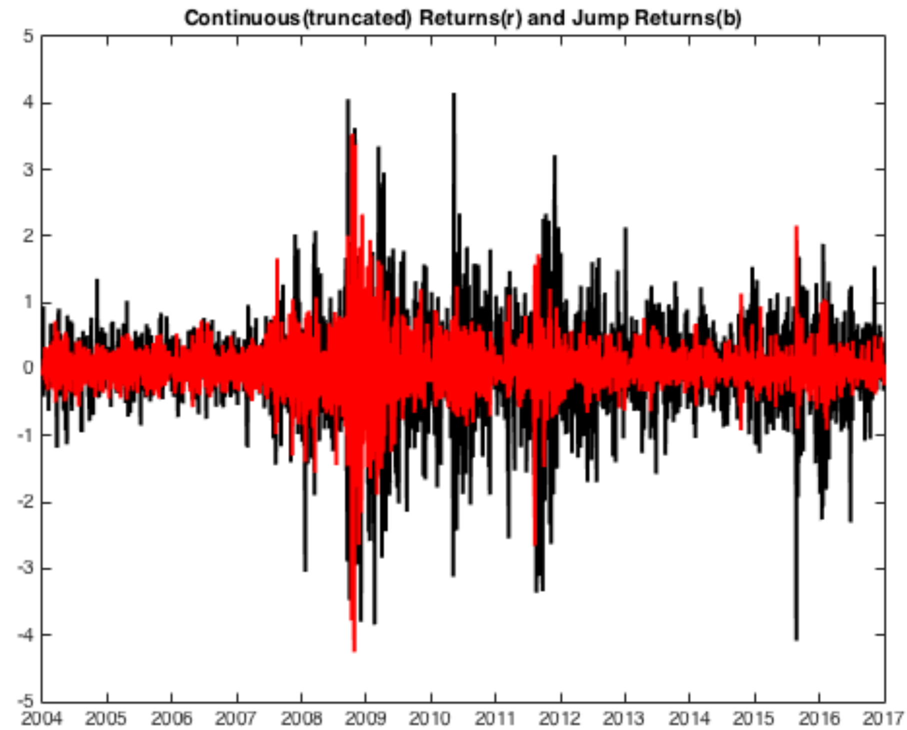
Table 5.7: Monte Carlo Results with Large Mean and Variance

Parameter	True Value	Mean	RMSE	5th percentile	Median	95th percentile
Panel A: large mean and variance without distribution assumption						
θ	0.6	0.550898	0.0491	0.550768	0.550891	0.551026
μ	5e-3	4.303e-3	6.97e-4	4.267e-3	4.309e-3	4.337e-3
ν	6e-3	4.300e-3	1.7e-3	4.273e-3	4.300e-3	4.333e-3
κ	2.5	2.5001	3.73e-3	2.4943	2.4998	2.5073
$\int_{\mathbb{R}} J^2G(dx)$	1.0025e-5	1.4972e-5	4.95e-6	1.4970e-5	1.4972e-5	1.4973e-5
$\int_{\mathbb{R}} J^4G(dx)$	3.02e-9	2.5e-8	2.198e-8	2.5e-8	2.5e-8	2.5e-8
$\int_{\mathbb{R}} QG(dx)$	5e-4	3.2415e-4	1.7587e-4	3.1964e-4	3.2434e-4	3.2846e-4
$\int_{\mathbb{R}} Q^2G(dx)$	4.25e-5	5.2358e-5	9.87e-6	5.1549e-5	5.2346e-5	5.3247e-5
$\int_{\mathbb{R}} QJ^2G(dx)$	6.388e-8	4e-8	2.39e-8	4e-8	4e-8	4e-8
Panel B: large mean and variance with distribution assumption						
θ	0.6	0.5061	0.096	0.4750	0.5028	0.5383
μ	0.005	4.863e-3	1.37e-4	4.851e-3	4.862e-3	4.872e-3
ν	0.006	4.861e-3	1.14e-3	4.841e-3	4.864e-3	4.872e-3
κ	2.5	2.4749	0.026	2.4576	2.4751	2.4862
λ	0.1	0.1159	0.016	0.1136	0.1156	0.1178
μ_J	5e-4	3.999e-4	1e-4	3.999e-4	3.999e-4	3.999e-4
σ_J	0.01	0.014769	4.77e-3	0.014553	0.014771	0.015024
μ_Q	5e-3	3.546e-3	1.45e-3	3.521e-3	3.546e-3	3.571e-3
σ_Q	0.02	0.01870	1.31e-3	0.01846	0.01867	0.01898
ρ	-0.7	-0.7512	0.0513	-0.7554	-0.7518	-0.7462

Figure 1: The Power&Bipower Variations and Truncated Returns



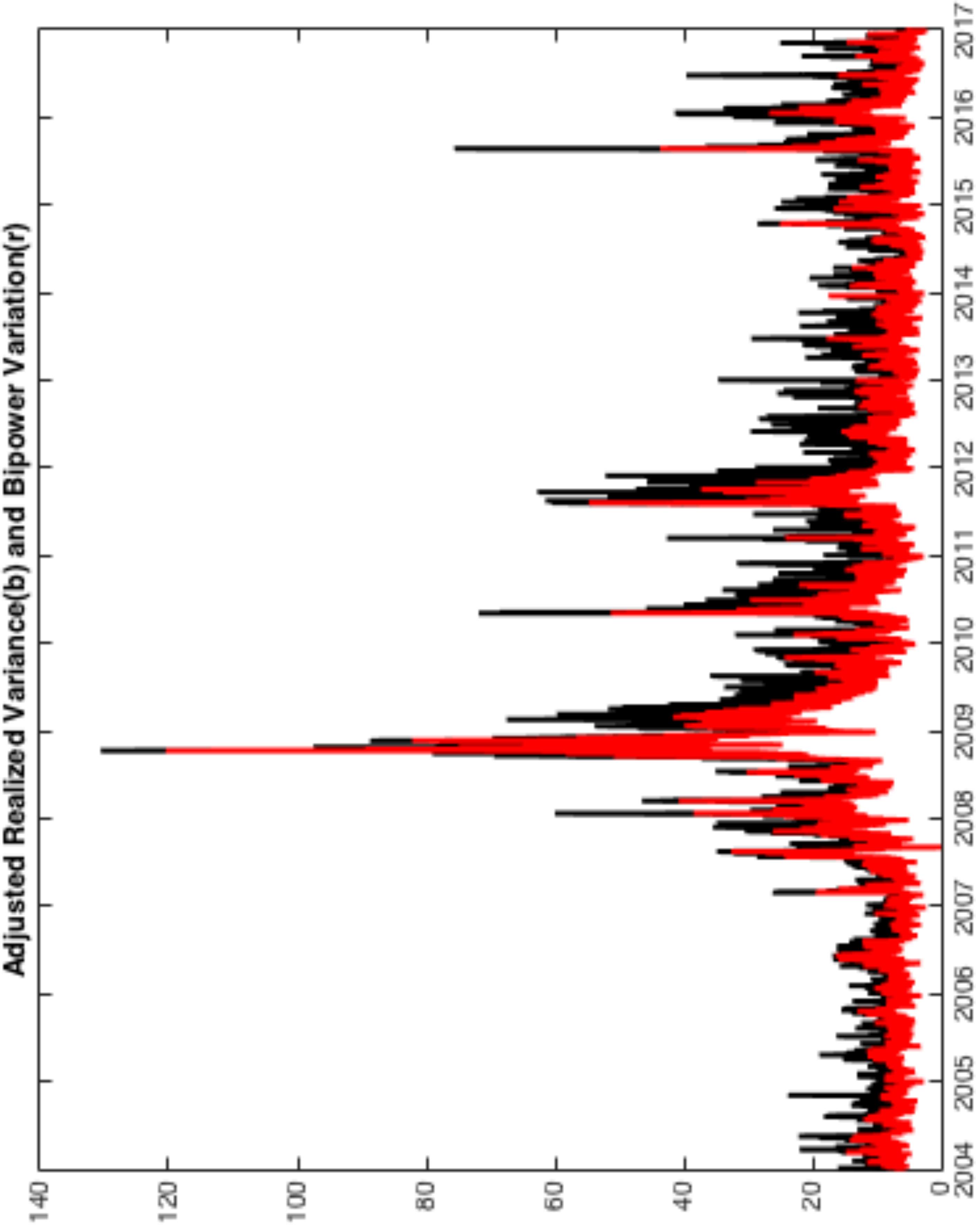
The daily adjusted Realized Variance (black) and the Bipower Variation (red) (The Realized Variance is adjusted to be no less than the Bipower Variation)



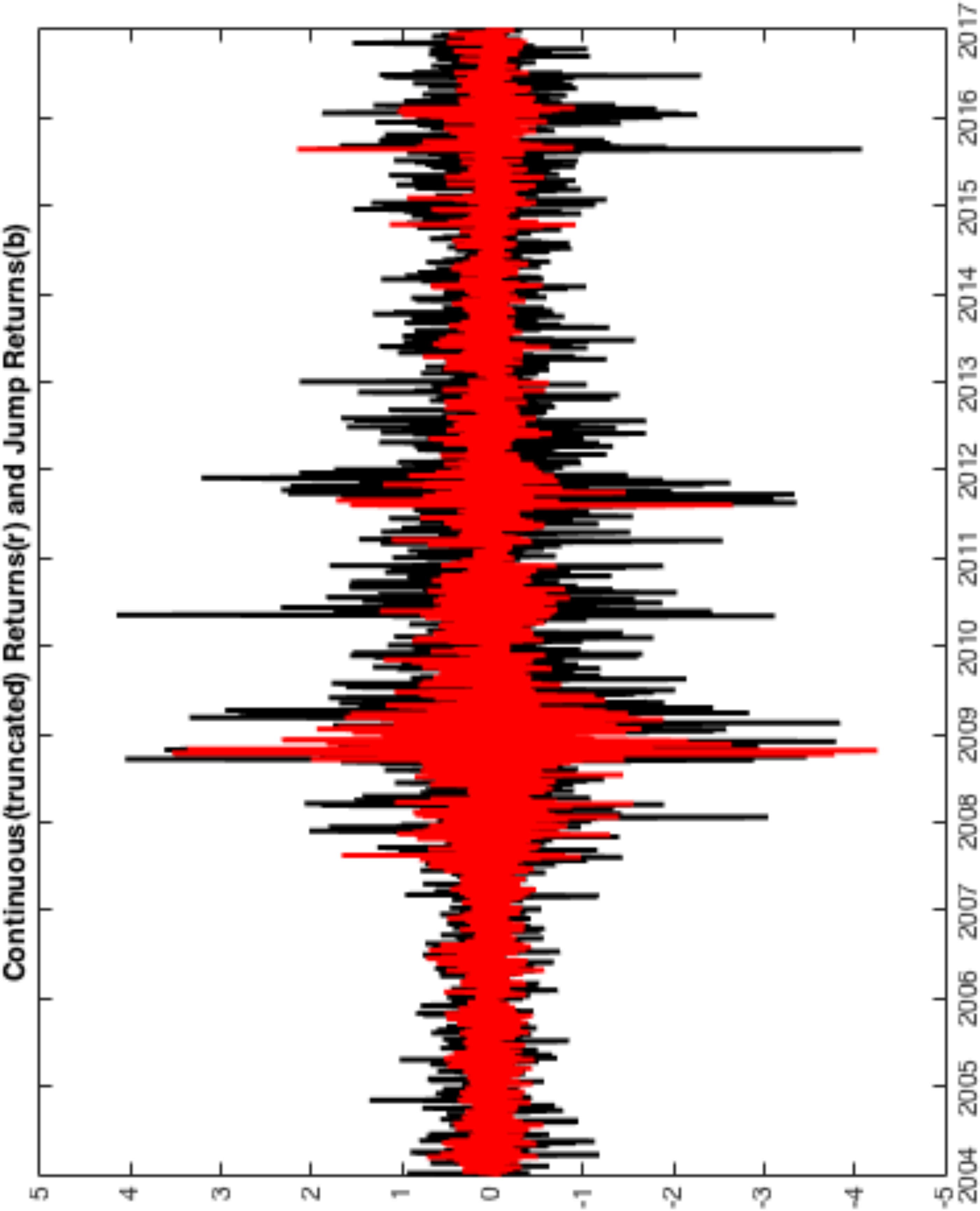
Continuous returns (red) and jump returns (black) disentangled by truncation method ($\alpha = 4.5$)

Abstract

In this article we introduce a linear quadratic volatility model with co-jumps and show how to calibrate this model to a rich dataset. We apply GMM and more specifically match the moments of realized power and multi-power variations, which are obtained from high-frequency stock market data. Our model incorporates two salient features: the setting of simultaneous jumps in both return process and volatility process and the superposition structure of a continuous linear quadratic volatility process and a Lévy-driven Ornstein-Uhlenbeck process. We compare the quality of fit for several models, and show that our model outperforms the conventional jump diffusion or Bates model. Besides that, we find evidence that the jump sizes are not normally distributed and that our model performs best when the distribution of jump-sizes is only specified through certain (co-) moment conditions. Monte Carlo experiments are employed to confirm this.


Adjusted Realized Variance(b) and Bipower Variation(r)
0
20
40
60
80
100
120
140
2004
2005
2006
2007
2008
2009
2010
2011
2012
2013
2014
2015
2016
2017

Figure(s) Click here to access/download;Figure(s);CR_TR.eps



Stochastic Volatility: A Tale of Co-Jumps Non-Normality, GMM and High Frequency Data

Supplementary Appendix

Details of the Asymptotic Variance-covariance Matrix of the GMM Estimator

We present some details of the covariance matrix W for selected moment conditions f_t,

$$W = \sum_{l=-\infty}^{\infty} \mathbb{E}(g_t g'_{t-l}) = \sum_{l=-\infty}^{\infty} Cov(f_t f'_{t-l}).$$

We derive the elements of the main covariance functions $Cov(f_t, f'_{t-l})$, which is required for the computation of the infinite sum. We do not attempt the summation, as these expressions are extremely complex and not necessary for this article. It is in fact far more practical and common to use a kernel estimator to approximate the infinite sum. However, we still hope that future research can benefit from these derivations and therefore we choose to present these in this supplementary appendix.[1]

First, we present higher moment conditions for integrals with respect to compensated Poisson random measure and integrated variance (IV) that are necessary for the covariance functions. Then we state the analytical expressions for elements of those covariance functions.

The proof of higher moment conditions for integrals regarding compensated Poisson random measure begins with proving the case where the integrands are left-continuous simple predictable functions, and then generalizing to predictable functions.

Definition .1. *Write $\mathscr{L}_0$ as the collection of bounded, left-continuous, simple predictable processes. That is, a function of the form $\Phi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is in $\mathscr{L}_0$ if there is a sequence of stopping times $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and a sequence of bounded $\mathcal{F}_t$-measurable random variables $\{\phi_i(\omega, x)\}_{i=0}^n$ such that*

$$\Phi(0, x) = \phi_0, \text{ and } \Phi(t, x) = \phi_i \text{ for } t \in (t_i, t_{i+1}],\ i = 0, 1, \ldots, n-1.$$

Clearly we can express Φ as

$$\Phi(t, x) = \sum_{j=1}^{m} \phi_{i=0,j} \mathbb{1}_{i=0}(t) \mathbb{1}_{A_j}(x) + \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} \mathbb{1}_{(t_i, t_{i+1}]}(t) \mathbb{1}_{A_j}(x), \tag{1}$$

here $\{\phi_{ij}\}_{j=1}^m$ are $\mathcal{F}_{t_i}$-measurable random variables and $\{A_j\}_{j=1}^m$ are disjoint subsets of $\mathbb{R}$.

[1] We agree with and would like to thank the anonymous referee and the editors for suggesting to add this part.

Naturally, the stochastic integral of Φ with respect to the compensated Poisson random measure $\tilde{\mu}$ is

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(dt, dx) &= \sum_{i,j=1}^{n,m} \phi_{ij} \tilde{\mu}((t_i \wedge t, t_{i+1} \wedge t] \times A_j) \\ &= \sum_{i,j=1}^{n,m} \phi_{ij} (\mu((t_i \wedge t, t_{i+1} \wedge t] \times A_j) - \nu((t_i \wedge t, t_{i+1} \wedge t] \times A_j)), \end{aligned} \tag{2}$$

where we skip the first term (t=0) on the right hand side of the above equation with a slight abuse of notation, as this simplifies but does not affect the following proof.

Based on this definition, we can now present higher moments of jump processes that appear in this context. We start from the case that the integrand is a left-continuous simple predictable function, and extend to the case of general predictable functions. Then the higher moments conditions of jump processes in this appendix is just a special case, since $J(\cdot)$ and $Q(\cdot)$ are deterministic left-continuous functions by definition. The lemma below is proved similarly to the isometry formula in Proposition 8.7 of Cont & Tankov (2003).

Lemma .1. *For any $\Phi \in \mathscr{L}_0$, the third and fourth absolute moment of the process $\{X_t\}_{0 \le t \le T}$ defined by*

$$X_t = \int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$$

satisfy the equations

$$\mathbb{E}(|X_t|^3) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx) \Big), \tag{3}$$

$$\mathbb{E}(|X_t|^4) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^4 \nu(ds, dx) \Big) + 3\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^2 \nu(ds, dx) \Big)^2, \tag{4}$$

provided they are bounded.

Proof. For the third absolute moment $k = 3$, we define $Y_t^j = \tilde{\mu}((0, t] \times A_j)$, then $\tilde{\mu}((t_i \wedge t, t_{i+1} \wedge t] \times A_j) = Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}$, where $Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}$ can be viewed as a compensated compound Poisson process from $(t_i \wedge t, t_{i+1} \wedge t]$ with jump sizes in A_j. Then we can express X_t as

$$X_t = \sum_{i,j=1}^{n,m} \phi_{ij} (Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}),$$

and

$$\begin{aligned}\mathbb{E}(|X_T|^3) &= \mathbb{E}\Big(\sum_{i,j=1}^{n,m} \phi_{ij}^3 (Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t})^3 \Big) \\ &= \sum_{i,j=1}^{n,m} \mathbb{E}(\mathbb{E}((Y_{t_{i+1}}(\phi_{ij}) - Y_{t_i}(\phi_{ij}))^3 | \mathcal{F}_{t_i})) \\ &= \sum_{i,j=1}^{n,m} \mathbb{E}(\phi_{ij}^3 \nu((t_i, t_{i+1}] \times A_j) \end{aligned}$$

$$= \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx) \Big),$$

where $Y_{t_{i+1}}(\phi_{ij}) - Y_{t_i}(\phi_{ij})$ is the increment of a compensated compound Poisson process from $(t_i, t_{i+1}]$ with jump size ϕ_{ij}. Note that the Poisson random measure $\mu(\omega, (t_i, t_{i+1}] \times A_j) = \mu((t_i, t_{i+1}] \times A_j)$ is a Poisson random variable with intensity parameter $\nu((t_i, t_{i+1}] \times A_j)$ for each rectangle $(t_i, t_{i+1}] \times A_j \in [0, T] \times \mathbb{R}$, hence the first equality holds due to the fact that Poisson distributions are independent for disjoint $(t_i, t_{i+1}] \times A_j$. The equality even holds when the jumps are infinite Lévy processes. In that case, infinite jumps can be interpreted as an infinite superposition of independent Poisson processes.(Cont & Tankov, 2003) The third equality is a direct result of the third moment of compensated compound Poisson processes.[2] The fourth absolute moment follows similarly. □

We extend the previous lemma to general predictable integrand functions. Let $\mathscr{L}_p$ ($p \in [1, \infty)$) be the space of predictable processes $\Phi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$||\Phi||_\nu^p = \mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} |\Phi|^p \nu(ds, dx) \Big) < \infty.$$

We need to show that given a predictable function $\Phi \in \mathscr{L}_3$, there exists a sequence of simple predictable functions $\Phi_n \in \mathscr{L}_0$ converging to Φ in the sense that

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} |\Phi - \Phi_n|^3 \nu(ds, dx) \Big) \to 0, \text{ as } n \to \infty,$$

and therefore the process $\int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$ satisfies similar equality to (3). The fourth moment case follows in $\mathscr{L}_4$.

Proposition .1. *For any $\Phi \in \mathscr{L}_4$, the third and fourth absolute moment of the process $\{X_t\}_{0 \le t \le T}$ defined by*

$$X_t = \int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$$

satisfy the equations

$$\mathbb{E}(|X_t|^3) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx) \Big), \tag{5}$$

$$\mathbb{E}(|X_t|^4) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^4 \nu(ds, dx) \Big) + 3\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^2 \nu(ds, dx) \Big)^2. \tag{6}$$

Proof. We sketch the proof of for the case of the third moment, i.e. $\Phi \in \mathscr{L}_3$ and the case of the fourth moment follows similarly.

$\mathscr{L}_3$ is the standard L^3 space under the measure $\nu \times dP$ of the predictable σ-algebra.(See Remark 11.3.7 of Cohen & Elliott (2015)) We know that the space $\mathscr{L}_0$ of left-continuous simple predictable processes generates the predictable σ-algebra, and the predictable process Φ is measurable with respect to the predictable σ-algebra by definition. Hence by the *simple approximation theorem* and the *dominated convergence theorem*, $\mathscr{L}_0$ is dense in $\mathscr{L}_3$ (and $\mathscr{L}_4$). Thus,

[2] See Barndorff-Nielsen & Shephard (2006) for calculation of higher moments of Lévy processes.

for any $\Phi \in \mathscr{L}_3$, there is a sequence of functions $\Phi_n \in \mathscr{L}_0$ converging to Φ in the sense that $||\Phi - \Phi_n||_\nu^3 \to 0$, and hence

$$\mathbb{E}\Big(\Big(\int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx) - \int_0^t \int_{\mathbb{R}} \Phi_n \tilde{\mu}(ds, dx)\Big)^3\Big) \to 0 \text{ as } n \to \infty.$$

Therefore we have the moment condition (5). □

Remark .1. *In our manuscript, $J(\cdot)$ and $Q(\cdot)$ are deterministic left-continuous functions with $J : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}^+$ and the jumps associated with price process and volatility process are simultaneous, and by definition $\nu(dt, dx) = G(dx)dt$. Thus, we have the following*

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} J^2 \tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q \tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q \tilde{\mu}(ds, dx)\Big) = t \int_{\mathbb{R}} J^2 Q^2 G(dx), \tag{7}$$

and

$$\begin{aligned}\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} J^2 \tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q \tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q \tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q \tilde{\mu}(ds, dx)\Big) \\ = t \int_{\mathbb{R}} J^2 Q^3 G(dx) + 3t^2 \Big(\int_{\mathbb{R}} (J^2 Q^3)^{\frac{1}{2}} G(dx)\Big)^2.\end{aligned} \tag{8}$$

The next part is higher (conditional) moment conditions for the LQJD volatility process, derived by the iterated general formula.

Proposition .2 (Higher Moments of the Linear Quadratic Volatility Process)**.** *Given the initial value σ_0, for a Linear Quadratic Volatility process of the type $V(t) = \sigma^2(t)$ with $d\sigma(t) = \theta(\mu - \sigma(t))dt + \nu dW(t)$ ($\theta > 0, \mu > 0, \nu > 0$ and $W(t)$ is a Wiener Process), we have the following moments for $\sigma(t)$*

$$\begin{aligned}\mathbb{E}(\sigma_T^3|\mathcal{F}_t) = e^{-3\theta T}\Big(\sigma_t^3 e^{3\theta t} - \frac{1}{2\theta}\Big(\Big(e^{\theta t} - e^{\theta T}\Big)\Big(e^{\theta(t+T)}\Big(3\sigma_t\Big(2\theta\mu^2 + \nu^2\Big) - 4\theta\mu^3\Big) \\ + e^{2\theta t}\Big(3\nu^2(\mu + \sigma_t) - 2\theta\mu\Big(5\mu^2 - 3\sigma_t^2 + 3\mu\sigma_t\Big)\Big) + \mu e^{2\theta T}\Big(2\theta\mu^2 + 3\nu^2\Big)\Big)\Big)\Big),\end{aligned} \tag{9}$$

$$\begin{aligned}\mathbb{E}(\sigma_T^4|\mathcal{F}_t) = e^{-4\theta T}\Big(\frac{1}{4\theta^2}\Big(6\Big(2\theta\mu^2 + \nu^2\Big)e^{2\theta(t+T)}\Big(\nu^2 + 2\theta\Big(-\mu^2 + \sigma_t^2 - 2\mu\sigma_t\Big)\Big) \\ + 8\theta\mu e^{\theta(3t+T)}\Big(2\theta\Big(5\mu^3 - 3\mu\sigma_t^2 + \sigma_t^3 + 3\mu^2\sigma_t\Big) - 3\nu^2(\mu + \sigma_t)\Big) \\ + 8\theta\mu(\sigma_t - \mu)\Big(2\theta\mu^2 + 3\nu^2\Big)e^{\theta(t+3T)} - e^{4\theta t}\Big(9\nu^4 + 4\theta^2\mu\Big(11\mu^3 - 6\mu\sigma_t^2 + 4\sigma_t^3 + 4\mu^2\sigma_t\Big) \\ + 12\theta\nu^2(\mu + \sigma_t)(\sigma_t - 3\mu)\Big) + e^{4\theta T}\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)\Big) + \sigma_t^4 e^{4\theta t}\Big),\end{aligned} \tag{10}$$

$$\begin{aligned}\mathbb{E}(\sigma_T^5|\mathcal{F}_t) = e^{-5\theta T}\Big(\frac{1}{4\theta^2}\Big(-5\mu e^{\theta(4t+T)}\Big(9\nu^4 - 4\theta^2\Big(-11\mu^4 + 6\mu^2\sigma_t^2 - 4\mu\sigma_t^3 + \sigma_t^4 - 4\mu^3\sigma_t\Big) \\ + 12\theta\nu^2(\mu + \sigma_t)(\sigma_t - 3\mu)\Big) + 10\mu\Big(2\theta\mu^2 + 3\nu^2\Big)e^{2\theta t + 3\theta T}\Big(\nu^2 + 2\theta\Big(-\mu^2 + \sigma_t^2 \\ - 2\mu\sigma_t\Big)\Big) + 10\Big(2\theta\mu^2 + \nu^2\Big)e^{3\theta t + 2\theta T}\Big(2\theta\Big(5\mu^3 - 3\mu\sigma_t^2 + \sigma_t^3 + 3\mu^2\sigma_t\Big) - 3\nu^2(\mu + \sigma_t)\Big) \\ + 5(\sigma_t - \mu)\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)e^{\theta(t+4T)} + e^{5\theta t}\Big(4\theta^2\mu\Big(19\mu^4 - 10\mu^2\sigma_t^2 \\ + 10\mu\sigma_t^3 - 5\sigma_t^4 + 5\mu^3\sigma_t\Big) - 20\theta\nu^2(\mu + \sigma_t)\Big(7\mu^2 + \sigma_t^2 - 4\mu\sigma_t\Big) + 15\nu^4(3\mu + \sigma_t)\Big) \\ + \mu e^{5\theta T}\Big(4\theta^2\mu^4 + 20\theta\mu^2\nu^2 + 15\nu^4\Big)\Big) + \sigma_t^5 e^{5\theta t}\Big),\end{aligned} \tag{11}$$

$$\mathbb{E}(\sigma_T^6|\mathcal{F}_t) = e^{-6\theta T}\Big(\frac{1}{8\theta^3}\Big(15\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)e^{2\theta(t+2T)}\Big(- 2\theta\mu^2 + \nu^2 + 2\theta\sigma_t^2 - 4\theta\mu\sigma_t\Big)$$
$$+ 12\theta\mu e^{\theta(5t+T)}\Big(76\theta^2\mu^5 - 140\theta\mu^3\nu^2 + 45\mu\nu^4 - 20\theta^2\mu\sigma_t^4 + 4\theta^2\sigma_t^5 + 20\theta\sigma_t^3\Big(2\theta\mu^2$$
$$- \nu^2\Big) + 20\theta\mu\sigma_t^2\Big(3\nu^2 - 2\theta\mu^2\Big) + 5\sigma_t\Big(4\theta^2\mu^4 - 12\theta\mu^2\nu^2 + 3\nu^4\Big)\Big)$$
$$+ 40\theta\mu e^{3\theta(t+T)}\Big(20\theta^2\mu^5 + 24\theta\mu^3\nu^2 - 9\mu\nu^4 - 6\theta\mu\sigma_t^2\Big(2\theta\mu^2 + 3\nu^2\Big) + 2\theta\sigma_t^3\Big(2\theta\mu^2$$
$$+ 3\nu^2\Big) + 3\sigma_t\Big(4\theta^2\mu^4 + 4\theta\mu^2\nu^2 - 3\nu^4\Big)\Big) + 15e^{2\theta(2t+T)}\Big(28\theta^2\mu^4\nu^2 - 88\theta^3\mu^6 + 18\theta\mu^2\nu^4$$
$$- 9\nu^6 - 16\theta^2\mu\sigma_t^3\Big(2\theta\mu^2 + \nu^2\Big) + 4\theta^2\sigma_t^4\Big(2\theta\mu^2 + \nu^2\Big) + 12\sigma_t^2\Big(4\theta^3\mu^4 - \theta\nu^4\Big)$$
$$- 8\theta\mu\sigma_t\Big(4\theta^2\mu^4 - 4\theta\mu^2\nu^2 - 3\nu^4\Big)\Big) + 12\theta\mu(\sigma_t - \mu)\Big(4\theta^2\mu^4 + 20\theta\mu^2\nu^2 + 15\nu^4\Big)e^{\theta(t+5T)}$$
$$+ e^{6\theta t}\Big(780\theta^2\mu^4\nu^2 - 232\theta^3\mu^6 - 450\theta\mu^2\nu^4 + 75\nu^6 - 80\theta^2\mu\sigma_t^3\Big(2\theta\mu^2 - 3\nu^2\Big)$$
$$+ 60\theta^2\sigma_t^4\Big(2\theta\mu^2 - \nu^2\Big) + 30\theta\sigma_t^2\Big(4\theta^2\mu^4 - 12\theta\mu^2\nu^2 + 3\nu^4\Big) - 48\theta^3\mu\sigma_t^5 - 12\theta\mu\sigma_t\Big(4\theta^2\mu^4$$
$$- 20\theta\mu^2\nu^2 + 15\nu^4\Big)\Big) + e^{6\theta T}\Big(60\theta^2\mu^4\nu^2 + 8\theta^3\mu^6 + 90\theta\mu^2\nu^4 + 15\nu^6\Big)\Big) + \sigma_t^6 e^{6\theta t}\Big), \tag{12}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^5) = \mu^5 + 10\mu^3\frac{\nu^2}{2\theta} + 15\mu(\frac{\nu^2}{2\theta})^2, \tag{13}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^6) = \mu^6 + 15\mu^4\frac{\nu^2}{2\theta} + 45\mu^2(\frac{\nu^2}{2\theta})^2 + 15(\frac{\nu^2}{2\theta})^3, \tag{14}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^7) = \mu^7 + 21\mu^5\frac{\nu^2}{2\theta} + 105\mu^3(\frac{\nu^2}{2\theta})^2 + 105(\frac{\nu^2}{2\theta})^3, \tag{15}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^8) = \mu^8 + 28\mu^6\frac{\nu^2}{2\theta} + 210\mu^4(\frac{\nu^2}{2\theta})^2 + 420\mu^2(\frac{\nu^2}{2\theta})^3 + 105(\frac{\nu^2}{2\theta})^4, \tag{16}$$

$$\mathbb{E}(\sigma_{T_1}^2\sigma_{T_2}^2|\mathcal{F}_t) = \frac{1}{4\theta^2}\Big(e^{-2\theta(T_1+T_2)}\Big(6\Big(2\theta\mu^2 + \nu^2\Big)e^{2\theta(t+T_1)}\Big(\nu^2 + 2\theta\Big(- \mu^2 + \sigma_t^2 - 2\mu\sigma_t\Big)\Big)$$
$$+ 8\theta\mu e^{\theta(3t+T_1)}\Big(2\theta\Big(5\mu^3 - 3\mu\sigma_t^2 + \sigma_t^3 + 3\mu^2\sigma_t\Big) - 3\nu^2(\mu + \sigma_t)\Big) + 8\theta\mu(\sigma_t - \mu)\Big(2\theta\mu^2$$
$$+ 3\nu^2\Big)e^{\theta(t+3T_1)} + e^{4\theta t}\Big(- 9\nu^4 + 4\theta^2\Big(- 11\mu^4 + 6\mu^2\sigma_t^2 - 4\mu\sigma_t^3 + \sigma_t^4 - 4\mu^3\sigma_t\Big)$$
$$- 12\theta\nu^2(\sigma_t - 3\mu)(\mu + \sigma_t)\Big) + e^{4\theta T_1}\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)\Big) + \Big(2\theta\mu^2 + \nu^2$$
$$+ e^{2\theta(T_1-T_2)}\Big(\nu^2 - 2\theta\mu^2\Big) - 4\theta\mu^2 e^{\theta(T_1-T_2)}\Big)\Big(2\theta\mu^2 + \nu^2 + e^{2\theta(t-T_1)}\Big(\nu^2$$
$$+ 2\theta\Big(- \mu^2 + \sigma_t^2 - 2\mu\sigma_t\Big)\Big) + 4\theta\mu(\sigma_t - \mu)e^{\theta(t-T_1)}\Big) - 4\theta\mu e^{-\theta(2T_1+T_2)}\Big(e^{\theta(T_1-T_2)}$$
$$- 1\Big)\Big(2\theta\sigma_t^3 e^{3\theta t} - \Big(e^{\theta t} - e^{\theta T_1}\Big)\Big(e^{\theta(t+T_1)}\Big(3\sigma_t\Big(2\theta\mu^2 + \nu^2\Big) - 4\theta\mu^3\Big) + e^{2\theta t}\Big(3\nu^2(\mu$$
$$+ \sigma_t) - 2\theta\mu\Big(5\mu^2 - 3\sigma_t^2 + 3\mu\sigma_t\Big)\Big) + \mu e^{2\theta T_1}\Big(2\theta\mu^2 + 3\nu^2\Big)\Big)\Big)\Big), \ \forall t \le T_1 \le T_2. \tag{17}$$

Proof. By Itô's lemma we have

$$de^{n\theta t}\sigma_t^n = n\theta e^{n\theta t}\sigma_t^n dt + n\sigma_t^{n-1}d\sigma_t + \frac{n(n-1)}{2}\sigma_t^{n-2}d[\sigma_t, \sigma_t], \ n \ge 2. \tag{18}$$

Express this in integral form from [t,T] and take conditional expectation of $\mathcal{F}_t$ on both sides,

$$\mathbb{E}(\sigma_T^n|\mathcal{F}_t) = e^{n\theta(t-T)}\sigma_t^n + \mu(1 - e^{\theta n(t-T)})\mathbb{E}(\sigma_T^{n-1}|\mathcal{F}_t) + \frac{\nu^2(n-1)(1 - e^{\theta n(t-T)})}{2\theta}\mathbb{E}(\sigma_T^{n-2}|\mathcal{F}_t). \tag{19}$$

Thus, we can obtain the above (conditional) moments for σ_t by iteration. □

Now we can directly compute the covariance functions $Cov(f_t, f'_{t-l})$. In the manuscript, we have already derived the covariance of the first moment of integrated variance i.e. $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$, $l = ia$, $i \in \mathbb{N}$. When $l = 0$, it is just the variance of $IV_{(t,t+a]}$. Notably that the sum of infinite covariance series is certainly convergent due to the fact

$$\sum_{l=0}^{\infty} e^{-c|l|} = 1 + \frac{1}{e^c - 1},\ c > 0,$$

where $e^{-c|l|}$ can be observed as a common coefficient of all individual terms. Although we do not sum up every covariance function, it is expected to appear in each function after simplifying expressions by canceling other terms.

We discuss the elements in the variance-covariance in two case hereafter, that is, when $l = 0$ and $l \neq 0$, because for some expectations of jump processes we may have very different results in those two cases. We start with the the covariance of the first moment of quadratic variation, i.e. $QV_{(t,t+a]}$ and $QV_{(t+l,t+a+l]}$. The variance of $QV_{(t,t+a]}$ is derived in the manuscript. When $l \neq 0$, we have

$$\begin{aligned}
Cov(QV_{(t,t+a]}, QV_{(t+l,t+a+l]}) &= \mathbb{E}(QV_{(t,t+a]}QV_{(t+l,t+a+l]}) - \Big(\mathbb{E}(IV_{(t,t+a]})^2 \\
&+ 2\mathbb{E}(IV_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big)^2\Big) \\
&= Cov(IV_{(t,t+a]}, IV_{(t+l,t+a+l]}) + \mathbb{E}\Big(IV_{(t+l,t+a+l]}\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) \\
&+ \mathbb{E}\Big(IV_{(t,t+a]}\int_{t+l}^{t+a+l}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\int_{t+l}^{t+a+l}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) \\
&- \Big(2\mathbb{E}\Big(IV_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big)^2\Big),
\end{aligned} \tag{20}$$

taking the limit of $t \to \infty$,

$$\begin{aligned}
\lim_{t\to\infty} Cov(QV_{(t,t+a]}, QV_{(t+l,t+a+l]}) &= a\Big(\frac{\nu^2}{2\theta} + \mu^2\Big)\frac{(\nu^2 - 2\theta\nu^2)(e^{-2\theta l} - e^{-2\theta(l-a)})}{4\theta^2} + \Big(a\Big(\frac{\nu^2}{2\theta} + \mu^2\Big)^2 \\
&+ \frac{2\mu^2\nu^2(1 - e^{-\theta a})}{\theta^2} + \frac{\nu^4(1 - e^{-2\theta a})}{4\theta^3}\Big) \cdot \left(\frac{e^{-2\theta(l-a)} - e^{-\theta l}}{2\theta}\right) \\
&+ \left(\frac{1 - e^{-\theta a}}{\theta}\left(\mu^3 + \frac{3\mu\nu^2}{2\theta}\right) + \mu\left(a - \frac{1 - e^{-\theta a}}{\theta}\right)\left(\frac{\nu^2}{2\theta} + \mu^2\right)\right) \\
&\cdot \left(\frac{e^{-\theta(l-a)} - e^{-\theta l}}{\theta} - \frac{e^{-2\theta(l-a)} - e^{-2\theta l}}{2\theta}\right) 2\mu + \frac{(e^{\kappa a} - 1)e^{-\kappa(2a+l)}}{2\kappa^3}\int_{\mathbb{R}} Q^2 G(dx) \\
&+ \frac{e^{-\kappa(l+a)} + e^{-\kappa(l-a)}}{\kappa^2}\int_{\mathbb{R}} J^2 Q G(dx).
\end{aligned} \tag{21}$$

Then we deal with the covariance between the second central moment of $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$, $l = 0, 1a, 2a, ...$ From now on, we state expressions without taking limits as $t \to \infty$ for the sake of simplicity. The final expressions for the stationary covariance can then be obtained by taking limits. We have

$$
\begin{aligned}
&Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2)=\\
&\quad=\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]})-2\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+3\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2\\
&\quad-2\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+4\mathbb{E}(IV_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2\\
&\quad+\mathbb{E}(IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2-3\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2\\
&\quad=\mathbb{1}_{\{l=0\}}Var((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2)\\
&\quad+\mathbb{1}_{\{l\neq0\}}Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2). \qquad (22)
\end{aligned}
$$

Compute in the first case, i.e. $l=0$,

$$
\begin{aligned}
Var((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2)&=\mathbb{E}(IV^4_{(t,t+a]})-4\mathbb{E}(IV^3_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})\\
&+8\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2-4\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2. \qquad (23)
\end{aligned}
$$

As some of the elements are already computed analytically, we are left with two elements, that is, $\mathbb{E}(IV^4_{(t,t+a]})$ and $\mathbb{E}(IV^3_{(t,t+a]})$.

$$
\begin{aligned}
\mathbb{E}(IV^4_{(t,t+a]})&=\mathbb{E}\Big(\Big(\int_t^{t+a}V_d d\tau\Big)^4+4\Big(\int_t^{t+a}V_d d\tau\Big)^3\Big(\int_t^{t+a}V_c d\tau\Big)+6\Big(\int_t^{t+a}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)^2\\
&+4\Big(\int_t^{t+a}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)^3+\Big(\int_t^{t+a}V_c d\tau\Big)^4\Big), \qquad (24)\\
\mathbb{E}(IV^3_{(t,t+a]})&=\mathbb{E}\Big(\Big(\int_t^{t+a}V_d d\tau\Big)^3+3\Big(\int_t^{t+a}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)+3\Big(\int_t^{t+a}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)^2\\
&+\Big(\int_t^{t+a}V_c d\tau\Big)^3\Big). \qquad (25)
\end{aligned}
$$

Recall equation (23) for $(\int_t^{t+a}V_d d\tau)$ in the manuscript. We know that it can be decomposed into three elements, i.e. two independent integrals with respect to a compensated Poisson random measure and one deterministic part. We can use the fact that integrals with respect to compensated Poisson random measure and integrated variance are independent.

We start with the expectations of powers of the continuous part of the variance. Following the approach for computing $\lim_{t\to\infty}\mathbb{E}((\int_t^{t+a}V_c d\tau)^2)$ and by *Fubini's theorem*,

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)^3\Big)&=\int_0^a\int_0^a\int_0^a\mathbb{E}\Big(\sigma_c^2(s)\sigma_c^2(u)\sigma_c^2(k)\Big)dsdudk\\
&=\int_0^a\int_0^a\int_0^a\mathbb{E}\Big(\sigma_c^2(s)\mathbb{E}(\sigma_c^2(u)\mathbb{E}(\sigma_c^2(k)|\mathcal{F}_u)|\mathcal{F}_s)\Big)dsdudk, \qquad (26)
\end{aligned}
$$

provided the integrand is Lebesgue integrable. By using Proposition .2 we can compute it analytically. The procedure for $\lim_{t\to\infty}\mathbb{E}((\int_t^{t+a}V_c d\tau)^4)$ is similar. As for $\mathbb{E}((\int_t^{t+a}V_d d\tau))^4$ and $\mathbb{E}((\int_t^{t+a}V_d d\tau))^3$, we expand the formula and integrate each part, following equation (25) in the manuscript and the previous Proposition .1. We have

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^3\Big)=\frac{1}{16\theta^6}\Big(2a\theta\Big(12\theta^2\mu^4\nu^2\Big(a^2\theta^2-5\Big)+6\theta\mu^2\nu^4\Big(a^2\theta^2-14\Big)$$
$$+8\theta^3\mu^6\Big(a^2\theta^2+1\Big)+\nu^6\Big(a^2\theta^2-18\Big)\Big)+16a\theta^2\mu^2\nu^2\Big(e^{-a\theta}$$
$$+e^{a\theta}\Big)\Big(6\theta\mu^2+5\nu^2\Big)+16\theta\mu^2\Big(e^{3a\theta}-e^{-3a\theta}\Big)\Big(4\theta^2\mu^4-\nu^4\Big)$$
$$+16\theta\mu^2\Big(e^{a\theta}-e^{-a\theta}\Big)\Big(4\theta^2\mu^4+4\theta\mu^2\nu^2-7\nu^4\Big)-2a\theta\Big(e^{-2a\theta}$$
$$+e^{2a\theta}\Big)\Big(2\theta\mu^2+9\nu^2\Big)\Big(2\theta^2\mu^4-\nu^4\Big)+\Big(e^{2a\theta}-e^{-2a\theta}\Big)\Big(-44\theta^2\mu^4\nu^2$$
$$-104\theta^3\mu^6+74\theta\mu^2\nu^4+3\nu^6\Big)-\frac{3}{2}\Big(e^{4a\theta}-e^{-4a\theta}\Big)\Big(\nu^2-2\theta\mu^2\Big)^2\Big(2\theta\mu^2+\nu^2\Big)\Big),\qquad(27)$$

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^4\Big)=\frac{1}{7680\theta^9}\Big(30\Big(8a^4\theta^4-372a^2\theta^2-13\Big)\nu^{10}+60\theta\Big(4a^2\Big(\Big(10a^2\theta^2$$
$$-103\Big)\mu^2-171\Big)\theta^2-19\mu^2+31\Big)\nu^8+80\theta^2\mu\Big(\Big(120a^4\theta^4+792a^2\theta^2$$
$$-2483\Big)\mu^3+36\Big(67-49a^2\theta^2\Big)\mu-441\Big)\nu^6+32\theta^3\mu^4\Big(60a^2\Big(2\Big(5a^2\theta^2$$
$$+57\Big)\mu^2-169\Big)\theta^2-22509\mu^2+39640\Big)\nu^4+32\theta^4\mu^6\Big(60a^2\Big(\Big(10a^2\theta^2$$
$$+113\Big)\mu^2-174\Big)\theta^2-15711\mu^2+898\Big)\nu^2+64\theta^5\mu^8\Big(60a^2\Big(\Big(2a^2\theta^2$$
$$+27\Big)\mu^2-23\Big)\theta^2+4567\mu^2-8437\Big)+320a\Big(-e^{-3a\theta}+e^{3a\theta}\Big)\theta^2\mu^2\Big(2\theta\mu^2$$
$$+\nu^2\Big)\Big(-45\nu^6+2\theta\Big(14\mu^2-39\Big)\nu^4+4\theta^2\mu^2\Big(75\mu^2+7\Big)\nu^2+16\theta^3\Big(8\mu^6$$
$$+\mu^4\Big)\Big)-15a\Big(-e^{-4a\theta}+e^{4a\theta}\Big)\theta\Big(2\theta\mu^2+\nu^2\Big)\Big(165\nu^8+6\theta\Big(17-48\mu^2\Big)\nu^6$$
$$-4\theta^2\mu^2\Big(152\mu^2+99\Big)\nu^4+8\theta^3\mu^4\Big(180\mu^2+11\Big)\nu^2+16\theta^4\Big(23\mu^8+\mu^6\Big)\Big)$$
$$+144\Big(e^{-5a\theta}+e^{5a\theta}\Big)\theta\mu\Big(45\mu\nu^8+10\theta\Big(4\mu^3-18\mu+21\Big)\nu^6+4\theta^2\mu^3\Big(5$$
$$-76\mu^2\Big)\nu^4-8\theta^3\mu^5\Big(32\mu^2+9\Big)\nu^2+16\theta^4\Big(19\mu^9+\mu^7\Big)\Big)-320a\Big(-e^{-a\theta}$$
$$+e^{a\theta}\Big)\theta^2\mu^2\Big(-63\nu^8+10\theta\Big(111-17\mu^2\Big)\nu^6+4\theta^2\mu^2\Big(23\mu^2-44\Big)\nu^4$$
$$+8\theta^3\mu^4\Big(49\mu^2-129\Big)\nu^2+32\theta^4\mu^6\Big(2\mu^2-5\Big)\Big)+32\Big(e^{-3a\theta}+e^{3a\theta}\Big)\theta\mu\Big($$
$$-15\mu\nu^8-10\theta\Big(44\mu^3-336\mu+189\Big)\nu^6-4\theta^2\mu^3\Big(101\mu^2+665\Big)\nu^4$$
$$+16\theta^3\mu^5\Big(199\mu^2+8\Big)\nu^2+96\theta^4\mu^7\Big(49\mu^2+36\Big)\Big)+16\Big(e^{-a\theta}+e^{a\theta}\Big)\theta\mu\Big($$
$$-15\Big(12a^2\theta^2+25\Big)\mu\nu^8+10\theta\Big(\Big(820-144a^2\theta^2\Big)\mu^3+6\Big(78a^2\theta^2-85\Big)\mu$$
$$+189\Big)\nu^6+4\theta^2\mu^3\Big(60a^2\Big(49-18\mu^2\Big)\theta^2+8246\mu^2-14075\Big)\nu^4$$
$$+8\theta^3\mu^5\Big(180a^2\Big(7-4\mu^2\Big)\theta^2+2692\mu^2+369\Big)\nu^2+16\theta^4\mu^7\Big(-180a^2\Big(\mu^2-1\Big)\theta^2$$
$$-1079\mu^2+1799\Big)\Big)-\Big(e^{-4a\theta}+e^{4a\theta}\Big)\Big(1065\nu^{10}+30\theta\Big(361\mu^2+3\Big)\nu^8$$
$$-40\theta^2\mu\Big(127\mu^3-492\mu+189\Big)\nu^6-16\theta^3\mu^4\Big(4961\mu^2+1240\Big)\nu^4$$

$$+16\theta^4\mu^6\Big(301\mu^2-1878\Big)\nu^2+32\theta^5\mu^8\Big(4603\mu^2+767\Big)\Big)-5\Big(e^{-6a\theta}+e^{6a\theta}\Big)\Big(
-75\nu^{10}+30\theta\Big(5\mu^2-1\Big)\nu^8+120\theta^2\mu\Big(6\mu^3-20\mu+21\Big)\nu^6+16\theta^3\mu^4\Big(15
-68\mu^2\Big)\nu^4-16\theta^4\mu^6\Big(137\mu^2+14\Big)\nu^2+32\theta^5\Big(29\mu^{10}+\mu^8\Big)\Big)-10a\Big(-e^{-2a\theta}
+e^{2a\theta}\Big)\theta\Big(-495\nu^{10}-18\theta\Big(71\mu^2+17\Big)\nu^8+32\theta^2\mu^2\Big(103\mu^2-654\Big)\nu^6
+32\theta^3\mu^4\Big(467\mu^2-38\Big)\nu^4+16\theta^4\mu^6\Big(1183\mu^2+1212\Big)\nu^2+32\theta^5\mu^8\Big(283\mu^2
+125\Big)\Big)-\Big(e^{-2a\theta}+e^{2a\theta}\Big)\Big(-15\Big(372a^2\theta^2+59\Big)\nu^{10}-30\theta\Big(4a^2\Big(127\mu^2
+171\Big)\theta^2+405\mu^2-33\Big)\nu^8+40\theta^2\mu\Big(\Big(216a^2\theta^2+626\Big)\mu^3+12\Big(9a^2\theta^2
+185\Big)\mu-567\Big)\nu^6+16\theta^3\mu^4\Big(180a^2\Big(14\mu^2+9\Big)\theta^2+12232\mu^2-20635\Big)\nu^4
+16\theta^4\mu^6\Big(60a^2\Big(17\mu^2-6\Big)\theta^2+10273\mu^2+5406\Big)\nu^2+32\theta^5\mu^8\Big(60a^2\Big(3\mu^2
+1\Big)\theta^2-2741\mu^2+8711\Big)\Big)\Big)\Big), \tag{28}$$

$$\mathbb{E}\Big(\Big(\int_t^{t+a}V_d(\tau)d\tau\Big)^3\Big)=\Big(\frac{1}{3\kappa^4}\Big(e^{-3a\kappa}\Big(e^{a\kappa}-1\Big)^3\Big)+\frac{1}{6\kappa^4}\Big(e^{-3a\kappa}\Big(-9e^{a\kappa}+18e^{2a\kappa}+2\Big)+6a\kappa
-11\Big)\Big)\int_{\mathbb{R}}Q^3G(dx)+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}
+3\Big)\Big)\frac{3a}{\kappa}\int_{\mathbb{R}}Q^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)++\frac{a^3}{\kappa^3}\Big(\int_{\mathbb{R}}QG(dx)\Big)^3, \tag{29}$$

$$\mathbb{E}\Big(\Big(\int_t^{t+a}V_d(\tau)d\tau\Big)^4\Big)=\Big(\frac{1}{4\kappa^5}\Big(e^{-4a\kappa}\Big(e^{a\kappa}-1\Big)^4\Big)+\frac{1}{12\kappa^5}\Big(4e^{-3a\kappa}\Big(-9e^{a\kappa}+12e^{2a\kappa}+4\Big)-3e^{-4a\kappa}
+12a\kappa-25\Big)\Big)\int_{\mathbb{R}}Q^4G(dx)+\Big(\frac{1}{3\kappa^4}\Big(e^{-3a\kappa}\Big(e^{a\kappa}-1\Big)^3\Big)+\frac{1}{6\kappa^4}\Big(e^{-3a\kappa}\Big(-9e^{a\kappa}
+18e^{2a\kappa}+2\Big)+6a\kappa-11\Big)\Big)\frac{4a}{\kappa}\int_{\mathbb{R}}Q^3G(dx)\int_{\mathbb{R}}QG(dx)+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}
-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3\Big)\Big)\frac{6a^2}{\kappa^2}\int_{\mathbb{R}}Q^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2
+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3\Big)\Big)3\int_{\mathbb{R}}Q^2G(dx)
+\frac{a^4}{\kappa^4}\Big(\int_{\mathbb{R}}QG(dx)\Big)^4. \tag{30}$$

In the second case, i.e. $l \neq 0$,

$$Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2)=
=\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]})-2\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+3\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2
-2\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+4\mathbb{E}(IV_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2
+\mathbb{E}(IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2-3\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2. \tag{31}$$

Start from the first element that has not been computed,

$$\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]}) = \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big) + 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big) + \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big) + \Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big) + 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big) + \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big), \tag{32}$$

where

$$\begin{aligned}\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2 \mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau|\mathcal{F}_{t+a}\Big)\Big)\\ &= \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\frac{1}{4\theta^2}\Big(e^{-2\theta l}\Big(-2\theta\mu^2\Big(e^{2a\theta}-1\Big)+\nu^2\Big(e^{2a\theta}-1\Big)\\ &+ 2a\theta e^{2\theta l}\Big(2\theta\mu^2+\nu^2\Big) - 8\theta\mu^2\Big(e^{a\theta}-1\Big)e^{\theta l}\Big)\Big)\\ &+ \frac{\sigma_c^2(t+a)}{2\theta}\Big(e^{2a\theta}-1\Big)e^{-2\theta l} + \frac{\sigma_c(t+a)}{4\theta^2}\Big(e^{-2\theta l}\Big(8\theta\mu\Big(e^{a\theta}-1\Big)e^{\theta l}\\ &- 4\theta\mu\Big(e^{2a\theta}-1\Big)\Big)\Big)\Big)\Big)\\ &= \mathbb{E}\Big(\int_t^{t+a}\int_t^{t+a}\mathbb{E}\Big(\sigma_c^2(\tau_1)\sigma_c^2(\tau_2)|\mathcal{F}_t\Big)d\tau_1 d\tau_2\frac{1}{4\theta^2}\\ &\Big(e^{-2\theta(a+l)}\Big(\nu^2\Big(-e^{2a\theta}+2e^{4a\theta}-1\Big)+4a\theta^2\mu^2e^{2\theta(a+l)}\\ &+ 2\theta\Big(\mu^2\Big(e^{2a\theta}-2e^{4a\theta}+4e^{\theta(a+l)}-4e^{\theta(2a+l)}+1\Big)+a\nu^2e^{2\theta(a+l)}\Big)\Big)\Big)\\ &+ \frac{1}{2\theta}\Big(\sigma_c(t)^2\Big(e^{2a\theta}-1\Big)e^{-2\theta(a+l)}\Big) - \frac{1}{\theta}\Big(\mu\sigma_c(t)\Big(e^{a\theta}-1\Big)e^{-2\theta(a+l)}\Big(e^{a\theta}\\ &- 2e^{\theta(a+l)}+1\Big)\Big)\Big).\end{aligned} \tag{33}$$

By computing that explicitly with the previous result, we have

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big) &= \frac{1}{8\theta^6}e^{-2(3a+l)\theta}\Big(8a^3e^{2(3a+l)\theta}\theta^6\mu^6 + 4a^2\theta^5\Big(3ae^{2(3a+l)\theta}\nu^2\\ &- 2e^{4a\theta}\Big(-1+e^{4a\theta}\Big)\mu^2\Big)\mu^4 + 2a\theta^4\Big(-12e^{2(a+l)\theta}\mu^4 - 2e^{2(4a+l)\theta}\mu^4\\ &+ 48e^{3a\theta+2l\theta}\mu^4 + 16e^{5a\theta+2l\theta}\mu^4 + 2ae^{4a\theta}\nu^2\mu^2 - 2ae^{8a\theta}\nu^2\mu^2\\ &+ e^{2(2a+l)\theta}\Big(16a\mu^2\nu^2 - 58\mu^4\Big) + e^{2(3a+l)\theta}\Big(8\mu^4 - 16a\nu^2\mu^2\\ &+ 3a^2\nu^4\Big)\Big)\mu^2 + \Big(-3+e^{2a\theta}\Big)\Big(-1+e^{2a\theta}\Big)^3\Big(1+e^{2a\theta}\Big)\nu^6 - \Big(-1\\ &+ e^{a\theta}\Big)^2\Big(1+e^{a\theta}\Big)\theta\nu^4\Big(-2\Big(-1+e^{a\theta}\Big)\Big(9+6e^{a\theta}+17e^{2a\theta}+8e^{3a\theta}\\ &+ 3e^{4a\theta}+2e^{5a\theta}-e^{6a\theta}+8e^{(3a+l)\theta}\Big)\mu^2 - ae^{2a\theta}\Big(1+e^{a\theta}\Big)\Big(12+8e^{2a\theta}\\ &- 3e^{2l\theta}+e^{2(a+l)\theta}\Big)\nu^2\Big) + \theta^3\Big(4\Big(-1+e^{a\theta}\Big)^3\Big(1+e^{a\theta}\Big)\Big(1+e^{2a\theta}\Big)\end{aligned}$$

$$
\begin{aligned}
&\Big(2e^{3a\theta}\Big(\frac{1}{2}\Big(-e^{-a\theta}+e^{a\theta}\Big)+1\Big)-12e^{a\theta}+6\Big)\mu^6+2ae^{2a\theta}\Big(-1+e^{a\theta}\Big)^2\\
&\Big(16+32e^{a\theta}+32e^{2a\theta}+32e^{3a\theta}+16e^{4a\theta}+6e^{2l\theta}+5e^{2(a+l)\theta}\\
&+16e^{(2a+l)\theta}-e^{2(2a+l)\theta}+12e^{(a+2l)\theta}+6e^{3a\theta+2l\theta}\Big)\nu^2\mu^4\\
&+2a^2e^{4a\theta}\Big(-1+e^{4a\theta}\Big)\nu^4\mu^2+a^3e^{2(3a+l)\theta}\nu^6\Big)+\theta^2\Big(a^2e^{4a\theta}\Big(-1\\
&+e^{2a\theta}\Big)\Big(1+e^{2a\theta}+8e^{2l\theta}\Big)\nu^6-2ae^{2a\theta}\Big(-1+e^{a\theta}\Big)^2\Big(16\\
&+32e^{a\theta}+32e^{2a\theta}+32e^{3a\theta}+16e^{4a\theta}-3e^{2l\theta}-16e^{(2a+l)\theta}\\
&-e^{2(2a+l)\theta}+6e^{(a+2l)\theta}-6e^{3a\theta+2l\theta}\Big)\mu^2\nu^4-2\Big(-1+e^{a\theta}\Big)^3\Big(1\\
&+e^{a\theta}\Big)\Big(18-12e^{a\theta}+21e^{2a\theta}-2e^{3a\theta}+4e^{4a\theta}+10e^{5a\theta}+e^{6a\theta}\\
&+8e^{(3a+l)\theta}\Big)\mu^4\nu^2\Big)\Big). \qquad (34)
\end{aligned}
$$

The second element is

$$
\begin{aligned}
\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]}) = \mathbb{E}\Big(&\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)\\
&+2\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)\\
&+\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)+\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big(\int_t^{t+a}V_d d\tau\Big)\\
&+2\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big(\int_t^{t+a}V_d d\tau\Big)\\
&+\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_d d\tau\Big)\Big), \qquad (35)
\end{aligned}
$$

where

$$
\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l}\int_{t+l}^{t+a+l}V_c(\tau_1)V_c(\tau_2)d\tau_1 d\tau_2|\mathcal{F}_{t+a}\Big)\Big). \qquad (36)
$$

By taking limits on both sides we get

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big) &= \frac{1}{8\theta^6}e^{-14a\theta-13l\theta}\Big(-3e^{8a\theta+9l\theta}\Big(2\theta\mu^2-\nu^2\Big)^3\\
&+8e^{11a\theta+13l\theta}\theta\mu^2\nu^2\Big(2\theta\mu^2-\nu^2\Big)-24e^{2(8a+5l)\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\\
&\Big(2\theta\mu^2-\nu^2\Big)+24e^{17a\theta+10l\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)\\
&+32e^{15a\theta+11l\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-8e^{16a\theta+11l\theta}\theta\Big(2\mu^2\\
&+a\nu^2\Big)\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-24e^{5(3a+2l)\theta}\theta\mu^2\Big(2\theta(a\theta\\
&-2)\mu^2+(a\theta+2)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)+24e^{11a\theta+10l\theta}\theta\mu^2\Big(2\theta(a\theta-2)\mu^2+(a\theta\\
&+2)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-24e^{2(6a+5l)\theta}\theta\mu^2\Big(2\theta(a\theta-3)\mu^2+(a\theta+3)\nu^2\Big)\Big(2\theta\mu^2
\end{aligned}
$$

$$
\begin{aligned}
&- \nu^2\Big) + 24e^{2(7a+5l)\theta}\theta\mu^2\Big(2\theta(2a\theta-3)\mu^2 + (2a\theta+3)\nu^2\Big)\Big(2\theta\mu^2 - \nu^2\Big)\\
&- e^{10a\theta+13l\theta}\Big(2\theta^2\mu^4 - \nu^4\Big)\Big(2\theta\mu^2 - \nu^2\Big) - 24e^{10(a+l)\theta}\theta\Big(\mu\nu^2 - 2\theta\mu^3\Big)^2\\
&+ 24e^{9a\theta+10l\theta}\theta\Big(\mu\nu^2 - 2\theta\mu^3\Big)^2 - 16e^{3(5a+4l)\theta}\theta\mu^2\nu^2\Big(\theta\mu^2 - \nu^2\Big) + 16e^{11a\theta+12l\theta}\theta\mu^2\nu^2\\
&\Big(\theta\mu^2 - \nu^2\Big) - 64ae^{13a\theta+12l\theta}\theta^2\mu^2\nu^2\Big(\theta\mu^2 + \nu^2\Big) + 8ae^{13(a+l)\theta}\theta^2\mu^2\nu^2\Big(2\theta\mu^2 + \nu^2\Big)\\
&+ 32e^{2(7a+6l)\theta}\theta\mu^2\nu^2\Big(\theta(a\theta+1)\mu^2 + (a\theta-1)\nu^2\Big) + 32e^{12(a+l)\theta}\theta\mu^2\nu^2\Big(\theta(a\theta\\
&- 1)\mu^2 + (a\theta+1)\nu^2\Big) + 8e^{15a\theta+13l\theta}\theta\mu^2\nu^2\Big(2\theta(a\theta-1)\mu^2 + (a\theta+1)\nu^2\Big)\\
&- 3e^{9(2a+l)\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-1)\mu^2 + (a\theta+1)\nu^2\Big) + 6e^{12a\theta+9l\theta}\Big(\nu^2\\
&- 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-2)\mu^2 + (a\theta+2)\nu^2\Big) - 6e^{14a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta\\
&- 2)\mu^2 + (a\theta+2)\nu^2\Big) - 3e^{10a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-3)\mu^2 + (a\theta+3)\nu^2\Big)\\
&+ 3e^{16a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(2a\theta-3)\mu^2 + (2a\theta+3)\nu^2\Big) - e^{16a\theta+13l\theta}\Big(2\theta(a\theta\\
&- 1)\mu^2 + (a\theta+1)\nu^2\Big)\Big(2\theta^2\mu^4 - \nu^4\Big) - 24ae^{13a\theta+10l\theta}\theta^2\mu^2\Big(4\theta^2\mu^4 - \nu^4\Big)\\
&+ 16e^{11(a+l)\theta}\theta\mu^2\Big(8\theta^2\mu^4 - 8\theta\nu^2\mu^2 + 3\nu^4\Big) - 4e^{10a\theta+11l\theta}\theta\Big(2\mu^2 - a\nu^2\Big)\Big(8\theta^2\mu^4\\
&- 8\theta\nu^2\mu^2 + 3\nu^4\Big) + 16e^{13a\theta+11l\theta}\theta\mu^2\Big(8a\theta^3\mu^4 - (2a\theta+1)\nu^4\Big) - 4e^{14a\theta+11l\theta}\theta\\
&\Big(16\theta^2(2a\theta-1)\mu^6 + 8\theta(a\theta+2)\nu^2\mu^4 - 2(8a\theta+3)\nu^4\mu^2 + a\nu^6\Big) - 8e^{12a\theta+11l\theta}\theta\\
&\Big(8\theta^2(a\theta+1)\mu^6 - 4\theta(a\theta(a\theta-1)+2)\nu^2\mu^4 + 2(1-3a\theta)\nu^4\mu^2 + a(a\theta+2)\nu^6\Big)\\
&+ e^{14a\theta+13l\theta}\Big(4\theta^3(2a\theta(a\theta(a\theta-1)+1)-3)\mu^6 + 2\theta^2(2a\theta(a\theta(3a\theta-1)-7)\\
&+ 19)\nu^2\mu^4 + 2\theta(a\theta(a\theta+2)(3a\theta-5)-5)\nu^4\mu^2 + (a\theta(a\theta-1)(a\theta+2)-3)\nu^6\Big)\\
&+ e^{12a\theta+13l\theta}\Big(4\theta^3(a\theta(2a\theta-1)+3)\mu^6 + 2\theta^2(a\theta(2a\theta-1)-19)\nu^2\mu^4\\
&+ 2\theta(a\theta(1-a\theta)+5)\nu^4\mu^2 + (a\theta(1-a\theta)+3)\nu^6\Big)\Big).
\end{aligned}
\tag{37}
$$

The last "unknown" element in equation (31) is

$$
\begin{aligned}
\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]}) = \mathbb{E}\Big(&\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\\
&+ \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2 + \Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)
\end{aligned}
$$

$$+4\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)$$
$$+2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big), \tag{38}$$

where

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\Big) = \frac{1}{32\theta^8}e^{-16a\theta-13l\theta}\Big(-48e^{17a\theta+9l\theta}\theta\mu^2\Big(2\theta\mu^2-\nu^2\Big)^3$$
$$+48e^{10a\theta+11l\theta}\theta\Big(2\mu^2-a\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)^3-48e^{21a\theta+10l\theta}\theta\mu^2$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+16e^{20a\theta+11l\theta}\theta\Big(2\mu^2+a\nu^2\Big)$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)-64e^{19a\theta+11l\theta}\theta\mu^2\Big(2\theta^2\mu^4+4\theta\nu^2\mu^2+(2a\theta-1)\nu^4\Big)$$
$$\Big(2\theta\mu^2-\nu^2\Big)-48e^{2(9a+5l)\theta}\theta\mu^2\Big(4\theta^2\Big(a^2\theta^2-1\Big)\mu^4+4\theta(a\theta(a\theta-4)-1)\nu^2\mu^2$$
$$+(a\theta(a\theta+8)-7)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+48e^{19a\theta+10l\theta}\theta\mu^2\Big(2\theta^2\Big(2a^2\theta^2+5\Big)\mu^4$$
$$+4\theta(a\theta-5)(a\theta+1)\nu^2\mu^2+(a\theta(a\theta+8)-6)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+64e^{17a\theta+11l\theta}\theta\mu^2$$
$$\Big(2\theta^2\Big(2a^2\theta^2-1\Big)\mu^4+4\theta(a\theta(a\theta-5)-5)\nu^2\mu^2+(a\theta(a\theta+8)-4)\nu^4\Big)$$
$$\Big(2\theta\mu^2-\nu^2\Big)-48e^{17a\theta+10l\theta}\theta\mu^2\Big(4\theta^2\Big(a^2\theta^2+20\Big)\mu^4+4\theta(a\theta(a\theta-8)-11)\nu^2\mu^2$$
$$+(a\theta(a\theta+16)-11)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)-16e^{18a\theta+11l\theta}\theta\Big(8\theta^2\Big(a^2\theta^2+1\Big)\mu^6$$
$$+4\theta(a\theta(a\theta(a\theta+2)-6)-14)\nu^2\mu^4+2(a\theta(a\theta(2a\theta-7)+2)-3)\nu^4\mu^2$$
$$+a(a\theta(a\theta+8)-5)\nu^6\Big)\Big(2\theta\mu^2-\nu^2\Big)-96e^{19a\theta+9l\theta}\theta\Big(\theta\mu^2-\nu^2\Big)$$
$$\Big(\mu\nu^2-2\theta\mu^3\Big)^2-192e^{11(a+l)\theta}\theta\Big(6\theta\mu^2-(2a\theta+1)\nu^2\Big)$$
$$\Big(\mu\nu^2-2\theta\mu^3\Big)^2-48e^{21a\theta+9l\theta}\theta\Big(\mu\nu^3-2\theta\mu^3\nu\Big)^2+64e^{12(a+l)\theta}\theta$$
$$\Big(\theta\mu^3\nu-\mu\nu^3\Big)^2+64e^{6(3a+2l)\theta}\theta\Big(\theta\mu^3\nu-\mu\nu^3\Big)^2+2e^{20a\theta+13l\theta}$$
$$\Big(\nu^4-2\theta^2\mu^4\Big)^2-128e^{17a\theta+12l\theta}\theta\mu^2\nu^2\Big(\theta\mu^2-\nu^2\Big)\Big(\theta(2a\theta+1)\mu^2+(2a\theta-1)\nu^2\Big)$$
$$+128e^{13a\theta+12l\theta}\theta\mu^2\nu^2\Big(\theta\mu^2-\nu^2\Big)\Big(\theta(2a\theta-1)\mu^2+(2a\theta+1)\nu^2\Big)$$
$$+6e^{22a\theta+9l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2\Big(2\theta^2\mu^4-\nu^4\Big)+6e^{10a\theta+13l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)+32e^{19a\theta+13l\theta}\theta\mu^2\nu^2\Big(\nu^4-2\theta^2\mu^4\Big)-256e^{3(5a+4l)\theta}\theta\mu^2\nu^2$$
$$\Big(\theta^2\Big(2a^2\theta^2-1\Big)\mu^4+2\theta\Big(2a^2\theta^2+1\Big)\nu^2\mu^2+\Big(2a^2\theta^2-1\Big)\nu^4\Big)$$
$$+64e^{4(4a+3l)\theta}\theta\mu^2\nu^2\Big(\theta^2(4a\theta(a\theta+2)-1)\mu^4+2\theta\Big(4a^2\theta^2+1\Big)\nu^2\mu^2$$
$$+(4a\theta(a\theta-2)-1)\nu^4\Big)+64e^{2(7a+6l)\theta}\theta\mu^2\nu^2\Big(\theta^2(4a\theta(a\theta-2)-1)\mu^4+2\theta$$
$$\Big(4a^2\theta^2+1\Big)\nu^2\mu^2+(4a\theta(a\theta+2)-1)\nu^4\Big)-6e^{20a\theta+9l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2$$
$$\Big(4\theta^2\Big(a^2\theta^2+3\Big)\mu^4+4\theta(a\theta(a\theta-4)-3)\nu^2\mu^2+(a\theta(a\theta+8)-7)\nu^4\Big)+6e^{9(2a+l)\theta}$$

$$\left(\nu^2-2\theta\mu^2\right)^2\left(2\theta^2\left(4a^2\theta^2+39\right)\mu^4+8\theta(a\theta(a\theta-6)-5)\nu^2\mu^2+(2a\theta(a\theta+12)\right.$$
$$\left.-19)\nu^4\right)-96e^{11a\theta+13l\theta}\theta\mu^2\left(2\theta^3\mu^6+\theta^2\nu^2\mu^4-3\theta\nu^4\mu^2+\nu^6\right)+48e^{10(2a+l)\theta}\theta\mu^2$$
$$\left(4\theta^3\mu^6+14\theta^2\nu^2\mu^4-10\theta\nu^4\mu^2+\nu^6\right)+16e^{15a\theta+13l\theta}\theta^2\mu^2\left(4\theta^2\left(2a^2\theta^2-1\right)\mu^6\right.$$
$$\left.+6\theta\left(2a^2\theta^2-13\right)\nu^2\mu^4+6\left(a^2\theta^2+1\right)\nu^4\mu^2+a^2\theta\nu^6\right)-128e^{13a\theta+11l\theta}\theta\mu^2$$
$$\left(126\theta^3\mu^6-5\theta^2(4a\theta+23)\nu^2\mu^4+6\theta(3a\theta+5)\nu^4\mu^2+2(1-4a\theta)\nu^6\right)+32e^{17a\theta+13l\theta}$$
$$\theta\mu^2\left(-2\theta^3\mu^6+\theta^2\left(4a^2\theta^2+5\right)\nu^2\mu^4+\theta(4a\theta(a\theta-2)-13)\nu^4\mu^2\right.$$
$$\left.+(a\theta(a\theta+4)-3)\nu^6\right)+16e^{13(a+l)\theta}\theta\mu^2\left(4\theta^3\left(6a^2\theta^2+5\right)\mu^6+2\theta^2\left(6a^2\theta^2-89\right)\right.$$
$$\left.\nu^2\mu^4+2\theta(a\theta(8-3a\theta)+33)\nu^4\mu^2+(10-a\theta(3a\theta+8))\nu^6\right)+6e^{15a\theta+9l\theta}\theta\mu$$
$$\left(256\theta^3\mu^7-384\theta^2\nu^2\mu^5+192\theta\nu^4\mu^3+(3\mu-35)\nu^6\right)-6e^{9(a+l)\theta}\theta\mu$$
$$\left(192\theta^3\mu^7-288\theta^2\nu^2\mu^5+144\theta\nu^4\mu^3+(11\mu-35)\nu^6\right)-6e^{9a\theta+10l\theta}\theta\mu$$
$$\left(192\theta^3\mu^7-288\theta^2\nu^2\mu^5+144\theta\nu^4\mu^3+(11\mu-35)\nu^6\right)-6e^{13a\theta+9l\theta}\theta\mu$$
$$\left(256\theta^3\mu^7-416\theta^2\nu^2\mu^5+224\theta\nu^4\mu^3+5(13\mu-21)\nu^6\right)+6e^{11a\theta+9l\theta}\theta\mu$$
$$\left(320\theta^3\mu^7-512\theta^2\nu^2\mu^5+272\theta\nu^4\mu^3+3(19\mu-35)\nu^6\right)+6e^{10(a+l)\theta}\theta\mu\left(960\theta^3\mu^7\right.$$
$$\left.-1056\theta^2\nu^2\mu^5+336\theta\nu^4\mu^3+(27\mu-35)\nu^6\right)-6e^{11a\theta+10l\theta}\theta\mu\left(1504\theta^3\mu^7\right.$$
$$\left.-16\theta^2(16a\theta+75)\nu^2\mu^5+64\theta(4a\theta+1)\nu^4\mu^3+((7-64a\theta)\mu+105)\nu^6\right)$$
$$+6e^{2(8a+5l)\theta}\theta\mu\left(64\theta^3\left(2a^2\theta^2+27\right)\mu^7+32\theta^2(2a\theta(a\theta-12)-63)\nu^2\mu^5\right.$$
$$\left.+32\theta(a\theta(24-a\theta)+9)\nu^4\mu^3((125-16a\theta(a\theta+12))\mu+35)\nu^6\right)+6e^{13a\theta+10l\theta}\theta\mu$$
$$\left(32\theta^3\left(2a^2\theta^2+79\right)\mu^7+16\theta^2(2a\theta(a\theta-16)-167)\nu^2\mu^5+16\theta(a\theta(32-a\theta)\right.$$
$$\left.+35)\nu^4\mu^3+((31-8a\theta(a\theta+16))\mu+105)\nu^6\right)$$
$$-6e^{2(7a+5l)\theta}\theta\mu\left(64\theta^3\left(a^2\theta^2+44\right)\mu^7+32\theta^2(a\theta(a\theta-24)-76)\nu^2\mu^5\right.$$
$$\left.+16\theta(a\theta(48-a\theta)+10)\nu^4\mu^3+((87-8a\theta(a\theta+24))\mu+105)\nu^6\right)$$
$$+6e^{2(6a+5l)\theta}\theta\mu\left(32\theta^3\mu^7+16\theta^2(31-16a\theta)\nu^2\mu^5+16\theta(16a\theta-29)\nu^4\mu^3+(105\right.$$
$$\left.-(64a\theta+17)\mu)\nu^6\right)-6e^{5(3a+2l)\theta}\theta\mu$$
$$\left(64\theta^3\left(a^2\theta^2-5\right)\mu^7+32\theta^2\left(a^2\theta^2-7\right)\nu^2\mu^5-16\theta\left(a^2\theta^2-16\right)\nu^4\mu^3\right.$$
$$\left.+\left(35-\left(8a^2\theta^2+35\right)\mu\right)\nu^6\right)+64e^{15a\theta+11l\theta}\theta\mu^2\left(4\theta^3\left(2a^2\theta^2-45\right)\right.$$
$$\mu^6+2\theta^2(2a\theta(a\theta-6)+57)\nu^2\mu^4-2\theta(a\theta(a\theta-14)+12)\nu^4\mu^2-(a\theta(a\theta+16)$$
$$\left.+2)\nu^6\right)+3e^{8a\theta+9l\theta}\left(96\theta^4\mu^8-192\theta^3\nu^2\mu^6+144\theta^2\nu^4\mu^4+2\theta(11\mu-35)\nu^6\mu+7\nu^8\right)$$
$$-32e^{16a\theta+11l\theta}\theta\left(8\theta^3\left(2a^2\theta^2-19\right)\mu^8+4\theta^2(2a\theta(a\theta-11)+7)\nu^2\mu^6+4\theta\right.$$
$$\left.(a\theta(3a\theta+19)+3)\nu^4\mu^4-2(9a\theta(a\theta+1)-1)\nu^6\mu^2+a(2a\theta-3)\nu^8\right)+6e^{10a\theta+9l\theta}$$

$$\Big(136\theta^4\mu^8 - 8\theta^3(8a\theta+13)\nu^2\mu^6 + 2\theta^2(48a\theta-25)\nu^4\mu^4 - 3\theta((16a\theta+15)\mu-35)\nu^6\mu$$
$$+ (8a\theta-15)\nu^8\Big) + 16e^{12a\theta+11l\theta}\theta\Big(712\theta^3\mu^8 - 12\theta^2(15a\theta+59)\nu^2\mu^6 + 2\theta$$
$$(a\theta(16a\theta+77)+94)\nu^4\mu^4 - 2(a\theta(16a\theta+17)-1)\nu^6\mu^2 + a(12a\theta-7)\nu^8\Big) - 2e^{18a\theta+13l\theta}$$
$$\Big(8\theta^4\Big(2a^2\theta^2+3\Big)\mu^8 + 8\theta^3(2a\theta(a\theta-2)-7)\nu^2\mu^6 - 2\theta^2(2a\theta(a\theta-4)+45)\nu^4\mu^4$$
$$+ 4\theta(7-2a\theta(a\theta-2))\nu^6\mu^2 + (7-2a\theta(a\theta+4))\nu^8\Big) - 16e^{14a\theta+11l\theta}\theta\Big(16\theta^3$$
$$\Big(a^2\theta^2-62\Big)\mu^8 + 8\theta^2(a\theta(a\theta(1-a\theta)+3)+102)\nu^2\mu^6 + 4\theta(a\theta(a\theta(7-a\theta)+1)$$
$$- 58)\nu^4\mu^4 + 2(a\theta(a\theta(a\theta-17)-2)-6)\nu^6\mu^2 + a(a\theta(a\theta+16)-2)\nu^8\Big)$$
$$+ 6e^{14a\theta+9l\theta}\Big(8\theta^4\Big(4a^2\theta^2+55\Big)\mu^8$$
$$- 8\theta^3(32a\theta+71)\nu^2\mu^6 + 2\theta^2(57-8a\theta(a\theta-24))\nu^4\mu^4 + \theta((57-192a\theta)\mu+35)\nu^6\mu$$
$$+ (2a\theta(a\theta+16)-33)\nu^8\Big) - 3e^{16a\theta+9l\theta}\Big(32\theta^4\Big(2a^2\theta^2+31\Big)\mu^8 - 64\theta^3(8a\theta+21)\nu^2\mu^6$$
$$+ 16\theta^2(23-2a\theta(a\theta-24))\nu^4\mu^4 + 48\theta(3-8a\theta)\nu^6\mu^2 + (4a\theta(a\theta+16)-59)\nu^8\Big)$$
$$- 6e^{12a\theta+9l\theta}\Big(16\theta^4\Big(a^2\theta^2+25\Big)\mu^8 - 96\theta^3(2a\theta+5)\nu^2\mu^6 + 8\theta^2(a\theta(36-a\theta)$$
$$+ 10)\nu^4\mu^4 + \theta(105-(144a\theta+25)\mu)\nu^6\mu + (a\theta(a\theta+24)-28)\nu^8\Big) + 2e^{12a\theta+13l\theta}\Big(4\theta^4$$
$$\Big(17-12a^2\theta^2\Big)\mu^8 + 16\theta^3(37-2a\theta)\nu^2\mu^6 + 4\theta^2(2a\theta(3a\theta+2)-111)\nu^4\mu^4 + 8\theta(2a\theta$$
$$+ 5)\nu^6\mu^2 + (13-a\theta(3a\theta+8))\nu^8\Big) - 4e^{14a\theta+13l\theta}\Big(8\theta^4\Big(15a^2\theta^2+14\Big)\mu^8 - 8\theta^3(a\theta$$
$$(a\theta-2)(4a\theta-3)+94)\nu^2\mu^6 - 2\theta^2(a\theta(a\theta(8a\theta+9)-76)-73)\nu^4\mu^4 + 8\theta(a\theta(a\theta$$
$$(a\theta-3)-5)+9)\nu^6\mu^2 + (4a\theta(a\theta(a\theta-1)-3)+11)\nu^8\Big) + 2e^{16a\theta+13l\theta}\Big(8\theta^4\Big(2a^4\theta^4$$
$$+ 6a^2\theta^2+19\Big)\mu^8 + 16\theta^3(2a\theta(a\theta-3)(a\theta(a\theta+1)+1)-5)\nu^2\mu^6 + 8\theta^2(a\theta(a\theta$$
$$(a\theta(3a\theta-4)-16)+38)+18)\nu^4\mu^4 + 8\theta(a\theta(a\theta(a\theta(a\theta+2)-7)-10)+15)\nu^6\mu^2$$
$$+ (a\theta(a\theta(a\theta(a\theta+8)-7)-24)+18)\nu^8\Big)\Big). \tag{39}$$

The covariance of the second central moment of $QV_{(t,t+a]}$ and $QV_{(t+l,t+a+l]}$, $l = 0, 1a, 2a, ...$ is similar to that of $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$. Hence

$$\begin{aligned}
&Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2) = \\
&\quad = \mathbb{E}(QV^2_{(t,t+a]}QV^2_{(t+l,t+a+l]}) - 2\mathbb{E}(QV^2_{(t,t+a]}QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 3\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad - 2\mathbb{E}(QV_{(t,t+a]}QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 4\mathbb{E}(QV_{(t,t+a]}QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad + \mathbb{E}(QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 - 3\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2 \\
&\quad = \mathbb{1}_{\{l=0\}}Var((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2) \\
&\quad + \mathbb{1}_{\{l\neq 0\}}Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2).
\end{aligned} \tag{40}$$

Computation in the first case, i.e. $l = 0$, gives

$$Var((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2) = \mathbb{E}(QV^4_{(t,t+a]}) - 4\mathbb{E}(QV^3_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})$$

$$+ 8\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2 - 4\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2. \tag{41}$$

We know that $QV_{(t,t+a]} = IV_{(t,t+a]} + \int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)$, and abbreviate $\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)$ as $QV^d_{(t,t+a]}$. For $\mathbb{E}(QV^4_{(t,t+a]})$ and $\mathbb{E}(QV^3_{(t,t+a]})$ we have,

$$\begin{aligned}\mathbb{E}(QV^4_{(t,t+a]}) &= \mathbb{E}\Big(IV^4_{(t,t+a]} + 4(QV^d_{(t,t+a]})^3 IV_{(t,t+a]} + 6(QV^d_{(t,t+a]})^2 IV^2_{(t,t+a]} + 4QV^d_{(t,t+a]} IV^3_{(t,t+a]} \\ &\quad + (QV^d_{(t,t+a]})^4\Big), \end{aligned}\tag{42}$$

$$\mathbb{E}(QV^3_{(t,t+a]}) = \mathbb{E}\Big((QV^d_{(t,t+a]})^3 + 3(QV^d_{(t,t+a]})^2 IV_{(t,t+a]} + 3QV^d_{(t,t+a]} IV^2_{(t,t+a]} + IV^3_{(t,t+a]}\Big), \tag{43}$$

where

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})^3) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^3\Big) \\ &= t\int_{\mathbb{R}} J^6 G(dx) + 3t^2\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} J^2 G(dx) + t^3\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3, \end{aligned}\tag{44}$$

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})^4) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^4\Big) \\ &= \mathbb{E}\Big(t\int_{\mathbb{R}} J^8 G(dx) + 4t^2\int_{\mathbb{R}} J^6 G(dx)\int_{\mathbb{R}} J^2 G(dx) + 6t^3\int_{\mathbb{R}} J^4 G(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2 \\ &\quad + 4t^4\int_{\mathbb{R}} J^2 G(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3 + t^4\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^4\Big). \end{aligned}\tag{45}$$

Due to the independence of continuous variance and jump processes, we have the following equations

$$\mathbb{E}((QV^d_{(t,t+a]})^2 IV_{(t,t+a]}) = \mathbb{E}\Big((QV^d_{(t,t+a]})^2\int_t^{t+a} V_d d\tau\Big) + \mathbb{E}((QV^d_{(t,t+a]})^2)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big), \tag{46}$$

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})(IV_{(t,t+a]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\ &\quad + 2\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big) \\ &\quad + \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}(\int_t^{t+a} V_c d\tau)^2, \end{aligned}\tag{47}$$

$$\mathbb{E}((QV^d_{(t,t+a]})^3 IV_{(t,t+a]}) = \mathbb{E}\Big((QV^d_{(t,t+a]})^3\int_t^{t+a} V_d d\tau\Big) + \mathbb{E}((QV^d_{(t,t+a]})^3)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big), \tag{48}$$

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})^2(IV_{(t,t+a]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\ &\quad + 2\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big) \\ &\quad + \mathbb{E}(QV^d_{(t,t+a]})^2\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^2, \end{aligned}\tag{49}$$

$$\begin{aligned}\mathbb{E}(QV^d_{(t,t+a]}(IV_{(t,t+a]})^3) &= \mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big) \\ &\quad + 3\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big) \end{aligned}$$

$$
\begin{aligned}
&+ 3\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^2 \\
&+ \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^3. \qquad (50)
\end{aligned}
$$

Recall equation (23) for $\int_t^{t+a} V_d d\tau$ in the manuscript, we have

$$
\begin{aligned}
\mathbb{E}\Big((QV^d_{(t,t+a]})^2\int_t^{t+a} V_d d\tau\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2\Big(\int_t^{t+a} V_d d\tau\Big)\Big) \\
&= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2 \\
&\Big(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) + \int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) \\
&+ a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\Big)\Big) \\
&= \frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2}\int_{\mathbb{R}} J^4 QG(dx) + \frac{a^2}{\kappa}\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} QG(dx) \\
&+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^2 QG(dx)\int_{\mathbb{R}} J^2 G(dx) + \frac{a^3}{\kappa}\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2\int_{\mathbb{R}} QG(dx),
\end{aligned}
\tag{51}
$$

$$
\begin{aligned}
\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\
&= -\frac{-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3}{2\kappa^3}\int_{\mathbb{R}} J^2 Q^2 G(dx) + \frac{a^3}{\kappa^2}\int_{\mathbb{R}} J^2 G(dx)\Big(\int_{\mathbb{R}} QG(dx)\Big)^2 \\
&+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^3}\int_{\mathbb{R}} J^2 QG(dx)\int_{\mathbb{R}} QG(dx) \\
&- \frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3 - e^{-2a\kappa}(e^{a\kappa}-1)^2)}{2\kappa^3}\int_{\mathbb{R}} J^2 G(dx)\int_{\mathbb{R}} Q^2 G(dx),
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\mathbb{E}\Big((QV^d_{(t,t+a]})^3\int_t^{t+a} V_d d\tau\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^3\Big(\int_t^{t+a} V_d d\tau\Big)\Big) \\
&= \frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2}\int_{\mathbb{R}} J^6 QG(dx) + 3\Big(\int_t^{t+a}\Big(\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\Big)^{\frac{1}{2}}du\int_{\mathbb{R}}(J^6 Q)^{\frac{1}{2}}G(dx)\Big)^2 \\
&+ \frac{3a(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^4 QG(dx)\int_{\mathbb{R}} J^2 G(dx) + \frac{a^4}{\kappa}\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3\int_{\mathbb{R}} QG(dx) \\
&+ \frac{3a^2(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^2 QG(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2 \\
&+ \frac{a^2}{\kappa}\int_{\mathbb{R}} J^6 G(dx)\int_{\mathbb{R}} QG(dx) + \frac{3a^3}{\kappa}\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} J^2 G(dx)\int_{\mathbb{R}} QG(dx),
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big) \\
&= \frac{e^{-3a\kappa}\big(-9e^{a\kappa} + 18e^{2a\kappa} + 2\big) + 6a\kappa - 11}{6\kappa^4}\int_{\mathbb{R}} J^2 Q^3 G(dx) \\
&+ 3\Big(\int_t^{t+a}\Big(\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\Big)^{\frac{3}{2}}du\int_{\mathbb{R}}(J^2 Q^3)^{\frac{1}{2}}G(dx)\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&- \Big(\frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3)}{2\kappa^3} - \frac{a^3}{\kappa^2}\Big) \int_{\mathbb{R}} Q^2 G(dx) \int_{\mathbb{R}} J^2 G(dx) \\
&+ \frac{a e^{-3a\kappa} (e^{a\kappa} - 1)^3}{3\kappa^4} \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q^3 G(dx) \\
&+ \frac{3a^2 e^{-2a\kappa} (e^{a\kappa} - 1)^2}{2\kappa^4} \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q^2 G(dx) \int_{\mathbb{R}} Q G(dx),
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\mathbb{E}\Big((QV^d_{(t,t+a]})^2 \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a} \int_{\mathbb{R}} J^2 \tilde{\mu}(ds,dx) + \int_t^{t+a} \int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2 \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\
&= -\frac{-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3}{2\kappa^3} \int_{\mathbb{R}} J^4 Q^2 G(dx) + 3\Big(\frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2} \int_{\mathbb{R}} J^2 Q G(dx)\Big)^2 \\
&+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^3} \int_{\mathbb{R}} J^4 Q G(dx) \int_{\mathbb{R}} Q G(dx) + \frac{a^3}{\kappa^2} \int_{\mathbb{R}} J^4 Q G(dx) \Big(\int_{\mathbb{R}} Q G(dx)\Big)^2 \\
&- \frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3)}{\kappa^3} \int_{\mathbb{R}} J^2 Q^2 G(dx) \int_{\mathbb{R}} J^2 G(dx) \\
&+ \frac{4a^2(a\kappa + e^{-a\kappa} - 1)}{\kappa^3} \int_{\mathbb{R}} J^2 Q G(dx) \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q G(dx) \\
&+ \frac{a^2(e^{-2a\kappa} (e^{a\kappa} - 1)^2 + 2a\kappa + 4e^{-a\kappa} - e^{-2a\kappa} - 3))}{2\kappa^3} \Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2 \int_{\mathbb{R}} Q^2 G(dx) \\
&+ \frac{a^4}{\kappa^2} \Big(\int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q G(dx)\Big)^2 + \frac{a^2 e^{-2a\kappa} (e^{a\kappa} - 1)^2}{2\kappa^3} \int_{\mathbb{R}} J^4 G(dx) \int_{\mathbb{R}} Q^2 G(dx).
\end{aligned} \tag{55}
$$

When $l \neq 0$,

$$
\begin{aligned}
&Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2) = \\
&\quad = \mathbb{E}(QV^2_{(t,t+a]} QV^2_{(t+l,t+a+l]}) - 2\mathbb{E}(QV^2_{(t,t+a]} QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 3\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad - 2\mathbb{E}(QV_{(t,t+a]} QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 4\mathbb{E}(QV_{(t,t+a]} QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad + \mathbb{E}(QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 - 3\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2.
\end{aligned} \tag{56}
$$

Then we have to analytically compute the following

$$
\mathbb{E}((QV^d_{(t,t+a]})^2 IV_{(t+l,t+a+l]}) = \mathbb{E}\Big((QV^d_{(t,t+a]})^2 \int_{t+l}^{t+a+l} V_d d\tau\Big) + \mathbb{E}\Big((QV^d_{(t,t+a]})^2)\mathbb{E}(\int_{t+l}^{t+a+l} V_c d\tau\Big), \tag{57}
$$

$$
\begin{aligned}
\mathbb{E}((QV^d_{(t,t+a]})(IV_{(t+l,t+a+l]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\Big) + \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2 \\
&+ 2\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big),
\end{aligned} \tag{58}
$$

$$
\begin{aligned}
\mathbb{E}((QV^d_{(t,t+a]})^2(IV_{(t+l,t+a+l]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\Big) + \mathbb{E}(QV^d_{(t,t+a]})^2\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2 \\
&+ 2\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big),
\end{aligned} \tag{59}
$$

where

$$\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big)$$
$$=\frac{(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^2}\int_{\mathbb{R}}J^4QG(dx)+\frac{a^2}{\kappa}\int_{\mathbb{R}}J^4G(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{2a(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}J^2G(dx)+\frac{a^3}{\kappa}\Big(\int_{\mathbb{R}}J^2G(dx)\Big)^2\int_{\mathbb{R}}QG(dx), \tag{60}$$

$$\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)$$
$$=\frac{(e^{a\kappa}-1)^3(e^{a\kappa}+1)e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^2Q^2G(dx)+\frac{a^3}{\kappa^2}\int_{\mathbb{R}}J^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2$$
$$+\frac{2a(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{a(e^{-2a\kappa}(e^{a\kappa}-1)^2+2a\kappa+4e^{-a\kappa}-e^{-2a\kappa}-3)}{2\kappa^3}\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}Q^2G(dx), \tag{61}$$

$$\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)$$
$$=\frac{(e^{a\kappa}-1)^3(e^{a\kappa}+1)e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^4Q^2G(dx)+3\Big(\frac{(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^2}\int_{\mathbb{R}}J^2QG(dx)\Big)^2$$
$$+\frac{2a(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^4QG(dx)\int_{\mathbb{R}}QG(dx)+\frac{a^3}{\kappa^2}\int_{\mathbb{R}}J^4QG(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2$$
$$\frac{a(e^{a\kappa}-1)^3(e^{a\kappa}+1)e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^2Q^2G(dx)\int_{\mathbb{R}}J^2G(dx)$$
$$+\frac{2a(e^{a\kappa}-1)^2e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{a^2(e^{-2a\kappa}(e^{a\kappa}-1)^2+2a\kappa+4e^{-a\kappa}-e^{-2a\kappa}-3)}{2\kappa^3}\Big(\int_{\mathbb{R}}J^2G(dx)\Big)^2\int_{\mathbb{R}}Q^2G(dx)$$
$$+a^2\Big(-\frac{-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3}{2\kappa^3}+\frac{(e^{a\kappa}-1)^2e^{-2\kappa(a+l)}}{2\kappa^3}$$
$$+\frac{(e^{a\kappa}-1)^2(e^{-2a\kappa}-e^{-2\kappa l})}{2\kappa^3}\Big)\int_{\mathbb{R}}J^4G(dx)\int_{\mathbb{R}}Q^2G(dx)$$
$$+\frac{a^4}{\kappa^2}\Big(\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}QG(dx)\Big)^2. \tag{62}$$

The covariance function of covariance of IV and other moments can be computed analogically to that of the variance term. The main components of that should resemble those of the covariance of variance of IV, up to different time lags. As for the covariance of the fourth variation (FV) and other moments, we can see that FV approximates the term $\int_t^{t+a}\int_{\mathbb{R}}J^4\mu(ds,dx)$, which is in the same form of $QV^d_{(t,t+a]}$. The only difference is the jump size, which is J^4 and J^2, respectively. Thus, the covariance functions of $\int_t^{t+a}\int_{\mathbb{R}}J^4\mu(ds,dx)$ and other moments are straightforward by merely changing the jump size, and we do not state them here.

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Stochastic Volatility: A Tale of Co-Jumps Non-Normality, GMM and High Frequency Data

Supplementary Appendix

Details of the Asymptotic Variance-covariance Matrix of the GMM Estimator

We present some details of the covariance matrix W for selected moment conditions f_t,

$$W = \sum_{l=-\infty}^{\infty} \mathbb{E}(g_t g'_{t-l}) = \sum_{l=-\infty}^{\infty} Cov(f_t f'_{t-l}).$$

We derive the elements of the main covariance functions $Cov(f_t, f'_{t-l})$, which is required for the computation of the infinite sum. We do not attempt the summation, as these expressions are extremely complex and not necessary for this article. It is in fact far more practical and common to use a kernel estimator to approximate the infinite sum. However, we still hope that future research can benefit from these derivations and therefore we choose to present these in this supplementary appendix.[1]

First, we present higher moment conditions for integrals with respect to compensated Poisson random measure and integrated variance (IV) that are necessary for the covariance functions. Then we state the analytical expressions for elements of those covariance functions.

The proof of higher moment conditions for integrals regarding compensated Poisson random measure begins with proving the case where the integrands are left-continuous simple predictable functions, and then generalizing to predictable functions.

Definition .1. *Write $\mathscr{L}_0$ as the collection of bounded, left-continuous, simple predictable processes. That is, a function of the form $\Phi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is in $\mathscr{L}_0$ if there is a sequence of stopping times $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, and a sequence of bounded $\mathcal{F}_t$-measurable random variables $\{\phi_i(\omega, x)\}_{i=0}^{n}$ such that*

$$\Phi(0,x) = \phi_0, \text{ and } \Phi(t,x) = \phi_i \text{ for } t \in (t_i, t_{i+1}],\ i = 0, 1, \ldots, n-1.$$

Clearly we can express Φ as

$$\Phi(t,x) = \sum_{j=1}^{m} \phi_{i=0,j} \mathbb{1}_{i=0}(t) \mathbb{1}_{A_j}(x) + \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} \mathbb{1}_{(t_i,t_{i+1}]}(t) \mathbb{1}_{A_j}(x), \tag{1}$$

here $\{\phi_{ij}\}_{j=1}^{m}$ are $\mathcal{F}_{t_i}$-measurable random variables and $\{A_j\}_{j=1}^{m}$ are disjoint subsets of $\mathbb{R}$.

[1] We agree with and would like to thank the anonymous referee and the editors for suggesting to add this part.

Naturally, the stochastic integral of Φ with respect to the compensated Poisson random measure $\tilde{\mu}$ is

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(dt, dx) &= \sum_{i,j=1}^{n,m} \phi_{ij} \tilde{\mu}((t_i \wedge t, t_{i+1} \wedge t] \times A_j) \\ &= \sum_{i,j=1}^{n,m} \phi_{ij} (\mu((t_i \wedge t, t_{i+1} \wedge t] \times A_j) - \nu((t_i \wedge t, t_{i+1} \wedge t] \times A_j)), \end{aligned} \tag{2}$$

where we skip the first term (t=0) on the right hand side of the above equation with a slight abuse of notation, as this simplifies but does not affect the following proof.

Based on this definition, we can now present higher moments of jump processes that appear in this context. We start from the case that the integrand is a left-continuous simple predictable function, and extend to the case of general predictable functions. Then the higher moments conditions of jump processes in this appendix is just a special case, since $J(\cdot)$ and $Q(\cdot)$ are deterministic left-continuous functions by definition. The lemma below is proved similarly to the isometry formula in Proposition 8.7 of Cont & Tankov (2003).

Lemma .1. *For any $\Phi \in \mathscr{L}_0$, the third and fourth absolute moment of the process $\{X_t\}_{0 \leq t \leq T}$ defined by*

$$X_t = \int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$$

satisfy the equations

$$\mathbb{E}(|X_t|^3) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx) \Big), \tag{3}$$

$$\mathbb{E}(|X_t|^4) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^4 \nu(ds, dx) \Big) + 3\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^2 \nu(ds, dx) \Big)^2, \tag{4}$$

provided they are bounded.

Proof. For the third absolute moment $k = 3$, we define $Y_t^j = \tilde{\mu}((0, t] \times A_j)$, then $\tilde{\mu}((t_i \wedge t, t_{i+1} \wedge t] \times A_j) = Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}$, where $Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}$ can be viewed as a compensated compound Poisson process from $(t_i \wedge t, t_{i+1} \wedge t]$ with jump sizes in A_j. Then we can express X_t as

$$X_t = \sum_{i,j=1}^{n,m} \phi_{ij} (Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t}),$$

and

$$\begin{aligned}\mathbb{E}(|X_T|^3) &= \mathbb{E}\Big(\sum_{i,j=1}^{n,m} \phi_{ij}^3 (Y^j_{t_{i+1} \wedge t} - Y^j_{t_i \wedge t})^3 \Big) \\ &= \sum_{i,j=1}^{n,m} \mathbb{E}(\mathbb{E}((Y_{t_{i+1}}(\phi_{ij}) - Y_{t_i}(\phi_{ij}))^3 | \mathcal{F}_{t_i})) \\ &= \sum_{i,j=1}^{n,m} \mathbb{E}(\phi_{ij}^3 \nu((t_i, t_{i+1}] \times A_j) \end{aligned}$$

$$= \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx)\Big),$$

where $Y_{t_{i+1}}(\phi_{ij}) - Y_{t_i}(\phi_{ij})$ is the increment of a compensated compound Poisson process from $(t_i, t_{i+1}]$ with jump size ϕ_{ij}. Note that the Poisson random measure $\mu(\omega, (t_i, t_{i+1}] \times A_j) = \mu((t_i, t_{i+1}] \times A_j)$ is a Poisson random variable with intensity parameter $\nu((t_i, t_{i+1}] \times A_j)$ for each rectangle $(t_i, t_{i+1}] \times A_j \in [0, T] \times \mathbb{R}$, hence the first equality holds due to the fact that Poisson distributions are independent for disjoint $(t_i, t_{i+1}] \times A_j$. The equality even holds when the jumps are infinite Lévy processes. In that case, infinite jumps can be interpreted as an infinite superposition of independent Poisson processes.(Cont & Tankov, 2003) The third equality is a direct result of the third moment of compensated compound Poisson processes.[2] The fourth absolute moment follows similarly. □

We extend the previous lemma to general predictable integrand functions. Let $\mathscr{L}_p$ ($p \in [1, \infty)$) be the space of predictable processes $\Phi : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ such that

$$||\Phi||_\nu^p = \mathbb{E}\Big(\int_0^T \int_{\mathbb{R}} |\Phi|^p \nu(ds, dx)\Big) < \infty.$$

We need to show that given a predictable function $\Phi \in \mathscr{L}_3$, there exists a sequence of simple predictable functions $\Phi_n \in \mathscr{L}_0$ converging to Φ in the sense that

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} |\Phi - \Phi_n|^3 \nu(ds, dx)\Big) \to 0, \text{ as } n \to \infty,$$

and therefore the process $\int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$ satisfies similar equality to (3). The fourth moment case follows in $\mathscr{L}_4$.

Proposition .1. *For any $\Phi \in \mathscr{L}_4$, the third and fourth absolute moment of the process $\{X_t\}_{0 \le t \le T}$ defined by*

$$X_t = \int_0^t \int_{\mathbb{R}} \Phi \tilde{\mu}(ds, dx)$$

satisfy the equations

$$\mathbb{E}(|X_t|^3) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^3 \nu(ds, dx)\Big), \tag{5}$$

$$\mathbb{E}(|X_t|^4) = \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^4 \nu(ds, dx)\Big) + 3\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \Phi^2 \nu(ds, dx)\Big)^2. \tag{6}$$

Proof. We sketch the proof of for the case of the third moment, i.e. $\Phi \in \mathscr{L}_3$ and the case of the fourth moment follows similarly.

$\mathscr{L}_3$ is the standard L^3 space under the measure $\nu \times dP$ of the predictable σ-algebra.(See Remark 11.3.7 of Cohen & Elliott (2015)) We know that the space $\mathscr{L}_0$ of left-continuous simple predictable processes generates the predictable σ-algebra, and the predictable process Φ is measurable with respect to the predictable σ-algebra by definition. Hence by the *simple approximation theorem* and the *dominated convergence theorem*, $\mathscr{L}_0$ is dense in $\mathscr{L}_3$ (and $\mathscr{L}_4$). Thus,

[2] See Barndorff-Nielsen & Shephard (2006) for calculation of higher moments of Lévy processes.

for any $\Phi \in \mathscr{L}_3$, there is a sequence of functions $\Phi_n \in \mathscr{L}_0$ converging to Φ in the sense that $||\Phi - \Phi_n||_\nu^3 \to 0$, and hence

$$\mathbb{E}\Big(\Big(\int_0^t \int_{\mathbb{R}} \Phi\tilde{\mu}(ds, dx) - \int_0^t \int_{\mathbb{R}} \Phi_n\tilde{\mu}(ds, dx)\Big)^3\Big) \to 0 \text{ as } n \to \infty.$$

Therefore we have the moment condition (5). □

Remark .1. *In our manuscript, $J(\cdot)$ and $Q(\cdot)$ are deterministic left-continuous functions with $J : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}^+$ and the jumps associated with price process and volatility process are simultaneous, and by definition $\nu(dt, dx) = G(dx)dt$. Thus, we have the following*

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} J^2\tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q\tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q\tilde{\mu}(ds, dx)\Big) = t\int_{\mathbb{R}} J^2Q^2G(dx), \tag{7}$$

and

$$\mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} J^2\tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q\tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q\tilde{\mu}(ds, dx) \int_0^t \int_{\mathbb{R}} Q\tilde{\mu}(ds, dx)\Big) \\ = t\int_{\mathbb{R}} J^2Q^3G(dx) + 3t^2\Big(\int_{\mathbb{R}} (J^2Q^3)^{\frac{1}{2}}G(dx)\Big)^2. \tag{8}$$

The next part is higher (conditional) moment conditions for the LQJD volatility process, derived by the iterated general formula.

Proposition .2 (Higher Moments of the Linear Quadratic Volatility Process)**.** *Given the initial value σ_0, for a Linear Quadratic Volatility process of the type $V(t) = \sigma^2(t)$ with $d\sigma(t) = \theta(\mu - \sigma(t))dt + \nu dW(t)$ ($\theta > 0, \mu > 0, \nu > 0$ and $W(t)$ is a Wiener Process), we have the following moments for $\sigma(t)$*

$$\mathbb{E}(\sigma_T^3|\mathcal{F}_t) = e^{-3\theta T}\Big(\sigma_t^3 e^{3\theta t} - \frac{1}{2\theta}\Big(\Big(e^{\theta t} - e^{\theta T}\Big)\Big(e^{\theta(t+T)}\Big(3\sigma_t\Big(2\theta\mu^2 + \nu^2\Big) - 4\theta\mu^3\Big) \\ + e^{2\theta t}\Big(3\nu^2(\mu + \sigma_t) - 2\theta\mu\Big(5\mu^2 - 3\sigma_t^2 + 3\mu\sigma_t\Big)\Big) + \mu e^{2\theta T}\Big(2\theta\mu^2 + 3\nu^2\Big)\Big)\Big)\Big), \tag{9}$$

$$\mathbb{E}(\sigma_T^4|\mathcal{F}_t) = e^{-4\theta T}\Big(\frac{1}{4\theta^2}\Big(6\Big(2\theta\mu^2 + \nu^2\Big)e^{2\theta(t+T)}\Big(\nu^2 + 2\theta\Big(-\mu^2 + \sigma_t^2 - 2\mu\sigma_t\Big)\Big) \\ + 8\theta\mu e^{\theta(3t+T)}\Big(2\theta\Big(5\mu^3 - 3\mu\sigma_t^2 + \sigma_t^3 + 3\mu^2\sigma_t\Big) - 3\nu^2(\mu + \sigma_t)\Big) \\ + 8\theta\mu(\sigma_t - \mu)\Big(2\theta\mu^2 + 3\nu^2\Big)e^{\theta(t+3T)} - e^{4\theta t}\Big(9\nu^4 + 4\theta^2\mu\Big(11\mu^3 - 6\mu\sigma_t^2 + 4\sigma_t^3 + 4\mu^2\sigma_t\Big) \\ + 12\theta\nu^2(\mu + \sigma_t)(\sigma_t - 3\mu)\Big) + e^{4\theta T}\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)\Big) + \sigma_t^4 e^{4\theta t}\Big), \tag{10}$$

$$\mathbb{E}(\sigma_T^5|\mathcal{F}_t) = e^{-5\theta T}\Big(\frac{1}{4\theta^2}\Big(-5\mu e^{\theta(4t+T)}\Big(9\nu^4 - 4\theta^2\Big(-11\mu^4 + 6\mu^2\sigma_t^2 - 4\mu\sigma_t^3 + \sigma_t^4 - 4\mu^3\sigma_t\Big) \\ + 12\theta\nu^2(\mu + \sigma_t)(\sigma_t - 3\mu)\Big) + 10\mu\Big(2\theta\mu^2 + 3\nu^2\Big)e^{2\theta t + 3\theta T}\Big(\nu^2 + 2\theta\Big(-\mu^2 + \sigma_t^2 \\ - 2\mu\sigma_t\Big)\Big) + 10\Big(2\theta\mu^2 + \nu^2\Big)e^{3\theta t + 2\theta T}\Big(2\theta\Big(5\mu^3 - 3\mu\sigma_t^2 + \sigma_t^3 + 3\mu^2\sigma_t\Big) - 3\nu^2(\mu + \sigma_t)\Big) \\ + 5(\sigma_t - \mu)\Big(4\theta^2\mu^4 + 12\theta\mu^2\nu^2 + 3\nu^4\Big)e^{\theta(t+4T)} + e^{5\theta t}\Big(4\theta^2\mu\Big(19\mu^4 - 10\mu^2\sigma_t^2 \\ + 10\mu\sigma_t^3 - 5\sigma_t^4 + 5\mu^3\sigma_t\Big) - 20\theta\nu^2(\mu + \sigma_t)\Big(7\mu^2 + \sigma_t^2 - 4\mu\sigma_t\Big) + 15\nu^4(3\mu + \sigma_t)\Big) \\ + \mu e^{5\theta T}\Big(4\theta^2\mu^4 + 20\theta\mu^2\nu^2 + 15\nu^4\Big)\Big) + \sigma_t^5 e^{5\theta t}\Big), \tag{11}$$

$$
\begin{aligned}
\mathbb{E}(\sigma_T^6|\mathcal{F}_t) = e^{-6\theta T}\Big(&\frac{1}{8\theta^3}\Big(15\Big(4\theta^2\mu^4+12\theta\mu^2\nu^2+3\nu^4\Big)e^{2\theta(t+2T)}\Big(-2\theta\mu^2+\nu^2+2\theta\sigma_t^2-4\theta\mu\sigma_t\Big)\\
&+12\theta\mu e^{\theta(5t+T)}\Big(76\theta^2\mu^5-140\theta\mu^3\nu^2+45\mu\nu^4-20\theta^2\mu\sigma_t^4+4\theta^2\sigma_t^5+20\theta\sigma_t^3\Big(2\theta\mu^2\\
&-\nu^2\Big)+20\theta\mu\sigma_t^2\Big(3\nu^2-2\theta\mu^2\Big)+5\sigma_t\Big(4\theta^2\mu^4-12\theta\mu^2\nu^2+3\nu^4\Big)\Big)\\
&+40\theta\mu e^{3\theta(t+T)}\Big(20\theta^2\mu^5+24\theta\mu^3\nu^2-9\mu\nu^4-6\theta\mu\sigma_t^2\Big(2\theta\mu^2+3\nu^2\Big)+2\theta\sigma_t^3\Big(2\theta\mu^2\\
&+3\nu^2\Big)+3\sigma_t\Big(4\theta^2\mu^4+4\theta\mu^2\nu^2-3\nu^4\Big)\Big)+15e^{2\theta(2t+T)}\Big(28\theta^2\mu^4\nu^2-88\theta^3\mu^6+18\theta\mu^2\nu^4\\
&-9\nu^6-16\theta^2\mu\sigma_t^3\Big(2\theta\mu^2+\nu^2\Big)+4\theta^2\sigma_t^4\Big(2\theta\mu^2+\nu^2\Big)+12\sigma_t^2\Big(4\theta^3\mu^4-\theta\nu^4\Big)\\
&-8\theta\mu\sigma_t\Big(4\theta^2\mu^4-4\theta\mu^2\nu^2-3\nu^4\Big)\Big)+12\theta\mu(\sigma_t-\mu)\Big(4\theta^2\mu^4+20\theta\mu^2\nu^2+15\nu^4\Big)e^{\theta(t+5T)}\\
&+e^{6\theta t}\Big(780\theta^2\mu^4\nu^2-232\theta^3\mu^6-450\theta\mu^2\nu^4+75\nu^6-80\theta^2\mu\sigma_t^3\Big(2\theta\mu^2-3\nu^2\Big)\\
&+60\theta^2\sigma_t^4\Big(2\theta\mu^2-\nu^2\Big)+30\theta\sigma_t^2\Big(4\theta^2\mu^4-12\theta\mu^2\nu^2+3\nu^4\Big)-48\theta^3\mu\sigma_t^5-12\theta\mu\sigma_t\Big(4\theta^2\mu^4\\
&-20\theta\mu^2\nu^2+15\nu^4\Big)\Big)+e^{6\theta T}\Big(60\theta^2\mu^4\nu^2+8\theta^3\mu^6+90\theta\mu^2\nu^4+15\nu^6\Big)\Big)+\sigma_t^6e^{6\theta t}\Big),
\end{aligned}
\tag{12}
$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^5)=\mu^5+10\mu^3\frac{\nu^2}{2\theta}+15\mu(\frac{\nu^2}{2\theta})^2, \tag{13}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^6)=\mu^6+15\mu^4\frac{\nu^2}{2\theta}+45\mu^2(\frac{\nu^2}{2\theta})^2+15(\frac{\nu^2}{2\theta})^3, \tag{14}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^7)=\mu^7+21\mu^5\frac{\nu^2}{2\theta}+105\mu^3(\frac{\nu^2}{2\theta})^2+105(\frac{\nu^2}{2\theta})^3, \tag{15}$$

$$\lim_{t\to\infty}\mathbb{E}(\sigma_t^8)=\mu^8+28\mu^6\frac{\nu^2}{2\theta}+210\mu^4(\frac{\nu^2}{2\theta})^2+420\mu^2(\frac{\nu^2}{2\theta})^3+105(\frac{\nu^2}{2\theta})^4, \tag{16}$$

$$
\begin{aligned}
\mathbb{E}(\sigma_{T_1}^2\sigma_{T_2}^2|\mathcal{F}_t) = \frac{1}{4\theta^2}\Big(&e^{-2\theta(T_1+T_2)}\Big(6\Big(2\theta\mu^2+\nu^2\Big)e^{2\theta(t+T_1)}\Big(\nu^2+2\theta\Big(-\mu^2+\sigma_t^2-2\mu\sigma_t\Big)\Big)\\
&+8\theta\mu e^{\theta(3t+T_1)}\Big(2\theta\Big(5\mu^3-3\mu\sigma_t^2+\sigma_t^3+3\mu^2\sigma_t\Big)-3\nu^2(\mu+\sigma_t)\Big)+8\theta\mu(\sigma_t-\mu)\Big(2\theta\mu^2\\
&+3\nu^2\Big)e^{\theta(t+3T_1)}+e^{4\theta t}\Big(-9\nu^4+4\theta^2\Big(-11\mu^4+6\mu^2\sigma_t^2-4\mu\sigma_t^3+\sigma_t^4-4\mu^3\sigma_t\Big)\\
&-12\theta\nu^2(\sigma_t-3\mu)(\mu+\sigma_t)\Big)+e^{4\theta T_1}\Big(4\theta^2\mu^4+12\theta\mu^2\nu^2+3\nu^4\Big)\Big)+\Big(2\theta\mu^2+\nu^2\\
&+e^{2\theta(T_1-T_2)}\Big(\nu^2-2\theta\mu^2\Big)-4\theta\mu^2e^{\theta(T_1-T_2)}\Big)\Big(2\theta\mu^2+\nu^2+e^{2\theta(t-T_1)}\Big(\nu^2\\
&+2\theta\Big(-\mu^2+\sigma_t^2-2\mu\sigma_t\Big)\Big)+4\theta\mu(\sigma_t-\mu)e^{\theta(t-T_1)}\Big)-4\theta\mu e^{-\theta(2T_1+T_2)}\Big(e^{\theta(T_1-T_2)}\\
&-1\Big)\Big(2\theta\sigma_t^3e^{3\theta t}-\Big(e^{\theta t}-e^{\theta T_1}\Big)\Big(e^{\theta(t+T_1)}\Big(3\sigma_t\Big(2\theta\mu^2+\nu^2\Big)-4\theta\mu^3\Big)+e^{2\theta t}\Big(3\nu^2(\mu\\
&+\sigma_t)-2\theta\mu\Big(5\mu^2-3\sigma_t^2+3\mu\sigma_t\Big)\Big)+\mu e^{2\theta T_1}\Big(2\theta\mu^2+3\nu^2\Big)\Big)\Big)\Big),\ \forall t\le T_1\le T_2.
\end{aligned}
\tag{17}
$$

Proof. By Itô's lemma we have

$$de^{n\theta t}\sigma_t^n=n\theta e^{n\theta t}\sigma_t^n dt+n\sigma_t^{n-1}d\sigma_t+\frac{n(n-1)}{2}\sigma_t^{n-2}d[\sigma_t,\sigma_t],\ n\ge 2. \tag{18}$$

Express this in integral form from [t,T] and take conditional expectation of $\mathcal{F}_t$ on both sides,

$$\mathbb{E}(\sigma_T^n|\mathcal{F}_t)=e^{n\theta(t-T)}\sigma_t^n+\mu(1-e^{\theta n(t-T)})\mathbb{E}(\sigma_T^{n-1}|\mathcal{F}_t)+\frac{\nu^2(n-1)(1-e^{\theta n(t-T)})}{2\theta}\mathbb{E}(\sigma_T^{n-2}|\mathcal{F}_t). \tag{19}$$

Thus, we can obtain the above (conditional) moments for σ_t by iteration. □

Now we can directly compute the covariance functions $Cov(f_t, f'_{t-l})$. In the manuscript, we have already derived the covariance of the first moment of integrated variance i.e. $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$, $l = ia$, $i \in \mathbb{N}$. When $l = 0$, it is just the variance of $IV_{(t,t+a]}$. Notably that the sum of infinite covariance series is certainly convergent due to the fact

$$\sum_{l=0}^{\infty} e^{-c|l|} = 1 + \frac{1}{e^c - 1}, \; c > 0,$$

where $e^{-c|l|}$ can be observed as a common coefficient of all individual terms. Although we do not sum up every covariance function, it is expected to appear in each function after simplifying expressions by canceling other terms.

We discuss the elements in the variance-covariance in two case hereafter, that is, when $l = 0$ and $l \neq 0$, because for some expectations of jump processes we may have very different results in those two cases. We start with the the covariance of the first moment of quadratic variation, i.e. $QV_{(t,t+a]}$ and $QV_{(t+l,t+a+l]}$. The variance of $QV_{(t,t+a]}$ is derived in the manuscript. When $l \neq 0$, we have

$$\begin{aligned}
Cov(QV_{(t,t+a]}, QV_{(t+l,t+a+l]}) &= \mathbb{E}(QV_{(t,t+a]}QV_{(t+l,t+a+l]}) - \Big(\mathbb{E}(IV_{(t,t+a]})^2 \\
&+ 2\mathbb{E}(IV_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big)^2\Big) \\
&= Cov(IV_{(t,t+a]}, IV_{(t+l,t+a+l]}) + \mathbb{E}\Big(IV_{(t+l,t+a+l]}\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) \\
&+ \mathbb{E}\Big(IV_{(t,t+a]}\int_{t+l}^{t+a+l}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\int_{t+l}^{t+a+l}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) \\
&- \Big(2\mathbb{E}\Big(IV_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big) + \mathbb{E}\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)\Big)^2\Big),
\end{aligned} \tag{20}$$

taking the limit of $t \to \infty$,

$$\begin{aligned}
\lim_{t\to\infty} Cov(QV_{(t,t+a]}, QV_{(t+l,t+a+l]}) &= a\Big(\frac{\nu^2}{2\theta} + \mu^2\Big)\frac{(\nu^2 - 2\theta\nu^2)(e^{-2\theta l} - e^{-2\theta(l-a)})}{4\theta^2} + \Big(a\Big(\frac{\nu^2}{2\theta} + \mu^2\Big)^2 \\
&+ \frac{2\mu^2\nu^2(1-e^{-\theta a})}{\theta^2} + \frac{\nu^4(1-e^{-2\theta a})}{4\theta^3}\Big) \cdot \left(\frac{e^{-2\theta(l-a)} - e^{-\theta l}}{2\theta}\right) \\
&+ \left(\frac{1-e^{-\theta a}}{\theta}\left(\mu^3 + \frac{3\mu\nu^2}{2\theta}\right) + \mu\left(a - \frac{1-e^{-\theta a}}{\theta}\right)\left(\frac{\nu^2}{2\theta} + \mu^2\right)\right) \\
&\cdot \left(\frac{e^{-\theta(l-a)} - e^{-\theta l}}{\theta} - \frac{e^{-2\theta(l-a)} - e^{-2\theta l}}{2\theta}\right) 2\mu + \frac{(e^{\kappa a} - 1)e^{-\kappa(2a+l)}}{2\kappa^3}\int_{\mathbb{R}} Q^2 G(dx) \\
&+ \frac{e^{-\kappa(l+a)} + e^{-\kappa(l-a)}}{\kappa^2}\int_{\mathbb{R}} J^2 Q G(dx).
\end{aligned} \tag{21}$$

Then we deal with the covariance between the second central moment of $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$, $l = 0, 1a, 2a, \ldots$. From now on, we state expressions without taking limits as $t \to \infty$ for the sake of simplicity. The final expressions for the stationary covariance can then be obtained by taking limits. We have

$$
\begin{aligned}
&Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2)=\\
&\quad=\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]})-2\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+3\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2\\
&\quad-2\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+4\mathbb{E}(IV_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2\\
&\quad+\mathbb{E}(IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2-3\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2\\
&\quad=\mathbb{1}_{\{l=0\}}Var((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2)\\
&\quad+\mathbb{1}_{\{l\neq 0\}}Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2). \qquad (22)
\end{aligned}
$$

Compute in the first case, i.e. $l=0$,

$$
\begin{aligned}
Var((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2)&=\mathbb{E}(IV^4_{(t,t+a]})-4\mathbb{E}(IV^3_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})\\
&\quad+8\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2-4\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2. \qquad (23)
\end{aligned}
$$

As some of the elements are already computed analytically, we are left with two elements, that is, $\mathbb{E}(IV^4_{(t,t+a]})$ and $\mathbb{E}(IV^3_{(t,t+a]})$.

$$
\begin{aligned}
\mathbb{E}(IV^4_{(t,t+a]})&=\mathbb{E}\Big(\Big(\int_t^{t+a}V_d d\tau\Big)^4+4\Big(\int_t^{t+a}V_d d\tau\Big)^3\Big(\int_t^{t+a}V_c d\tau\Big)+6\Big(\int_t^{t+a}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)^2\\
&\quad+4\Big(\int_t^{t+a}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)^3+\Big(\int_t^{t+a}V_c d\tau\Big)^4\Big), \qquad (24)\\
\mathbb{E}(IV^3_{(t,t+a]})&=\mathbb{E}\Big(\Big(\int_t^{t+a}V_d d\tau\Big)^3+3\Big(\int_t^{t+a}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)+3\Big(\int_t^{t+a}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)^2\\
&\quad+\Big(\int_t^{t+a}V_c d\tau\Big)^3\Big). \qquad (25)
\end{aligned}
$$

Recall equation (23) for $(\int_t^{t+a}V_d d\tau)$ in the manuscript. We know that it can be decomposed into three elements, i.e. two independent integrals with respect to a compensated Poisson random measure and one deterministic part. We can use the fact that integrals with respect to compensated Poisson random measure and integrated variance are independent.

We start with the expectations of powers of the continuous part of the variance. Following the approach for computing $\lim_{t\to\infty}\mathbb{E}((\int_t^{t+a}V_c d\tau)^2)$ and by *Fubini's theorem*,

$$
\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)^3\Big)&=\int_0^a\int_0^a\int_0^a\mathbb{E}\Big(\sigma_c^2(s)\sigma_c^2(u)\sigma_c^2(k)\Big)dsdudk\\
&=\int_0^a\int_0^a\int_0^a\mathbb{E}\Big(\sigma_c^2(s)\mathbb{E}(\sigma_c^2(u)\mathbb{E}(\sigma_c^2(k)|\mathcal{F}_u)|\mathcal{F}_s)\Big)dsdudk, \qquad (26)
\end{aligned}
$$

provided the integrand is Lebesgue integrable. By using Proposition 2 we can compute it analytically. The procedure for $\lim_{t\to\infty}\mathbb{E}((\int_t^{t+a}V_c d\tau)^4)$ is similar. As for $\mathbb{E}((\int_t^{t+a}V_d d\tau))^4$ and $\mathbb{E}((\int_t^{t+a}V_d d\tau))^3$, we expand the formula and integrate each part, following equation (25) in the manuscript and the previous Proposition 1. We have

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^3\Big)=\frac{1}{16\theta^6}\Big(2a\theta\Big(12\theta^2\mu^4\nu^2\Big(a^2\theta^2-5\Big)+6\theta\mu^2\nu^4\Big(a^2\theta^2-14\Big)$$
$$+8\theta^3\mu^6\Big(a^2\theta^2+1\Big)+\nu^6\Big(a^2\theta^2-18\Big)\Big)+16a\theta^2\mu^2\nu^2\Big(e^{-a\theta}$$
$$+e^{a\theta}\Big)\Big(6\theta\mu^2+5\nu^2\Big)+16\theta\mu^2\Big(e^{3a\theta}-e^{-3a\theta}\Big)\Big(4\theta^2\mu^4-\nu^4\Big)$$
$$+16\theta\mu^2\Big(e^{a\theta}-e^{-a\theta}\Big)\Big(4\theta^2\mu^4+4\theta\mu^2\nu^2-7\nu^4\Big)-2a\theta\Big(e^{-2a\theta}$$
$$+e^{2a\theta}\Big)\Big(2\theta\mu^2+9\nu^2\Big)\Big(2\theta^2\mu^4-\nu^4\Big)+\Big(e^{2a\theta}-e^{-2a\theta}\Big)\Big(-44\theta^2\mu^4\nu^2$$
$$-104\theta^3\mu^6+74\theta\mu^2\nu^4+3\nu^6\Big)-\frac{3}{2}\Big(e^{4a\theta}-e^{-4a\theta}\Big)\Big(\nu^2-2\theta\mu^2\Big)^2\Big(2\theta\mu^2+\nu^2\Big)\Big),\qquad(27)$$

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c(\tau)d\tau\Big)^4\Big)=\frac{1}{7680\theta^9}\Big(30\Big(8a^4\theta^4-372a^2\theta^2-13\Big)\nu^{10}+60\theta\Big(4a^2\Big(\Big(10a^2\theta^2$$
$$-103\Big)\mu^2-171\Big)\theta^2-19\mu^2+31\Big)\nu^8+80\theta^2\mu\Big(\Big(120a^4\theta^4+792a^2\theta^2$$
$$-2483\Big)\mu^3+36\Big(67-49a^2\theta^2\Big)\mu-441\Big)\nu^6+32\theta^3\mu^4\Big(60a^2\Big(2\Big(5a^2\theta^2$$
$$+57\Big)\mu^2-169\Big)\theta^2-22509\mu^2+39640\Big)\nu^4+32\theta^4\mu^6\Big(60a^2\Big(\Big(10a^2\theta^2$$
$$+113\Big)\mu^2-174\Big)\theta^2-15711\mu^2+898\Big)\nu^2+64\theta^5\mu^8\Big(60a^2\Big(\Big(2a^2\theta^2$$
$$+27\Big)\mu^2-23\Big)\theta^2+4567\mu^2-8437\Big)+320a\Big(-e^{-3a\theta}+e^{3a\theta}\Big)\theta^2\mu^2\Big(2\theta\mu^2$$
$$+\nu^2\Big)\Big(-45\nu^6+2\theta\Big(14\mu^2-39\Big)\nu^4+4\theta^2\mu^2\Big(75\mu^2+7\Big)\nu^2+16\theta^3\Big(8\mu^6$$
$$+\mu^4\Big)\Big)-15a\Big(-e^{-4a\theta}+e^{4a\theta}\Big)\theta\Big(2\theta\mu^2+\nu^2\Big)\Big(165\nu^8+6\theta\Big(17-48\mu^2\Big)\nu^6$$
$$-4\theta^2\mu^2\Big(152\mu^2+99\Big)\nu^4+8\theta^3\mu^4\Big(180\mu^2+11\Big)\nu^2+16\theta^4\Big(23\mu^8+\mu^6\Big)\Big)$$
$$+144\Big(e^{-5a\theta}+e^{5a\theta}\Big)\theta\mu\Big(45\mu\nu^8+10\theta\Big(4\mu^3-18\mu+21\Big)\nu^6+4\theta^2\mu^3\Big(5$$
$$-76\mu^2\Big)\nu^4-8\theta^3\mu^5\Big(32\mu^2+9\Big)\nu^2+16\theta^4\Big(19\mu^9+\mu^7\Big)\Big)-320a\Big(-e^{-a\theta}$$
$$+e^{a\theta}\Big)\theta^2\mu^2\Big(-63\nu^8+10\theta\Big(111-17\mu^2\Big)\nu^6+4\theta^2\mu^2\Big(23\mu^2-44\Big)\nu^4$$
$$+8\theta^3\mu^4\Big(49\mu^2-129\Big)\nu^2+32\theta^4\mu^6\Big(2\mu^2-5\Big)\Big)+32\Big(e^{-3a\theta}+e^{3a\theta}\Big)\theta\mu\Big($$
$$-15\mu\nu^8-10\theta\Big(44\mu^3-336\mu+189\Big)\nu^6-4\theta^2\mu^3\Big(101\mu^2+665\Big)\nu^4$$
$$+16\theta^3\mu^5\Big(199\mu^2+8\Big)\nu^2+96\theta^4\mu^7\Big(49\mu^2+36\Big)\Big)+16\Big(e^{-a\theta}+e^{a\theta}\Big)\theta\mu\Big($$
$$-15\Big(12a^2\theta^2+25\Big)\mu\nu^8+10\theta\Big(\Big(820-144a^2\theta^2\Big)\mu^3+6\Big(78a^2\theta^2-85\Big)\mu$$
$$+189\Big)\nu^6+4\theta^2\mu^3\Big(60a^2\Big(49-18\mu^2\Big)\theta^2+8246\mu^2-14075\Big)\nu^4$$
$$+8\theta^3\mu^5\Big(180a^2\Big(7-4\mu^2\Big)\theta^2+2692\mu^2+369\Big)\nu^2+16\theta^4\mu^7\Big(-180a^2\Big(\mu^2-1\Big)\theta^2$$
$$-1079\mu^2+1799\Big)\Big)-\Big(e^{-4a\theta}+e^{4a\theta}\Big)\Big(1065\nu^{10}+30\theta\Big(361\mu^2+3\Big)\nu^8$$
$$-40\theta^2\mu\Big(127\mu^3-492\mu+189\Big)\nu^6-16\theta^3\mu^4\Big(4961\mu^2+1240\Big)\nu^4$$

$$+16\theta^4\mu^6\Big(301\mu^2-1878\Big)\nu^2+32\theta^5\mu^8\Big(4603\mu^2+767\Big)\Big)-5\Big(e^{-6a\theta}+e^{6a\theta}\Big)\Big(
-75\nu^{10}+30\theta\Big(5\mu^2-1\Big)\nu^8+120\theta^2\mu\Big(6\mu^3-20\mu+21\Big)\nu^6+16\theta^3\mu^4\Big(15
-68\mu^2\Big)\nu^4-16\theta^4\mu^6\Big(137\mu^2+14\Big)\nu^2+32\theta^5\Big(29\mu^{10}+\mu^8\Big)\Big)-10a\Big(-e^{-2a\theta}
+e^{2a\theta}\Big)\theta\Big(-495\nu^{10}-18\theta\Big(71\mu^2+17\Big)\nu^8+32\theta^2\mu^2\Big(103\mu^2-654\Big)\nu^6
+32\theta^3\mu^4\Big(467\mu^2-38\Big)\nu^4+16\theta^4\mu^6\Big(1183\mu^2+1212\Big)\nu^2+32\theta^5\mu^8\Big(283\mu^2
+125\Big)\Big)-\Big(e^{-2a\theta}+e^{2a\theta}\Big)\Big(-15\Big(372a^2\theta^2+59\Big)\nu^{10}-30\theta\Big(4a^2\Big(127\mu^2
+171\Big)\theta^2+405\mu^2-33\Big)\nu^8+40\theta^2\mu\Big(\Big(216a^2\theta^2+626\Big)\mu^3+12\Big(9a^2\theta^2
+185\Big)\mu-567\Big)\nu^6+16\theta^3\mu^4\Big(180a^2\Big(14\mu^2+9\Big)\theta^2+12232\mu^2-20635\Big)\nu^4
+16\theta^4\mu^6\Big(60a^2\Big(17\mu^2-6\Big)\theta^2+10273\mu^2+5406\Big)\nu^2+32\theta^5\mu^8\Big(60a^2\Big(3\mu^2
+1\Big)\theta^2-2741\mu^2+8711\Big)\Big)\Big)\Big), \tag{28}$$

$$\mathbb{E}\Big(\Big(\int_t^{t+a}V_d(\tau)d\tau\Big)^3\Big)=\Big(\frac{1}{3\kappa^4}\Big(e^{-3a\kappa}\Big(e^{a\kappa}-1\Big)^3\Big)+\frac{1}{6\kappa^4}\Big(e^{-3a\kappa}\Big(-9e^{a\kappa}+18e^{2a\kappa}+2\Big)+6a\kappa
-11\Big)\Big)\int_{\mathbb{R}}Q^3G(dx)+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}
+3\Big)\Big)\frac{3a}{\kappa}\int_{\mathbb{R}}Q^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)++\frac{a^3}{\kappa^3}\Big(\int_{\mathbb{R}}QG(dx)\Big)^3, \tag{29}$$

$$\mathbb{E}\Big(\Big(\int_t^{t+a}V_d(\tau)d\tau\Big)^4\Big)=\Big(\frac{1}{4\kappa^5}\Big(e^{-4a\kappa}\Big(e^{a\kappa}-1\Big)^4\Big)+\frac{1}{12\kappa^5}\Big(4e^{-3a\kappa}\Big(-9e^{a\kappa}+12e^{2a\kappa}+4\Big)-3e^{-4a\kappa}
+12a\kappa-25\Big)\Big)\int_{\mathbb{R}}Q^4G(dx)+\Big(\frac{1}{3\kappa^4}\Big(e^{-3a\kappa}\Big(e^{a\kappa}-1\Big)^3\Big)+\frac{1}{6\kappa^4}\Big(e^{-3a\kappa}\Big(-9e^{a\kappa}
+18e^{2a\kappa}+2\Big)+6a\kappa-11\Big)\Big)\frac{4a}{\kappa}\int_{\mathbb{R}}Q^3G(dx)\int_{\mathbb{R}}QG(dx)+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}
-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3\Big)\Big)\frac{6a^2}{\kappa^2}\int_{\mathbb{R}}Q^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2
+\Big(\frac{1}{2\kappa^3}\Big(e^{-2a\kappa}\Big(e^{a\kappa}-1\Big)\Big)-\frac{1}{2\kappa^3}\Big(-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3\Big)\Big)3\int_{\mathbb{R}}Q^2G(dx)
+\frac{a^4}{\kappa^4}\Big(\int_{\mathbb{R}}QG(dx)\Big)^4. \tag{30}$$

In the second case, i.e. $l\neq 0$,

$$Cov((IV_{(t,t+a]}-\mathbb{E}(IV_{(t,t+a]}))^2,(IV_{(t+l,t+a+l]}-\mathbb{E}(IV_{(t+l,t+a+l]}))^2)=
=\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]})-2\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+3\mathbb{E}(IV^2_{(t,t+a]})\mathbb{E}(IV_{(t,t+a]})^2
-2\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})+4\mathbb{E}(IV_{(t,t+a]}IV_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2
+\mathbb{E}(IV^2_{(t+l,t+a+l]})\mathbb{E}(IV_{(t,t+a]})^2-3\mathbb{E}(IV_{(t,t+a]})^4-\mathbb{E}(IV^2_{(t,t+a]})^2. \tag{31}$$

Start from the first element that has not been computed,

$$\mathbb{E}(IV^2_{(t,t+a]}IV_{(t+l,t+a+l]}) = \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)$$
$$+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)$$
$$+ \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big) + \Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)$$
$$+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big) + \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big), \tag{32}$$

where

$$\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big) = \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2 \mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau | \mathcal{F}_{t+a}\Big)\Big)$$
$$= \mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\frac{1}{4\theta^2}\Big(e^{-2\theta l}\Big(-2\theta\mu^2\Big(e^{2a\theta}-1\Big)+\nu^2\Big(e^{2a\theta}-1\Big)$$
$$+ 2a\theta e^{2\theta l}\Big(2\theta\mu^2+\nu^2\Big) - 8\theta\mu^2\Big(e^{a\theta}-1\Big)e^{\theta l}\Big)\Big)$$
$$+ \frac{\sigma_c^2(t+a)}{2\theta}\Big(e^{2a\theta}-1\Big)e^{-2\theta l} + \frac{\sigma_c(t+a)}{4\theta^2}\Big(e^{-2\theta l}\Big(8\theta\mu\Big(e^{a\theta}-1\Big)e^{\theta l}$$
$$- 4\theta\mu\Big(e^{2a\theta}-1\Big)\Big)\Big)\Big)\Big)$$
$$= \mathbb{E}\Big(\int_t^{t+a}\int_t^{t+a} \mathbb{E}\Big(\sigma_c^2(\tau_1)\sigma_c^2(\tau_2)|\mathcal{F}_t\Big)d\tau_1 d\tau_2 \frac{1}{4\theta^2}$$
$$\Big(e^{-2\theta(a+l)}\Big(\nu^2\Big(-e^{2a\theta}+2e^{4a\theta}-1\Big)+4a\theta^2\mu^2e^{2\theta(a+l)}$$
$$+ 2\theta\Big(\mu^2\Big(e^{2a\theta}-2e^{4a\theta}+4e^{\theta(a+l)}-4e^{\theta(2a+l)}+1\Big)+a\nu^2e^{2\theta(a+l)}\Big)\Big)\Big)$$
$$+ \frac{1}{2\theta}\Big(\sigma_c(t)^2\Big(e^{2a\theta}-1\Big)e^{-2\theta(a+l)}\Big) - \frac{1}{\theta}\Big(\mu\sigma_c(t)\Big(e^{a\theta}-1\Big)e^{-2\theta(a+l)}\Big(e^{a\theta}$$
$$- 2e^{\theta(a+l)}+1\Big)\Big)\Big). \tag{33}$$

By computing that explicitly with the previous result, we have

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big) = \frac{1}{8\theta^6}e^{-2(3a+l)\theta}\Big(8a^3e^{2(3a+l)\theta}\theta^6\mu^6 + 4a^2\theta^5\Big(3ae^{2(3a+l)\theta}\nu^2$$
$$- 2e^{4a\theta}\Big(-1+e^{4a\theta}\Big)\mu^2\Big)\mu^4 + 2a\theta^4\Big(-12e^{2(a+l)\theta}\mu^4 - 2e^{2(4a+l)\theta}\mu^4$$
$$+ 48e^{3a\theta+2l\theta}\mu^4 + 16e^{5a\theta+2l\theta}\mu^4 + 2ae^{4a\theta}\nu^2\mu^2 - 2ae^{8a\theta}\nu^2\mu^2$$
$$+ e^{2(2a+l)\theta}\Big(16a\mu^2\nu^2 - 58\mu^4\Big) + e^{2(3a+l)\theta}\Big(8\mu^4 - 16a\nu^2\mu^2$$
$$+ 3a^2\nu^4\Big)\Big)\mu^2 + \Big(-3+e^{2a\theta}\Big)\Big(-1+e^{2a\theta}\Big)^3\Big(1+e^{2a\theta}\Big)\nu^6 - \Big(-1$$
$$+ e^{a\theta}\Big)^2\Big(1+e^{a\theta}\Big)\theta\nu^4\Big(-2\Big(-1+e^{a\theta}\Big)\Big(9+6e^{a\theta}+17e^{2a\theta}+8e^{3a\theta}$$
$$+ 3e^{4a\theta}+2e^{5a\theta}-e^{6a\theta}+8e^{(3a+l)\theta}\Big)\mu^2 - ae^{2a\theta}\Big(1+e^{a\theta}\Big)\Big(12+8e^{2a\theta}$$
$$- 3e^{2l\theta}+e^{2(a+l)\theta}\Big)\nu^2\Big) + \theta^3\Big(4\Big(-1+e^{a\theta}\Big)^3\Big(1+e^{a\theta}\Big)\Big(1+e^{2a\theta}\Big)$$

$$\Big(2e^{3a\theta}\Big(\frac{1}{2}\Big(-e^{-a\theta}+e^{a\theta}\Big)+1\Big)-12e^{a\theta}+6\Big)\mu^6+2ae^{2a\theta}\Big(-1+e^{a\theta}\Big)^2$$
$$\Big(16+32e^{a\theta}+32e^{2a\theta}+32e^{3a\theta}+16e^{4a\theta}+6e^{2l\theta}+5e^{2(a+l)\theta}$$
$$+16e^{(2a+l)\theta}-e^{2(2a+l)\theta}+12e^{(a+2l)\theta}+6e^{3a\theta+2l\theta}\Big)\nu^2\mu^4$$
$$+2a^2e^{4a\theta}\Big(-1+e^{4a\theta}\Big)\nu^4\mu^2+a^3e^{2(3a+l)\theta}\nu^6\Big)+\theta^2\Big(a^2e^{4a\theta}\Big(-1$$
$$+e^{2a\theta}\Big)\Big(1+e^{2a\theta}+8e^{2l\theta}\Big)\nu^6-2ae^{2a\theta}\Big(-1+e^{a\theta}\Big)^2\Big(16$$
$$+32e^{a\theta}+32e^{2a\theta}+32e^{3a\theta}+16e^{4a\theta}-3e^{2l\theta}-16e^{(2a+l)\theta}$$
$$-e^{2(2a+l)\theta}+6e^{(a+2l)\theta}-6e^{3a\theta+2l\theta}\Big)\mu^2\nu^4-2\Big(-1+e^{a\theta}\Big)^3\Big(1$$
$$+e^{a\theta}\Big)\Big(18-12e^{a\theta}+21e^{2a\theta}-2e^{3a\theta}+4e^{4a\theta}+10e^{5a\theta}+e^{6a\theta}$$
$$+8e^{(3a+l)\theta}\Big)\mu^4\nu^2\Big)\Big). \tag{34}$$

The second element is

$$\begin{aligned}\mathbb{E}(IV_{(t,t+a]}IV^2_{(t+l,t+a+l]}) = \mathbb{E}\Big(&\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)\\
&+2\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big(\int_t^{t+a}V_c d\tau\Big)\\
&+\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_c d\tau\Big)+\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big(\int_t^{t+a}V_d d\tau\Big)\\
&+2\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big(\int_t^{t+a}V_d d\tau\Big)\\
&+\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big(\int_t^{t+a}V_d d\tau\Big)\Big),\end{aligned} \tag{35}$$

where

$$\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l}\int_{t+l}^{t+a+l}V_c(\tau_1)V_c(\tau_2)d\tau_1 d\tau_2|\mathcal{F}_{t+a}\Big)\Big). \tag{36}$$

By taking limits on both sides we get

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a}V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l}V_c d\tau\Big)^2\Big)=\frac{1}{8\theta^6}e^{-14a\theta-13l\theta}\Big(-3e^{8a\theta+9l\theta}\Big(2\theta\mu^2-\nu^2\Big)^3$$
$$+8e^{11a\theta+13l\theta}\theta\mu^2\nu^2\Big(2\theta\mu^2-\nu^2\Big)-24e^{2(8a+5l)\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)$$
$$\Big(2\theta\mu^2-\nu^2\Big)+24e^{17a\theta+10l\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)$$
$$+32e^{15a\theta+11l\theta}\theta\mu^2\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-8e^{16a\theta+11l\theta}\theta\Big(2\mu^2$$
$$+a\nu^2\Big)\Big(2\theta(a\theta-1)\mu^2+(a\theta+1)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-24e^{5(3a+2l)\theta}\theta\mu^2\Big(2\theta(a\theta$$
$$-2)\mu^2+(a\theta+2)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)+24e^{11a\theta+10l\theta}\theta\mu^2\Big(2\theta(a\theta-2)\mu^2+(a\theta$$
$$+2)\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)-24e^{2(6a+5l)\theta}\theta\mu^2\Big(2\theta(a\theta-3)\mu^2+(a\theta+3)\nu^2\Big)\Big(2\theta\mu^2$$

$$
\begin{aligned}
&- \nu^2\Big) + 24e^{2(7a+5l)\theta}\theta\mu^2\Big(2\theta(2a\theta-3)\mu^2 + (2a\theta+3)\nu^2\Big)\Big(2\theta\mu^2 - \nu^2\Big)\\
&- e^{10a\theta+13l\theta}\Big(2\theta^2\mu^4 - \nu^4\Big)\Big(2\theta\mu^2 - \nu^2\Big) - 24e^{10(a+l)\theta}\theta\Big(\mu\nu^2 - 2\theta\mu^3\Big)^2\\
&+ 24e^{9a\theta+10l\theta}\theta\Big(\mu\nu^2 - 2\theta\mu^3\Big)^2 - 16e^{3(5a+4l)\theta}\theta\mu^2\nu^2\Big(\theta\mu^2 - \nu^2\Big) + 16e^{11a\theta+12l\theta}\theta\mu^2\nu^2\\
&\Big(\theta\mu^2 - \nu^2\Big) - 64ae^{13a\theta+12l\theta}\theta^2\mu^2\nu^2\Big(\theta\mu^2 + \nu^2\Big) + 8ae^{13(a+l)\theta}\theta^2\mu^2\nu^2\Big(2\theta\mu^2 + \nu^2\Big)\\
&+ 32e^{2(7a+6l)\theta}\theta\mu^2\nu^2\Big(\theta(a\theta+1)\mu^2 + (a\theta-1)\nu^2\Big) + 32e^{12(a+l)\theta}\theta\mu^2\nu^2\Big(\theta(a\theta\\
&- 1)\mu^2 + (a\theta+1)\nu^2\Big) + 8e^{15a\theta+13l\theta}\theta\mu^2\nu^2\Big(2\theta(a\theta-1)\mu^2 + (a\theta+1)\nu^2\Big)\\
&- 3e^{9(2a+l)\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-1)\mu^2 + (a\theta+1)\nu^2\Big) + 6e^{12a\theta+9l\theta}\Big(\nu^2\\
&- 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-2)\mu^2 + (a\theta+2)\nu^2\Big) - 6e^{14a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta\\
&- 2)\mu^2 + (a\theta+2)\nu^2\Big) - 3e^{10a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(a\theta-3)\mu^2 + (a\theta+3)\nu^2\Big)\\
&+ 3e^{16a\theta+9l\theta}\Big(\nu^2 - 2\theta\mu^2\Big)^2\Big(2\theta(2a\theta-3)\mu^2 + (2a\theta+3)\nu^2\Big) - e^{16a\theta+13l\theta}\Big(2\theta(a\theta\\
&- 1)\mu^2 + (a\theta+1)\nu^2\Big)\Big(2\theta^2\mu^4 - \nu^4\Big) - 24ae^{13a\theta+10l\theta}\theta^2\mu^2\Big(4\theta^2\mu^4 - \nu^4\Big)\\
&+ 16e^{11(a+l)\theta}\theta\mu^2\Big(8\theta^2\mu^4 - 8\theta\nu^2\mu^2 + 3\nu^4\Big) - 4e^{10a\theta+11l\theta}\theta\Big(2\mu^2 - a\nu^2\Big)\Big(8\theta^2\mu^4\\
&- 8\theta\nu^2\mu^2 + 3\nu^4\Big) + 16e^{13a\theta+11l\theta}\theta\mu^2\Big(8a\theta^3\mu^4 - (2a\theta+1)\nu^4\Big) - 4e^{14a\theta+11l\theta}\theta\\
&\Big(16\theta^2(2a\theta-1)\mu^6 + 8\theta(a\theta+2)\nu^2\mu^4 - 2(8a\theta+3)\nu^4\mu^2 + a\nu^6\Big) - 8e^{12a\theta+11l\theta}\theta\\
&\Big(8\theta^2(a\theta+1)\mu^6 - 4\theta(a\theta(a\theta-1)+2)\nu^2\mu^4 + 2(1-3a\theta)\nu^4\mu^2 + a(a\theta+2)\nu^6\Big)\\
&+ e^{14a\theta+13l\theta}\Big(4\theta^3(2a\theta(a\theta(a\theta-1)+1)-3)\mu^6 + 2\theta^2(2a\theta(a\theta(3a\theta-1)-7)\\
&+ 19)\nu^2\mu^4 + 2\theta(a\theta(a\theta+2)(3a\theta-5)-5)\nu^4\mu^2 + (a\theta(a\theta-1)(a\theta+2)-3)\nu^6\Big)\\
&+ e^{12a\theta+13l\theta}\Big(4\theta^3(a\theta(2a\theta-1)+3)\mu^6 + 2\theta^2(a\theta(2a\theta-1)-19)\nu^2\mu^4\\
&+ 2\theta(a\theta(1-a\theta)+5)\nu^4\mu^2 + (a\theta(1-a\theta)+3)\nu^6\Big)\Big). \qquad (37)
\end{aligned}
$$

The last "unknown" element in equation (31) is

$$
\begin{aligned}
\mathbb{E}(IV^2_{(t,t+a]}IV^2_{(t+l,t+a+l]}) = \mathbb{E}\Big(&\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\\
&+ \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2 + \Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ \Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\\
&+ 2\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)
\end{aligned}
$$

$$+4\Big(\int_t^{t+a} V_c d\tau\Big)\Big(\int_t^{t+a} V_d d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)$$

$$+2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big), \tag{38}$$

where

$$\lim_{t\to\infty}\mathbb{E}\Big(\Big(\int_t^{t+a} V_c d\tau\Big)^2\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2\Big) = \frac{1}{32\theta^8}e^{-16a\theta-13l\theta}\Big(-48e^{17a\theta+9l\theta}\theta\mu^2\Big(2\theta\mu^2-\nu^2\Big)^3$$
$$+48e^{10a\theta+11l\theta}\theta\Big(2\mu^2-a\nu^2\Big)\Big(2\theta\mu^2-\nu^2\Big)^3-48e^{21a\theta+10l\theta}\theta\mu^2$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+16e^{20a\theta+11l\theta}\theta\Big(2\mu^2+a\nu^2\Big)$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)-64e^{19a\theta+11l\theta}\theta\mu^2\Big(2\theta^2\mu^4+4\theta\nu^2\mu^2+(2a\theta-1)\nu^4\Big)$$
$$\Big(2\theta\mu^2-\nu^2\Big)-48e^{2(9a+5l)\theta}\theta\mu^2\Big(4\theta^2\Big(a^2\theta^2-1\Big)\mu^4+4\theta(a\theta(a\theta-4)-1)\nu^2\mu^2$$
$$+(a\theta(a\theta+8)-7)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+48e^{19a\theta+10l\theta}\theta\mu^2\Big(2\theta^2\Big(2a^2\theta^2+5\Big)\mu^4$$
$$+4\theta(a\theta-5)(a\theta+1)\nu^2\mu^2+(a\theta(a\theta+8)-6)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)+64e^{17a\theta+11l\theta}\theta\mu^2$$
$$\Big(2\theta^2\Big(2a^2\theta^2-1\Big)\mu^4+4\theta(a\theta(a\theta-5)-5)\nu^2\mu^2+(a\theta(a\theta+8)-4)\nu^4\Big)$$
$$\Big(2\theta\mu^2-\nu^2\Big)-48e^{17a\theta+10l\theta}\theta\mu^2\Big(4\theta^2\Big(a^2\theta^2+20\Big)\mu^4+4\theta(a\theta(a\theta-8)-11)\nu^2\mu^2$$
$$+(a\theta(a\theta+16)-11)\nu^4\Big)\Big(2\theta\mu^2-\nu^2\Big)-16e^{18a\theta+11l\theta}\theta\Big(8\theta^2\Big(a^2\theta^2+1\Big)\mu^6$$
$$+4\theta(a\theta(a\theta(a\theta+2)-6)-14)\nu^2\mu^4+2(a\theta(a\theta(2a\theta-7)+2)-3)\nu^4\mu^2$$
$$+a(a\theta(a\theta+8)-5)\nu^6\Big)\Big(2\theta\mu^2-\nu^2\Big)-96e^{19a\theta+9l\theta}\theta\Big(\theta\mu^2-\nu^2\Big)$$
$$\Big(\mu\nu^2-2\theta\mu^3\Big)^2-192e^{11(a+l)\theta}\theta\Big(6\theta\mu^2-(2a\theta+1)\nu^2\Big)$$
$$\Big(\mu\nu^2-2\theta\mu^3\Big)^2-48e^{21a\theta+9l\theta}\theta\Big(\mu\nu^3-2\theta\mu^3\nu\Big)^2+64e^{12(a+l)\theta}\theta$$
$$\Big(\theta\mu^3\nu-\mu\nu^3\Big)^2+64e^{6(3a+2l)\theta}\theta\Big(\theta\mu^3\nu-\mu\nu^3\Big)^2+2e^{20a\theta+13l\theta}$$
$$\Big(\nu^4-2\theta^2\mu^4\Big)^2-128e^{17a\theta+12l\theta}\theta\mu^2\nu^2\Big(\theta\mu^2-\nu^2\Big)\Big(\theta(2a\theta+1)\mu^2+(2a\theta-1)\nu^2\Big)$$
$$+128e^{13a\theta+12l\theta}\theta\mu^2\nu^2\Big(\theta\mu^2-\nu^2\Big)\Big(\theta(2a\theta-1)\mu^2+(2a\theta+1)\nu^2\Big)$$
$$+6e^{22a\theta+9l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2\Big(2\theta^2\mu^4-\nu^4\Big)+6e^{10a\theta+13l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2$$
$$\Big(2\theta^2\mu^4-\nu^4\Big)+32e^{19a\theta+13l\theta}\theta\mu^2\nu^2\Big(\nu^4-2\theta^2\mu^4\Big)-256e^{3(5a+4l)\theta}\theta\mu^2\nu^2$$
$$\Big(\theta^2\Big(2a^2\theta^2-1\Big)\mu^4+2\theta\Big(2a^2\theta^2+1\Big)\nu^2\mu^2+\Big(2a^2\theta^2-1\Big)\nu^4\Big)$$
$$+64e^{4(4a+3l)\theta}\theta\mu^2\nu^2\Big(\theta^2(4a\theta(a\theta+2)-1)\mu^4+2\theta\Big(4a^2\theta^2+1\Big)\nu^2\mu^2$$
$$+(4a\theta(a\theta-2)-1)\nu^4\Big)+64e^{2(7a+6l)\theta}\theta\mu^2\nu^2\Big(\theta^2(4a\theta(a\theta-2)-1)\mu^4+2\theta$$
$$\Big(4a^2\theta^2+1\Big)\nu^2\mu^2+(4a\theta(a\theta+2)-1)\nu^4\Big)-6e^{20a\theta+9l\theta}\Big(\nu^2-2\theta\mu^2\Big)^2$$
$$\Big(4\theta^2\Big(a^2\theta^2+3\Big)\mu^4+4\theta(a\theta(a\theta-4)-3)\nu^2\mu^2+(a\theta(a\theta+8)-7)\nu^4\Big)+6e^{9(2a+l)\theta}$$

$$\left(\nu^2-2\theta\mu^2\right)^2\left(2\theta^2\left(4a^2\theta^2+39\right)\mu^4+8\theta(a\theta(a\theta-6)-5)\nu^2\mu^2+(2a\theta(a\theta+12)\right.$$
$$\left.-19)\nu^4\right)-96e^{11a\theta+13l\theta}\theta\mu^2\left(2\theta^3\mu^6+\theta^2\nu^2\mu^4-3\theta\nu^4\mu^2+\nu^6\right)+48e^{10(2a+l)\theta}\theta\mu^2$$
$$\left(4\theta^3\mu^6+14\theta^2\nu^2\mu^4-10\theta\nu^4\mu^2+\nu^6\right)+16e^{15a\theta+13l\theta}\theta^2\mu^2\left(4\theta^2\left(2a^2\theta^2-1\right)\mu^6\right.$$
$$\left.+6\theta\left(2a^2\theta^2-13\right)\nu^2\mu^4+6\left(a^2\theta^2+1\right)\nu^4\mu^2+a^2\theta\nu^6\right)-128e^{13a\theta+11l\theta}\theta\mu^2$$
$$\left(126\theta^3\mu^6-5\theta^2(4a\theta+23)\nu^2\mu^4+6\theta(3a\theta+5)\nu^4\mu^2+2(1-4a\theta)\nu^6\right)+32e^{17a\theta+13l\theta}$$
$$\theta\mu^2\left(-2\theta^3\mu^6+\theta^2\left(4a^2\theta^2+5\right)\nu^2\mu^4+\theta(4a\theta(a\theta-2)-13)\nu^4\mu^2\right.$$
$$\left.+(a\theta(a\theta+4)-3)\nu^6\right)+16e^{13(a+l)\theta}\theta\mu^2\left(4\theta^3\left(6a^2\theta^2+5\right)\mu^6+2\theta^2\left(6a^2\theta^2-89\right)\right.$$
$$\left.\nu^2\mu^4+2\theta(a\theta(8-3a\theta)+33)\nu^4\mu^2+(10-a\theta(3a\theta+8))\nu^6\right)+6e^{15a\theta+9l\theta}\theta\mu$$
$$\left(256\theta^3\mu^7-384\theta^2\nu^2\mu^5+192\theta\nu^4\mu^3+(3\mu-35)\nu^6\right)-6e^{9(a+l)\theta}\theta\mu$$
$$\left(192\theta^3\mu^7-288\theta^2\nu^2\mu^5+144\theta\nu^4\mu^3+(11\mu-35)\nu^6\right)-6e^{9a\theta+10l\theta}\theta\mu$$
$$\left(192\theta^3\mu^7-288\theta^2\nu^2\mu^5+144\theta\nu^4\mu^3+(11\mu-35)\nu^6\right)-6e^{13a\theta+9l\theta}\theta\mu$$
$$\left(256\theta^3\mu^7-416\theta^2\nu^2\mu^5+224\theta\nu^4\mu^3+5(13\mu-21)\nu^6\right)+6e^{11a\theta+9l\theta}\theta\mu$$
$$\left(320\theta^3\mu^7-512\theta^2\nu^2\mu^5+272\theta\nu^4\mu^3+3(19\mu-35)\nu^6\right)+6e^{10(a+l)\theta}\theta\mu\left(960\theta^3\mu^7\right.$$
$$\left.-1056\theta^2\nu^2\mu^5+336\theta\nu^4\mu^3+(27\mu-35)\nu^6\right)-6e^{11a\theta+10l\theta}\theta\mu\left(1504\theta^3\mu^7\right.$$
$$\left.-16\theta^2(16a\theta+75)\nu^2\mu^5+64\theta(4a\theta+1)\nu^4\mu^3+((7-64a\theta)\mu+105)\nu^6\right)$$
$$+6e^{2(8a+5l)\theta}\theta\mu\left(64\theta^3\left(2a^2\theta^2+27\right)\mu^7+32\theta^2(2a\theta(a\theta-12)-63)\nu^2\mu^5\right.$$
$$\left.+32\theta(a\theta(24-a\theta)+9)\nu^4\mu^3((125-16a\theta(a\theta+12))\mu+35)\nu^6\right)+6e^{13a\theta+10l\theta}\theta\mu$$
$$\left(32\theta^3\left(2a^2\theta^2+79\right)\mu^7+16\theta^2(2a\theta(a\theta-16)-167)\nu^2\mu^5+16\theta(a\theta(32-a\theta)\right.$$
$$\left.+35)\nu^4\mu^3+((31-8a\theta(a\theta+16))\mu+105)\nu^6\right)$$
$$-6e^{2(7a+5l)\theta}\theta\mu\left(64\theta^3\left(a^2\theta^2+44\right)\mu^7+32\theta^2(a\theta(a\theta-24)-76)\nu^2\mu^5\right.$$
$$\left.+16\theta(a\theta(48-a\theta)+10)\nu^4\mu^3+((87-8a\theta(a\theta+24))\mu+105)\nu^6\right)$$
$$+6e^{2(6a+5l)\theta}\theta\mu\left(32\theta^3\mu^7+16\theta^2(31-16a\theta)\nu^2\mu^5+16\theta(16a\theta-29)\nu^4\mu^3+(105\right.$$
$$\left.-(64a\theta+17)\mu)\nu^6\right)-6e^{5(3a+2l)\theta}\theta\mu$$
$$\left(64\theta^3\left(a^2\theta^2-5\right)\mu^7+32\theta^2\left(a^2\theta^2-7\right)\nu^2\mu^5-16\theta\left(a^2\theta^2-16\right)\nu^4\mu^3\right.$$
$$\left.+\left(35-\left(8a^2\theta^2+35\right)\mu\right)\nu^6\right)+64e^{15a\theta+11l\theta}\theta\mu^2\left(4\theta^3\left(2a^2\theta^2-45\right)\right.$$
$$\mu^6+2\theta^2(2a\theta(a\theta-6)+57)\nu^2\mu^4-2\theta(a\theta(a\theta-14)+12)\nu^4\mu^2-(a\theta(a\theta+16)$$
$$\left.+2)\nu^6\right)+3e^{8a\theta+9l\theta}\left(96\theta^4\mu^8-192\theta^3\nu^2\mu^6+144\theta^2\nu^4\mu^4+2\theta(11\mu-35)\nu^6\mu+7\nu^8\right)$$
$$-32e^{16a\theta+11l\theta}\theta\left(8\theta^3\left(2a^2\theta^2-19\right)\mu^8+4\theta^2(2a\theta(a\theta-11)+7)\nu^2\mu^6+4\theta\right.$$
$$\left.(a\theta(3a\theta+19)+3)\nu^4\mu^4-2(9a\theta(a\theta+1)-1)\nu^6\mu^2+a(2a\theta-3)\nu^8\right)+6e^{10a\theta+9l\theta}$$

$$\Big(136\theta^4\mu^8 - 8\theta^3(8a\theta+13)\nu^2\mu^6 + 2\theta^2(48a\theta-25)\nu^4\mu^4 - 3\theta((16a\theta+15)\mu-35)\nu^6\mu$$
$$+ (8a\theta-15)\nu^8\Big) + 16e^{12a\theta+11l\theta}\theta\Big(712\theta^3\mu^8 - 12\theta^2(15a\theta+59)\nu^2\mu^6 + 2\theta$$
$$(a\theta(16a\theta+77)+94)\nu^4\mu^4 - 2(a\theta(16a\theta+17)-1)\nu^6\mu^2 + a(12a\theta-7)\nu^8\Big) - 2e^{18a\theta+13l\theta}$$
$$\Big(8\theta^4\Big(2a^2\theta^2+3\Big)\mu^8 + 8\theta^3(2a\theta(a\theta-2)-7)\nu^2\mu^6 - 2\theta^2(2a\theta(a\theta-4)+45)\nu^4\mu^4$$
$$+ 4\theta(7-2a\theta(a\theta-2))\nu^6\mu^2 + (7-2a\theta(a\theta+4))\nu^8\Big) - 16e^{14a\theta+11l\theta}\theta\Big(16\theta^3$$
$$\Big(a^2\theta^2-62\Big)\mu^8 + 8\theta^2(a\theta(a\theta(1-a\theta)+3)+102)\nu^2\mu^6 + 4\theta(a\theta(a\theta(7-a\theta)+1)$$
$$- 58)\nu^4\mu^4 + 2(a\theta(a\theta(a\theta-17)-2)-6)\nu^6\mu^2 + a(a\theta(a\theta+16)-2)\nu^8\Big)$$
$$+ 6e^{14a\theta+9l\theta}\Big(8\theta^4\Big(4a^2\theta^2+55\Big)\mu^8$$
$$- 8\theta^3(32a\theta+71)\nu^2\mu^6 + 2\theta^2(57-8a\theta(a\theta-24))\nu^4\mu^4 + \theta((57-192a\theta)\mu+35)\nu^6\mu$$
$$+ (2a\theta(a\theta+16)-33)\nu^8\Big) - 3e^{16a\theta+9l\theta}\Big(32\theta^4\Big(2a^2\theta^2+31\Big)\mu^8 - 64\theta^3(8a\theta+21)\nu^2\mu^6$$
$$+ 16\theta^2(23-2a\theta(a\theta-24))\nu^4\mu^4 + 48\theta(3-8a\theta)\nu^6\mu^2 + (4a\theta(a\theta+16)-59)\nu^8\Big)$$
$$- 6e^{12a\theta+9l\theta}\Big(16\theta^4\Big(a^2\theta^2+25\Big)\mu^8 - 96\theta^3(2a\theta+5)\nu^2\mu^6 + 8\theta^2(a\theta(36-a\theta)$$
$$+ 10)\nu^4\mu^4 + \theta(105-(144a\theta+25)\mu)\nu^6\mu + (a\theta(a\theta+24)-28)\nu^8\Big) + 2e^{12a\theta+13l\theta}\Big(4\theta^4$$
$$\Big(17-12a^2\theta^2\Big)\mu^8 + 16\theta^3(37-2a\theta)\nu^2\mu^6 + 4\theta^2(2a\theta(3a\theta+2)-111)\nu^4\mu^4 + 8\theta(2a\theta$$
$$+ 5)\nu^6\mu^2 + (13-a\theta(3a\theta+8))\nu^8\Big) - 4e^{14a\theta+13l\theta}\Big(8\theta^4\Big(15a^2\theta^2+14\Big)\mu^8 - 8\theta^3(a\theta$$
$$(a\theta-2)(4a\theta-3)+94)\nu^2\mu^6 - 2\theta^2(a\theta(a\theta(8a\theta+9)-76)-73)\nu^4\mu^4 + 8\theta(a\theta(a\theta$$
$$(a\theta-3)-5)+9)\nu^6\mu^2 + (4a\theta(a\theta(a\theta-1)-3)+11)\nu^8\Big) + 2e^{16a\theta+13l\theta}\Big(8\theta^4\Big(2a^4\theta^4$$
$$+ 6a^2\theta^2+19\Big)\mu^8 + 16\theta^3(2a\theta(a\theta-3)(a\theta(a\theta+1)+1)-5)\nu^2\mu^6 + 8\theta^2(a\theta(a\theta$$
$$(a\theta(3a\theta-4)-16)+38)+18)\nu^4\mu^4 + 8\theta(a\theta(a\theta(a\theta(a\theta+2)-7)-10)+15)\nu^6\mu^2$$
$$+ (a\theta(a\theta(a\theta(a\theta+8)-7)-24)+18)\nu^8\Big)\Big). \tag{39}$$

The covariance of the second central moment of $QV_{(t,t+a]}$ and $QV_{(t+l,t+a+l]}$, $l = 0, 1a, 2a, ...$ is similar to that of $IV_{(t,t+a]}$ and $IV_{(t+l,t+a+l]}$. Hence

$$\begin{aligned}
&Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2) = \\
&\quad = \mathbb{E}(QV^2_{(t,t+a]}QV^2_{(t+l,t+a+l]}) - 2\mathbb{E}(QV^2_{(t,t+a]}QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 3\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad - 2\mathbb{E}(QV_{(t,t+a]}QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 4\mathbb{E}(QV_{(t,t+a]}QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 \\
&\quad + \mathbb{E}(QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 - 3\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2 \\
&\quad = \mathbb{1}_{\{l=0\}}Var((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2) \\
&\quad + \mathbb{1}_{\{l\neq 0\}}Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2).
\end{aligned} \tag{40}$$

Computation in the first case, i.e. $l = 0$, gives

$$Var((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2) = \mathbb{E}(QV^4_{(t,t+a]}) - 4\mathbb{E}(QV^3_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})$$

$$+ 8\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2 - 4\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2. \tag{41}$$

We know that $QV_{(t,t+a]} = IV_{(t,t+a]} + \int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)$, and abbreviate $\int_t^{t+a}\int_{\mathbb{R}} J^2\mu(ds,dx)$ as $QV^d_{(t,t+a]}$. For $\mathbb{E}(QV^4_{(t,t+a]})$ and $\mathbb{E}(QV^3_{(t,t+a]})$ we have,

$$\begin{aligned}\mathbb{E}(QV^4_{(t,t+a]}) &= \mathbb{E}\Big(IV^4_{(t,t+a]} + 4(QV^d_{(t,t+a]})^3 IV_{(t,t+a]} + 6(QV^d_{(t,t+a]})^2 IV^2_{(t,t+a]} + 4QV^d_{(t,t+a]} IV^3_{(t,t+a]} \\ &\quad + (QV^d_{(t,t+a]})^4\Big), \qquad (42)\\ \mathbb{E}(QV^3_{(t,t+a]}) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^3 + 3(QV^d_{(t,t+a]})^2 IV_{(t,t+a]} + 3QV^d_{(t,t+a]} IV^2_{(t,t+a]} + IV^3_{(t,t+a]}\Big), \qquad (43)\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})^3) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^3\Big) \\ &= t\int_{\mathbb{R}} J^6 G(dx) + 3t^2\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} J^2 G(dx) + t^3\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3, \qquad (44)\\ \mathbb{E}((QV^d_{(t,t+a]})^4) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^4\Big) \\ &= \mathbb{E}\Big(t\int_{\mathbb{R}} J^8 G(dx) + 4t^2\int_{\mathbb{R}} J^6 G(dx)\int_{\mathbb{R}} J^2 G(dx) + 6t^3\int_{\mathbb{R}} J^4 G(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2 \\ &\quad + 4t^4\int_{\mathbb{R}} J^2 G(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3 + t^4\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^4\Big). \qquad (45)\end{aligned}$$

Due to the independence of continuous variance and jump processes, we have the following equations

$$\begin{aligned}\mathbb{E}((QV^d_{(t,t+a]})^2 IV_{(t,t+a]}) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^2\int_t^{t+a} V_d d\tau\Big) + \mathbb{E}((QV^d_{(t,t+a]})^2)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big), \qquad (46)\\ \mathbb{E}((QV^d_{(t,t+a]})(IV_{(t,t+a]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\ &\quad + 2\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big) \\ &\quad + \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}(\int_t^{t+a} V_c d\tau)^2, \qquad (47)\\ \mathbb{E}((QV^d_{(t,t+a]})^3 IV_{(t,t+a]}) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^3\int_t^{t+a} V_d d\tau\Big) + \mathbb{E}((QV^d_{(t,t+a]})^3)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big), \qquad (48)\\ \mathbb{E}((QV^d_{(t,t+a]})^2 (IV_{(t,t+a]})^2) &= \mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) \\ &\quad + 2\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big) \\ &\quad + \mathbb{E}(QV^d_{(t,t+a]})^2\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^2, \qquad (49)\\ \mathbb{E}(QV^d_{(t,t+a]}(IV_{(t,t+a]})^3) &= \mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big) \\ &\quad + 3\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)\end{aligned}$$

$$+ 3\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^2$$
$$+ \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}\Big(\int_t^{t+a} V_c d\tau\Big)^3. \tag{50}$$

Recall equation (23) for $\int_t^{t+a} V_d d\tau$ in the manuscript, we have

$$\begin{aligned}
\mathbb{E}\Big((QV^d_{(t,t+a]})^2 \int_t^{t+a} V_d d\tau\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\\
&= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2\\
&\Big(\int_t^{t+a}\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx) + \int_{-\infty}^{t}\int_t^{t+a} e^{\kappa(u-\tau)}d\tau\int_{\mathbb{R}} Q\tilde{\mu}(du,dx)\\
&+ a\frac{\int_{\mathbb{R}} QG(dx)}{\kappa}\Big)\Big)\\
&= \frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2}\int_{\mathbb{R}} J^4 QG(dx) + \frac{a^2}{\kappa}\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} QG(dx)\\
&+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^2 QG(dx)\int_{\mathbb{R}} J^2 G(dx) + \frac{a^3}{\kappa}\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2\int_{\mathbb{R}} QG(dx),
\end{aligned} \tag{51}$$

$$\begin{aligned}
\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big)\\
&= -\frac{-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3}{2\kappa^3}\int_{\mathbb{R}} J^2 Q^2 G(dx) + \frac{a^3}{\kappa^2}\int_{\mathbb{R}} J^2 G(dx)\Big(\int_{\mathbb{R}} QG(dx)\Big)^2\\
&+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^3}\int_{\mathbb{R}} J^2 QG(dx)\int_{\mathbb{R}} QG(dx)\\
&- \frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3 - e^{-2a\kappa}(e^{a\kappa}-1)^2)}{2\kappa^3}\int_{\mathbb{R}} J^2 G(dx)\int_{\mathbb{R}} Q^2 G(dx),
\end{aligned} \tag{52}$$

$$\begin{aligned}
\mathbb{E}\Big((QV^d_{(t,t+a]})^3 \int_t^{t+a} V_d d\tau\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^3\Big(\int_t^{t+a} V_d d\tau\Big)\Big)\\
&= \frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2}\int_{\mathbb{R}} J^6 QG(dx) + 3\Big(\int_t^{t+a}\Big(\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\Big)^{\frac{1}{2}}du\int_{\mathbb{R}}(J^6 Q)^{\frac{1}{2}}G(dx)\Big)^2\\
&+ \frac{3a(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^4 QG(dx)\int_{\mathbb{R}} J^2 G(dx) + \frac{a^4}{\kappa}\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^3\int_{\mathbb{R}} QG(dx)\\
&+ \frac{3a^2(a\kappa + e^{-a\kappa} - 1)}{\kappa^2}\int_{\mathbb{R}} J^2 QG(dx)\Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2\\
&+ \frac{a^2}{\kappa}\int_{\mathbb{R}} J^6 G(dx)\int_{\mathbb{R}} QG(dx) + \frac{3a^3}{\kappa}\int_{\mathbb{R}} J^4 G(dx)\int_{\mathbb{R}} J^2 G(dx)\int_{\mathbb{R}} QG(dx),
\end{aligned} \tag{53}$$

$$\begin{aligned}
\mathbb{E}\Big(QV^d_{(t,t+a]}\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big) &= \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2\tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)\Big(\int_t^{t+a} V_d d\tau\Big)^3\Big)\\
&= \frac{e^{-3a\kappa}\big(-9e^{a\kappa} + 18e^{2a\kappa} + 2\big) + 6a\kappa - 11}{6\kappa^4}\int_{\mathbb{R}} J^2 Q^3 G(dx)\\
&+ 3\Big(\int_t^{t+a}\Big(\int_u^{t+a} e^{\kappa(u-\tau)}d\tau\Big)^{\frac{3}{2}}du\int_{\mathbb{R}}(J^2 Q^3)^{\frac{1}{2}}G(dx)\Big)^2
\end{aligned}$$

$$- \Big(\frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3)}{2\kappa^3} - \frac{a^3}{\kappa^2}\Big) \int_{\mathbb{R}} Q^2 G(dx) \int_{\mathbb{R}} J^2 G(dx)$$
$$+ \frac{ae^{-3a\kappa}(e^{a\kappa} - 1)^3}{3\kappa^4} \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q^3 G(dx)$$
$$+ \frac{3a^2 e^{-2a\kappa}(e^{a\kappa} - 1)^2}{2\kappa^4} \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} Q^2 G(dx) \int_{\mathbb{R}} QG(dx), \tag{54}$$

$$\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big) = \mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}} J^2 \tilde{\mu}(ds,dx) + \int_t^{t+a}\int_{\mathbb{R}} J^2 G(ds,dx)\Big)^2\Big(\int_t^{t+a} V_d d\tau\Big)^2\Big)$$
$$= -\frac{-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3}{2\kappa^3} \int_{\mathbb{R}} J^4 Q^2 G(dx) + 3\Big(\frac{a\kappa + e^{-a\kappa} - 1}{\kappa^2} \int_{\mathbb{R}} J^2 QG(dx)\Big)^2$$
$$+ \frac{2a(a\kappa + e^{-a\kappa} - 1)}{\kappa^3} \int_{\mathbb{R}} J^4 QG(dx) \int_{\mathbb{R}} QG(dx) + \frac{a^3}{\kappa^2} \int_{\mathbb{R}} J^4 QG(dx) \Big(\int_{\mathbb{R}} QG(dx)\Big)^2$$
$$- \frac{a(-2a\kappa - 4e^{-a\kappa} + e^{-2a\kappa} + 3)}{\kappa^3} \int_{\mathbb{R}} J^2 Q^2 G(dx) \int_{\mathbb{R}} J^2 G(dx)$$
$$+ \frac{4a^2(a\kappa + e^{-a\kappa} - 1)}{\kappa^3} \int_{\mathbb{R}} J^2 QG(dx) \int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} QG(dx)$$
$$+ \frac{a^2(e^{-2a\kappa}(e^{a\kappa} - 1)^2 + 2a\kappa + 4e^{-a\kappa} - e^{-2a\kappa} - 3))}{2\kappa^3} \Big(\int_{\mathbb{R}} J^2 G(dx)\Big)^2 \int_{\mathbb{R}} Q^2 G(dx)$$
$$+ \frac{a^4}{\kappa^2}\Big(\int_{\mathbb{R}} J^2 G(dx) \int_{\mathbb{R}} QG(dx)\Big)^2 + \frac{a^2 e^{-2a\kappa}(e^{a\kappa} - 1)^2}{2\kappa^3} \int_{\mathbb{R}} J^4 G(dx) \int_{\mathbb{R}} Q^2 G(dx). \tag{55}$$

When $l \neq 0$,

$$Cov((QV_{(t,t+a]} - \mathbb{E}(QV_{(t,t+a]}))^2, (QV_{(t+l,t+a+l]} - \mathbb{E}(QV_{(t+l,t+a+l]}))^2) =$$
$$= \mathbb{E}(QV^2_{(t,t+a]} QV^2_{(t+l,t+a+l]}) - 2\mathbb{E}(QV^2_{(t,t+a]} QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 3\mathbb{E}(QV^2_{(t,t+a]})\mathbb{E}(QV_{(t,t+a]})^2$$
$$- 2\mathbb{E}(QV_{(t,t+a]} QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]}) + 4\mathbb{E}(QV_{(t,t+a]} QV_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2$$
$$+ \mathbb{E}(QV^2_{(t+l,t+a+l]})\mathbb{E}(QV_{(t,t+a]})^2 - 3\mathbb{E}(QV_{(t,t+a]})^4 - \mathbb{E}(QV^2_{(t,t+a]})^2. \tag{56}$$

Then we have to analytically compute the following

$$\mathbb{E}((QV^d_{(t,t+a]})^2 IV_{(t+l,t+a+l]}) = \mathbb{E}\Big((QV^d_{(t,t+a]})^2 \int_{t+l}^{t+a+l} V_d d\tau\Big) + \mathbb{E}\Big((QV^d_{(t,t+a]})^2)\mathbb{E}(\int_{t+l}^{t+a+l} V_c d\tau\Big), \tag{57}$$

$$\mathbb{E}((QV^d_{(t,t+a]})(IV_{(t+l,t+a+l]})^2) = \mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\Big) + \mathbb{E}(QV^d_{(t,t+a]})\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2$$
$$+ 2\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big), \tag{58}$$

$$\mathbb{E}((QV^d_{(t,t+a]})^2(IV_{(t+l,t+a+l]})^2) = \mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)^2\Big) + \mathbb{E}(QV^d_{(t,t+a]})^2\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big)^2$$
$$+ 2\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l} V_d d\tau\Big)\Big)\mathbb{E}\Big(\int_{t+l}^{t+a+l} V_c d\tau\Big), \tag{59}$$

where

$$\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)\Big)$$
$$=\frac{(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^2}\int_{\mathbb{R}}J^4QG(dx)+\frac{a^2}{\kappa}\int_{\mathbb{R}}J^4G(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{2a\,(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}J^2G(dx)+\frac{a^3}{\kappa}\Big(\int_{\mathbb{R}}J^2G(dx)\Big)^2\int_{\mathbb{R}}QG(dx),\tag{60}$$

$$\mathbb{E}\Big((QV^d_{(t,t+a]})\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)$$
$$=\frac{(e^{a\kappa}-1)^3\,(e^{a\kappa}+1)\,e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^2Q^2G(dx)+\frac{a^3}{\kappa^2}\int_{\mathbb{R}}J^2G(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2$$
$$+\frac{2a\,(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{a(e^{-2a\kappa}\,(e^{a\kappa}-1)^2+2a\kappa+4e^{-a\kappa}-e^{-2a\kappa}-3)}{2\kappa^3}\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}Q^2G(dx),\tag{61}$$

$$\mathbb{E}\Big((QV^d_{(t,t+a]})^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)=\mathbb{E}\Big(\Big(\int_t^{t+a}\int_{\mathbb{R}}J^2\tilde{\mu}(ds,dx)+\int_t^{t+a}\int_{\mathbb{R}}J^2G(ds,dx)\Big)^2\Big(\int_{t+l}^{t+a+l}V_d d\tau\Big)^2\Big)$$
$$=\frac{(e^{a\kappa}-1)^3\,(e^{a\kappa}+1)\,e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^4Q^2G(dx)+3\Big(\frac{(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^2}\int_{\mathbb{R}}J^2QG(dx)\Big)^2$$
$$+\frac{2a\,(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^4QG(dx)\int_{\mathbb{R}}QG(dx)+\frac{a^3}{\kappa^2}\int_{\mathbb{R}}J^4QG(dx)\Big(\int_{\mathbb{R}}QG(dx)\Big)^2$$
$$\frac{a\,(e^{a\kappa}-1)^3\,(e^{a\kappa}+1)\,e^{-2\kappa(a+l)}}{2\kappa^3}\int_{\mathbb{R}}J^2Q^2G(dx)\int_{\mathbb{R}}J^2G(dx)$$
$$+\frac{2a\,(e^{a\kappa}-1)^2\,e^{-(\kappa(a+l))}}{\kappa^3}\int_{\mathbb{R}}J^2QG(dx)\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}QG(dx)$$
$$+\frac{a^2(e^{-2a\kappa}\,(e^{a\kappa}-1)^2+2a\kappa+4e^{-a\kappa}-e^{-2a\kappa}-3)}{2\kappa^3}\Big(\int_{\mathbb{R}}J^2G(dx)\Big)^2\int_{\mathbb{R}}Q^2G(dx)$$
$$+a^2\Big(-\frac{-2a\kappa-4e^{-a\kappa}+e^{-2a\kappa}+3}{2\kappa^3}+\frac{(e^{a\kappa}-1)^2\,e^{-2\kappa(a+l)}}{2\kappa^3}$$
$$+\frac{(e^{a\kappa}-1)^2\,(e^{-2a\kappa}-e^{-2\kappa l})}{2\kappa^3}\Big)\int_{\mathbb{R}}J^4G(dx)\int_{\mathbb{R}}Q^2G(dx)$$
$$+\frac{a^4}{\kappa^2}\Big(\int_{\mathbb{R}}J^2G(dx)\int_{\mathbb{R}}QG(dx)\Big)^2.\tag{62}$$

The covariance function of covariance of IV and other moments can be computed analogically to that of the variance term. The main components of that should resemble those of the covariance of variance of IV, up to different time lags. As for the covariance of the fourth variation (FV) and other moments, we can see that FV approximates the term $\int_t^{t+a}\int_{\mathbb{R}}J^4\mu(ds,dx)$, which is in the same form of $QV^d_{(t,t+a]}$. The only difference is the jump size, which is J^4 and J^2, respectively. Thus, the covariance functions of $\int_t^{t+a}\int_{\mathbb{R}}J^4\mu(ds,dx)$ and other moments are straightforward by merely changing the jump size, and we do not state them here.

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