FEUILLES SYMPLECTIQUES POUR LES TRANCHES DANS LES GRASSMANNIENNES AFFINES GÉNÉRALISÉES

SYMPLECTIC LEAVES FOR GENERALIZED AFFINE GRASSMANNIAN SLICES

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Résumé. Les tranches dans les grassmanniennes affines généralisées $\overline{\mathcal{W}}^{\lambda}_{\mu}$ sont des variétés affines introduites par Braverman, Finkelberg et Nakajima au cours de leur étude des branches de Coulomb pour les théories de jauge 3d $\mathcal{N} = 4$ de type carquois. Nous prouvons une de leurs conjectures, en montrant que l'ouvert affine $\mathcal{W}^{\lambda}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$ est lisse. Une décomposition précise en feuilles symplectiques en découle. Notre preuve est valable sur un anneau arbitraire et, en particulier, nous montrons que l'ensemble des points complexes $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ forme une variété complexe. Un aspect subtil de notre méthode est l'emploi essentiel des ind-schémas qui sont formellement lisses mais qui ne sont pas lisses.

Abstract. The generalized affine Grassmannian slices $\overline{\mathcal{W}}^{\lambda}_{\mu}$ are algebraic varieties introduced by Braverman, Finkelberg, and Nakajima in their study of Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories. We prove a conjecture of theirs by showing that the dense open subset $\mathcal{W}^{\lambda}_{\mu} \subseteq \overline{\mathcal{W}}^{\lambda}_{\mu}$ is smooth. An explicit decomposition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$ into symplectic leaves follows as a corollary. Our argument works over an arbitrary ring and in particular implies that the complex points $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ form a smooth holomorphic symplectic manifold. A subtle aspect of the method is our essential use of ind-schemes that are formally smooth but not smooth.

1. Introduction

Affine Grassmannian slices for a reductive group G are objects of considerable interest. As transversal slices to spherical Schubert varieties, they capture information about singularities in the affine Grassmannian. By the geometric Satake correspondence, these singularities are known to carry representation-theoretic information about the Langlands dual group of G. Additionally, they have a Poisson structure that quantizes to the truncated shifted Yangians [KWWY14]. Furthermore, they form a large class of conical symplectic singularities that do not admit symplectic resolutions in general. As such, they form an important test ground for ideas in symplectic algebraic geometry and representation theory.

Recently, Braverman, Finkelberg, and Nakajima [BFN19] showed that affine Grassmannian slices arise as Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories. Their construction of affine Grassmannian slices is particularly satisfying because: (1) the quantization comes essentially for free in their construction, (2) their construction works for arbitrary symmetric Kac-Moody type. Because of point (2), the Coulomb branch perspective seems to be a fruitful path toward understanding the geometric Satake correspondence in affine type and beyond (see e.g. [Fin18, Nak]).

However, their construction produces more than just affine Grassmannian slices: usual affine Grassmannian slices are indexed by a pair of dominant coweights λ and μ, but their construction does not constrain μ to be dominant. Rather, they construct **generalized affine Grassmannian slices** denoted $\overline{\mathcal{W}}^{\lambda}_{\mu}$ for λ constrained to be dominant but for *arbitrary* $\mu \leqslant \lambda$.

The geometry of the generalized affine Grassmannian slices for μ non-dominant is less understood than the case of μ dominant. For example, there is a disjoint decomposition

$$\overline{\mathcal{W}}^{\lambda}_{\mu} = \bigsqcup_{\substack{\nu \text{ dominant}, \\ \mu \leqslant \nu \leqslant \lambda}} \mathcal{W}^{\nu}_{\mu} \tag{1.1}$$

that Braverman, Finkelberg and Nakajima conjecture ([BFN19, Remark 3.19]) to be a decomposition of $\overline{\mathcal{W}}^{\lambda}_{\mu}$ into symplectic leaves. They show that this would follow if one could show that the subvarieties $\mathcal{W}^{\lambda}_{\mu}$ are smooth for all λ and μ. In this note, we show the following, which proves their conjecture.

Theorem 1.2 (Corollary 3.17). *For any $\lambda \geqslant \mu$ with λ dominant, the variety $\mathcal{W}^{\lambda}_{\mu}$ is smooth.*

In particular, it follows that the set of $\mathbb{C}$–points $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ is a smooth holomorphic symplectic manifold. This verifies part of an expectation that it is also hyper-Kähler, since $\mathcal{W}^{\lambda}_{\mu}(\mathbb{C})$ should be identified with a moduli space of singular instantons on $\mathbb{R}^3$, see [BFN19], [BDG17].

1.1. **Previously known cases.** Theorem 1.2 is known when μ is dominant because in this case $\overline{\mathcal{W}}^{\lambda}_{\mu}$ is a usual affine Grassmannian slice. It is also known for $\mu \leqslant w_0(\lambda)$ where w_0 is the longest element of the Weyl group [BFN19, Remark 3.19]. In type A, all cases are known by work of Nakajima and Takayama on Cherkis bow varieties [NT17, Theorem 7.13]. In [KP19], Krylov and Perunov prove Theorem 1.2 in general type for λ minuscule and μ lying in the orbit of λ under the Weyl group. In fact, they prove more: they show that $\overline{\mathcal{W}}^{\lambda}_{\mu} = \mathcal{W}^{\lambda}_{\mu}$ is an affine space in this case.

We note that our main theorem has been expected by physicists, since $\mathcal{W}^{\lambda}_{\mu}$ should be a space of singular instantons as mentioned above, and that the decomposition (1.1) is an instance of *monopole bubbling*. We refer the reader again to [BDG17], and to the references in the physics literature given in the introduction of [BFN19], as well as in [Nak16].

Finally, generalized affine Grassmannian slices of the form $\mathcal{W}^{0}_{\mu}$ had been previously considered in a different guise: these are the so called "open Zastava spaces" whose smoothness is deduced by a certain cohomology vanishing computation [FM99]. Our approach gives a direct group-theoretic proof of this smoothness. It would be interesting to understand how these two approaches are precisely related. We elaborate on this briefly in §3.3.3.

1.2. **Our approach.** There is a group theoretic construction of generalized affine Grassmannian slices $\overline{\mathcal{W}}^{\lambda}_{\mu}$ and their open subschemes $\mathcal{W}^{\lambda}_{\mu}$ given in [BFN19, Section 2(xi)], inspired by the scattering matrix description of singular monopoles from [BDG17, Section 6.4.1]. We slightly modify this group-theoretic construction to produce spaces that we call $\overline{\mathcal{X}}^{\lambda}_{\mu}$ and $\mathcal{X}^{\lambda}_{\mu}$. We show these spaces are products of the corresponding $\mathcal{W}$-versions and an infinite dimensional affine space (Proposition 3.8). We then show that the spaces $\mathcal{X}^{\lambda}_{\mu}$ are formally smooth (Theorem 3.14), from which we conclude that the spaces $\mathcal{W}^{\lambda}_{\mu}$ are formally smooth. Because the $\overline{\mathcal{W}}^{\lambda}_{\mu}$ (and hence $\mathcal{W}^{\lambda}_{\mu}$) are known to be finitely presented (see Proposition 2.10 for a direct group-theoretic proof), we conclude that $\mathcal{W}^{\lambda}_{\mu}$ is in fact smooth. A subtle point in our approach is that we make use of the formal smoothness of an ind-scheme $\mathcal{X}^{\lambda}$ that is *not* smooth, so the use of infinite-dimensional spaces and formal smoothness appears essential in our approach (see Remark 3.10).

We note that our approach to smoothness is analogous to a standard approach to the smoothness of usual affine Grassmannian slices (and in fact general Schubert slices for partial flag varieties, see e.g.[KL80, §1.4])). Our space $\mathcal{X}_\mu^\lambda$ is constructed using an auxiliary space $\mathcal{X}_\mu$ that plays the role of an open chart in the affine Grassmannian. We explain this briefly in §3.3.2.

1.3. **Acknowledgements.** We thank Michael Finkelberg, Hiraku Nakajima, and Oded Yacobi for helpful comments. D.M. was supported by JSPS KAKENHI Grant Number JP19K14495. A.W. is grateful to Kavli IPMU for hosting him during the workshop "Representation theory, gauge theory, and integrable systems", where this project was started. This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade.

2. Preliminaries

2.1. **Schemes and functors.** Let $\mathbb{k}$ be a commutative ring. Let $\mathrm{Alg}_\mathbb{k}$ be the category of commutative $\mathbb{k}$-algebras, and let $\mathrm{Sch}_\mathbb{k}$ be the category of $\mathbb{k}$-schemes. We define the category $\mathrm{Func}_\mathbb{k}$ of $\mathbb{k}$-functors to be the category of functors $\mathrm{Alg}_\mathbb{k} \to \mathrm{Set}$. Recall that there is a fully-faithful embedding $\mathrm{Sch}_\mathbb{k} \hookrightarrow \mathrm{Func}_\mathbb{k}$ coming from the Yoneda Lemma and the fact that morphisms of schemes are local for the Zariski topology. Following the usual terminology, we will call this inclusion the map sending a scheme to the functor it represents. All the functors we consider will be ind-schemes (of possibly ind-infinite type), so it is not strictly necessary to consider them as functors. However, we will be focused on questions of formal smoothness, so the functorial viewpoint will be essential.

There are notions of open and closed subfunctors in $\mathrm{Func}_\mathbb{k}$, see [EH00, VI.1.1]. We note that they behave well with respect to base change, and agree with the usual notions of open and closed subscheme in the case of a functor represented by a scheme.

2.2. **Formal smoothness.** Let $\varphi : \widetilde{A} \to A$ be a morphism in $\mathrm{Alg}_\mathbb{k}$. Recall that we say φ is a **square-zero extension** if φ is surjective and the ideal $I = \ker(\varphi)$ satisfies $I^2 = 0$.

Let $X \in \mathrm{Func}_\mathbb{k}$. We say that X is **formally smooth** if for every square zero extension $\widetilde{A} \to A$, the map $X(\widetilde{A}) \to X(A)$ is surjective. The relevance of formal smoothness is the following theorem of Grothendieck (see e.g. [Sta19, Lemma 02H6]).

Proposition 2.1. *Let X be a locally finitely presented $\mathbb{k}$-scheme. Then X is smooth if and only if it is formally smooth.*

We record the following lemma for use later.

Lemma 2.2. *Let $X, Y \in \mathrm{Func}_\mathbb{k}$. Suppose $X \times Y$ is formally smooth and $X(\mathbb{k}) \neq \varnothing$. Then Y is formally smooth.*

Proof. Let $\varphi : \widetilde{A} \to A$ be a square-zero extension, and let $y \in Y(A)$. By assumption there exists $x \in X(\mathbb{k})$, which we may view as an A-point. Then $(x, y) \in (X \times Y)(A)$. By formal smoothness, this point has a lift $(\widetilde{x}, \widetilde{y}) \in (X \times Y)(\widetilde{A})$. But then $\widetilde{y} \in Y(\widetilde{A})$ is a lift of y, as desired. $\square$

2.3. **Group theoretic data.** Let G be a connected split reductive group (see e.g. [Jan03, Section II.1] for an overview). In particular, G is defined over $\mathbb{Z}$. Let T be a maximal torus, and let U^+ and U^- be opposite unipotent subgroups (i.e. U^-T and U^+T are opposite Borel subgroups). Let P be the coweight lattice of T, and let Q be the coroot lattice. We write P_{++} for the dominant cone and write Q_+ for the positive coroot cone. Recall the dominance order where for $\lambda, \mu \in P$, we write $\mu \leqslant \lambda$ if $\lambda - \mu \in Q_+$.

Let $P^\vee$ be the weight lattice of T, and $P^\vee_{++}$ its cone of dominant weights. For each $\Lambda \in P^\vee_{++}$, let $V(-\Lambda)$ be the Weyl module of G with lowest weight $-\Lambda$. This is a free $\Bbbk$-module, and the lowest weight space $V(-\Lambda)_{-\Lambda}$ is a rank-one free $\Bbbk$-module. Let $v_{-\Lambda}$ be a generator of this free module, and let $v^*_{-\Lambda}$ be the linear functional on $V(-\Lambda)$ that is equal to one on $v_{-\Lambda}$ and is zero on all other weight spaces. Let Δ_Λ be the regular function on G defined by $\Delta_\Lambda(g) = \langle v^*_{-\Lambda}, g v_{-\Lambda}\rangle$.

Recall that the big cell of G is the open subscheme $U^+TU^- \subset G$. It is isomorphic to the product $U^+ \times T \times U^-$, via the multiplication map.

Lemma 2.3. *Let $\Lambda, \Lambda' \in P^\vee_{++}$. For $t \in T$ and $g \in U^+TU^-$, we have:*

(a) $\Delta_\Lambda(tg) = \Delta_\Lambda(gt) = \Delta_\Lambda(t)\Delta_\Lambda(g)$,
(b) $\Delta_{\Lambda+\Lambda'}(g) = \Delta_\Lambda(g)\Delta_{\Lambda'}(g)$.

The coordinate ring of T is the group algebra of $P^\vee$, and thus for any $A \in \mathrm{Alg}_\Bbbk$ there is a natural bijection $T(A) \longleftrightarrow \mathrm{Hom}_{groups}(P^\vee, A^\times)$ where $A^\times \subset A$ are the units. Since the Grothendieck group of the semigroup $P^\vee_{++}$ is canonically isomorphic to $P^\vee$, we have:

Lemma 2.4. *For any $A \in \mathrm{Alg}_\Bbbk$ there is a natural bijection*

$$T(A) \longleftrightarrow \mathrm{Hom}_{semigroups}(P^\vee_{++}, A^\times),$$

such that $t \in T(A)$ corresponds to the homomorphism $\Lambda \mapsto \Delta_\Lambda(t)$.

2.4. **Arcs and loops.** Let z be a formal variable. Let $G[z]$ be the group object in $\mathrm{Func}_\Bbbk$ defined by $G[z](A) = G(A[z])$ for each $A \in \mathrm{Alg}_\Bbbk$. Similarly we define $G((z^{-1}))$ and $G[[z^{-1}]]$. We have closed embeddings $G[z] \hookrightarrow G((z^{-1}))$ and $G[[z^{-1}]] \hookrightarrow G((z^{-1}))$. Observe that $G[[z^{-1}]]$ is a scheme of infinite type, $G[z]$ is an ind-scheme of ind-finite type, and $G((z^{-1}))$ is an ind-scheme of ind-infinite type.

Let $\mathrm{Gr}_G^{\mathrm{thick}}$ be the thick affine Grassmannian, which is a scheme of infinite type. It can be defined as the moduli of G-torsors on $\mathbb{P}^1$ with a trivialization on the formal neighbourhood of $\infty \in \mathbb{P}^1$, see [Zhu17, Remark 2.3.6]. There is a unit point $1 \in \mathrm{Gr}_G^{\mathrm{thick}}$ corresponding to the trivial bundle, and a $G((z^{-1}))$-action on $\mathrm{Gr}_G^{\mathrm{thick}}$ changing trivializations. The stabilizer of the point $1 \in \mathrm{Gr}_G^{\mathrm{thick}}$ is $G[z]$, and the map

$$G((z^{-1})) \to \mathrm{Gr}_G^{\mathrm{thick}} \tag{2.5}$$

obtained by acting on $1 \in \mathrm{Gr}_G^{\mathrm{thick}}$ is a Zariski locally trivial principal bundle for the group $G[z]$. For this reason, one often writes $\mathrm{Gr}_G^{\mathrm{thick}} = G((z^{-1}))/G[z]$. However, one needs to be careful with this notation because the naïve functor indicated by $G((z^{-1}))/G[z]$ is not the functor that represents $\mathrm{Gr}_G^{\mathrm{thick}}$: one needs to appropriately sheafify it.

2.5. **Generalized affine Grassmannian slices.** For $\lambda \in P$, we can consider the point $z^\lambda \in G((z^{-1}))$ and the corresponding point $z^\lambda \in \mathrm{Gr}_G^{\mathrm{thick}}$. We will restrict our attention to $\lambda \in P^{++}$. Then we write Gr^λ for the orbit inside of $\mathrm{Gr}_G^{\mathrm{thick}}$ of the point z^λ under the group $G[z]$. Let $\overline{\mathrm{Gr}^\lambda}$ be the closure of this orbit (with its reduced scheme structure). It is well known that both Gr^λ and $\overline{\mathrm{Gr}^\lambda}$ are finite type schemes, and that Gr^λ is a smooth scheme.

We define the closed subfunctor $\overline{\mathcal{X}^\lambda}$ of $G((z^{-1}))$ to be the preimage of $\overline{\mathrm{Gr}^\lambda}$ under the surjective map $G((z^{-1})) \to \mathrm{Gr}_G^{\mathrm{thick}}$. We define $\mathcal{X}^\lambda$ to be the open subfunctor of $\overline{\mathcal{X}^\lambda}$ that is the preimage of Gr^λ. Suggestively, we will write $\mathcal{X}^\lambda = G[z]z^\lambda G[z]$ and $\overline{\mathcal{X}^\lambda} = \overline{G[z]z^\lambda G[z]}$. Observe that both $\overline{\mathcal{X}^\lambda}$ and $\mathcal{X}^\lambda$ have $G[z] \times G[z]$ actions given by left and right multiplications.

Let $G_1[[z^{-1}]]$ be the closed subscheme of $G[[z^{-1}]]$ consisting of elements that evaluate to $1 \in G$ modulo z^{-1}. Then we have the Gauss decomposition (in $\mathbb{k}$-functors):

$$G_1[[z^{-1}]] = U_1^+[[z^{-1}]] \cdot T_1[[z^{-1}]] \cdot U_1^-[[z^{-1}]] \tag{2.6}$$

where the factors on the right hand side are defined exactly as for G. For each $\mu \in P$, we will consider

$$\mathcal{W}_\mu = U_1^+[[z^{-1}]] \cdot z^\mu T_1[[z^{-1}]] \cdot U_1^-[[z^{-1}]] \tag{2.7}$$

which is a closed sub-scheme of $G((z^{-1}))$. Note that this is also a product of $\mathbb{k}$–functors. We will restrict attention to μ with $\mu \leqslant \lambda$.

Following Braverman, Finkelberg, and Nakajima [BFN19, Section 2(xi)], we define:

$$\overline{\mathcal{W}}_\mu^\lambda = \overline{\mathcal{X}^\lambda} \cap \mathcal{W}_\mu \tag{2.8}$$

and

$$\mathcal{W}_\mu^\lambda = \mathcal{X}^\lambda \cap \mathcal{W}_\mu. \tag{2.9}$$

As they explain, $\overline{\mathcal{W}}_\mu^\lambda$ is a finite-type affine scheme with a Poisson structure. When μ is also dominant they show that $\overline{\mathcal{W}}_\mu^\lambda$ is isomorphic to a transversal slice in the thick affine Grassmannian, under the map $G((z^{-1})) \to \mathrm{Gr}_G^{\mathrm{thick}}$. (In fact it is isomorphic to a slice in the thin affine Grassmannian, see *loc. cit.*). For this reason, $\overline{\mathcal{W}}_\mu^\lambda$ is called a **generalized affine Grassmannian slice**. Note that $\mathcal{W}_\mu^\lambda \subset \overline{\mathcal{W}}_\mu^\lambda$ is open, as $\mathcal{X}^\lambda \subset \overline{\mathcal{X}^\lambda}$ is open.

The following is [BFN19, Lemma 2.5]; we record the following elementary proof for the benefit of the reader.

Proposition 2.10. *For any $\lambda \geqslant \mu$ with λ dominant, $\overline{\mathcal{W}}_\mu^\lambda$ is a finitely presented affine $\mathbb{k}$-scheme.*

Proof. Observe that $\overline{\mathcal{W}}_\mu^\lambda$ is defined over $\mathbb{Z}$, so it suffices to show that $\overline{\mathcal{W}}_\mu^\lambda$ is a finite type affine scheme over $\mathbb{Z}$ [Sta19, Lemma 01TX].

Embed G as a closed subgroup of GL_n such that T and $U^\pm$ are compatible with the standard torus and unipotents in GL_n. We see that $\overline{\mathcal{W}}_\mu^\lambda$ is a closed subfunctor of a generalized affine Grassmannian slice for GL_n. Therefore, we are reduced to considering the case of $G = GL_n$.

Let $\overline{\mathbb{X}^\lambda}$ be the closed subfunctor of $G((z^{-1}))$ consisting of elements g such that for any $\Lambda \in P_{++}^\vee$, the matrix entries of g acting on $V(-\Lambda)$ have poles in z of order no worse than $-\langle \lambda, \Lambda \rangle$. The functor $\overline{\mathbb{X}^\lambda}$ is the preimage of the "moduli version" of $\overline{\mathrm{Gr}^\lambda}$ under the map $G((z^{-1})) \to \mathrm{Gr}_G^{\mathrm{thick}}$ (see [KMW18] for a detailed discussion about the "moduli version" of $\overline{\mathrm{Gr}^\lambda}$). In particular, $\overline{\mathcal{X}^\lambda}$ is a closed subfunctor of $\overline{\mathbb{X}^\lambda}$. Therefore $\overline{\mathcal{W}}_\mu^\lambda = \mathcal{W}_\mu \cap \overline{\mathcal{X}^\lambda} \subset \mathcal{W}_\mu \cap \overline{\mathbb{X}^\lambda}$ is a closed subfunctor, so we are further reduced to proving that $\mathcal{W}_\mu \cap \overline{\mathbb{X}^\lambda}$ is of finite type.

Let $g = xz^\mu ty \in \mathcal{W}_\mu$, i.e. $x \in U_1^+[[z^{-1}]]$, $t \in T_1[[z^{-1}]]$, and $y \in U_1^-[[z^{-1}]]$. Suppose further that $xz^\mu ty \in \overline{\mathbb{X}^\lambda}$. As we are considering GL_n, t is a diagonal matrix with diagonal entries $t_1, \dots, t_n$, each of which is a series in z^{-1} with constant term 1,

For each i, let $a_i = t_i \cdots t_n$. Computing $\Delta_\Lambda(g)$ for Λ the fundamental weights and the determinant character, we see that each a_i is in fact a polynomial in z^{-1} with an explicit bound on the degree coming from λ. Furthermore, the coefficients of each t_i are polynomials in the finitely many coefficients of $a_1, \dots, a_n$.

For integers i, j with $1 \leqslant i < j \leqslant n$, consider x_{ij}, the (i,j)-th entry of the unipotent upper triangular matrix x. Each x_{ij} is a power series in z^{-1} with constant term 0. If we act by g on the lowest weight vector of the $(n-j+1)$-th fundamental representation, we see that $a_j x_{ij}$ appears as a matrix coefficient. Therefore, $b_{ij} = a_j x_{ij}$ is a polynomial in z^{-1} with an explicit bound on the degree, and the coefficients of x_{ij} are polynomials in the finitely many coefficients of a_j and b_{ij}. A similar consideration applies to the matrix coefficients of y, and therefore $\mathcal{W}_\mu \cap \overline{\mathbb{X}^\lambda}$ is a closed subscheme of a finite-dimensional affine space, so is of finite type. $\square$

3. Main Result

3.1. **The space $\mathcal{X}_\mu^\lambda$.** Let λ and μ be as above. We define the following variant of the space $\mathcal{W}_\mu$:

$$\mathcal{X}_\mu = U^+((z^{-1})) \cdot z^\mu T_1[[z^{-1}]] \cdot U^-((z^{-1})). \tag{3.1}$$

Lemma 3.2. *For any $A \in \mathrm{Alg}_\Bbbk$, we have:*

$$\mathcal{X}_\mu(A) = \left\{ g \in G\big(A((z^{-1}))\big) \;\middle|\; \Delta_\Lambda(g) \in z^{-\langle \mu, \Lambda\rangle} \cdot \left(1 + z^{-1}A[[z^{-1}]]\right), \quad \forall \Lambda \in P_{++}^\vee \right\}. \tag{3.3}$$

Proof. It is easy to see that any $g \in \mathcal{X}_\mu(A)$ satisfies the conditions on $\Delta_\Lambda(g)$ above. Conversely, if $g \in G(A((z^{-1})))$ satisfies these conditions, then in particular all $\Delta_\Lambda(g)$ are units in $A((z^{-1}))$. This implies that g is an $A((z^{-1}))$–point of the big cell $U^+TU^- \subset G$, so we can write $g = u^+tu^-$ with $u^\pm \in U^\pm(A((z^{-1})))$ and $t \in T(A((z^{-1})))$. In order to conclude that $g \in \mathcal{X}_\mu$, it remains to show that its factor t has the correct form. But since $\Delta_\Lambda(g) = \Delta_\Lambda(t)$, this follows from Lemma 2.4. $\square$

Consider the quotient $U^+((z^{-1}))/U_1^+[[z^{-1}]]$. Unlike the case of G, this quotient can be naïvely interpreted because the natural map

$$U^+[z] \xrightarrow{\sim} U^+((z^{-1}))/U_1^+[[z^{-1}]] \tag{3.4}$$

is an isomorphism. In particular, $U^+[z]$ gives us a section of the quotient map $U^+((z^{-1})) \to U^+((z^{-1}))/U_1^+[[z^{-1}]]$. Similarly, there is an isomorphism $U^-[z] \xrightarrow{\sim} U_1^-[[z^{-1}]]\backslash U^-((z^{-1}))$.

We therefore obtain a map

$$\pi_\mu : \mathcal{X}_\mu \to U^+[z] \times U^-[z] \tag{3.5}$$

given by sending a point $xz^\mu ty \in \mathcal{X}_\mu$ to $(x,y) \in U^+((z^{-1}))/U_1^+[[z^{-1}]] \times U_1^-[[z^{-1}]]\backslash U^-((z^{-1})) \cong U^+[z] \times U^-[z]$.

Definition 3.6. Let $\lambda \in P^{++}$ and $\mu \in P$. Define:

$$\mathcal{X}_\mu^\lambda = \mathcal{X}^\lambda \cap \mathcal{X}_\mu. \tag{3.7}$$

Proposition 3.8. *The ind-scheme $\mathcal{X}_\mu^\lambda$ is isomorphic to the product $U^+[z] \times \mathcal{W}_\mu^\lambda \times U^-[z]$.*

Proof. The map (3.5) is our required map $\mathcal{X}_\mu^\lambda \to U^+[z] \times U^-[z]$. We need to construct a map $\mathcal{X}_\mu^\lambda \to \mathcal{W}_\mu^\lambda$. Observe that $\mathcal{X}_\mu^\lambda$ has a $U^+[z] \times U^-[z]$-action where the $U^+[z]$-factor acts by left multiplication, and the $U^-[z]$-factor acts by right multiplication. Furthermore, the map π_μ is equivariant for this action. Given $g \in \mathcal{X}_\mu^\lambda$, consider $(x, y) = \pi_\mu(g)$. Then $x^{-1}\pi_\mu(g)y^{-1} \in \mathcal{W}_\mu^\lambda$. This defines our map $\mathcal{X}_\mu^\lambda \to \mathcal{W}_\mu^\lambda$, and it is clear that this and π_μ realize $\mathcal{X}_\mu^\lambda$ as the above product. □

3.2. Formal smoothness of $\mathcal{X}_\mu^\lambda$.

Proposition 3.9. *The ind-scheme $\mathcal{X}^\lambda$ is formally smooth.*

Proof. By definition, we have a Zariski-locally-trivial principal bundle $\mathcal{X}^\lambda \to \mathrm{Gr}^\lambda$ for the group $G[z]$ over the smooth base Gr^λ. So we are reduced to showing that the ind-scheme $G[z]$ is formally smooth.

We need to show that for any square-zero extension $\widetilde{A} \twoheadrightarrow A$, the map $G[z](\widetilde{A}) \to G[z](A)$ is surjective. This follows because $\widetilde{A}[z] \twoheadrightarrow A[z]$ is also a square-zero extension and G is smooth (and hence formally smooth). □

Remark 3.10. Although the ind-scheme is $\mathcal{X}^\lambda$ is formally smooth, it is *not* smooth, i.e. it cannot be written as a an increasing union of smooth varieties (not even locally in the analytic topology). Fishel, Grojnowski, and Teleman show that thin affine Grassmannians are not smooth in this sense (even though they are formally smooth) [FGT08, Theorem 5.4]. Locally in the Zariski topology, $\mathcal{X}^\lambda$ is isomorphic to a smooth variety times the big cell in the thin affine Grassmannian: it is locally the product of a smooth scheme times $G[z]$ as explained in the proof above, while $G[z]$ is the product of G times the big cell. Adapting the argument in *loc. cit.*, we can conclude that $\mathcal{X}^\lambda$ is not smooth.

Nonetheless, we will use the formal smoothness of $\mathcal{X}^\lambda$ to deduce the smoothness of the finitely presented scheme $\mathcal{W}_\mu^\lambda$ below. Because $\mathcal{X}^\lambda$ is not smooth, we cannot naïvely truncate the argument to a finite-dimensional situation. This is a subtle point of our approach: the use of infinite dimensional ind-schemes and formal smoothness seems to be essential.

The following lemma is a slight variation of the classical Weierstraß Preparation Theorem (see e.g. [Bou06, Ch. VII, §3.8]).

Lemma 3.11. *Let $\widetilde{A} \to A$ be a square-zero extension with kernel I. Let $I[z]$ denote the set of polynomials in $\widetilde{A}[z]$ having all coefficients lie in I. Let $D \in I[z] + 1 + z^{-1}\widetilde{A}[[z^{-1}]]$. Then there exists a unique polynomial $\gamma \in 1 + I[z]$ such that*

$$\gamma D \in 1 + z^{-1}\widetilde{A}[[z^{-1}]]. \tag{3.12}$$

Proof. For a series $s \in \widetilde{A}((z^{-1}))$, denote by $\mathrm{reg}(s) \in \widetilde{A}[z]$ its polynomial part. Write $D = x + 1 + a$ with $x \in I[z]$ and $a \in z^{-1}\widetilde{A}[[z^{-1}]]$, and consider a general element $\gamma = 1 + y \in 1 + I[z]$. Then $xy = 0$ since $I[z]^2 = 0$, and we have

$$\gamma D = x + 1 + y + ya + a. \tag{3.13}$$

Therefore $\mathrm{reg}(\gamma D) = x + 1 + y + \mathrm{reg}(ya)$. Since $a \in z^{-1}\widetilde{A}[[z^{-1}]]$, the coefficients of $y + \mathrm{reg}(ya)$ have an upper-triangularity property with respect to the coefficients of y: for $n \geqslant 0$ the coefficient of z^n in $y + \mathrm{reg}(ya)$ equals $y_n + \sum_{k>0} a_{-k}y_{n+k}$, where y_ℓ, a_ℓ denote the coefficients of z^ℓ in y and a, respectively. Thus, by starting with the leading degree coefficient of y and working downwards inductively, we can solve uniquely for y such that $\mathrm{reg}(\gamma D) = 1$ (observe in particular that the degree of y must be equal to the degree of x). This proves the claim. □

Theorem 3.14. *The ind-scheme $\mathcal{X}_\mu^\lambda$ is formally smooth.*

Proof. Let $\widetilde{A} \twoheadrightarrow A$ be a square-zero extension with kernel I, and let $g \in \mathcal{X}_\mu^\lambda(A) = \mathcal{X}^\lambda(A) \cap \mathcal{X}_\mu(A)$. Because $\mathcal{X}^\lambda$ is formally smooth, we can find $g' \in \mathcal{X}^\lambda(\widetilde{A})$ lifting g. Let $\Lambda \in P_{++}^\vee$. It may not be the case that $\Delta_\Lambda(g')$ lies in $z^{-\langle\mu,\Lambda\rangle} \cdot \left(1 + z^{-1}\widetilde{A}[[z^{-1}]]\right)$, but we know at least that $\Delta_\Lambda(g') \in z^{-\langle\mu,\Lambda\rangle} \cdot \left(I[z] + 1 + z^{-1}\widetilde{A}[[z^{-1}]]\right)$. Using Lemma 3.11, we can find a unique $\gamma_\Lambda \in 1 + I[z]$ so that

$$\gamma_\Lambda \Delta_\Lambda(g') \in z^{-\langle\mu,\Lambda\rangle} \cdot \left(1 + z^{-1}\widetilde{A}[[z^{-1}]]\right). \tag{3.15}$$

Since $\Delta_\Lambda(g)\Delta_{\Lambda'}(g) = \Delta_{\Lambda+\Lambda'}(g)$ by Lemma 2.3(b), we must have $\gamma_{\Lambda+\Lambda'} = \gamma_\Lambda\gamma_{\Lambda'}$ by uniqueness. Also note that $1 + I[z] \subset \widetilde{A}[z]^\times$ since $I[z]^2 = 0$. Therefore the map $\Lambda \mapsto \gamma_\Lambda$ defines an element of $\mathrm{Hom}_{\mathrm{semigroups}}(P_{++}^\vee, \widetilde{A}[z]^\times)$, and thus a point $t \in T(\widetilde{A}[z])$ by Lemma 2.4. Note that t is a lift of the identity element in $T(A[z])$.

Define $\widetilde{g} = t \cdot g'$. Because $t \in T(\widetilde{A}[z])$ and $\mathcal{X}^\lambda$ is invariant under $G[z]$-multiplication, $\widetilde{g} \in \mathcal{X}^\lambda(\widetilde{A})$. By Lemma 2.3(a) and Lemma 2.4, for each $\Lambda \in P_{++}^\vee$, we have:

$$\Delta_\Lambda(\widetilde{g}) = \Delta_\Lambda(t)\Delta_\Lambda(g') = \gamma_\Lambda\Delta_\Lambda(g') \in z^{-\langle\mu,\Lambda\rangle} \cdot \left(1 + z^{-1}\widetilde{A}[[z^{-1}]]\right). \tag{3.16}$$

Therefore $\widetilde{g} \in \mathcal{X}_\mu^\lambda(\widetilde{A})$ by Lemma 3.2. Since $\widetilde{g}$ is a lift of g, this shows that $\mathcal{X}_\mu^\lambda$ is formally smooth. □

Using Proposition 3.8, Lemma 2.2, and observing that $U^+[z] \times U^-[z]$ has a $\Bbbk$-point, we conclude that $\mathcal{W}_\mu^\lambda$ is formally smooth. As $\mathcal{W}_\mu^\lambda \subset \overline{\mathcal{W}}_\mu^\lambda$ is an open subscheme and $\overline{\mathcal{W}}_\mu^\lambda$ is finitely-presented (by Lemma 2.10), $\mathcal{W}_\mu^\lambda$ is locally of finite presentation [Sta19, Lemma 01TQ]. Applying Proposition 2.1, we conclude the following:

Corollary 3.17. *The scheme $\mathcal{W}_\mu^\lambda$ is smooth.*

3.3. **Concluding remarks.**

3.3.1. *Generalization.* In this section we will describe a more general framework in which our arguments above apply. Let Z be a scheme equipped with a $G[z]$–action, along with a $G[z]$–equivariant map $Z \to \mathrm{Gr}_G^{\mathrm{thick}}$. Consider the following diagram, where the top row is defined via fiber products:

$$\begin{array}{ccccccc} \mathcal{W}_\mu^Z & \longrightarrow & \mathcal{X}_\mu^Z & \longrightarrow & \mathcal{X}^Z & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{W}_\mu & \longrightarrow & \mathcal{X}_\mu & \longrightarrow & G((z^{-1})) & \longrightarrow & \mathrm{Gr}_G^{\mathrm{thick}} \end{array} \tag{3.18}$$

In particular, taking $Z = \mathrm{Gr}^\lambda \hookrightarrow \mathrm{Gr}_G^{\mathrm{thick}}$ the spaces in the top row are simply $\mathcal{W}_\mu^\lambda$, $\mathcal{X}_\mu^\lambda$, and $\mathcal{X}^\lambda$. Assuming that Z is formally smooth, the same arguments given above show that $\mathcal{W}_\mu^Z$ is formally smooth. We can apply this discussion to two important variations on the varieties $\mathcal{W}_\mu^\lambda$. They both depend on a choice of tuple of dominant coweights $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$ such that $\lambda = \lambda_1 + \ldots + \lambda_N$.

(a) For the first, we pick a point $\underline{z} = (z_1, \ldots, z_N) \in \Bbbk^N$. Define Z to be the $G[z]$–orbit through the point $\prod_{s=1}^N (z - z_s)^{\lambda_s} \in \mathrm{Gr}_G^{\mathrm{thick}}$. Then $\mathcal{W}_\mu^Z$ is an open subscheme of the slice $\overline{\mathcal{W}}_\mu^{\underline{\lambda},\underline{z}}$ from [BFN19, §2(xi)], which is a fiber of a Beilinson-Drinfeld deformation of $\overline{\mathcal{W}}_\mu^\lambda$. Note that $\overline{\mathcal{W}}_\mu^{\underline{\lambda},\underline{z}}$ itself is obtained from (3.18) by taking the closure $\overline{Z} \subset \mathrm{Gr}_G^{\mathrm{thick}}$ in place of Z.

(b) For the second, we take the (open) convolution variety $Z = \mathrm{Gr}^{\lambda_1} \widetilde{\times} \dots \widetilde{\times} \mathrm{Gr}^{\lambda_N}$ (for example, as defined as in the paragraph preceding [Zhu17, Equation (2.1.17)]). Then $\mathcal{W}_\mu^Z$ is an open subscheme of the space $\widetilde{\mathcal{W}}_\mu^\lambda$ from [BFN18, §5(i)], which is a partial resolution of $\overline{\mathcal{W}}_\mu^\lambda$. Note that $\widetilde{\mathcal{W}}_\mu^\lambda$ itself is obtained from (3.18) by taking the (closed) convolution variety $\overline{Z} = \overline{\mathrm{Gr}^{\lambda_1}} \widetilde{\times} \dots \widetilde{\times} \overline{\mathrm{Gr}^{\lambda_N}}$ in place of Z. Also note that $Z \subset \overline{Z}$ is a smooth open dense subvariety.

By similar arguments to Proposition 2.10, one can see that $\overline{\mathcal{W}}_\mu^{\lambda, \underline{z}}$ and $\widetilde{\mathcal{W}}_\mu^\lambda$ are finitely presented $\Bbbk$–schemes. (Alternatively this follows from [BFN19, §2(xi)] and [BFN18, §5], respectively.) Thus in either case, smoothness and formal smoothness coincide for their open subschemes $\mathcal{W}_\mu^Z$. Since Z is smooth in both cases, we conclude that $\mathcal{W}_\mu^Z$ is also smooth.

3.3.2. *Comparison to smoothness for open affine Grassmannian slices.* Our proof is inspired by the usual approach to showing smoothness of open affine Grassmannian slices. We will quickly review this. Let λ and μ be dominant weights with $\mu \leqslant \lambda$. In this case, $\overline{\mathcal{W}}_\mu^\lambda$ is a closed subscheme of the affine Grassmannian with open subscheme $\mathcal{W}_\mu^\lambda$. Let $\mathcal{A}_0$ denote the "big cell" of the affine Grassmannian. Left multiplying $\mathcal{A}_0$ by z^μ, we get an open subset $\mathcal{A}_\mu$ of the affine Grassmannian that contains the point z^μ.

Consider the intersection $\mathcal{A}_\mu \cap \mathrm{Gr}^\lambda$, which is a smooth variety because it is an open subset of the smooth variety Gr^λ. Furthermore, there is a map $\mathcal{A}_\mu \cap \mathrm{Gr}^\lambda \to V$, where V is the stabilizer inside of $U_1^+[z^{-1}]$ of the point z^μ. Observe that V is isomorphic to a finite dimensional affine space. This map realizes $\mathcal{A}_\mu \cap \mathrm{Gr}^\lambda$ as $V \times \mathcal{W}_\mu^\lambda$. Therefore, we conclude that $\mathcal{W}_\mu^\lambda$ is smooth. We mention that this is a general calculation that works for arbitrary Schubert slices (see e.g.[KL80, §1.4])).

For us, the space $\mathcal{X}_\mu$ plays the role of $\mathcal{A}_\mu$, and $\mathcal{X}^\lambda$ plays the role of Gr^λ. However, because $\mathcal{X}^\lambda$ is infinite-dimensional, smoothness is more subtle: hence our approach through formal smoothness.

3.3.3. *Open Zastava.* In the case $\lambda = 0$, we have $\overline{\mathcal{W}}_\mu^0 = \mathcal{W}_\mu^0$, and the space $\mathcal{W}_\mu^0$ has been considered previously: it is precisely the "open Zastava" space consisting of degree $-\mu$ based maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}$ of G. In this case, smoothness was previously known by work of Finkelberg and Mirković [FM99]. Let $\varphi : \mathbb{P}^1 \to \mathcal{B}$ be a degree $-\mu$ based map. They argue that φ is a smooth point of the open Zastava if and only if we have $H^1(\mathbb{P}^1, \varphi^* \mathcal{T}_\mathcal{B}) = 0$, where $\mathcal{T}_\mathcal{B}$ is the tangent sheaf of $\mathcal{B}$. Because $\mathcal{T}_\mathcal{B}$ is globally generated, and all globally generated vector bundles on $\mathbb{P}^1$ have vanishing higher cohomology, they deduce the necessary vanishing.

Our work gives another proof of the smoothness of the open Zastava space. It would be very interesting to understand precisely how the two calculations correspond.

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