

PROJECTIVE GENERATION FOR EQUIVARIANT \mathcal{D} -MODULES

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Abstract. We investigate compact projective generators in the category of equivariant \mathcal{D} -modules on a smooth affine variety. For a reductive group G acting on a smooth affine variety X , there is a natural countable set of compact projective generators indexed by finite dimensional representations of G . We show that only finitely many of these objects are required to generate; thus the category has a single compact projective generator. The proof goes via an analogous statement about compact generators in the equivariant derived category, which holds in much greater generality and may be of independent interest.

1. Introduction

Let X be a smooth complex affine variety equipped with an action of a complex reductive group G . Let \mathcal{D}_X denote the ring of (algebraic) differential operators on X . We consider the category $\mathrm{QCoh}(\mathcal{D}_X, G)$ of (strongly) G -equivariant \mathcal{D}_X -modules; see Section 2 for the definition.

Consider the \mathcal{D}_X -module

$$\mathcal{P}_X = \mathcal{D}_X / \mathcal{D}_X \mathfrak{g},$$

where $\mathfrak{g} \rightarrow \mathcal{D}_X$ is the infinitesimal action map. This object is naturally G -equivariant. It represents the functor of invariants on $\mathrm{QCoh}(\mathcal{D}_X, G)$; known as the functor

of quantum Hamiltonian reduction (see for instance [11]). In particular, it is a compact projective object. One may think of \mathcal{P}_X as global differential operators on the quotient stack $[X/G]$.

Recall that a smooth variety Y is called D -affine if the object \mathcal{D}_Y is a projective generator of the category $\text{QCoh}(\mathcal{D}_Y)$. Analogously, we say that the G -space X (or rather the stack $[X/G]$) is D -affine if \mathcal{P}_X generates the category $\text{QCoh}(\mathcal{D}_X, G)$. Equivalently, if every equivariant module has a non-zero invariant element. In this case, we have an equivalence of categories

$$\text{Hom}(\mathcal{P}_X, -) : \text{QCoh}(\mathcal{D}_X, G) \xrightarrow{\sim} R_X\text{-Mod},$$

where $R_X = \text{End}(\mathcal{P}_X) \cong (\mathcal{P}_X)^G$.

In general, it is a subtle problem to determine whether a given affine G -variety is D -affine. For example:

- The adjoint action of GL_n on \mathfrak{gl}_n is D -affine; however, the adjoint action of SL_n on \mathfrak{sl}_n is not D -affine.
- The scaling action of \mathbb{C}^\times on \mathbb{A}^1 is not D -affine.

The first claim follows from [13, Prop. 4.8] and [14, Thm. A], together with the fact that there are no cuspidal \mathcal{D} -modules for $(\text{GL}_n, \mathfrak{gl}_n)$ but there are for $(\text{SL}_n, \mathfrak{sl}_n)$, as shown by Lusztig [19]. Thus, instead of asking whether a particular projective module is a compact generator for $\text{QCoh}(\mathcal{D}_X, G)$, a more fruitful approach to understanding this category would be to ask whether it admits *some* compact projective generator.

Question 1. For any given reductive group G and smooth affine G -variety X , does the category $\text{QCoh}(\mathcal{D}_X, G)$ admit a compact projective generator?

1.1. Induced modules

Let $\text{Rep}(G)$ denote the category of finite dimensional representations of G . A countable projective generating set for $\text{QCoh}(\mathcal{D}_X, G)$ can be constructed by inducing representations of G . More specifically, given $V \in \text{Rep}(G)$, we define

$$\mathcal{P}_X(V) := \mathcal{D}_X \otimes_{\mathcal{U}\mathfrak{g}} V.$$

These objects represent the functor that assigns the V -multiplicity space of an equivariant \mathcal{D}_X -module; as such, they are compact and projective. It is clear from this characterization that the collection of objects $\mathcal{P}_X(V)$, as V ranges over the set of isomorphism classes of irreducible representations, generate $\text{QCoh}(\mathcal{D}_X, G)$.

Question 1 is equivalent to:

Question 2. For any given reductive group G and smooth affine G -variety X , does there exist a finite dimensional representation $V \in \text{Rep}(G)$ such that $\mathcal{P}_X(V)$ generates $\text{QCoh}(\mathcal{D}_X, G)$?

We will show that the answer to Question 2 (and thus also to Question 1) is yes:

Theorem 1. *Let X be a smooth affine G -variety. Then there exists $V \in \text{Rep}(G)$ such that $\mathcal{P}_X(V)$ generates $\text{QCoh}(\mathcal{D}_X, G)$.*

Theorem 1 implies that

$$\text{QCoh}(\mathcal{D}_X, G) \simeq R_X(V)\text{-Mod},$$

where $R_X(V) = \text{End}_{\text{QCoh}(\mathcal{D}_X, G)}(\mathcal{P}_X(V))^{\text{op}}$ is a noetherian \mathbb{C} -algebra. Concretely, this means that for each smooth affine G -variety X , there is a finite set $\{V_1, \dots, V_k\}$ of irreducible representations of G such that each equivariant \mathcal{D} -module \mathcal{M} contains some V_i appearing with non-zero multiplicity. Here we consider \mathcal{M} as a G -module using the equivariant structure. As a consequence, $\text{QCoh}(\mathcal{D}_X, G)$ admits a finite block decomposition, which is a priori not obvious.

Remark 1. We note that the answer to the analogous question for the category $\text{QCoh}(\mathcal{O}_X, G)$ of equivariant quasi-coherent sheaves on X is a clear no. Equivariant structures on skyscraper sheaves at a point $x \in X$ correspond to representations of $H = \text{Stab}_G(x)$. Thus, as long as there is a point where the stabilizer H contains a torus, the category $\text{QCoh}(\mathcal{O}_X, G)$ cannot admit a compact projective generator.

1.2. The equivariant derived category

Let $\mathbf{D}(X)^G$ be the equivariant derived category, as specified in Section 3.1. Analogous to the objects $\mathcal{P}_X(V)$ in $\text{QCoh}(\mathcal{D}_X, G)$, we have a family of compact objects $\mathbf{P}_X(V)$ of the triangulated category $\mathbf{D}(X)^G$; see Section 3.2 for further details. The proof of Theorem 1 is via the following analogous statement in the equivariant derived category.

Theorem 2. *Let G be a reductive group and X a smooth affine G -variety. Then, there is a finite dimensional representation V such that $\mathbf{P}_X(V)$ is a compact generator of $\mathbf{D}(X)^G$.*

We prove Theorem 2 in Section 3. The key observation is that the variety X can be stratified by locally closed G -stable subvarieties X_i , each of which admits a compact generator for the equivariant derived category. This collection is then used to build a compact generator for X .

In fact, Theorem 2 is a special case of the following more general result.

Theorem 3. *Let \mathcal{X} be an Artin stack of finite type with affine diagonal. Then the derived category of \mathcal{D} -modules $\mathbf{D}(\mathcal{X})$ has a compact generator.*

Remark 2. It is known that $\mathbf{D}(\mathcal{X})$ is compactly generated (see [7]). The assertion above is that a single compact object suffices to generate.

We present a proof of Theorem 3 in Section 4.

Remark 3. The abelian category $\text{QCoh}(\mathcal{D}_X, G)$ and the induced modules $\mathcal{P}_X(V)$ are also extensively studied in a paper by Lőrincz and Walther [17]. The questions studied in that paper are of a somewhat different nature to those in the present paper. In particular, the primary focus in loc. cit. is on G -varieties X with finitely many orbits, and certain subclasses thereof (e.g., spherical G -varieties). In the finite orbit case, the category of equivariant \mathcal{D} -modules is equivalent to modules

for a finite dimensional algebra (see Theorem 3.4 of loc. cit.) and the corresponding quiver is computed in numerous cases. Theorem 1 in the present paper is straightforward to see in the finite orbit case. Thus, the focus of the present paper is somewhat complementary to that of loc. cit.

1.3. Special cases

Let us briefly examine a few special cases of Theorem 1. Some of these cases were proved separately in an earlier version of this paper, before a general proof was found.

Linear actions. One may consider the case when X is a linear representation of a reductive group G . Luna's slice theory [18] effectively allows one to reduce the study of G -equivariant \mathcal{D} -modules on smooth affine varieties to the linear case.

Visible actions. Let $\pi: X \rightarrow X//G = \text{Spec}(\mathbb{C}[X]^G)$ denote the quotient map. Then X is said to be a *visible* G -space if every fibre of π has finitely many G -orbits. An earlier version of this paper gave a different proof of Theorem 1 in the visible case (without using derived category techniques). Essentially, one reduces to \mathcal{D} -modules supported on nullcones associated to each Luna slice; the result then follows from the fact that these nullcones have finitely many orbits.

A classical example of a visible action is the adjoint action of G on \mathfrak{g} . This example has been studied by the second author in [14]. It was shown that there is a finite orthogonal decomposition of $\text{QCoh}(\mathcal{D}_{\mathfrak{g}}, G)$ indexed by the *cuspidal data* associated to G (in the sense of Lusztig's generalized Springer correspondence [19]). This result gives an explicit finite set of compact projective generators in this case.

Torus actions. Consider the case of an algebraic torus T acting linearly on a vector space X . In this case, the statement of Theorem 1 can be phrased as follows: there is a finite subset $S \subseteq \text{Hom}(T, \mathbb{C}^\times)$ of characters of T such that for every object $\mathcal{M} \in \text{QCoh}(\mathcal{D}_X, T)$ there exists $\lambda \in S$ for which the λ -weight space \mathcal{M}_λ in \mathcal{M} is non-zero. An elementary proof of Theorem 1 in this case, based on an application of Kashiwara's Lemma (see, e.g., [16, Thm. 1.6.1]), appeared in an earlier version of this paper.

1.4. Motivation: quantization of cotangent stacks

The category $\text{QCoh}(\mathcal{D}_X, G)$ should be thought of as a quantization of the cotangent stack

$$T^*([X/G]) = [\mu_X^{-1}(0)/G],$$

where $\mu_X: T^*X \rightarrow \mathfrak{g}^*$ is the moment map. Such examples arise naturally in geometric and symplectic representation theory. Here are some motivating examples to keep in mind:

- If $X = \mathfrak{g}$ (or G) with the adjoint (conjugation) action, then the category $\text{QCoh}(\mathcal{D}_X, G)$ is the home of Lusztig's character sheaves, studied in their \mathcal{D} -module incarnation by Ginzburg [12]. Of particular interest are the cuspidal \mathcal{D} -modules, which correspond to certain IC sheaves on distinguished nilpotent orbits [19].

- In the case $G = \mathrm{GL}_n$, one can modify the previous example by taking $X = \mathfrak{gl}_n \times \mathbb{C}^n$. Objects of $\mathrm{QCoh}(\mathcal{D}_X, G)$ are known as *mirabolic* \mathcal{D} -modules. A certain localization of this category is closely related to representations of the rational Cherednik algebra and the geometry of the Hilbert scheme of points in the plane (see, e.g., [11], [8], [2]).
- Given a quiver $Q = (Q_1, Q_0)$ and fixing a dimension vector $\alpha \in \mathbb{N}^{Q_0}$ gives rise to another example of a vector space

$$X = \bigoplus_{f \in Q_1} \mathrm{Hom}(\mathbb{C}^{\alpha_{s(f)}}, \mathbb{C}^{\alpha_{t(f)}})$$

equipped with the action of $G = \prod_{i \in Q_0} \mathrm{GL}_{\alpha_i}$. The category $\mathrm{QCoh}(\mathcal{D}_X, G)$ is then related to quantizations of Nakajima quiver varieties (see, e.g., [6], [4, Sect. 1.4]).

- In the case where $G = T$ acts linearly on a vector space X , localizations of the category $\mathrm{QCoh}(\mathcal{D}_X, G)$ are related to quantizations of hypertoric varieties (see, e.g., [6]).
- Many more interesting examples can be found in [17].

1.5. Further questions

It is conjectured that if X is a smooth complex projective variety that is \mathcal{D} -affine, then $X \cong G/P$ for some reductive group G and parabolic subgroup P . Based on the results of this article, it is natural to ask:

- For which smooth complex quasi-projective varieties X does the category $\mathrm{Coh}(\mathcal{D}_X, G)$ admit a projective generator?

In general, the modules $\mathcal{P}_X(V)$ appear hard to compute. For a given affine G -variety X and representation V , it would be desirable to have a criterion to check whether $\mathcal{P}_X(V)$

- is non-zero;
- is indecomposable;
- is holonomic, or has a holonomic direct summand or submodule.

Some results in this direction have been obtained by Lőrincz and Walther [17]. For example, it is shown in Theorem 3.22 of *loc. cit.* that (for V irreducible) the objects $\mathcal{P}_X(V)$ are indecomposable when X is a D -affine spherical G -variety of Capelli type. Moreover, as explained in *loc. cit.*, these modules can be given an explicit presentation. This means that computer algebra systems can be used to check each of the above properties in concrete examples.

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2. Projective generators in the abelian category

Let G be a complex reductive algebraic group and X a smooth complex affine variety with a regular action of G , that is, a G -variety. Let \mathcal{D}_X denote the ring of

global algebraic differential operators on X . There is a canonical comoment map

$$\nu : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}_X,$$

extending the infinitesimal action $\mathfrak{g} \rightarrow \text{Vect}_X$.

Definition 1. A weakly G -equivariant \mathcal{D} -module on X is a vector space \mathcal{M} that is simultaneously a \mathcal{D}_X -module \mathcal{M} and a rational representation of G , such that the action map for the module structure $\mathcal{D}_X \otimes \mathcal{M} \rightarrow \mathcal{M}$ is a homomorphism of rational G -representations. A weakly G -equivariant \mathcal{D}_X -module is said to be strongly equivariant (or simply equivariant) if \mathfrak{g} -module structure on \mathcal{M} arising via the infinitesimal action map $\nu : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}_X$ coincides with the structure obtained by differentiating the rational G -action. A morphism of equivariant \mathcal{D}_X -modules (either weak or strong) is simply a morphism of \mathcal{D}_X -modules that intertwines the G -action.

The category of all G -equivariant \mathcal{D} -modules on X is denoted $\text{QCoh}(\mathcal{D}_X, G)$ and morphisms in $\text{QCoh}(\mathcal{D}_X, G)$ will be denoted $\text{Hom}_{(\mathcal{D}_X, G)}(L, M)$.

Remark 4. In the case when the group G is connected, the action of G on a (strongly) equivariant \mathcal{D}_X -module \mathcal{M} is uniquely determined by the \mathfrak{g} -module structure, which in turn is determined by the \mathcal{D}_X -module structure. It follows that $\text{QCoh}(\mathcal{D}_X, G)$ is a full subcategory of $\text{QCoh}(\mathcal{D}_X)$ in this case.

Remark 5. Let $\sigma : G \times X \rightarrow X$ be the action map, $p_2 : G \times X \rightarrow X$ projection onto the second factor and $s : X \hookrightarrow G \times X$ the embedding $s(x) = (e, x)$. One may alternatively define a G -equivariant \mathcal{D} -module on X as a pair (\mathcal{M}, θ) , where \mathcal{M} is a quasi-coherent \mathcal{D}_X -module and $\varphi : p_2^* \mathcal{M} \xrightarrow{\sim} \sigma^* \mathcal{M}$ is an isomorphism of $\mathcal{D}_{G \times X}$ -modules, satisfying both the cocycle condition of [16, Def. 11.5.2], and the rigidity condition $s^* \varphi = \text{Id}_{\mathcal{M}}$. For the equivalence of the two definitions see, for instance, [20, Prop. 2.2].

Definition 2. For each finite dimensional G -module V , we define

$$\mathcal{P}_X(V) = \mathcal{D}_X \otimes V / \mathcal{D}_X \{ \nu(x) \otimes v - 1 \otimes \Phi(x)(v) \mid v \in V \},$$

where $\Phi : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(V)$ is the action.

Lemma 4 ([1, Lem. 3.1], [17, Lem. 2.1], Proposition 2.7). *Let X be a smooth affine G -variety.*

- (i) $\mathcal{P}_X(V)$ has a canonical G -equivariant structure and is a projective object in the category $\text{QCoh}(\mathcal{D}_X, G)$.
- (ii) The category $\text{QCoh}(\mathcal{D}_X, G)$ has enough projectives.

Proof. Both statements follow from the fact that

$$\text{Hom}_{(\mathcal{D}_X, G)}(\mathcal{P}_X(V), L) = \text{Hom}_G(V, \Gamma(X, L)),$$

for any L in $\text{QCoh}(\mathcal{D}_X, G)$. \square

An important consequence of Lemma 4 is that $\text{QCoh}(\mathcal{D}_X, G)$ is a Grothendieck category. This is presumably well known, but we were unable find a suitable reference so we include a complete proof.

Lemma 5. *The category $\mathrm{QCoh}(\mathcal{D}_X, G)$ is a Grothendieck category.*

Proof. Explicitly we must show that in the category $\mathrm{QCoh}(\mathcal{D}_X, G)$, the following properties hold:

- (1) arbitrary direct sums exist,
- (2) direct limits of short exact sequences in are left-exact, and
- (3) there is a generator.

In this case, it is clear that $\mathrm{QCoh}(\mathcal{D}_X, G)$ admits arbitrary direct sums since both the category of $\mathcal{D}(X)$ -modules and the category of rational G -modules admit arbitrary direct sums, which in both cases commute with the forgetful functor to vector spaces. Moreover, if $(A_i)_{i \in I}$ is an increasing directed family of subobjects of $A \in \mathrm{QCoh}(\mathcal{D}_X, G)$ and B any G -equivariant submodule of A then $(\sum_{i \in I} A_i) \cap B = \sum_{i \in I} (A_i \cap B)$ since this already holds in the category of $\mathcal{D}(X)$ -modules. Finally, as noted in Lemma 4, the object

$$\mathcal{P}_X := \bigoplus_{V \in \check{G}} \mathcal{P}_X(V)$$

is a generator in $\mathrm{QCoh}(\mathcal{D}_X, G)$. \square

Remark 6. Lemma 5 implies that $\mathrm{QCoh}(\mathcal{D}_X, G)$ admits arbitrary products, or more generally limits. However, the reader is cautioned that these products differ from the corresponding products in $\mathrm{QCoh}(\mathcal{D}_X)$.

Recall that a subset Z of X is G -saturated if it is the preimage of a subset of $X//G$. If Z is a closed G -saturated subset of X then we say that V is a projective generator relative to Z if

$$\mathrm{Hom}_{(\mathcal{D}_X, G)}(\mathcal{P}_X(V), L) = \mathrm{Hom}_G(V, \Gamma(X, L)) \neq 0,$$

for all $L \in \mathrm{QCoh}(\mathcal{D}_X, G)$ supported on Z . In the above, it suffices to consider $L \in \mathrm{Coh}(\mathcal{D}_X, G)$; in fact we may assume that L is irreducible.

3. Compact generators in the derived category

3.1. Background: the equivariant derived category

Given an affine algebraic group G acting on a smooth quasi-projective algebraic variety X , we denote by $\mathbf{D}(X)^G$ the (unbounded) equivariant derived category of \mathcal{D} -modules (or equivalently, the derived category of \mathcal{D} -modules on the quotient stack $[X/G]$). This is a triangulated category with a t -structure whose heart is $\mathrm{QCoh}(\mathcal{D}_X, G)$. The definition may be found in [3, Subsect. 2.11] (see also [10, Vol. II, Part I, Chapt. 4] for another approach, applicable to a more general class of stacks). For the purposes of this paper, we will only need some formal properties, namely:

- Given a G -equivariant morphism $f : X \rightarrow Y$, we have a pair of functors

$$f_* : \mathbf{D}(X)^G \rightleftarrows \mathbf{D}(Y)^G : f^!$$

If f is an open embedding then $f^! = f^*$ is left adjoint to f_* , and if f is a closed embedding then $f^!$ is right adjoint to $f_* = f_!$.

- If $i : Z \hookrightarrow X$ is a G -equivariant closed embedding and $j : U \rightarrow X$ the complementary open embedding, then for each object $\mathcal{M} \in \mathbf{D}(X)^G$, there is an exact triangle:

$$i_*i^!\mathcal{M} \rightarrow \mathcal{M} \rightarrow j_*j^!\mathcal{M} \xrightarrow{[1]}.$$

- There are adjoint induction and restriction functors:¹

$$\mathbf{ind} : \mathbf{QC}(X)^G \rightleftarrows \mathbf{D}(X)^G : \mathbf{res}$$

(\mathbf{ind} is left adjoint). Here, $\mathbf{QC}(X)^G$ denotes the derived category of G -equivariant quasi-coherent sheaves on X .

3.2. Compact generators

Recall that an object d of a triangulated category \mathbf{D} is called compact if $\mathrm{Hom}_{\mathbf{D}}(d, -)$ preserves direct sums. A collection of objects $d_i, i \in I$, is said to generate \mathbf{D} if, for any non-zero object e of \mathbf{D} , we have

$$\mathrm{Hom}_{\mathbf{D}}(d_i, e[k]) \neq 0,$$

for some $i \in I$ and some integer $k \in \mathbb{Z}$ (denoting cohomological shift).

Now suppose G is a reductive group and X a smooth affine G -variety. Given a representation $V \in \mathrm{Rep}(G)$, there is a derived analogue of the modules $\mathcal{P}_X(V)$, which we denote $\mathbf{P}_X(V)$, defined by

$$\mathbf{P}_X(V) = \mathbf{ind}(V \otimes \mathcal{O}_X).$$

As in the abelian category setting, the objects $\mathbf{P}_X(V)$ form a set of compact generators of $\mathbf{D}(X)^G$ as V ranges over finite dimensional representations of G . In fact, this holds even under the weaker assumptions that X is quasi-affine and G a general affine algebraic group (as the objects $\mathcal{O}_X \otimes V$ still form a set of compact generators for $\mathbf{QC}(X)^G$ in this generality).

The following observation is immediate from the defining adjunction properties of $\mathbf{P}_X(V)$ and $\mathcal{P}_X(V)$, together with the fact that the object $V \in \mathrm{Rep}(G)$ is projective.

Lemma 6. *For any $V \in \mathrm{Rep}(G)$ and $\mathcal{M} \in \mathrm{QCoh}(\mathcal{D}_X, G)$, we have:*

$$\mathrm{Hom}_{\mathbf{D}(X)^G}(\mathbf{P}_X(V), \mathcal{M}[k]) \cong \begin{cases} \mathrm{Hom}_{\mathrm{Rep}(G)}(V, \mathcal{M}), & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, we obtain the following result relating generators in the abelian and derived categories:

¹In fact, there are two such pairs of functors corresponding to the realization of $\mathbf{D}(X)^G$ as either left or right \mathcal{D} -modules.

Proposition 7. *Suppose $V \in \text{Rep}(G)$ is a finite dimensional representation with the property that $\mathbf{P}_X(V)$ is a (compact) generator of $\mathbf{D}(X)^G$. Then $\mathcal{P}_X(V)$ is a (compact, projective) generator of $\text{QCoh}(\mathcal{D}_X, G)$.*

It follows that, to prove Theorem 1, it suffices to prove the derived analogue, Theorem 2.

3.3. Proof of Theorem 2

The basic idea behind the proof is to stratify the variety X into G -stable, locally closed, subvarieties X_i , each of which admits a compact generator for the equivariant derived category. Then we use this collection to build a compact generator for X .

Proposition 8. *Let X be a smooth affine G -variety. There is a sequence of G -stable, closed, subvarieties*

$$Y_0 \subseteq \dots \subseteq Y_n = X,$$

where the successive complements $X_i = Y_i - Y_{i-1}$ are quasi-affine, smooth subvarieties with the following property:

- *there is a finite dimensional representation $W_i \in \text{Rep}(G)$ such that the object $\mathbf{P}_{X_i}(W_i) \in \mathbf{D}(X_i)^G$ is a compact generator.*

Proposition 8 is a special case of Proposition 10 applied to the quotient stack $[X/G]$.

Remark 7. One can also prove Proposition 8 using the classical theory of Luna slices [18] and Hesselink’s stratification of the nullcone [15, Thm. 4.7].

For each $i = 0, \dots, n$, let $\alpha_i: Y_i \hookrightarrow X$ denote the closed embedding, $\hat{\alpha}_i: X - Y_i \hookrightarrow X$ the complementary open embedding, and $\beta_i: X_i \hookrightarrow X$ the locally closed embedding. We can now find a suitable representation V as follows. Let $Q_i \in \mathbf{D}(X)^G$ be a compact object extending $\mathbf{P}_{X_i}(W_i)$; that is, $\beta_i^!(Q_i) = \mathbf{P}_{X_i}(W_i)$. For example, one can take the induced \mathcal{D} -module associated to the coherent sheaf $W_i \otimes \mathcal{O}_{Y_i}$. As with any finite collection of compact objects, we can find a single finite dimensional representation V such that $Q_i \in \langle \mathbf{P}_X(V) \rangle$ for each $i = 0, \dots, n$. In other words, each Q_i can be expressed as a direct summand of a complex of copies of $\mathbf{P}_X(V)$.

Now suppose that $\mathcal{M} \in \mathbf{D}(X)^G$ is such that $R\text{Hom}_{\mathbf{D}(X)^G}(\mathbf{P}_X(V), \mathcal{M}) \simeq 0$. By construction, we must have $R\text{Hom}_{\mathbf{D}(X)}(Q_i, \mathcal{M}) \simeq 0$ for all $i = 0, \dots, n$. We will show by induction that $\alpha_{i*}\alpha_i^!(\mathcal{M}) \simeq 0$ for each $i = 0, \dots, n$.

To begin with, note that $\alpha_0 = \beta_0$ is a closed inclusion of a smooth subvariety Y_0 , and $Q_0 = (\alpha_0)_*\mathbf{P}_{Y_0}(W_0)$. There is an exact triangle:

$$(\alpha_0)_*(\alpha_0)^!\mathcal{M} \rightarrow \mathcal{M} \rightarrow (\hat{\alpha}_0)_*(\hat{\alpha}_0)^!\mathcal{M} \xrightarrow{[1]}.$$

Applying the functor $R\text{Hom}(Q_0, -)$ to this triangle, and noting that the middle term (by assumption) and the right hand term (as Q_0 is supported on Y_0) must vanish, we see that

$$0 \simeq R\text{Hom}(Q_0, (\alpha_0)_*(\alpha_0)^!\mathcal{M}) \simeq R\text{Hom}(\mathbf{P}_{X_0}(W_0), (\alpha_0)^!\mathcal{M}).$$

Thus $(\alpha_0)!\mathcal{M} \simeq 0$ as desired.

Now we want to show that $(\alpha_i)!\mathcal{M} \simeq 0$ for $i > 0$. First we have the exact triangle

$$(\alpha_i)_*(\alpha_i)!\mathcal{M} \rightarrow \mathcal{M} \rightarrow (\hat{\alpha}_i)_*(\hat{\alpha}_i)!\mathcal{M} \xrightarrow{[1]}.$$

Applying $R\mathrm{Hom}_{\mathbf{D}(X)}(Q_i, -)$ as before, we see that the middle- and right-hand terms vanish. Thus,

$$R\mathrm{Hom}_{\mathbf{D}(X)}(Q_i, (\alpha_i)_*(\alpha_i)!\mathcal{M}) \simeq 0.$$

On the other hand, there is another exact triangle

$$(\alpha_{i-1})_*(\alpha_{i-1})!\mathcal{M} \rightarrow (\alpha_i)_*(\alpha_i)!\mathcal{M} \rightarrow (\beta_i)_*(\beta_i)!\mathcal{M} \xrightarrow{[1]}.$$

By the inductive hypothesis, the left-hand term vanishes, and hence the right-hand arrow is an isomorphism. Thus, we have that

$$\begin{aligned} 0 &\simeq R\mathrm{Hom}_{\mathbf{D}(X)^G}(Q_i, (\alpha_i)_*(\alpha_i)!\mathcal{M}) \\ &\simeq R\mathrm{Hom}_{\mathbf{D}(X)^G}(Q_i, (\beta_i)_*(\beta_i)!\mathcal{M}) \\ &\simeq R\mathrm{Hom}_{\mathbf{D}(X)^G}(\mathbf{P}_{X_i}(W_i), (\beta_i)!\mathcal{M}) \end{aligned}$$

where the last isomorphism uses that everything is supported on Y_i and hence β_i behaves as an open embedding. By the assumption on W_i , we see that $\beta_i!\mathcal{M} \simeq 0$, and consequently $(\alpha_i)!\mathcal{M} \simeq 0$, as required.

4. Compact generators for the category of \mathcal{D} -modules on a finite type Artin stack

In this section we give a proof of Theorem 3. This proof is very similar to that of Theorem 2, but expressed in more abstract terms. First we record the following:

Lemma 9 ([7, Lem. 10.3.9]). *Given a finite type Artin stack \mathcal{Y} with affine diagonal, there is a diagram*

$$\mathcal{Y} \supset \mathring{\mathcal{Y}} \leftarrow \mathcal{Z} \rightarrow X \times BG,$$

where:

- $\mathring{\mathcal{Y}} \supset \mathcal{Y}$ is a non-empty open substack,
- $\mathcal{Z} \rightarrow \mathring{\mathcal{Y}}$ is a finite étale covering,
- X is a scheme and G a reductive group,
- The morphism $\mathcal{Z} \rightarrow X \times BG$ is a unipotent gerbe.

Proposition 10. *Suppose \mathcal{X} is a finite type Artin stack with affine diagonal. Then there is a sequence of closed substacks*

$$\mathcal{Y}_0 \subseteq \dots \subseteq \mathcal{Y}_n = \mathcal{X},$$

such that the successive complements $\mathcal{X}_i := \mathcal{Y}_i - \mathcal{Y}_{i-1}$ have the property that $\mathbf{D}(\mathcal{X}_i)$ admits a compact generator.

Proof. We define the substacks \mathcal{Y}_i and \mathcal{X}_i inductively as follows. We set $\mathcal{Y}_n = \mathcal{X}$. Then define \mathcal{X}_i to be the open substack \mathcal{Y}_i guaranteed by Lemma 9 and $\mathcal{Y}_{i-1} = \mathcal{Y}_i - \mathcal{X}_i$ to be the complementary (reduced) closed substack.

Now set $\mathcal{Y} = \mathcal{Y}_i$ and use the notation in Lemma 9. By Lemma 11 below, $\mathbf{D}(X \times BG)$ admits a compact generator (we take the external tensor product of the compact generator in each factor). As the morphism $\mathcal{Z} \rightarrow X \times BG$ is a unipotent gerbe, it induces an equivalence $\mathbf{D}(\mathcal{Z}) \simeq \mathbf{D}(X \times BG)$ (see [7, Lem. 10.3.6]). Finally, as f is a finite étale cover, the functor $f^! \simeq f^* : \mathbf{D}(\mathcal{Y}) \rightarrow \mathbf{D}(\mathcal{Z})$ is conservative, and thus f_* takes the compact generator of $\mathbf{D}(\mathcal{Z})$ to the compact generator of $\mathbf{D}(\mathcal{Y})$, as required. \square

Lemma 11. *The DG category $\mathbf{D}(\mathcal{X})$ has a compact generator in the following cases:*

- (1) $\mathcal{X} = X$ is a finite type scheme;
- (2) $\mathcal{X} = BG$ is the classifying stack of an affine algebraic group.

Proof. (1) First we recall that the derived category $\mathbf{QC}(X)$ of a quasi-coherent sheaves has a compact generator \mathcal{F} by [5, Thm. 3.1.1]. By [9, Prop. 3.4.11], the forgetful functor $\mathbf{res} : \mathbf{D}(X) \rightarrow \mathbf{QC}(X)$ is conservative and colimit preserving. It follows that the left adjoint \mathbf{ind} takes the compact generator \mathcal{F} to a compact generator of $\mathbf{D}(X)$.

(2) It is well known that $\pi_*(\mathcal{G}_{pt})$ is a compact generator, where $\pi : pt \rightarrow BG$ is the tautological G -bundle. Indeed, we have $R\mathrm{End}_{\mathbf{D}(BG)}(\pi_*(\mathcal{G}_{pt})) \simeq C_*(G)$ and thus $\mathbf{D}(BG) \simeq C_*(G)\text{-Mod}$ is identified with the category of dg modules for the dg algebra of chains on G (see [7, Sect. 7.2.2]). \square

The inductive step is then taken care of by the following result.

Lemma 12. *Let \mathcal{Z} be an closed substack of \mathcal{X} and \mathcal{U} the open complement. If the DG categories $\mathbf{D}(\mathcal{U})$ and $\mathbf{D}(\mathcal{Z})$ have a compact generator then so does $\mathbf{D}(\mathcal{X})$.*

Proof. Let $j : \mathcal{U} \hookrightarrow \mathcal{X}$ and $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ denote the inclusion morphisms. By assumption, $\mathbf{D}(\mathcal{U})$ has a compact generator \mathcal{M} , and $\mathbf{D}(\mathcal{Z})$ has a compact generator \mathcal{N} . By [7, Thm. 0.2.2], $\mathbf{D}(\mathcal{X})$ has a collection of compact generators $\{L_i\}_{i \in I}$ indexed by a set I . It follows immediately from the adjunction properties that $\{j^*(L_i)\}_{i \in I}$ form a set of compact generators of $\mathbf{D}(\mathcal{U})$. Moreover, as the object \mathcal{M} is compact, it must be contained in the Karoubian-triangulated envelope of some finite subset of generators $\{j^*L_{i_1}, \dots, j^*L_{i_n}\}$, which in turn must generate $\mathbf{D}(\mathcal{U})$.

We will show that the object

$$K := i_*\mathcal{N} \oplus L_{i_1} \oplus \dots \oplus L_{i_n}$$

is a compact generator of $\mathbf{D}(\mathcal{X})$. Indeed, given any object $S \in \mathbf{D}(\mathcal{X})$, there is an exact triangle

$$i_*i^!S \rightarrow S \rightarrow j_*j^*S \xrightarrow{[1]}.$$

We need to show that $R\mathrm{Hom}(K, S) \neq 0$. First suppose that $i^!S \neq 0$. Then

$$R\mathrm{Hom}(i_*\mathcal{N}, S) = R\mathrm{Hom}(\mathcal{N}, i^!S) \neq 0,$$

as \mathcal{N} is a generator of $\mathbf{D}(\mathcal{Z})$. Now suppose that $i^!S = 0$. Then $S \simeq j_*j^*S$, so we have

$$R\mathrm{Hom}(L_{i_1} \oplus \cdots \oplus L_{i_n}, S) \simeq R\mathrm{Hom}(j^*L_{i_1} \oplus \cdots \oplus j^*L_{i_n}, j^*S) \neq 0,$$

as $L_{i_1} \oplus \cdots \oplus L_{i_n}$ is a generator of $\mathbf{D}(\mathcal{U})$. In either case we have $R\mathrm{Hom}(K, S) \neq 0$ as required. \square

Theorem 3 now follows directly from Proposition 10 together with Lemma 12.

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